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Shift spaces and attractors in noninvertible horseshoes

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Abstract. As is well known, a horseshoe map, i.e. a special injective reimbedding of the unit square I^2 in $\mathbb{R}^2$ (or more generally, of the cube I^m in $\mathbb{R}^m$) as considered first by S. Smale [5], defines a shift dynamics on the maximal invariant subset of I^2 (or I^m). It is shown that this remains true almost surely for noninjective maps provided the contraction rate of the mapping in the stable direction is sufficiently strong, and bounds for this rate are given.

1. Definitions and results. For an integer $\theta \geq 2$ the set Σ_θ of all doubly infinite sequences $\underline{i} = (\ldots, i_{-1}, i_0, i_1, \ldots)$, where $i_l \in \{1, \ldots, \theta\}$, equipped with the metric

$$d((\ldots, i_{-1}, i_0, i_1, \ldots), (\ldots, j_{-1}, j_0, j_1, \ldots)) = \sum_{l=-\infty}^{\infty} 2^{-|l|} |i_l - j_l|$$

is a Cantor set. The shift mapping $\sigma : \Sigma_\theta \to \Sigma_\theta$ given by

$$\sigma(\ldots, i_{-1}, i_0, i_1, \ldots) = (\ldots, j_{-1}, j_0, j_1, \ldots) \quad \text{with } j_l = i_{l+1}$$

is a homeomorphism which defines a simple but nevertheless nontrivial dynamics on Σ_θ; e.g. its periodic points are dense, and there are dense orbits. Therefore, to ask whether or not a given discrete dynamical system contains a subsystem conjugate to a shift space of this kind is a natural question.

Let R be a topological space with metric d, R^* a compact subset of R and $f : R^* \to R$ continuous. For $k \geq 1$ we define the compact sets

$$R_k^* = \{p \in R \mid f^k(p) \text{ is defined}\}, \qquad A_k = f^k(R_k^*).$$

Then $R_1^* = R^* \supset R_2^* \supset R_3^* \supset \ldots$, $A_1 \supset A_2 \supset \ldots$, and we consider the compact sets

$$R_\infty^* = \bigcap_{k=1}^{\infty} R_k^*, \quad A = \bigcap_{k=1}^{\infty} A_k, \quad Z = R_\infty^* \cap A.$$

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The set A, if not empty, can be regarded as a global attractor of f. Indeed, $f(A \cap R^*) = A$, and there is a sequence $\varepsilon_1 > \varepsilon_2 > \dots$ of real numbers tending to 0 such that for any $k > 1$ and any $p \in R_k^*$ we have $d(f^k(p), A) \leq \varepsilon_k$. The set Z is the maximal invariant subset of R, i.e. the maximal set on which f is defined, and $f(Z) = Z$.

A subset S of R^* will be called a *shift space in* R if for some $\theta \geq 2$ there is a homeomorphism $h : \Sigma_\theta \to S$ such that $h\sigma = fh$. Obviously, if S is a shift space in R then $S \subset Z$. If Z itself is a shift space in R then we say that f *concentrates to a shift space*.

Among the best known examples of mappings which concentrate to a shift space are the so called horseshoe mappings (introduced by S. Smale in [5]) which can be defined as follows. Let $R_0 = \mathbb{R}^{m+1}$ $(m \geq 1)$ and let $R_0^* = I^{m+1} = I \times I^m$ be the $(m+1)$-dimensional unit cube in $\mathbb{R}^{m+1}$ which is regarded as the cartesian product of the unit interval $I = [0,1]$ with the m-dimensional unit cube. To define a horseshoe mapping we fix disjoint subintervals $I_1, \dots, I_\theta$ in I $(\theta \geq 2)$ and choose $f : R_0^* \to R_0$ so that the following conditions are satisfied, where $I^* = I_1 \cup \dots \cup I_\theta$:

(i) $f(R_0^*) \cap R_0^* = f(I^* \times I^m)$.

(ii) For some $\lambda \in (0,1)$ there are a C^1 mapping $\varphi : I^* \to I$ whose restriction to each component I_i of I^* is an expanding C^1 mapping onto I and a C^0 mapping $\psi : I^* \to [0, 1-\lambda]^m$ such that

$$f(t,x) = (\varphi(t), \psi(t) + \lambda x) \qquad ((t,x) \in I^* \times I^m). \tag{1}$$

(iii) f is injective on $I^* \times I^m$.

(See Fig. 1, where $m = 2$, $\theta = 3$.)

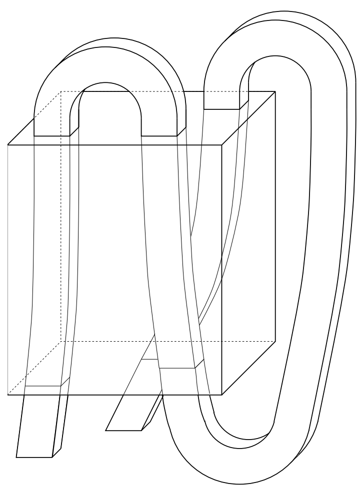

Fig. 1

It is well known (and not hard to prove) that f concentrates to a shift space Z. Moreover, the global attractor A of f is homeomorphic to the cartesian product $I \times C^0$ of I with a Cantor set C^0, and each component of A is a C^0 arc running upwards from the bottom $\{0\} \times I^m$ of R_0^* to the top $\{1\} \times I^m$. These facts remain true for more general mappings f (see e.g. [3], Ch. III), but they may fail to hold if (iii) is dropped from our assumptions (see Fig. 2, where $m = 2$, $\theta = 2$).

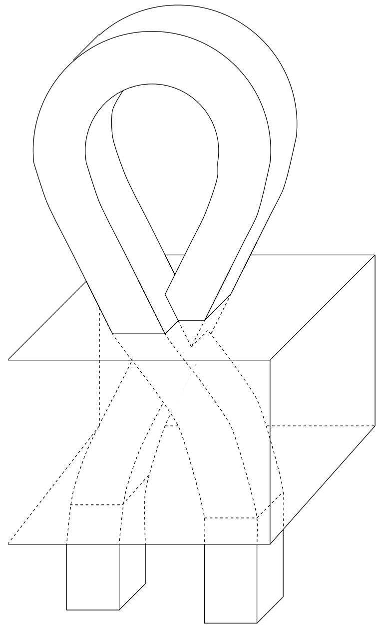

Fig. 2

This paper is concerned with mappings f satisfying (i) and (ii). If θ and φ are fixed we shall show that for "almost all" ψ the mapping f concentrates to a shift space and A has the structure mentioned above even if f is not injective on $I^* \times I^m$, provided λ is sufficiently small.

A natural technical simplification in the definition is obtained by neglecting the part of $R_0 = \mathbb{R}^{m+1}$ outside $R_0^* = I \times I^m$, i.e., we shall start with $R = I \times I^m$, $R^* = I^* \times I^m$ and the restriction of the original f to $f : R^* \to R$. Then the whole mapping f is defined by (1). We shall assume that θ, I^*, $\varphi : I^* \to I$ and $\lambda \in (0,1)$ are fixed while $\psi : I^* \to [0, 1-\lambda]^m$ is variable. Then f is determined by ψ, and sometimes instead of f we shall write f_ψ. The interval $[0, 1-\lambda]$ will be denoted by J.

The maximal subset I_k^* of I on which φ^k is defined ($k = 0, 1, 2, \ldots$) consists of θ^k disjoint intervals, where $I_0^* = I \supset I_1^* = I^* \supset I_2^* \supset I_3^* \supset \ldots,$

and

$$I_\infty^* = \bigcap_{k=0}^{\infty} I_k^*$$

is a Cantor set in I. The Hausdorff dimension $\dim_{\mathrm{H}} I_\infty^*$ of I_∞^* coincides with the box counting dimension $\dim_{\mathrm{B}} I_\infty^*$ (see [2]) and will be denoted by d^*. The end points of the interval I_i $(1 \le i \le \theta)$ will be denoted by s_i, t_i so that $\varphi(s_i) = 0$ and $\varphi(t_i) = 1$.

To avoid considerable technical difficulties (as e.g. piecewise linear approximations of φ and ψ) Theorem 1, Theorem 2 and its corollaries will be restricted to the piecewise linear case; i.e., we shall assume that the restrictions of φ and ψ to the components I_i of I^* are linear mappings onto I or into $[0, 1-\lambda]^m$, respectively. (See [1], where for nonlinear mapppings in a similar situation the attractor A is considered. Indeed, using the techniques applied there, facts analogous to those stated as Corollary 1 and Corollary 2 can be proved in the nonlinear case provided "full measure in $J^{2\theta m}$" is replaced by "open and dense in the space of all C^0 mappings $\psi : I^* \to J^m$".)

Now we consider the piecewise linear case. Here d^* is determined by $|I_1|^{d^*} + \ldots + |I_\theta|^{d^*} = 1$, where $|I_i|$ denotes the length of I_i. Since the mappings ψ are linear on each interval I_i they are completely determined by the 2θ points $a_i = \psi(s_i)$, $b_i = \psi(t_i)$ in J^m or, equivalently, by the point $(a_1, b_1, a_2, b_2, \ldots, a_\theta, b_\theta)$ in $J^{2\theta m}$. So all possible mappings ψ are in 1-to-1 correspondence with the points in $J^{2\theta m}$, and we shall not distinguish between ψ and the corresponding point.

The following sets will play an important role in the piecewise linear case. (A denotes the global attractor of f_ψ.)

$$\begin{aligned} \Psi &= \{\psi \in J^{2\theta m} \mid f_\psi \text{ does not concentrate to a shift space}\}, \\ \Psi_A &= \{\psi \in J^{2\theta m} \mid f_\psi|_{A\cap R^*} \text{ is not injective}\}. \end{aligned}$$

In Section 2 (Proposition 3) we shall see that Ψ and Ψ_A are compact, $\Psi \subset \Psi_A$ and that for $\psi \in J^{2\theta m} \setminus \Psi_A$ the global attractor A of f_ψ is homeomorphic to the cartesian product of an interval with a Cantor set. Moreover, since $A\cap R^*$ is compact and $f_\psi(A\cap R^*) = A$, for each $\psi \in J^{2\theta m} \setminus \Psi_A$ the restriction $f|_{A\cap R^*} : A\cap R^* \to A$ is a homeomorphism. The main results of this paper are stated in the following two theorems concerning the Hausdorff dimensions of Ψ and Ψ_A.

THEOREM 1. *If* $\lambda < 1/2$ *then*

$$\dim_{\mathrm{H}} \Psi \le 2\theta m - \left(m - d^* - \frac{2\log\theta}{\log(1/\lambda)}\right).$$

THEOREM 2. *If $\lambda < 1/2$ then*

$$\dim_{\mathrm{H}} \Psi_A \le 2\theta m - \left(m - 1 - \frac{2\log\theta}{\log(1/\lambda)}\right).$$

COROLLARY 1. *If $\lambda < \theta^{-2/(m-d^*)}$ and $\lambda < 1/2$, then the set of all those $\psi \in J^{2\theta m}$ for which f_ψ concentrates to a shift space is open in $J^{2\theta m}$ and has full measure $(1-\lambda)^{2\theta m}$.*

COROLLARY 2. *If $m > 1$, $\lambda < \theta^{-2/(m-1)}$ and $\lambda < 1/2$, then for all ψ in an open subset of $J^{2\theta m}$ with full measure $(1-\lambda)^{2\theta m}$ the global attractor A of f_ψ is the cartesian product of an interval with a Cantor set, and $f_\psi|_{A\cap R^*} : A \cap R^* \to A$ is a homeomorphism.*

This corollary can be regarded as a partial answer to a problem of F. Przytycki in [4].

P r o o f   o f   t h e   c o r o l l a r i e s. In these cases $\dim_{\mathrm{H}} \Psi < 2\theta m$ or $\dim_{\mathrm{H}} \Psi_A < 2\theta m$, respectively, and, by Proposition 3 of Section 2, Ψ and Ψ_A are compact. ∎

Propositions 1 and 2 in Section 2 will yield some further details.

R e m a r k 1. Our condition $\lambda < 1/2$ reflects the fact that two m-dimensional cubes in I^m of edge length at least $1/2$ and with edges parallel to those of I^m must intersect. We do not know whether it is necessary. (Here it is essentially used only in the proof of Lemma 1.)

R e m a r k 2. We do not know whether the bounds for $\dim_{\mathrm{B}} \Psi$ and $\dim_{\mathrm{B}} \Psi_A$ in the theorems are sharp. As is easily seen all points

$$\psi = \left(a_1, a_1, a_2, \frac{1}{t}a_1 + \left(1 - \frac{1}{t}\right)a_2, a_3, b_3, \ldots, a_\theta, b_\theta\right)$$

belong to Ψ if $t \in I^*_\infty \setminus \{0\}$ and to Ψ_A if $t \in (0, 1]$. Therefore

$$\dim_{\mathrm{H}} \Psi \ge 2\theta m - (2m - d^*), \quad \dim_{\mathrm{H}} \Psi_A \ge 2\theta m - (2m - 1),$$

but these lower bounds are rather weak, and they do not depend on λ.

The fact stated in Theorem 3 below seems to be more interesting than Remark 2. Though for small m the bounds for θ and λ are by no means exciting (owing to the factor 12 in the theorem) this theorem shows that the exponent $-2/(m-1)$ in Corollary 2 is optimal at least if m is odd. Here it is possible, and by the topological methods used in the proof even natural, to consider the general case, where φ and ψ are not necessarily linear. (Concerning the piecewise linear case see Remark 4 below.) With m, λ, I^* and $\varphi : I^* \to I$ fixed (φ not necessarily piecewise linear) we consider the space $\mathcal{F}$ of all C^0 mappings $\psi : I^* \to J^m$ with the subspace $\mathcal{G}$ consisting of all those $\psi \in \mathcal{F}$ for which the restriction of f_ψ to $A \cap R^*$ is not injective and the attractor A does not have the structure mentioned above. Therefore $\mathcal{G}$ corresponds to the set Ψ_A in the restricted case. By Proposition 3 of Section 2 the set $\mathcal{G}$ is closed in $\mathcal{F}$.

THEOREM 3. *If $m \geq 3$ is odd and $\lambda > 12\theta^{-2/(m-1)}$ then $\mathcal{G}$ has interior points.*

For $m \geq 3$ and $\theta \geq 2$ let $\alpha(m,\theta)$ be the infimum of all real $\alpha' > 0$ with the following property. For each I^* consisting of θ components, each $\varphi : I^* \to I$ (not necessarily piecewise linear) and each $\lambda \in (\alpha'\theta^{-2/(m-1)}, 1)$ the set $\mathcal{G}$ has interior points in $\mathcal{F}$. Then Theorem 3 is equivalent to

$$\alpha(m,\theta) \leq 12 \quad (m \geq 3 \text{ odd}). \tag{2}$$

Remark 3. Besides (2) the proof of Theorem 3 will show for odd positive integers m,

$$\lim_{m\to\infty} \lim_{\theta\to\infty} \alpha(m,\theta) \leq 8, \tag{3}$$

so that for each $\varepsilon > 0$ the factor 12 in the theorem can be replaced by $8 + \varepsilon$ provided m and θ are sufficiently large.

Remark 4. The proof of Theorem 3 can be modified to show that for m, θ, λ as in the theorem the set $\mathcal{G} \cap J^{2\theta m}$ ($J^{2\theta m} \subset \mathcal{F}$ in the obvious way) has interior points in $J^{2\theta m}$.

2. Preliminaries. For integers $\theta \geq 2$ and $k' \leq k''$ let $\theta^{[k',k'']}$ be the set of all sequences $(i_{k'}, i_{k'+1}, \ldots, i_{k''})$ where $i_l \in \{1, \ldots, \theta\}$, and let $\theta^{[-\infty,k'']}$, $\theta^{[k',\infty]}$, $\theta^{[-\infty,\infty]}$ consist of the sequences which are infinite to the left, to the right or in both directions, respectively. So $\theta^{[-\infty,\infty]}$ coincides with the Cantor set Σ_θ of Section 1, and $\theta^{[-\infty,k'']}$, $\theta^{[k',\infty]}$ also have a natural Cantor set structure. The shift map $\sigma : \theta^{[k',k'']} \to \theta^{[k'-1,k''-1]}$ is defined in the obvious way.

As in Section 1 we assume that $I^* = I_1 \cup \ldots \cup I_\theta$ ($\theta \geq 2$) is the union of θ disjoint closed subintervals of I and that $\varphi : I^* \to I$ is a C^1 mapping whose restrictions to the intervals I_i are expanding mappings onto I. Moreover, for some continuous $\psi : I^* \to J^m$ let $f : R^* = I^* \times I^m \to R = I \times I^m$ be defined by (1).

The θ^k components of the domain I_k^* of φ^k ($k \geq 1$) will be denoted by $I_{\underline{i}}$ ($\underline{i} \in \theta^{[1,k]}$), where the indices are chosen so that for $k > 1$,

$$I_{(i_1,\ldots,i_k)} \subset I_{(i_1,\ldots,i_{k-1})}, \quad \varphi(I_{(i_1,\ldots,i_k)}) = I_{(j_1,\ldots,j_{k-1})}, \quad \text{where } j_l = i_{l+1}.$$

For $\underline{i} = (i_1, i_2, \ldots) \in \theta^{[1,\infty]}$ the intersection $\bigcap_{k=1}^{\infty} I_{(i_1,\ldots,i_k)}$ contains exactly one point which will be denoted by $t_{\underline{i}}$. The sets $R_{\underline{i}} = I_{\underline{i}} \times I^m$ ($\underline{i} \in \theta^{[1,k]}$,

$1 \le k < \infty$) are slices of $R = I \times I^m$ while for $\underline{i} = (i_1, i_2, \ldots) \in \theta^{[1,\infty]}$,

$$R_{\underline{i}} = \bigcap_{k=1}^{\infty} R_{(i_1,\ldots,i_k)}$$

is the m-dimensional cube $\{t_{\underline{i}}\} \times I^m$.

For $\underline{i} \in \theta^{[1,k'']}$ ($1 \le k'' \le \infty$) and $1 \le k' \le k''$, $k' < \infty$, the image $f^{k'}(R_{\underline{i}})$ is well defined and will be denoted by $R_{\sigma^{k'}(\underline{i})}$. So $R_{\underline{i}}$ is now defined for all $\underline{i} \in \theta^{[k',k'']}$ provided $k' \le k''$, and $-\infty < k' \le 1$ and $0 \le k'' \le \infty$. By setting

$$R_{\underline{i}} = \bigcap_{k=0}^{-\infty} R_{(i_k,\ldots,i_0,\ldots)}$$

for $\underline{i} = (\ldots, i_{-1}, i_0, \ldots) \in \theta^{[-\infty,k'']}$ ($0 \le k'' \le \infty$) we include the case $k' = -\infty$ into our definition.

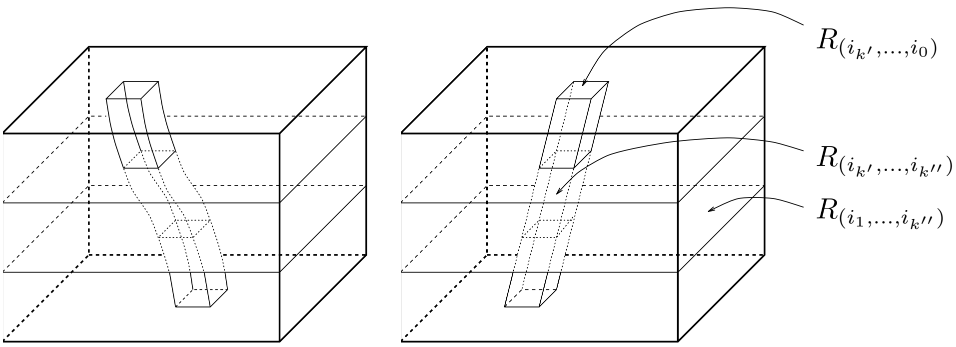

Fig. 3

For k' and k'' finite with $k' \le 0$ the set $R_{\underline{i}}$ is an $(m+1)$-dimensional curved prism over an m-dimensional cube with edge length $\lambda^{-k'+1}$, which for $k'' = 0$ has its bottom in $\{0\} \times I^m$ and its top in $\{1\} \times I^m$, while for $k' \le 0$ and $k'' \ge 1$,

$$R_{(i_{k'},\ldots,i_{k''})} = R_{(i_{k'},\ldots,i_0)} \cap R_{(i_1,\ldots,i_{k''})}$$

(see Fig. 3). In the piecewise linear case all these prisms are straight. For $\underline{i} \in \theta^{[-\infty,0]}$ the set $R_{\underline{i}}$ is an arc (or a straight segment in the piecewise linear case) running upwards from a point in $\{0\} \times I^m$ to a point on $\{1\} \times I^m$, and if $\underline{i} \in \theta^{[-\infty,\infty]}$ then $R_{\underline{i}}$ contains exactly one point which will be denoted by $p_{\underline{i}}$. As is easily seen,

$$f(R_{\underline{i}}) = R_{\sigma(\underline{i})} \tag{4}$$

wherever $R_{\underline{i}}$ and $R_{\sigma(\underline{i})}$ are defined. Moreover, $R_{\underline{j}} \subset R_{\underline{i}}$ provided $\underline{i}$ is a part of $\underline{j}$, i.e., if $\underline{i}$ can be obtained from $\underline{j}$ by cancelling digits on one or both ends. The domain of f^k $(k \geq 1)$ is

$$R_k^* = I_k^* \times I^m = \bigcup_{\underline{i} \in \theta^{[1,k]}} R_{\underline{i}},$$

and

$$R_\infty^* = I_\infty^* \times I^m = \bigcap_{k=1}^{\infty} R_k^*$$

is the maximal set on which all iterations f^k $(k \geq 1)$ are defined.

The global attractor of f is given by

$$A = \bigcup_{\underline{i} \in \theta^{[-\infty,0]}} R_{\underline{i}}.$$

The maximal invariant set of f is

$$Z = \bigcup_{\underline{i} \in \theta^{[-\infty,\infty]}} R_{\underline{i}},$$

i.e., Z consists of the points $p_{\underline{i}}$ $(\underline{i} \in \theta^{[-\infty,\infty]})$, and by setting $h(\underline{i}) = p_{\underline{i}}$ we get a surjective mapping $h : \Sigma_\theta = \theta^{[-\infty,\infty]} \to Z$. As is easily seen, h is continuous, and (4) implies $h\sigma = fh$. For $t \in I$ and $\underline{i} \in \theta^{[-\infty,0]}$ we define $g(t,\underline{i})$ to be the intersection point of $\{t\} \times I^m$ and $R_{\underline{i}}$. So we get a surjective continuous mapping $g : I \times \theta^{[-\infty,0]} \to A$.

PROPOSITION 1. *The following conditions are equivalent.*

(i) *f concentrates to a shift space.*
(ii) *$h : \Sigma_\theta \to Z$ is a homeomorphism.*
(iii) *If $\underline{i}, \underline{j} \in \theta^{[-\infty,0]}$, $\underline{i} \neq \underline{j}$, then $R_{\underline{i}} \cap R_{\underline{j}} \cap R_\infty^* = \emptyset$.*

Proof. The equivalence of (ii) and (iii) is an immediate consequence of the following fact. If $\underline{i}^- = (\ldots, i_{-1}, i_0) \in \theta^{[-\infty,0]}$ then the mapping $h_{\underline{i}^-} : \theta^{[1,\infty]} \to R_{\underline{i}^-} \cap R_\infty^*$ given by $h_{\underline{i}^-}(i_1, i_2, \ldots) = h(\ldots, i_{-1}, i_0, i_1, \ldots)$ is a homeomorphism.

The implication (ii)⇒(i) follows from (4).

To complete the proof we assume (i) and prove (ii). Since Σ_θ is compact and h is surjective it is sufficient to show that h is injective.

If for $\underline{i} = (\ldots, i_{-1}, i_0, i_1, \ldots), \underline{j} = (\ldots, j_{-1}, j_0, j_1, \ldots) \in \theta^{[-\infty,\infty]}$ the positive halves $\underline{i}^+ = (i_1, i_2, \ldots)$ and $\underline{j}^+ = (j_1, j_2, \ldots)$ are different, then $h(\underline{i}) \in R_{\underline{i}^+}$, $h(\underline{j}) \in R_{\underline{j}^+}$, $R_{\underline{i}^+} \cap R_{\underline{j}^+} = \emptyset$ implies $h(\underline{i}) \neq h(\underline{j})$. If $\underline{i}^+ = \underline{j}^+$ but $\underline{i} \neq \underline{j}$ then for some $k < 0$ the positive halves $\sigma^k(\underline{i})^+$ and $\sigma^k(\underline{j})^+$ of $\sigma^k(\underline{i})$ and $\sigma^k(\underline{j})$ will differ, and we get

$$h(\sigma^k(\underline{i})) \neq h(\sigma^k(\underline{j})).$$

By (i), $f|_Z : Z \to Z$ is a homeomorphism, and $(f|_Z)^k h = h\sigma^k$ for our negative exponent k. So we get $(f|_Z)^k h(\underline{i}) \neq (f|_Z)^k h(\underline{j})$ and therefore $h(\underline{i}) \neq h(\underline{j})$. ∎

PROPOSITION 2. *The following conditions are equivalent.*

(i) $f|_{A\cap R^*} : A \cap R^* \to A$ *is a homeomorphism.*
(ii) $g : I \times \theta^{[-\infty,0]} \to A$ *is a homeomorphism.*
(iii) *If* $\underline{i}, \underline{j} \in \theta^{[-\infty,0]}$, $\underline{i} \neq \underline{j}$, *then* $R_{\underline{i}} \cap R_{\underline{j}} = \emptyset$.

Proof. Since g maps each interval $I \times \{\underline{i}\}$ injectively onto $R_{\underline{i}}$, the equivalence of (ii) and (iii) is obvious.

Now we prove (i)⇒(iii). By (i) for $k \geq 1$ the mapping $f^k : A \cap R_k^* \to A$ is a homeomorphism. To prove (iii) we show that for $\underline{i} = (\ldots, i_{-1}, i_0), \underline{j} = (\ldots, j_{-1}, j_0) \in \theta^{[-\infty,0]}$ the existence of a common point $p = (t, x)$ of $R_{\underline{i}}$ and $R_{\underline{j}}$ ($t \in I$, $x \in I^m$) implies $\underline{i} = \underline{j}$.

For $k \geq 1$ there is a unique $p^* = (t^*, x) \in A \cap R_k^*$ such that $f^k(p^*) = p$. Here $t^* \in I_{\underline{i}^*}$, where $\underline{i}^* = (i_1^*, \ldots, i_k^*) \in \theta^{[1,k]}$ with $i_l^* = i_{l-k} = j_{l-k}$ ($1 \leq l \leq k$). Since $k \geq 1$ is arbitrary this shows $i_n = j_n$ for all $n \leq 0$.

To prove (iii)⇒(i) we assume that all arcs $R_{\underline{i}}$ ($\underline{i} \in \theta^{[-\infty,0]}$) are disjoint. Then each component of $A \cap R^*$ is an arc $R_{\underline{i}} \cap R_i$ ($\underline{i} = (\ldots, i_{-1}, i_0) \in \theta^{[-\infty,0]}$, $1 \leq i \leq \theta$), and f maps this arc injectively onto $R_{\underline{j}}$, where $\underline{j} = (\ldots, j_{-1}, j_0) \in \theta^{[-\infty,0]}$ is given by $j_l = j_{l+1}$ if $l < 0$ and $j_0 = i$. So f is injective on each component of $A \cap R^*$, and by (iii) different components have disjoint images. Since $A \cap R^*$ is compact, injectivity of $f|_{A\cap R^*}$ together with $f(A \cap R^*) = A$ implies (i). ∎

PROPOSITION 3. Ψ *and* Ψ_A *are compact*, *and* $\mathcal{G}$ *is closed in* $\mathcal{F}$.

Proof. Since the proofs of the three assertions are similar we consider Ψ (in the piecewise linear case) only. For $\psi \in J^{2\theta m}$, $f = f_\psi : R^* \to R$ the corresponding mapping and $1 \leq i \leq \theta$ let $Z_i(\psi)$ denote the union of all $R_{\underline{i}} \cap R_\infty^*$, where $\underline{i} = (\ldots, i_{-1}, i_0) \in \theta^{[-\infty,0]}$ and $i_0 = i$. Obviously $Z_1(\psi), \ldots, Z_\theta(\psi)$ are compact and their union is the set Z belonging to f_ψ.

If f_ψ concentrates to a shift space then (by Proposition 1(iii)) the sets $Z_i(\psi)$ are disjoint. We show that the converse also holds. Suppose that $Z_1(\psi), \ldots, Z_\theta(\psi)$ are disjoint, and let $\underline{i} = (\ldots, i_{-1}, i_0, i_1, \ldots), \underline{j} = (\ldots, j_{-1}, j_0, j_1, \ldots) \in \theta^{[-\infty,\infty]}$, $\underline{i} \neq \underline{j}$, be given. We have to show that $h(\underline{i}) \neq h(\underline{j})$. If $i_l \neq j_l$ for some $l \geq 1$, then $h(\underline{i})$ and $h(\underline{j})$ lie in different components of R_∞^*, and $h(\underline{i}) \neq h(\underline{j})$ is obvious. Now we assume that $l_0 \leq 0$ is the maximal index with $i_{l_0} \neq j_{l_0}$. Then for $\underline{i}' = (\ldots, i'_{-1}, i'_0, i'_1, \ldots) = \sigma^{l_0}(\underline{i})$ and

$\underline{j}' = (\ldots, j'_{-1}, j'_0, j'_1, \ldots) = \sigma^{l_0}(\underline{j})$ we have $i'_0 \neq j'_0$ but $i'_l = j'_l$ if $l \geq 1$. The points $h(\underline{i}')$ and $h(\underline{j}')$ lie in the same component $\{t\} \times I^m$ of R^*_∞ but in different and therefore disjoint sets $Z_{i'_0}(\psi)$ and $Z_{j'_0}(\psi)$. So $h(\underline{i}') \neq h(\underline{j}')$, and since f^{-l_0} is injective on $\{t\} \times I^m$ this gives

$$h(\underline{i}) = h\sigma^{-l_0}(\underline{i}') = f^{-l_0}h(\underline{i}') \neq f^{-l_0}h(\underline{j}') = h\sigma^{-l_0}(\underline{j}') = h(\underline{j}).$$

To prove that Ψ is compact we show that each point $\psi \in J^{2\theta m} \setminus \Psi$ has a neighbourhood which does not intersect Ψ. If $\psi \notin \Psi$ the corresponding sets $Z_1(\psi), \ldots, Z_\theta(\psi)$ are disjoint and since they are compact there is a positive ε such that the distance between any two of them is at least ε. As is easily seen, the segments $R_{\underline{i}}$ ($\underline{i} \in \theta^{[-\infty,0]}$) depend continuously on ψ, and this continuity is uniform with respect to $\underline{i}$. Therefore, if $\psi' \in J^{2\theta m}$ is sufficiently close to ψ the sets $Z_i(\psi')$ belonging to ψ' are close to the sets $Z_i(\psi)$ and hence mutually disjoint. This proves $\psi' \notin \Psi$. ∎

3. Proof of Theorems 1 and 2. We assume that $\varphi : I^* \to I$, $\lambda \in (0, 1/2)$ and therefore θ, I^*_k ($1 \leq k \leq \infty$), $I_{\underline{i}}$, $R_{\underline{i}}$ ($\underline{i} \in \theta^{[1,k]}$, $1 \leq k \leq \infty$) and $t_{\underline{i}}$ ($\underline{i} \in \theta^{[1,\infty]}$) are fixed. Let H denote one of the sets I^*_∞ or I, and let $q^* = \dim_{\mathrm{H}} H = \dim_{\mathrm{B}} H$. We define

$$\Psi^* = \{\psi \in J^{2\theta m} \mid R_{\underline{i}}(\psi) \cap R_{\underline{j}}(\psi) \cap (H \times I^m) \neq 0 \quad \text{for at least one pair } \underline{i} \neq \underline{j} \in \theta^{[-\infty,0]}\},$$

where $R_{\underline{i}}(\psi)$ denotes the set $R_{\underline{i}}$ which is constructed with the mapping ψ. Looking at the equivalences between (i) and (iii) of the first two propositions in Section 2 we see that both theorems of Section 1 are combined in

$$\dim_{\mathrm{H}} \Psi^* \leq 2\theta m - \left(m - q^* - \frac{2\log\theta}{\log(1/\lambda)}\right). \tag{5}$$

We shall prove (5) at the end of this section after some lemmas are stated and proved.

Besides Ψ^*, for $1 \leq k < \infty$, $\underline{i} = (i_1, \ldots, i_k), \underline{j} = (j_1, \ldots, j_k) \in \theta^{[1,k]}$, $\underline{i} \neq \underline{j}$, we shall consider the sets

$$\begin{aligned} \Psi^*_{\underline{i},\underline{j}} &= \{\psi \in J^{2\theta m} \mid R_{\sigma^k(\underline{i})}(\psi) \cap R_{\sigma^k(\underline{j})}(\psi) \cap (H \times I^m) \neq \emptyset\}, \\ \Psi^*_k &= \bigcup_{\substack{\underline{i},\underline{j} \in \theta^{[1,k]} \\ i_k \neq j_k}} \Psi^*_{\underline{i},\underline{j}}. \end{aligned} \tag{6}$$

Since $R_{(l_{-k},\dots,l_0)} \subset R_{(l_{-k+1},\dots,l_0)}$, we have $\Psi_1^* \supset \Psi_2^* \supset \dots$, and together with

$$\Psi^* = \bigcap_{k=1}^{\infty} \bigcup_{\substack{\underline{i},\underline{j}\in\theta^{[1,k]}\\ \underline{i}\neq\underline{j}}} \Psi^*_{\underline{i},\underline{j}}$$

the proof of Proposition 3 implies

$$\Psi^* = \bigcap_{k=1}^{\infty} \Psi_k^*. \tag{7}$$

For $k \ge 1$ and $\underline{i},\underline{j} \in \theta^{[1,k]}$, $\underline{i} \neq \underline{j}$, we define the mapping

$$\pi_{\underline{i},\underline{j}} : J^{2\theta m} \to I^{4m} = (I^m)^4$$

by $\pi_{\underline{i},\underline{j}}(\psi) = (a,b,c,d)$, where the points $a,b,c,d \in I^m$ are determined by

$$\begin{aligned} f_\psi^k(s_{\underline{i}}, o) &= (0,a), \quad & f_\psi^k(t_{\underline{i}}, o) &= (1,b),\\ f_\psi^k(s_{\underline{j}}, o) &= (0,c), \quad & f_\psi^k(t_{\underline{j}}, o) &= (1,d), \end{aligned}$$

with $s_{\underline{i}}$, $t_{\underline{i}}$ the end points of $I_{\underline{i}}$ such that $\varphi^k(s_{\underline{i}}) = 0$ and $\varphi^k(t_{\underline{i}}) = 1$, and $o = (0,\dots,0) \in I^m$. Therefore $(0,a),(1,b)$ are the end points of the segment $f_\psi^k(I_{\underline{i}} \times \{o\})$ and $(0,c),(1,d)$ those of $f_\psi^k(I_{\underline{j}} \times \{o\})$. Moreover, the segments $[(0,a),(1,b)]$ and $[(0,c),(1,d)]$ are edges of the prisms $f^k(R_{\underline{i}}) = R_{\sigma^k(\underline{i})}$ and $f^k(R_{\underline{j}}) = R_{\sigma^k(\underline{j})}$, respectively, such that for $(t,y) \in [(0,a),(1,b)]$ and $(t,z) \in [(0,c),(1,d)]$ we have the cubes

$$\begin{aligned} R_{\sigma^k(\underline{i})} \cap (\{t\} \times I^m) &= \{t\} \times (y + [0,\lambda^k]^m),\\ R_{\sigma^k(\underline{j})} \cap (\{t\} \times I^m) &= \{t\} \times (z + [0,\lambda^k]^m). \end{aligned} \tag{8}$$

For $(a,b,c,d) \in (I^m)^4 = I^{4m}$ we define

$$\pi(a,b,c,d) = (c-a, d-b)$$

and get a mapping $\pi : I^{4m} \to [-1,1]^{2m}$. Finally, we consider the composition

$$\varrho_{\underline{i},\underline{j}} = \pi\pi_{\underline{i},\underline{j}} : J^{2\theta m} \to I^{2m}.$$

LEMMA 1. *There is a real $\alpha_1 > 0$ not depending on k, $\underline{i} = (i_1,\dots,i_k)$, $\underline{j} = (j_1,\dots,j_k) \in \theta^{[1,k]}$ such that for any measurable set X in I^{4m},*

$$\mathrm{vol}^{2\theta m}(\pi^{-1}_{\underline{i},\underline{j}}(X)) \le \alpha_1 \,\mathrm{vol}^{4m}(X)$$

provided $i_k \neq j_k$. (By vol^p we denote the p-dimensional Lebesgue measure in $\mathbb{R}^p$.)

LEMMA 2. *There is a real $\alpha_2 > 0$ such that for any measurable set X in $[-1,1]^{2m}$,*

$$\mathrm{vol}^{4m}(\pi^{-1}(X)) \le \alpha_2 \,\mathrm{vol}^{2m}(X).$$

COROLLARY. *There is a real $\alpha > 0$ not depending on k, $\underline{i} = (i_1,\dots,i_k)$, $\underline{j} = (j_1,\dots,j_k) \in \theta^{[1,k]}$ such that for any measurable set X in $[-1,1]^{2m}$,*

$$\mathrm{vol}^{2\theta m}(\varrho_{\underline{i},\underline{j}}^{-1}(X)) \le \alpha \,\mathrm{vol}^{2m}(X)$$

provided $i_k \ne j_k$.

Since the proof of Lemma 2 is trivial it is sufficient to prove Lemma 1.

Proof of Lemma 1. We start with the remark that $\pi_{\underline{i},\underline{j}}$ can be extended to a linear mapping $\overline{\pi}_{\underline{i},\underline{j}} : \mathbb{R}^{2\theta m} \to \mathbb{R}^{4m}$.

The proof will proceed as follows. We define a $4m$-dimensional linear subspace L of $\mathbb{R}^{2\theta m}$ (depending on $\underline{i}$, $\underline{j}$) such that $\overline{\pi}_{\underline{i},\underline{j}}|_L : L \to \mathbb{R}^{4m}$ is a linear isomorphism and that for any measurable set X in $\mathbb{R}^{4m}$ we have

$$\mathrm{vol}^{4m}((\overline{\pi}_{\underline{i},\underline{j}}|_L)^{-1}(X)) \le \alpha^* \,\mathrm{vol}^{4m}(X), \tag{9}$$

where

$$\alpha^* = \left(\frac{1-\lambda}{1-2\lambda}\right)^{4m}.$$

(This is the point where we need $\lambda < 1/2$.) Obviously $\overline{\pi}_{\underline{i},\underline{j}} = \overline{\pi}_{\underline{i},\underline{j}}|_L \pi^*$ with a linear projection $\pi^* : \mathbb{R}^{2\theta m} \to L$, and therefore, if $X \subset I^{4m}$ then

$$\begin{aligned}
\mathrm{vol}^{2\theta m}(\pi_{\underline{i},\underline{j}}^{-1}(X)) &= \mathrm{vol}^{2\theta m}(\overline{\pi}_{\underline{i},\underline{j}}^{-1}(X) \cap J^{2\theta m}) \\
&= \mathrm{vol}^{2\theta m}(\pi^{*-1}(\overline{\pi}_{\underline{i},\underline{j}}|_L)^{-1}(X) \cap J^{2\theta m}) \\
&\le (\mathrm{diam}\, J^{2\theta m})^{2\theta m - 4m}\, \mathrm{vol}^{4m}((\overline{\pi}_{\underline{i},\underline{j}}|_L)^{-1}(X)) \\
&\le (\mathrm{diam}\, J^{2\theta m})^{2\theta m - 4m} \alpha^* \,\mathrm{vol}^{4m}(X),
\end{aligned}$$

so that the lemma will be proved with

$$\alpha_2 = (\mathrm{diam}\, J^{2\theta m})^{2\theta m - 4m} \left(\frac{1-\lambda}{1-2\lambda}\right)^{4m},$$

provided (9) is proved.

Thinking of our identification of the mappings $\psi : I^* \to J^m$ with the points in $J^{2\theta m}$ we regard $J^{2\theta m}$ as $(J^m)^{2\theta}$ and its points as sequences $(a_1, b_1, \dots, a_\theta, b_\theta)$, where $a_i, b_i \in J^m$. Let $J^{4m}_{\underline{i},\underline{j}}$ denote the $4m$-dimensional face of $J^{2\theta m}$ consisting of all $(a_1, b_1, \dots, a_\theta, b_\theta)$ with $a_i = b_i = o$ for $i_k \ne i \ne j_k$. (Here i_k, j_k are the last digits of $\underline{i}$, $\underline{j}$, respectively, and o denotes the point $(0,\dots,0)$ in $\mathbb{R}^m$.) Then L is defined to be the $4m$-dimensional linear subspace of $\mathbb{R}^{2\theta m}$ which contains $J^{4m}_{\underline{i},\underline{j}}$.

Since $\overline{\pi}_{\underline{i},\underline{j}}$ is linear there is a real δ such that for any measurable Y in L we have

$$\mathrm{vol}^{4m}(\overline{\pi}_{\underline{i},\underline{j}}(Y)) = \delta\,\mathrm{vol}^{4m}(Y),$$

and, since $\mathrm{vol}^{4m}\, J^{4m}_{\underline{i},\underline{j}} = (1-\lambda)^{4m}$, to prove (9) it is sufficient to show that

$$\mathrm{vol}^{4m}(\overline{\pi}_{\underline{i},\underline{j}}(J^{4m}_{\underline{i},\underline{j}})) \geq \left(\frac{1-2\lambda}{1-\lambda}\right)^{4m}(1-\lambda)^{4m} = (1-2\lambda)^{4m}$$

or that $\overline{\pi}_{\underline{i},\underline{j}}(J^{4m}_{\underline{i},\underline{j}})$ contains the cube $Q = [\lambda, 1-\lambda]^{4m}$.

It will be convenient to identify L with $\mathbb{R}^{4m}$ via the mapping $L \to \mathbb{R}^{4m}$ which is obtained by neglecting in points $(x_1,\dots,x_{2\theta m}) = (a_1, b_1, \dots, a_\theta, b_\theta) \in L$ $(a_i, b_i \in \mathbb{R}^m)$ all coordinates not belonging to a_{i_k}, b_{i_k}, a_{j_k}, b_{j_k}. Then $J^{4m}_{\underline{i},\underline{j}} = J^{4m}$ and we have to show

$$\overline{\pi}_{\underline{i},\underline{j}}(J^{4m}) \supset Q. \tag{10}$$

Starting with the cube $Q^* = [0,\lambda]^{4m}$ for each vertex ψ of J^{4m} we define the cube $Q^*_\psi = \psi + Q^*$. By a simple geometric argument illustrated in Figure 4 it can be proved that any convex set which intersects all 2^{4m} cubes Q^*_ψ must contain Q. Therefore to prove (10) it is sufficient to show that for any vertex

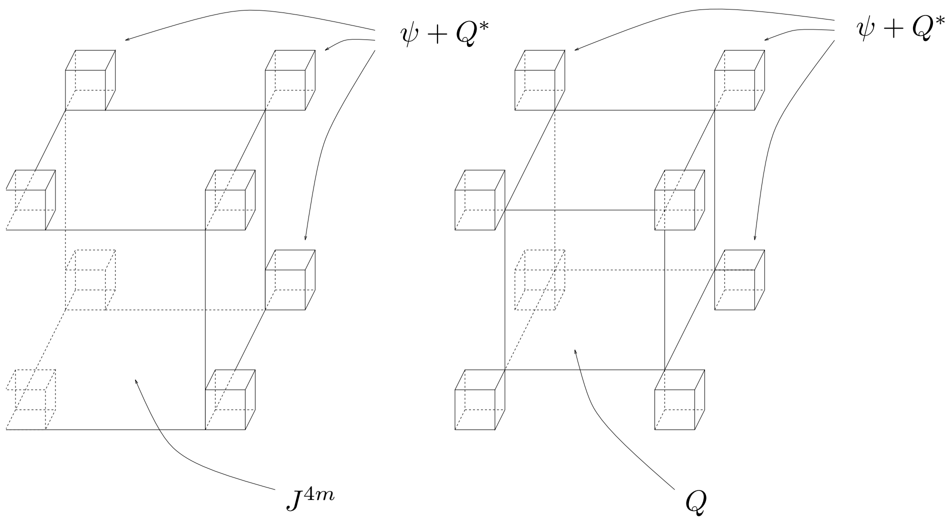

Fig. 4

ψ of J^{4m}, $\overline{\pi}_{\underline{i},\underline{j}}(\psi) \in Q^*_\psi$, or, equivalently,

$$\overline{\pi}_{\underline{i},\underline{j}}(\psi) - \psi \in [0,\lambda]^{4m}. \tag{11}$$

Assume $i_k < j_k$. For a vertex $\psi = (a_{i_k}, b_{i_k}, a_{j_k}, b_{j_k})$ of J^{4m} we shall write $\overline{\pi}_{\underline{i},\underline{j}}(\psi) = \pi_{\underline{i},\underline{j}}(\psi) = (a,b,c,d)$. To prove (11) it is sufficient to prove

$$a - a_{i_k}, b - b_{i_k}, c - a_{j_k}, d - b_{j_k} \in [0,\lambda]^m. \tag{12}$$

We consider $a - a_{i_k}$; the remaining cases are analogous. Our identification $\psi = (a_1, b_1, \ldots, a_\theta, b_\theta)$ made in Section 1 implies, for $1 \le i \le \theta$,

$$f_\psi(R_i) \cap (\{0\} \times I^m) = f_\psi(\{s_i\} \times I^m) = \{0\} \times (a_i + [0,\lambda]^m).$$

Therefore we have by the definition of $\pi_{\underline{i},\underline{j}}$,

$$(0,a) = f^k_\psi(s_{\underline{i}}, o) = f_\psi f^{k-1}_\psi(s_{\underline{i}}, o)$$

and, since $\varphi(s_{(i_1,\ldots,i_l)}) = s_{(i_2,\ldots,i_l)}$ ($(i_2,\ldots,i_l)$ regarded as element of $\theta^{[1,l-1]}$),

$$f^{k-1}_\psi(s_{\underline{i}}, o) \in \{\varphi^{k-1}(s_{\underline{i}})\} \times I^m = \{s_{i_k}\} \times I^m \subset R_{i_k}.$$

Therefore

$$(0,a) \in f_\psi(R_{i_k}) \cap (\{0\} \times I^m) = \{0\} \times (a_{i_k} + [0,\lambda]^m),$$

which proves (12) for $a - a_{i_k}$ and hence the lemma. ∎

We consider the compact subset

$$K = \{(a,b) \in ([-1,1]^m)^2 = [-1,1]^{2m} \mid (1-t)a + tb = o \text{ for some } t \in H\}$$

of $[-1,1]^{2m}$.

LEMMA 3. *Let $(a,b,c,d) \in I^{4m}$. Then the segments $[(0,a),(1,b)]$ and $[(0,c),(1,d)]$ intersect in a point (t,x) with $t \in H$ and $x \in I^m$ if and only if $\pi(a,b,c,d) \in K$.*

This lemma is an immediate consequence of the definitions of π and K. ∎

LEMMA 4. *There is a real $\beta > 0$ such that for any $k \ge 1$ and $\underline{i},\underline{j} \in \theta^{[1,k]}$, $\underline{i} \ne \underline{j}$, we have*

$$N_{\lambda^k}(\Psi^*_{\underline{i},\underline{j}}) \subset \varrho^{-1}_{\underline{i},\underline{j}}(N_{\beta\lambda^k}(K)),$$

where $N_{\lambda^k}(\Psi^*_{\underline{i},\underline{j}})$ denotes the λ^k-neighbourhood of $\Psi^*_{\underline{i},\underline{j}}$ in $J^{2\theta m}$ while $N_{\beta\lambda^k}(K)$ is the $\beta\lambda^k$-neighbourhood of K in $[-1,1]^{2m}$.

Proof. For an arbitrarily given $\psi = (a_1, b_1, \ldots, a_\theta, b_\theta) \in N_{\lambda^k}(\Psi^*_{\underline{i},\underline{j}})$ we choose $\psi' = (a'_1, b'_1, \ldots, a'_\theta, b'_\theta) \in \Psi^*_{\underline{i},\underline{j}}$ so that

$$|a'_i - a_i| \le \lambda^k, \quad |b'_i - b_i| \le \lambda^k \quad (1 \le i \le \theta).$$

A simple geometric argument (by induction with respect to k) shows that for

$$(a,b,c,d) = \pi_{\underline{i},\underline{j}}(\psi), \qquad (a',b',c',d') = \pi_{\underline{i},\underline{j}}(\psi')$$

each of the distances $|a'-a|$, $|b'-b|$, $|c'-c|$, $|d'-d|$ is at most

$$\lambda^k \sum_{i=0}^{k-1} \lambda^i < \frac{\lambda^k}{1-\lambda} < 2\lambda^k. \tag{13}$$

(The last inequality is a consequence of our assumption $\lambda < 1/2$. Instead of applying this assumption we could proceed with $1/(1-\lambda)$ instead of 2 and choose $\beta = 4/(1-\lambda) + 4\sqrt{m}$. Therefore in this proof $\lambda < 1/2$ is not essential.) As an immediate consequence of (13) we have

$$|\pi_{\underline{i},\underline{j}}(\psi') - \pi_{\underline{i},\underline{j}}(\psi)| < 4\lambda^k$$

and from $|\pi(p) - \pi(q)| < 2|p-q|$ we get

$$|\varrho_{\underline{i},\underline{j}}(\psi') - \varrho_{\underline{i},\underline{j}}(\psi)| < 8\lambda^k. \tag{14}$$

Since $\psi' \in \Psi^*_{\underline{i},\underline{j}}$, we can find points $t \in H$ and $x \in I^m$ such that

$$(t,x) \in R_{\sigma^k(\underline{i})}(\psi') \cap R_{\sigma^k(\underline{j})}(\psi'). \tag{15}$$

Let (t,y) and (t,z) be the points at which $\{t\} \times I^m$ intersects the segments $f^k_{\psi'}(I_{\underline{i}} \times \{o\})$ and $f^k_{\psi'}(I_{\underline{j}} \times \{o\})$, respectively. The end points of these segments are $(0,a'),(1,b')$ and $(0,c'),(1,d')$ respectively, and (8) together with $f^k_{\psi'}(I_{\underline{i}} \times \{o\}) \subset R_{\sigma^k(\underline{i})}(\psi')$, $f^k_{\psi'}(I_{\underline{j}} \times \{o\}) \subset R_{\sigma^k(\underline{j})}(\psi')$ and (15) implies

$$|x-y| \le \sqrt{m}\,\lambda^k, \qquad |x-z| \le \sqrt{m}\,\lambda^k. \tag{16}$$

Let $a^* = a' + x - y$, $b^* = b' + x - y$, $c^* = c' + x - z$ and $d^* = d' + x - z$. Then $(a^*,b^*,c^*,d^*) \in I^{4m}$, and since $(t,x) \in [(0,a^*),(1,b^*)] \cap [(0,c^*),(1,d^*)]$ and $t \in H$, by Lemma 3 we have $\pi(a^*,b^*,c^*,d^*) \in K$.

Applying (16) we get

$$|(a',b',c',d') - (a^*,b^*,c^*,d^*)| \le 2\sqrt{m}\,\lambda^k$$

and therefore, by the definition of π,

$$\operatorname{dist}(\varrho_{\underline{i},\underline{j}}(\psi'),K) \le |\pi(a',b',c',d') - \pi(a^*,b^*,c^*,d^*)| \le 4\sqrt{m}\,\lambda^k.$$

This together with (14) shows $\varrho_{\underline{i},\underline{j}}(\psi) \in N_{\beta\lambda^k}(K)$, where $\beta = 8 + 4\sqrt{m}$. ∎

Lemma 5. $\dim_{\mathrm{B}} K = m + q^*$.

Proof. K is the intersection of a cone with $[-1,1]^{2m}$, i.e., if $v \in K$, $\gamma \in \mathbb{R}$ and $\gamma v \in [-1,1]^{2m}$, then $\gamma v \in K$. The full cone is

$$\begin{aligned}\overline{K} &= \{\gamma v \mid v \in K,\ \gamma \in \mathbb{R}\}\\ &= \{(a,b) \in (\mathbb{R}^m)^2 = \mathbb{R}^{2m} \mid (1-t)a + tb = 0 \text{ for some } t \in H\},\end{aligned}$$

and $K = \overline{K} \cap [-1,1]^{2m}$. So it is sufficient to prove

$$\dim_{\mathrm{B}} \overline{K} = m + q^*.$$

To describe $\overline{K}$ we consider the boundary $\partial(\mathbb{D}^m \times \mathbb{D}^m) = (\mathbb{S}^{m-1} \times \mathbb{D}^m) \cup (\mathbb{D}^m \times \mathbb{S}^{m-1})$ of the topological ball $\mathbb{D}^m \times \mathbb{D}^m$ in $\mathbb{R}^{2m}$, where $\mathbb{D}^m = \{a \in \mathbb{R}^m \mid |a| \le 1\}$ and $S^{m-1} = \{a \in \mathbb{R}^m \mid |a| = 1\}$. Then, since

$$\dim_{\mathrm{B}} \overline{K} = 1 + \dim_{\mathrm{B}}(\partial(\mathbb{D}^m \times \mathbb{D}^m) \cap \overline{K}),$$

it is sufficient to show

$$\max[\dim_{\mathrm{B}}((\mathbb{S}^{m-1} \times \mathbb{D}^m) \cap \overline{K}),\ \dim_{\mathrm{B}}((\mathbb{D}^m \times \mathbb{S}^{m-1}) \cap \overline{K})] = m - 1 + q^*. \tag{17}$$

We consider the first term

$$(\mathbb{S}^{m-1} \times \mathbb{D}^m) \cap \overline{K} = \left\{\left(a, \frac{t-1}{t}a\right) \,\middle|\, a \in \mathbb{S}^{m-1},\ t \in H \cap [1/2,1]\right\}.$$

Let $F = \mathbb{S}^{m-1} \times [1/2,1]$, and let $\chi : F \to \mathbb{S}^{m-1} \times \mathbb{D}^m$ be the mapping given by

$$\chi(a,t) = \left(a, \frac{t-1}{t}a\right).$$

Obviously, χ is an injective C^∞ embedding satisfying $\chi(\mathbb{S}^{m-1} \times (H \cap [1/2,1])) = (\mathbb{S}^{m-1} \times \mathbb{D}^m) \cap \overline{K}$. Then since

$$\dim_{\mathrm{B}}(\mathbb{S}^{m-1} \times (H \cap [1/2,1])) = m - 1 + \dim_{\mathrm{B}}(H \cap (1/2,1]),$$

we have

$$\dim_{\mathrm{B}}((\mathbb{S}^{m-1} \times \mathbb{D}^m) \cap \overline{K}) = m - 1 + \dim_{\mathrm{B}}(H \cap [1/2,1]) \quad \text{if } H \cap [1/2,1] \ne \emptyset.$$

In the same way we get

$$(\mathbb{D}^m \times \mathbb{S}^{m-1}) \cap \overline{K} = \left\{\left(\frac{t}{t-1}b, b\right) \,\middle|\, b \in \mathbb{S}^{m-1},\ t \in H \cap [0,1/2]\right\},$$

$$\dim_{\mathrm{B}}((\mathbb{D}^m \times \mathbb{S}^{m-1}) \cap \overline{K}) = m - 1 + \dim_{\mathrm{B}}(H \cap [0,1/2]) \text{ if } H \cap [0,1/2] \ne \emptyset.$$

Since $q^* = \max(\dim_{\mathrm{B}}(H \cap [0,1/2]),\ \dim_{\mathrm{B}}(H \cap [1/2,1]))$, this implies (17). ∎

To prove (5) we apply the following result of C. Tricot Jr. [6], in which $\overline{\dim}_{\mathrm{B}}$ and $\underline{\dim}_{\mathrm{B}}$ denote the upper and the lower box counting dimension, respectively (see e.g. [2]).

LEMMA 6. *If X is a bounded subset of $\mathbb{R}^p$ then*

$$\overline{\dim}_{\mathrm{B}} X = p - \liminf_{\varepsilon\to 0} \frac{\log \mathrm{vol}^p N_\varepsilon(X)}{\log \varepsilon}, \tag{18}$$

$$\underline{\dim}_{\mathrm{B}} X = p - \limsup_{\varepsilon\to 0} \frac{\log \mathrm{vol}^p N_\varepsilon(X)}{\log \varepsilon}, \tag{19}$$

where $N_\varepsilon(X)$ denotes the ε-neighbourhood of X in $\mathbb{R}^p$. ∎

Proof of (5). Lemma 6 for $X = K$ together with Lemma 5 implies

$$2m - \lim_{\varepsilon\to 0} \frac{\log \mathrm{vol}^{2m} N_\varepsilon(K)}{\log \varepsilon} = m + q^*,$$

$$\lim_{\varepsilon\to 0} \frac{\log \mathrm{vol}^{2m} N_\varepsilon(K)}{\log \varepsilon} = m - q^*. \tag{20}$$

Applying Lemma 4 and the corollary to Lemmas 1 and 2 we get for $k \ge 1$ and $\underline{i} = (i_1, \ldots, i_k), \underline{j} = (j_1, \ldots, j_k) \in \theta^{[1,k]}$, $i_k \ne j_k$,

$$\mathrm{vol}^{2\theta m} N_{\lambda^k}(\Psi^*_{\underline{i},\underline{j}}) \le \alpha \, \mathrm{vol}^{2m} N_{\beta\lambda^k}(K),$$

where α, β do not depend on k, $\underline{i}$, $\underline{j}$. By (6) and (7) we have, for $k \ge 1$,

$$N_\varepsilon(\Psi^*) \subset N_\varepsilon(\Psi^*_k) = \bigcup_{\substack{\underline{i},\underline{j} \in \theta^{[1,k]} \\ i_k \ne j_k}} N_\varepsilon(\Psi^*_{\underline{i},\underline{j}}).$$

There are less than θ^{2k} summands on the right hand side, and therefore

$$\mathrm{vol}^{2\theta m} N_{\lambda^k}(\Psi^*) \le \theta^{2k} \alpha \, \mathrm{vol}^{2m}(N_{\beta\lambda^k}(K)).$$

Since $\lambda < 1$, i.e., $\log \lambda < 0$, this together with (20) implies

$$\begin{aligned}
&\limsup_{k\to\infty} \frac{\log \mathrm{vol}^{2\theta m} N_{\lambda^k}(\Psi^*)}{\log \lambda^k} \\
&\quad \ge \frac{2\log\theta}{\log\lambda} + \lim_{k\to\infty} \frac{\log\alpha}{\log\lambda^k} + \limsup_{k\to\infty} \frac{\log \mathrm{vol}^{2m} N_{\beta\lambda^k}(K)}{\log \lambda^k} \\
&\quad = \frac{2\log\theta}{\log\lambda} + \lim_{k\to\infty} \frac{\log \mathrm{vol}^{2m} N_{\beta\lambda^k}(K)}{\log\lambda^k - \log\beta} = \frac{2\log\theta}{\log\lambda} + m - q^*,
\end{aligned}$$

and a fortiori

$$\limsup_{\varepsilon\to 0} \frac{\log \mathrm{vol}^{2\theta m} N_\varepsilon(\Psi^*)}{\log\varepsilon} \ge \frac{2\log\theta}{\log\lambda} + m - q^*.$$

Then

$$2\theta m - \limsup_{\varepsilon\to 0} \frac{\log \mathrm{vol}^{2\theta m} N_\varepsilon(\Psi^*)}{\log\varepsilon} \le 2\theta m - m + q^* - \frac{2\log\theta}{\log\lambda},$$

and, since Ψ^* lies in $\mathbb{R}^{2\theta m}$, (19) implies

$$\underline{\dim}_{\mathrm{B}} \Psi^* \le 2\theta m - m + q^* + \frac{2\log\theta}{\log(1/\lambda)}.$$

Now (5) is a consequence of the well known inequality $\dim_{\mathrm{H}} \le \underline{\dim}_{\mathrm{B}}$. ∎

4. Proof of Theorem 3. In this proof the dimension m of the cube I^m is odd and at least 3. So we shall write $m = 2n+1$, where $n \ge 1$. Let Q be an m-dimensional cube. The k-dimensional skeleton $(0 \le k \le m)$ of Q, i.e. the union of all k-dimensional faces of Q, will be denoted by $\mathrm{Sk}_k Q$. For $r \ge 3$ we consider the subdivision of Q into r^m congruent cubes and the family $\mathfrak{P}_r(Q) = \{Q_1, \ldots, Q_{\theta_{m,r}}\}$ of all those cubes of this subdivision which intersect $\mathrm{Sk}_n Q$. The number $\theta_{m,r}$ of cubes in $\mathfrak{P}_r(Q)$ satisfies

$$\theta_{m,r} = \sum_{k=0}^{n} 2^{2n+1-k}\binom{2n+1}{k}(r-2)^k < 2^{n+1}\binom{2n+1}{n} r^n. \tag{21}$$

(Since Q has $2^{m-k}\binom{m}{k}$ k-dimensional faces the kth summand in the sum above is the number of cubes which intersect $\mathrm{Sk}_k Q$ but are disjoint from $\mathrm{Sk}_{k-1} Q$. The upper bound becomes clear by the fact that each of the $2^{n+1}\binom{2n+1}{n}$ faces in $\mathrm{Sk}_n Q$ intersects exactly r^n cubes of $\mathfrak{P}_r(Q)$.)

In the following lemma $\alpha(m,\theta)$ is the real number defined in Section 1; here $m = 2n+1$ or, equivalently, $n = (m-1)/2$ $(n = 1, 2, \ldots)$.

Lemma 7.

$$\alpha(m,\theta) \le \theta^{(n+1)/[n(2n+1)]}. \tag{22}$$

Lemma 8.

$$\alpha(m,\theta_{m,r}) \le \frac{1}{r}\theta_{m,r}^{1/n} \le 2^{1+1/n}\binom{2n+1}{n}^{1/n}. \tag{23}$$

Before proving these lemmas (the second of which is the crucial point of the whole proof) we show how they imply Theorem 3, i.e. (2), and Remark 3, i.e. (3).

Let $m = 2n+1$ $(n \ge 1)$ be fixed. Obviously $\theta_{m,3} < \theta_{m,4} < \ldots$ The proof will be divided into the two cases $\theta \ge \theta_{m,3}$ and $\theta < \theta_{m,3}$.

We begin with $\theta \ge \theta_{m,3}$ and choose r so that $\theta_{m,r} \le \theta < \theta_{m,r+1}$. As an immediate consequence of the definition of $\alpha(m,\theta)$ in Section 1 we have the implication $\theta' \le \theta'' \Rightarrow \alpha(m,\theta'){\theta'}^{-1/n} \ge \alpha(m,\theta''){\theta''}^{-1/n}$ (since $m = 2n+1$ we have $-1/n = -2/(m-1)$). Therefore

$$\begin{aligned}\alpha(m,\theta_{m,r})\biggl(\frac{\theta_{m,r+1}}{\theta_{m,r}}\biggr)^{1/n}\theta^{-1/n} &> \alpha(m,\theta_{m,r})\biggl(\frac{\theta}{\theta_{m,r}}\biggr)^{1/n}\theta^{-1/n}\\ &= \alpha(m,\theta_{m,r})\theta_{m,r}^{-1/n} \ge \alpha(m,\theta)\theta^{-1/n},\end{aligned}$$

and by Lemma 8 and (21),

$$\begin{aligned}\text{(24)}\qquad \alpha(m,\theta) &\le \alpha(m,\theta_{m,r})\Big(\frac{\theta_{m,r+1}}{\theta_{m,r}}\Big)^{1/n}\\ &\le \frac{1}{r}\theta_{m,r+1}^{1/n} = \frac{1}{r}\Big[\sum_{k=0}^{n} 2^{2n+1-k}\binom{2n+1}{k}(r-1)^k\Big]^{1/n}.\end{aligned}$$

As is easily seen, if n is fixed and $r\to\infty$ the last term is increasing with limit 12 for $n=1$ and smaller than 12 for $n>1$. This proves (2) for $\theta\ge\theta_{m,3}$.

To prove (2) for $2\le\theta<\theta_{m,3}$ we apply Lemma 7 and (21). So we get

$$\begin{aligned}\alpha(m,\theta) &\le \theta^{(n+1)/[n(2n+1)]} < \theta_{m,3}^{(n+1)/[n(2n+1)]}\\ &< \Big[2^{n+1}\binom{2n+1}{n}3^n\Big]^{(n+1)/[n(2n+1)]}.\end{aligned}$$

This easily shows $\alpha(m,\theta)\le 12$ in this case.

Now we prove (3). Applying (24), (23) and (21) we get

$$\begin{aligned}\lim_{\theta\to\infty}\alpha(m,\theta) &\le \lim_{r\to\infty}\frac{1}{r}\theta_{m,r+1}^{1/n} = \lim_{r\to\infty}\frac{r+1}{r}\cdot\frac{1}{r+1}\theta_{m,r+1}^{1/n}\\ &\le \lim_{r\to\infty} 2^{1+1/n}\frac{r+1}{r}\binom{2n+1}{n}^{1/n} = 2^{1+1/n}\binom{2n+1}{n}^{1/n}\end{aligned}$$

and by Stirling's formula we have $\lim_{n\to\infty}\binom{2n+1}{n}^{1/n}=4$. ∎

P r o o f o f L e m m a 7. For $m=2n+1$, $\varphi: I^*\to I$, θ the number of components of I^* and $\lambda\in(0,1)$ we consider the space $\mathcal{F}$ with the subset $\mathcal{G}$ as defined in Section 1 in connection with Theorem 3. To prove the lemma it must be shown that $\mathcal{G}$ has interior points provided

$$\text{(25)}\qquad \lambda > \theta^{(n+1)/[n(2n+1)]}\theta^{-1/n}.$$

We shall even prove that (25) implies $\mathcal{G}=\mathcal{F}$ or, equivalently, that the existence of a mapping $\psi\in\mathcal{F}\setminus\mathcal{G}$ implies

$$\text{(26)}\qquad \lambda \le \theta^{(n+1)/[n(2n+1)]}\theta^{-1/n}.$$

Let $\psi\in\mathcal{F}\setminus\mathcal{G}$ be fixed. By the proof of Proposition 2 in Section 2 we can find an integer $q\ge 0$ such that for $k\le -q$ and any two sequences $\underline{i}=(i_k,\dots,i_0),\underline{j}=(j_k,\dots,j_0)\in\theta^{[k,0]}$ satisfying $(i_{k+q},\dots,i_0)\ne(j_{k+q},\dots,j_0)$ we have $R_{\underline{i}}\cap R_{\underline{j}}=\emptyset$, where $R_{\underline{i}}$, $R_{\underline{j}}$ are defined as in Section 2 with the mapping $f_\psi: R^*\to R$.

Then using standard methods for any $t\in I$ the Hausdorff dimension of the sets $A\cap(\{t\}\times I^m)$ can be calculated to be

$$\dim_{\mathrm{H}}(A\cap(\{t\}\times I^m)) = -\log\theta/\log\lambda.$$

Obviously this dimension does not exceed $m = 2n + 1$, i.e., $-\log\theta/\log\lambda \le 2n + 1$, which is equivalent to (26). ∎

The proof of Lemma 8 will use the following elementary fact concerning linking of n-skeletons of $(2n + 1)$-dimensional cubes.

LEMMA 9. *Let $Q'(t)$, $Q''(t)$ $(t \in I)$ be two continuous families of m-dimensional cubes in $\mathbb{R}^m$, where the edges of all these cubes are of equal length and parallel to the axes of $\mathbb{R}^m$. Assume $Q'(0)\cap Q''(0) \neq \emptyset$ and $Q'(1)\cap Q''(1) = \emptyset$. Then for some $t' \in I$ the skeletons $\mathrm{Sk}_n(Q'(t'))$ and $\mathrm{Sk}_n(Q''(t'))$ intersect.*

Proof. The topological background of this lemma is the fact that for $t = 0$ the skeletons $\mathrm{Sk}_n(Q'(0))$ and $\mathrm{Sk}_n(Q''(0))$ either intersect or are linked as indicated for $n = 1$ and $m = 3$ in Fig. 5, while by disjointness of $Q'(1)$ and $Q''(1)$, the skeletons $\mathrm{Sk}_n(Q'(1))$ and $\mathrm{Sk}_n(Q''(1))$ are unlinked.

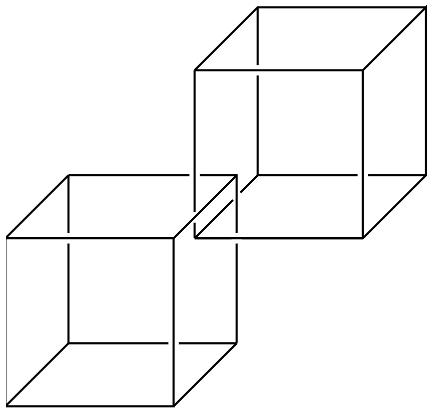

Fig. 5

We shall apply the following simple fact concerning two cubes F' and F'' of dimension $m - 1 = 2n$ lying in $\mathbb{R}^{m-1}$. If the edges of F'' are parallel to and of the same length as the edges of F' then $F' \cap F'' \neq 0$ implies $\mathrm{Sk}_n F' \cap \mathrm{Sk}_n F'' \neq \emptyset$. Therefore to prove the lemma it is sufficient to find some $t' \in I$ and $(m - 1)$-dimensional faces F' and F'' of $Q'(t')$ and $Q''(t')$, respectively, such that F' and F'' intersect and lie in a common hyperplane of $\mathbb{R}^m$. We define

$$t' = \sup\{t \in I \mid Q'(t) \cap Q''(t) \neq \emptyset\}.$$

Then $Q'(t') \cap Q''(t') \neq \emptyset$, but $\mathrm{Int}\, Q'(t') \cap \mathrm{Int}\, Q''(t') = \emptyset$. If $H_1, \ldots, H_{2m}$ are the hyperplanes in $\mathbb{R}^m$ each of which contains an $(m-1)$-dimensional face of $Q'(t')$, then there is at least one H_i such that $Q'(t')$ and $Q''(t')$ lie on opposite sides of H_i. (Otherwise the interiors of the two cubes would intersect.) Let

F' and F'' be the $(m-1)$-faces of $Q'(t')$ and $Q''(t')$, respectively, which lie in H_i. Then $F' \cap F'' \neq \emptyset$. ∎

Proof of Lemma 8. For $m = 2n+1 \geq 3$, $r \geq 3$, $\varphi : I^* \to I$, $\theta_{m,r}$ the number of components of I^* and $\lambda \in (0,1)$ we consider the space $\mathcal{F}$ with the subset $\mathcal{G}$.

We shall prove that $\mathcal{G}$ has interior points provided $\lambda > 1/2$. By the definition of $\alpha(m, \theta_{m,r})$ in Section 1 this implies the first inequality in (23). The second follows from (21).

The equality of (21) together with $r \geq 3$ implies

$$\theta_{m,r}^{(n+1)/[n(2n+1)]} \theta_{m,r}^{-1/n} = \theta_{m,r}^{-1/(2n+1)} < 1/2,$$

and using Lemma 7 we get

$$\alpha(m, \theta_{m,r}) \theta_{m,r}^{-2/(m-1)} < 1/2.$$

Then by the definition of $\alpha(m, \theta_{m,r})$ the set $\mathcal{G}$ has interior points provided $\lambda \geq 1/2$. In the remaining crucial case $\lambda \in (1/r, 1/2)$ we proceed as follows.

For m, r, I^*, φ and $\theta_{m,r}$ as above and $\lambda \in (1/r, 1/2)$ fixed we define an open subset $\mathcal{H} \neq \emptyset$ of $\mathcal{F}$. Then for each $\psi \in \mathcal{H}$ we construct two sequences $\underline{i} = (\ldots, i_{-1}, i_0)$ and $\underline{j} = (\ldots, j_{-1}, j_0)$ in $\theta_{m,r}^{[-\infty,0]}$ such that $i_0 = 1$, $j_0 \neq 1$ and for any $k \geq 0$ the two curved prisms $R_{(i_{-k},\ldots,i_0)}$ and $R_{(j_{-k},\ldots,j_0)}$ corresponding to f_ψ intersect, i.e.,

$$R_{(i_{-k},\ldots,i_0)} \cap R_{(j_{-k},\ldots,j_0)} \neq \emptyset. \tag{27}$$

Since

$$R_{(i_{-k-1},\ldots,i_0)} \subset R_{(i_{-k},\ldots,i_0)}, \qquad R_{(j_{-k-1},\ldots,j_0)} \subset R_{(j_{-k},\ldots,j_0)},$$

and the prisms are compact the arcs

$$R_{\underline{i}} = \bigcap_{k=0}^{\infty} R_{(i_{-k},\ldots,i_0)}, \qquad R_{\underline{j}} = \bigcap_{k=0}^{\infty} R_{(j_{-k},\ldots,j_0)}$$

also intersect, and we have $\psi \in \mathcal{G}$. This implies $\mathcal{H} \subset \mathcal{G}$, and $\mathcal{G}$ is shown to have interior points.

Construction of $\mathcal{H}$. Let $\overline{Q} = [\delta, 1-\delta]^m$ be a subcube of I^m, and let $\overline{\mathfrak{P}}$ be the set of all those cubes from the subdivision of $\overline{Q}$ into r^m cubes with edge length $(1-2\delta)r^{-1}$ which intersect the skeleton $\mathrm{Sk}_n \overline{Q}$. By the definition of the number $\theta_{m,r}$ at the beginning of this section, $\overline{\mathfrak{P}}$ consists of $\theta_{m,r}$ cubes and these will be denoted by $\overline{Q}_1, \ldots, \overline{Q}_{\theta_{m,r}}$.

Then for an arbitrarily chosen point t^+ in $I \setminus I^*$ the set $\mathcal{H}$ consists of all $\psi \in \mathcal{F}$ for which the corresponding map $f_\psi : I^* \times I^m \to R$ has the following two properties, where the points $t_i^+ \in I_i$ are determined by $\varphi(t_i^+) = t^+$:

(A) If $t \in I_i$, $\varphi(t) \in I^*$ then $\{\varphi(t)\} \times \overline{Q}_i \subset f_\psi(\{t\} \times \mathrm{Int}\, \overline{Q})$ $(i = 1, \ldots, \theta_{m,r})$.
(B) $f_\psi(\{t_1^+\} \times I^m) \cap f_\psi(\{t_i^+\} \times I^m) = \emptyset$ $(i = 2, \ldots, \theta_{m,r})$.

Obviously $\mathcal{H}$ is open. To show $\mathcal{H} \neq \emptyset$ we remark that using $\lambda < 1/2$ it is not hard to find a mapping $\psi \in \mathcal{F}$ which satisfies (A) and (B) and is constant on $I^* \setminus \varphi^{-1}(I^+)$, where I^+ denotes the component of $I \setminus I^*$ containing t^+. Then, as indicated in Fig. 6, the curved prisms $R_{\underline{i}}$ ($\underline{i} \in \theta_{m,r}^{[0,0]}$) are straight outside $I^+ \times I^m$ while possibly bended inside $I^+ \times I^m$ so that (B) holds. In order that this construction is possible (i.e. that these prisms lie in R) we have to assume that $\overline{Q}$ is sufficiently small.

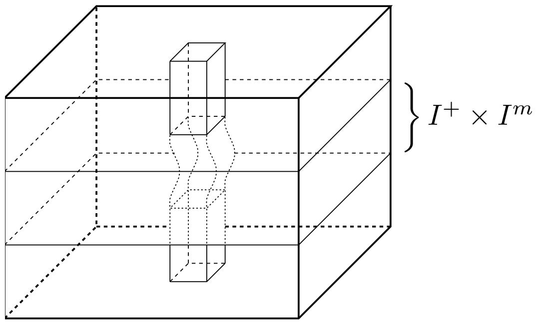

Fig. 6

Construction of $\underline{i}, \underline{j}$. With $\psi \in \mathcal{H}$ fixed we shall define $i_0, i_{-1}, \ldots, j_0, j_{-1}, \ldots$ successively together with points $t_0, t_1, \ldots \in I$ such that for any $k \geq 0$,

(28) $\quad (\operatorname{Int} \overline{R}_{(i_{-k},\ldots,i_0)} \cap (\{t_k\} \times I^m)) \cap (\operatorname{Int} \overline{R}_{(j_{-k},\ldots,j_0)} \cap (\{t_k\} \times I^m)) \neq \emptyset,$

where

$$\overline{R}_{(i_{-k},\ldots,i_0)} = f^{k+1}(I_{\sigma^{-k-1}(i_{-k},\ldots,i_0)} \times \overline{Q}),$$
$$\overline{R}_{(j_{-k},\ldots,j_0)} = f^{k+1}(I_{\sigma^{-k-1}(j_{-k},\ldots,j_0)} \times \overline{Q}),$$

are the prisms contained in and concentric to $R_{(i_{-k},\ldots,i_0)}$, $R_{(j_{-k},\ldots,j_0)}$, respectively, and which intersect the cubes $\{t\} \times I^m$ ($t \in I$) in cubes with edge length $\lambda^{k+1}(1-2\delta)$.

We start with $i_0 = 1$ and fix $j_0 \in \{2, \ldots, \theta_{m,r}\}$ so that $\overline{Q}_1$, $\overline{Q}_{j_0}$ are neighbours in $\mathfrak{P}$, i.e. $\overline{Q}_1 \cap \overline{Q}_{j_0} \neq \emptyset$.

The point t_0 can be arbitrarily chosen in I^*. Then property (A) of ψ implies

$$\operatorname{Int} \overline{R}_{(i_0)} \cap (\{t_0\} \times I^m) \supset \overline{Q}_1 \times (\{t_0\} \times I^m),$$
$$\operatorname{Int} \overline{R}_{(j_0)} \cap (\{t_0\} \times I^m) \supset \overline{Q}_{j_0} \times (\{t_0\} \times I^m),$$

and we get (28) for $k = 0$.

Now we assume that $t_0, \ldots, t_k$, $(i_{-k}, \ldots, i_0)$ and $(j_{-k}, \ldots, j_0)$ are defined where $\overline{R}_{(i_{-k},\ldots,i_0)} \cap (\{t_k\} \times I^m)$ and $\overline{R}_{(j_{-k},\ldots,j_0)} \cap (\{t_k\} \times I^m)$ have common interior points. By the definition of t^+ and since $i_0 = 1$ and $j_0 \neq 1$ the cubes $\overline{R}_{(i_{-k},\ldots,i_0)} \cap (\{t^+\} \times I^m)$ and $\overline{R}_{(j_{-k},\ldots,j_0)} \cap (\{t^+\} \times I^m)$ are disjoint. (Here we use the property (B) of ψ.) Applying Lemma 9 we get a point t_{k+1} in the subinterval of I with end points t^+ and t_k such that the k-skeletons of the cubes $\overline{R}_{(i_{-k},\ldots,i_0)} \cap (\{t_{k+1}\} \times I^m)$ and $\overline{R}_{(j_{-k},\ldots,j_0)} \cap (\{t_{k+1}\} \times I^m)$ intersect. Using the fact that these skeletons are covered by the interiors of the cubes $\overline{R}_{(i,i_{-k},\ldots,i_0)} \cap (\{t_{k+1}\} \times I^m)$ and $\overline{R}_{(j,j_{-k},\ldots,j_0)} \cap (\{t_{k+1}\} \times I^m)$ $(1 \le i, j \le \theta_{m,r})$, respectively, we can find indices $i_{-(k+1)}$ and $j_{-(k+1)}$ such that $\overline{R}_{(i_{-(k+1)},\ldots,i_0)} \cap (\{t_{k+1}\} \times I^m)$ and $\overline{R}_{(j_{-(k+1)},\ldots,j_0)} \cap (\{t_{k+1}\} \times I^m)$ have common interior points. So our induction is complete. ∎

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