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Connected covers and Neisendorfer's localization theorem

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To the Rochester Mathematicians, in admiration and solidarity

Abstract. Our point of departure is J. Neisendorfer's localization theorem which reveals a subtle connection between some simply connected finite complexes and their connected covers. We show that even though the connected covers do not forget that they came from a finite complex their homotopy-theoretic properties are drastically different from those of finite complexes. For instance, connected covers of finite complexes may have uncountable genus or nontrivial SNT sets, their Lusternik–Schnirelmann category may be infinite, and they may serve as domains for nontrivial phantom maps.

1. Introduction. Let X be a connected CW-complex and let $X\langle n\rangle$ denote its n-connected cover. The 1-connected cover, $X\langle 1\rangle$, of a space is usually referred to as its universal cover and is familiar to most first year topology students. However, for $n > 1$, the space $X\langle n\rangle$ is less familiar and not much has been said about it in the literature. Strictly speaking, $X\langle n\rangle$ is not a covering space of X in the usual sense when $n \geq 2$, but it is an n-connected space and there is a map $X\langle n\rangle \to X$ which induces an isomorphism on all homotopy groups above dimension n. This map can be regarded as the inclusion of the fiber in the fibration sequence

$$X\langle n\rangle \to X \to X^{(n)},$$

whose base space is the Postnikov approximation of X through dimension n.

Recently Neisendorfer has proved a remarkable result about the n-connected covers of certain finite complexes. To describe it, fix a rational prime p and let $\mathcal{L}_p$ denote the homotopy functor defined by localizing with respect to the constant map $\varphi : B\mathbb{Z}/p \to \bullet$, in the sense of Dror Farjoun [8], and then completing at the prime p in the sense of Bousfield–Kan [3]. In symbols, $\mathcal{L}_p(X) = (L_\varphi(X))_p$. Now if X is a finite-dimensional CW-complex, it follows

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from Miller's solution to the Sullivan conjecture that $\mathcal{L}_p(X) \simeq X_p$. At first glance, this would suggest that the functor $\mathcal{L}_p$ is unlikely to yield any new information. However, in [23], Neisendorfer showed that this functor has a remarkable property when applied to certain n-connected covers. His main result was the following.

THEOREM 1. *Let X be a 1-connected finite complex with $\pi_2 X$ a finite group. Then $\mathcal{L}_p(X\langle n\rangle) \simeq X_p$ for any positive integer n.*

Thus, up to p-completion, no information is lost when one passes to the n-connected cover of such a complex! Of course, this is false for more general spaces, where the first n homotopy groups and the corresponding k-invariants are irretrievably lost in such a process. Thus Theorem 1 reveals a subtle homotopy property of certain finite-dimensional complexes and their connected covers.

This paper deals with a number of questions about connected covers of finite complexes. These questions were inspired by Neisendorfer's result and, not surprising, most of their answers involve applications of his theorem. We start with perhaps the most basic question.

QUESTION 1. *When is the n-connected cover of a finite complex a finite-dimensional space*?

Assume throughout this section that X is a finite complex which satisfies the conditions of Neisendorfer's theorem. It then follows that *every* nontrivial connected cover of X has nonzero mod p homology, for some prime p, in infinitely many dimensions. The proof is easy: suppose that $X\langle n\rangle$ is a nontrivial connected cover of X. Then there is a prime p such that the completions X_p and $X\langle n\rangle_p$ are different up to homotopy. Now if $X\langle n\rangle$ were a finite complex then $\mathcal{L}_p(X\langle n\rangle)$ would equal $X\langle n\rangle_p$. Since $\mathcal{L}_p(X\langle n\rangle) = X_p$ instead, we conclude that $X\langle n\rangle$ is not finite-dimensional.

On the other hand, suppose that Y is a 1-connected finite complex such that $\pi_2 Y$ is free of rank $r \geq 1$. It follows that there is a principal fibration

$$\underbrace{S^1 \times \ldots \times S^1}_{r} \to Y\langle 2\rangle \to Y.$$

A glance at the Serre spectral sequence for this fibration shows that the dimension of $Y\langle 2\rangle$ equals $r + \dim(Y)$. Thus $Y\langle 2\rangle$ has the homotopy type of a 2-connected finite complex and so $\mathcal{L}_p(Y\langle 2\rangle) = Y\langle 2\rangle_p \neq Y_p$. Thus in Theorem 1 the conditions on $\pi_2 X$ cannot be dropped entirely. For some mild generalizations of Theorem 1 see Section 3.

The following questions deal with those cases where $X\langle n\rangle$ is an infinite-dimensional space. Theorem 1 says that $X\langle n\rangle$ does not forget that it came from a finite complex and so it is natural to wonder if $X\langle n\rangle$ shares some of the homotopy-theoretic properties of finite complexes. For example:

Question 2. *Is the cohomology $H^*(X\langle n\rangle;\mathbb{Z}/p)$ necessarily locally finite as a module over the Steenrod algebra*?

The answer is no! For each prime p at which $X\langle n\rangle$ and X are different, the mod p cohomology of $X\langle n\rangle$ is *not* locally finite as a module over the mod p Steenrod algebra. (If it were then $\mathcal{L}_p(X\langle n\rangle)$ would equal $X\langle n\rangle_p$ by results of Lannes and Schwartz [15].)

Question 3. *Is the Lusternik–Schnirelmann category of $X\langle n\rangle$ necessarily finite*?

We have some partial answers. The first one is a rational result which is very different from the mod p results which follow it.

Proposition 3.1. *For all integers n, the rational category of $X\langle n\rangle$ is at most* $\mathrm{cat}(X)$ *and hence is finite.*

Since the natural map $X\langle n\rangle \to X$ induces a monomorphism on homotopy groups, this result follows from the mapping theorem of Felix and Halperin; see James ([12], page 1307) for an elegant proof of it. The next three results prompt us to conjecture that the answer to Question 3 is almost always no. Their proofs will be given in §4.

Proposition 3.2. *Let b be the smallest positive degree q such that $H_q(X;\mathbb{Z}) \neq 0$. Then the category of $X\langle b\rangle$ is infinite. Indeed, the mod p cohomology of $X\langle b\rangle$, for some prime p, contains an element of infinite height.*

Proposition 3.3. *If the Postnikov approximaton $X^{(n)}$ is rationally nontrivial, then the category of $X\langle n\rangle$ is infinite. Indeed, the reduced cohomology algebra $\widetilde{H}^*(X\langle n\rangle;\mathbb{Z}/p)$ contains elements of infinite height for all sufficiently large primes p.*

Proposition 3.4. *Assume also that X is an H-space and let b be defined as in* 3.2. *Then the category of $X\langle m\rangle$ is infinite for every integer $m \geq b$. Indeed, the reduced Morava k-theory $K(n)^*X\langle m\rangle$ has elements of infinite height, for any $n \geq 1$ and any prime p.*

Recall that the *Mislin genus* of a space Y is defined to be the pointed set $\mathcal{G}(Y)$ of homotopy types $[Z]$, where Z runs through those finite type spaces which are locally homotopy equivalent to Y; in symbols, $Z_{(p)} \simeq Y_{(p)}$ for each prime p. When Y is a 1-connected finite CW-complex, the genus set $\mathcal{G}(Y)$ is finite, according to Wilkerson [30]. This prompts the following question.

Question 4. *Is the Mislin genus of $X\langle n\rangle$ necessarily a finite set*?

The answer is no, but the biggest surprise is how simple the necessary example turned out to be.

Example 4.1. *If $n \geq 2$, then the Mislin genus of $S^{2n}\langle 2n\rangle$ is uncountably large.*

This reminds us very much of a famous example—the genus of $\mathbb{H}P^\infty$, which was first described by D. Rector [26]. It too is uncountably large. In both cases there is a homotopy-theoretic recognition principle for the most distinguished member; $\mathbb{H}P^\infty$ is the only member of its genus which has a maximal torus in the sense of Rector, see ([16], §9), while $S^{2n}\langle 2n\rangle$ is the only member of its genus which is the connected cover of a finite complex. Moreover, both genus sets are very rigid in the sense that there are *no* essential maps between different members of the same genus. This phenomenon within the genus of $\mathbb{H}P^\infty$ was first discovered by Møller in [22]. The properties just mentioned of $S^{2n}\langle 2n\rangle$ and its genus will be verified in Section 4.

Given a space Y, let $\mathrm{SNT}(Y)$ denote the pointed set of homotopy types $[Z]$ of spaces with the same n-type as Y; that is, the Postnikov approximations $Z^{(n)}$ and $Y^{(n)}$ are homotopy equivalent for each n, but not necessarily in any coherent manner [29]. When Y is a finite-dimensional space it is easy to see that $\mathrm{SNT}(Y)$ has just one member, namely $[Y]$. Thus we ask

Question 5. *Is $\mathrm{SNT}(X\langle n\rangle)$ necessarily the singleton set when X is a finite complex?*

We know of one special case where the answer is yes. Recall that a space Y is called an *H_0-space* if its rationalization Y_0 is homotopy equivalent to a product of rational Eilenberg–MacLane spaces. Obviously, every H-space is an H_0-space. The sphere S^5 is perhaps the simplest H_0-space which is not an H-space. Other familiar H_0-spaces include the complex and quaternionic Stiefel manifolds. On the other hand, the even-dimensional sphere, S^{2n} when $n \geq 1$, is perhaps the simplest example of a space which is not H_0. In [19] we showed that if Y is a nilpotent H_0-space and $\mathrm{SNT}(Y)$ has just one member, then the same is true of $Y\langle n\rangle$ for any positive integer n. Thus it follows, for example, that if X is any 1-connected compact Lie group, then $\mathrm{SNT}(X\langle n\rangle)$ has just one element for any n. The following example shows that the answer to Question 4 is no, in general. It also shows that the H_0-hypothesis cannot be dropped in the special result just cited.

Example 5.1. *Let X denote the r-fold product $S^n \times \ldots \times S^n$, where $r \geq 2$ and n is even and greater than 2. Then $\mathrm{SNT}(X\langle n\rangle)$ is uncountably large.*

Given a CW-complex Y, recall that a *phantom map* $Y \to Z$ is a pointed map whose restriction to each n-skeleton Y_n is null-homotopic. Obviously, if the domain Y is a finite-dimensional complex then any phantom map out of it must be homotopic to the constant map. This observation prompts the following question.

QUESTION 6. *Do there exist essential phantom maps out of the connected cover of a finite complex*?

The answer is almost always yes! In [11] it was shown that for a pointed finite type space Y, the universal phantom map out of Y is null-homotopic at a prime p if and only if the suspension ΣY is p-equivalent to a bouquet of finite-dimensional complexes. But if the cohomology $H^*(Y;\mathbb{Z}/p)$ is not locally finite, as a module over the Steenrod algebra, then it is not possible for ΣY to decompose as a bouquet of finite-dimensional retracts. Thus, in view of the answer to Question 2, the universal phantom map out of $X\langle n\rangle$ is essential at every prime at which $X\langle n\rangle$ and X are different.

Do there exist essential phantom maps from $X\langle n\rangle$ into targets of finite type? Since the universal phantom map $Y \to \bigvee \Sigma Y_n$ takes values in a space which does not have finite type, the answer does not follow from the observations made in the preceding paragraph ([1]). However, one does not have to look far to see that the answer is again yes.

EXAMPLE 6.1. *For each $n \geq 2$ there are uncountably many different homotopy classes of phantom maps from $S^{2n}\langle 2n\rangle$ to S^{4n} which are essential when localized at any prime p.*

This is a special case of Proposition 6.0 below, which deals with the set $[X\langle n\rangle, Y]$ for certain finite complexes X and Y. See §4 for its statement and proof.

Closely related to phantom maps is the notion of a *weak identity*. This is a self-map of a space Y which, up to homotopy, projects to the identity on each Postnikov approximation of Y. Obviously, on a finite complex, there is only one weak identity, up to homotopy. However, the following example shows that this need not be true for connected covers of finite complexes.

EXAMPLE 6.2. *Let $X = S^{2n} \vee S^{4n}$. Then for each $n \geq 2$, there are uncountably many homotopy classes of weak identities on $X\langle 2n\rangle$.*

2. Variants of Theorem 1. The following theorem is the most general version of Neisendorfer's theorem we know. Let us say that a space X is *$B\mathbb{Z}/p$-null* (or *B-null* for short) if the function space of based maps $\mathrm{map}_*(B\mathbb{Z}/p, X)$ is weakly contractible. By Miller's theorem, the class of B-null spaces includes all finite-dimensional spaces as well as their iterated loop spaces. Thus in the following theorem the spaces are not necessarily finite-dimensional; nor do they necessarily have finite type.

THEOREM 7.1. *Let X be a B-null space. Let Y be a 1-connected space such that $\mathcal{L}_p(\Omega Y) \simeq \bullet$. In particular*, *this holds if $\pi_2 Y$ is torsion and $\pi_n Y =$*

([1]) For example, the universal phantom map out of $\mathbb{R}P^\infty$ is essential, but there are no essential phantom maps from this space into any target of finite type [11].

0 for n sufficiently large. If $f : X \to Y$ is any continuous map and F is its homotopy fiber, then $\mathcal{L}_p(F) = X_p$.

Of course, Theorem 1 follows at once by taking $f : X \to Y$ to be the Postnikov approximation $X \to X^{(n)}$. The proof that we give follows the one Casacuberta gave in [4]. The key ingredient in the proof is that $\mathcal{L}_p(\Omega Y) \simeq \bullet$ when Y is an appropriate Postnikov section. Recently, McGibbon found a different case of this phenomenon; he found that $\mathcal{L}_p(E) \simeq \bullet$ whenever E is a connected infinite loop space with a torsion fundamental group [18]. As a consequence, he obtained the following perturbation of Theorem 1. As usual, $QX = \lim \Omega^n \Sigma^n X$.

THEOREM 7.2. *Let X be a 1-connected finite-dimensional complex with $\pi_2 X$ torsion. If F denotes the fiber of the infinite suspension $X \to QX$, then $\mathcal{L}_p(F) \simeq X_p$.*

Hopkins and Ravenel obtained the following stable version of Theorem 1 as a consequence of showing that all suspension spectra are harmonic [13].

THEOREM 7.3. *Let $\mathbf{X}$ be a suspension spectrum with $\pi_*\mathbf{X} \otimes \mathbb{Q} = 0$. Let $\mathbf{X}\langle n\rangle$ denote the n-connected cover of $\mathbf{X}$ (as a spectrum). Then the $\mathbf{E}_*$-localization of $\mathbf{X}\langle n\rangle$ is X, where $\mathbf{E}$ denotes the wedge of Morava K-theories $K(n)$ over all $n \geq 0$ and all primes p.*

Thus a rationally trivial suspension spectrum can be fully recovered from any one of its connected covers—no completion is necessary. This is also true unstably as the next result shows. We remind the reader that $L_\varphi(\ )$ denotes localization with respect to the constant map $B\mathbb{Z}/p \to \bullet$.

THEOREM 7.4. *Assume that X is a 1-connected, p-local, B-null space with $\pi_2 X$ torsion. Then for each n there is a homotopy fiber sequence*

$$L_\varphi(X\langle n\rangle) \to X \to X_0^{(n)}.$$

In particular, if X is rationally trivial, then $L_\varphi(X\langle n\rangle) \simeq X$ for each n.

Of course, if a space X is 1-connected and rationally trivial, then it is homotopy equivalent to the wedge of its p-primary pieces. If X is also $B\mathbb{Z}/p$-null for each prime p, then it is uniquely determined by any one of its connected covers, using the above result, one prime at a time. However, when X is not rationally trivial, it is not uniquely determined by any one of its connected covers. At the end of the next section we will take a close look at the indeterminacy. We conclude the present section with the observation that even though Neisendorfer's theorem fails (as he noted in [23]) when the condition on $\pi_2 X$ is dropped, all is not lost—there is the following result.

Theorem 7.5. *Let X be a space which is* 1*-connected. Over X there is a* 1*-connected "cover" $E \to X$ which identifies $\pi_2 E$ with the torsion subgroup of $\pi_2 X$, and induces an isomorphism on all higher homotopy groups. If X is B-null, then so is E. Moreover, in this case, $\mathcal{L}_p(X\langle n\rangle) = E_p$ for each $n \geq 2$.*

3. Other properties of $X\langle n\rangle$. Suppose that X is a 1-connected finite complex with $\pi_n X \otimes \mathbb{Q} = 0$ for n sufficiently large. Such a space is sometimes said to be *rationally elliptic*. Homogeneous spaces provide natural examples of such spaces. J. C. Moore has conjectured that for such a space X, the order of the p-torsion in $\pi_* X$ has a finite upper bound—for each prime p. Although this conjecture is known to be true for almost all primes for any given X (cf. [20]), it is still an open problem for "small" primes. One method of attacking it is to pose a more geometric question.

Question 8. *Given a rationally elliptic complex X, does it follow that some iterated loop space $\Omega^k X_p\langle n\rangle$ has a null-homotopic power map* (*i.e. a geometric exponent*) *for some k and n sufficiently large*?

When X is the sphere S^{2n+1} and p is an odd prime, it is the celebrated result of Cohen, Moore and Neisendorfer [5] that the p-torsion in $\pi_* S^{2n+1}$ has exponent p^n and that this is best possible (2). In [6] those authors showed that the loop space $\Omega^m S^m\langle m\rangle$, where $m = 2n+1$, has a geometric exponent at each prime p; it is exactly p^n when p is odd and at most 4^n at $p = 2$. On the other hand, Neisendorfer and Selick proved in [24] that the loop space $\Omega^{2n-2} S^{2n+1}\langle 2n+1\rangle$ has no geometric exponent at any prime p. In other words, they showed that every nonzero power map on this loop space is essential at each prime p. They used a clever argument which involved the K-theory of $\mathbb{C}P^\infty$. However, their conclusion was essentially limited to one particular connected cover of one particular space. The following result deals with *all* connected covers of a large class of spaces.

Proposition 8.1. *Let X be a* 1*-connected finite complex and assume that $t > 2$ is an integer such that $\pi_t X \otimes \mathbb{Q} \neq 0$. Then the loop space $\Omega^k X\langle n\rangle$ has no geometric exponent at any prime p, for any pair (n, k) where $n \geq 1$ and $0 < k < t - 2$.*

It should be noted that although we improve the Neisendorfer–Selick result in one direction—namely in showing that *no* connected cover of $\Omega^{2n-2} S^{2n+1}$ has a geometric exponent—we are unable to increase the number of loops in their result. In particular, whether or not some connected

(2) The precise exponent $2^{e(n)}$ for S^{2n+1} at $p = 2$ is still unknown. However, it is known that $n + \varepsilon \leq e(n) \leq 2n - [n/2]$, where $\varepsilon = 1$ if n is congruent to 1 or 2 mod 4 and is zero otherwise. This lower bound is due to Mahowald; the upper bound is due to Selick.

cover of $\Omega^{2n-1}S^{2n+1}$ has a geometric exponent is still an open question. For another example, let X be a homogeneous space of the form $\mathrm{Sp}(n)/K$, where $n \geq 2$. Letting $m > t = 4n - 1$, it follows from 8.1 that the torsion space $\Omega^{4n-4}X\langle m\rangle$ has no geometric exponent at any prime p.

If X is an H-space with the higher homotopy associativity of an A_n-space in the sense of Stasheff [27], then it is well known that the same is true of $X^{(n)}$ and $X\langle n\rangle$. Indeed, it is often the case that the Postnikov approximation $X^{(n)}$ carries more multiplicative structure than X does—at least for small values of n. This raises the following

QUESTION 9. *Given a finite complex X, what (if any) additional multiplicative structure does there exist on $X\langle n\rangle$?*

For example, it once seemed plausible that $S^n\langle m\rangle$ might be a mod 2 H-space for sufficiently large m and for some values of n other than the classical $1, 3$, and 7. For another example, a theorem of Hubbuck asserts there is no homotopy commutative multiplication (at $p = 2$) on a 1-connected nontrivial finite H-space X. But what about on some connected cover of this H-space; might not a homotopy commutative multiplication exist there? The following result puts an end to such speculation.

PROPOSITION 9.1. *If $X\langle n\rangle$ has the structure of an H-space, then so does X_p. If $X\langle n\rangle$ is also homotopy commutative or homotopy associative, then so is X_p.*

A space X is said to be *irreducible* (up to homotopy) if any essential map $K \to X$ which has a left inverse is a homotopy equivalence. Thus such an X has no retracts which are nontrivial in the homotopy sense. A special case of the following result was first observed by Zabrodsky in [32].

PROPOSITION 9.2. *Given X as in Theorem 1, the completion $X\langle n\rangle_p$ is irreducible if and only if X_p is.*

Our final result deals with the extent to which a space X is determined by any one of its connected covers $X\langle n\rangle$. A special case of this problem was treated in Theorem 7.4. Here we show that, under certain restrictions, the indeterminacy involved is finite and, in some cases, we can give a lower bound on this indeterminacy in terms of the completion genus of the space X.

THEOREM 10.1. *Let $\mathcal{C}$ be the class of all 1-connected finite CW-complexes with π_2X torsion. Then for each $X \in \mathcal{C}$ and for each n, there are, up to homotopy, at most a finite number of $Y \in \mathcal{C}$ such that $X\langle n\rangle = Y\langle n\rangle$. Moreover, if $X \in \mathcal{C}$ with $\pi_nX \otimes \mathbb{Q} = 0$ for n sufficiently large, then for each $Y \in \mathcal{C}$, it follows that $X\langle n\rangle = Y\langle n\rangle$ if and only if $X_p = Y_p$ for each prime p.*

The Lie group $\mathrm{SU}(n)$ is a good example to consider here. It is known that when $n \geq 3$, the genus of $\mathrm{SU}(n+1)$ has order at least $\frac{\phi(6)}{2} \cdot \frac{\phi(24)}{2} \cdot \ldots \cdot \frac{\phi(n!)}{2}$, where ϕ denotes the Euler ϕ function [31]. A little arithmetic then shows that, up to homotopy, there are at least 6,144 different finite complexes X such that $X\langle m\rangle = \mathrm{SU}(7)\langle m\rangle$ when $m \geq 13$.

This concludes the discussion of the results in this paper. We now turn to their proofs.

4. Proofs. The following result is an immediate consequence of Neisendorfer's theorem; it will be used in a few of the proofs which follow.

COROLLARY 1.1. *Let X and Y denote the p-completions of two spaces which satisfy the hypothesis of Theorem* 1. *Then the pointed mapping spaces* $\mathrm{map}_*(X, Y)$ *and* $\mathrm{map}_*(X\langle n\rangle, Y\langle n\rangle)$ *are homotopy equivalent for all $n \geq 2$*; *in particular*, *there is a bijection of pointed sets*

$$[X, Y] \approx [X\langle n\rangle, Y\langle n\rangle]$$

given by $f \mapsto f\langle n\rangle$ with inverse $g \mapsto \mathcal{L}_p(g)$.

Proof of Proposition 3.2. By hypothesis, there is a fibration

$$K(\pi, b-1) \to X\langle b\rangle \to X$$

and a prime p such that $H^*(K(\pi, b-1); \mathbb{Z}/p)$ contains an element, say x, of infinite height. The results of Serre and Cartan on the cohomology of Eilenberg–MacLane spaces are relevant here, of course. Consider the Serre spectral sequence in mod p cohomology for this fibration, and regard x as an element of $E_2^{0,*}$. Since the differentials are derivations it follows that x^p survives to E_3, and that $(x^p)^p$ survives to E_4, and so on. However, since the base X is a finite complex, there can only be a finite number of nonzero differentials. Thus some finite power of x is an infinite cycle. Using the edge homomorphism it follows that there exists a class $y \in H^*(X\langle n\rangle; \mathbb{Z}/p)$ which maps to a nonzero power of x. Since x has infinite height, so must y.

Proof of Proposition 3.3. Consider the fibration

$$\Omega X^{(n)} \to X\langle n\rangle \to X$$

and note that the fibre at a large enough prime p decomposes into a product of Eilenberg–MacLane spaces, at least one of which is nontrivial by assumption. This follows because the fibre, being an H-space of finite type, has k-invariants of finite order. A Serre spectral sequence argument, similar to the one that occurred in the proof of Proposition 3.2, now shows that the reduced cohomology $\widetilde{H}^*(X\langle n\rangle; \mathbb{Z}/p)$ contains an element of infinite height.

Proof of Proposition 3.4. We use the fibration

$$\Omega X^{(m)} \to X\langle m\rangle \to X,$$

where X is a nontrivial 1-connected finite H-space and where m is large enough that $X^{(m)}$ is nontrivial. It follows that $X^{(m)}$ is rationally nontrivial as well, by the loop theorem of Lin and Kane. The Atiyah–Hirzebruch–Serre spectral sequence (AHSSS) for this fibration, with coefficients in the Morava K-theory, has the E_2 term

$$E_2^{p,q} = H^p(X; K(n)^q \Omega X^{(m)})$$

and it converges to $K(n)^* X\langle m\rangle$. From the results of [25] and [14] it follows that the reduced Morava K-theory of $\Omega X^{(m)}$ contains elements of infinite height. Now the AHSSS with coefficients in a multiplicative cohomology theory is multiplicative; see e.g. [7]. The rest of the proof then proceeds just as in 3.2.

Proof of Example 4.1. Fix $n \geq 2$ and, to simplify notation, let $W = S^{2n}\langle 2n\rangle$. Then, of course, W is a $2n$-connected space of finite type with the rational homotopy type of S^{4n-1}. Each member of its Mislin genus can be obtained as a homotopy pullback of a diagram of the following sort:

$$\begin{array}{ccc} & & \widehat{W} \\ & & \downarrow{\scriptstyle j} \\ W_0 & \xrightarrow{f} & \overline{W} \end{array}$$

Here W_0 denotes the rationalization, $\widehat{W}$ is the profinite completion and $\overline{W}$ denotes Sullivan's formal completion [28]. The vertical map j is fixed. It first rationalizes and then identifies $(\widehat{W})_0$ with $\overline{W}$. This identification is valid for 1-connected spaces of finite type. The horizontal map is the standard inclusion $i : W_0 \to \overline{W}$ followed by a suitable self equivalence of $\overline{W}$. Here suitable means that the induced automorphism on homotopy groups is a $\widehat{\mathbb{Q}}$-module isomorphism ([3]), where $\widehat{\mathbb{Q}} = \mathbb{Q} \otimes \widehat{\mathbb{Z}}$. The group of such self-equivalences is denoted by $\mathrm{CAut}(\overline{W})$. The following double coset formula, due to Wilkerson [30],

$$\mathcal{G}(W) \approx i_* \operatorname{Aut}(W_0) \backslash \mathrm{CAut}(\overline{W}) / j_* \operatorname{Aut}(\widehat{W})$$

enables one to describe this genus set algebraically ([4]). Notice that

$$W_0 \simeq K(\mathbb{Q}, 4n-1) \quad \text{and} \quad \overline{W} \simeq K(\widehat{\mathbb{Q}}, 4n-1).$$

Thus $\operatorname{Aut}(W_0)$ is isomorphic to the multiplicative group of nonzero rationals $\mathbb{Q}^*$, while $\mathrm{CAut}(\overline{W})$ is isomorphic to the group of units in $\widehat{\mathbb{Q}}$. In particular,

([3]) The profinite completion of the integers is denoted here by $\widehat{\mathbb{Z}}$. It is isomorphic to the product $\prod_p \mathbb{Z}_p$, over all primes, of the p-adics.

([4]) This formula actually determines the set $\widehat{G}_0(W)$ which contains the Mislin genus. In this special case the two sets can be seen to coincide using the methods of [19].

these groups are abelian and so this double coset space has a natural group structure. The induced inclusion $\mathbb{Q}^* \to \widehat{\mathbb{Q}}^*$ is essentially the diagonal embedding. This makes sense since each nonzero rational is a p-adic unit for almost all primes p.

Since $\mathrm{Aut}(\widehat{W}) = \prod_p \mathrm{Aut}(W_p)$, it is necessary to determine the image of $\mathrm{Aut}(W_p) \to \mathrm{Aut}((W_p)_0)$ for each prime. The function

$$\mathbb{Z} \approx [S^{2n}, S^{2n}] \xrightarrow{f \mapsto (f\langle 2n\rangle)_0} [W_0, W_0] \approx \mathbb{Q}$$

is easily seen to be the squaring map, $d \mapsto d^2$. This implies that the image of $\mathrm{Aut}(W_p)$ in $\mathrm{Aut}((W_p)_0)$ contains the squares of the p-adic units. On the other hand, if one completes at p, then the first step $f \mapsto f\langle 2n\rangle$ has an inverse by Corollary 1.1. Consequently, the image of $\mathrm{Aut}(W_p)$ is precisely the group of squares $\mathcal{U}_p^2 = \{u^2 \mid u \in \mathbb{Z}_p^*\}$. So we are led to consider the double coset space $\Delta(\mathbb{Q}^*)\backslash\widehat{\mathbb{Q}}^*/\prod_p \mathcal{U}_p^2$.

Consider a unit in $\widehat{\mathbb{Q}}$; it can be viewed as a sequence

$$u = (2^{\varepsilon_2}u_2, 3^{\varepsilon_3}u_3, \ldots, p^{\varepsilon_p}u_p, \ldots),$$

where the integer exponents ε_p are zero for almost all primes and where each u_p lies in $\mathbb{Z}_p^*$. Thus if $r = 2^{-\varepsilon_2}3^{-\varepsilon_3}\ldots p^{-\varepsilon_p}\ldots$, then r is a rational number and every component of ru is integral. To put it another way, the obvious map

$$\widehat{\mathbb{Z}}^* \to \mathbb{Q}^*\backslash\widehat{\mathbb{Q}}^*$$

is surjective. The kernel here is clearly $\{\pm 1\}$. Let $\mathcal{P}$ denote the set of all rational primes and consider the homomorphism

$$\Phi : \widehat{\mathbb{Z}}^* \to (\mathbb{Z}/2)^{\mathcal{P}}$$

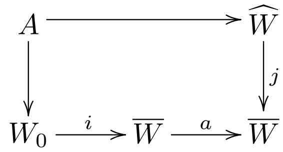
whose pth coordinate is the Legendre symbol (u/p) if p is odd and whose first coordinate is ± 1 depending upon the mod 8 reduction of u_2. This homomorphism is surjective and its kernel is $\prod_p \mathcal{U}_p^2$. It follows that the double coset space in question is uncountably large.

Suppose that Y is the m-connected cover of a finite complex K, and that Y is in the same Mislin genus as W. We intend to show that $Y \simeq W$. First, it is obvious that $m = 2n$. Since K is a finite complex, its universal cover $K\langle 1\rangle$ is finite-dimensional and hence is B-null by Miller's theorem. Then, by Theorem 7.5, there is a space E over K which is 1-connected with $\pi_2 E$ torsion and with $\mathcal{L}_p(Y) = E_p$ for each prime. It follows that

$$E_p \simeq \mathcal{L}_p(Y) \simeq \mathcal{L}_p(W) \simeq S_p^{2n}$$

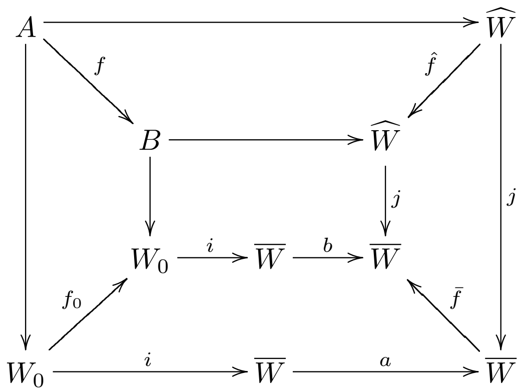
for each p. If E has finite type, then it follows easily that $E \simeq S^{2n}$ and hence $Y \simeq E\langle 2n\rangle \simeq W$. If E does not have finite type, then the group $\pi_{2n}E$ cannot be finitely generated (because the other homotopy groups clearly are). The only way this could happen would be if $\pi_{2n}E$ contains elements which are infinitely divisible. The Whitehead pairing would then imply the same is true of $\pi_{4n-1}E$. But this group is isomorphic to $\pi_{4n-1}Y$, which is finitely generated. Thus E must have finite type and we conclude that W is the only member of its genus which covers a finite complex.

We now investigate the maps within the genus of $S^{2n}\langle 2n\rangle$. Suppose that A is a member of this genus. Then A corresponds to a unit, say a, in $\widehat{\mathbb{Q}}$ in the sense that there is a homotopy pullback diagram



In this diagram we have identified a with the self-map of $\overline{W} = K(\widehat{\mathbb{Q}}, 4n-1)$ which induces multiplication by a on $\pi_{4n-1}\overline{W}$. The unlabeled vertical map in this diagram rationalizes A and also identifies A_0 with W_0. Similar remarks apply to the unlabeled horizontal map.

Given another member B in the genus of W and a map $f : A \to B$, it follows easily that there is a diagram



which commutes up to homotopy. The map $\widehat{f}$ induces multiplication by d^2 on $\pi_{4n-1}\widehat{W} \otimes \mathbb{Q}$ for some $d \in \widehat{\mathbb{Z}}$; similarly, f_0 induces multiplication by some rational r on $\pi_{4n-1}W_0$. The commutativity of the big diagram implies that $ad^2 = br$.

Assume now that the map f is essential. It is easy to check that there are no essential phantom maps between A and B and so the completion $\widehat{f}$ must be essential. It then follows from Corollary 1.1 that $d \neq 0$, and hence $r \neq 0$ as well. Since a, b, and r are units in $\widehat{\mathbb{Q}}$, so is d. Thus $d = su$ for some nonzero rational s and some unit $u \in \widehat{\mathbb{Z}}$. Solving for b, we get

$$\frac{s^2}{r} \cdot a \cdot u^2 = b.$$

But this means that a and b are in the same double coset, and so $A = B$, up to homotopy. This completes our analysis of Example 4.1.

Proof of Example 5.1. The space $X\langle n\rangle$ has the rational homotopy type of the product of r copies of S^{2n-1} and so it is an H_0-space. For such a space Y, the main result of [19] states that $\mathrm{SNT}(Y)$ is the one-element set if and only if the canonical map $\mathrm{Aut}(Y) \to \mathrm{Aut}(Y^{(m)})$ has a finite cokernel for each m. In particular, let $Y = X\langle n\rangle$ and let $X\langle n, 2n\rangle$ denote the Postnikov approximation $Y^{(2n-1)}$, which therefore has nonzero homotopy groups only in dimensions q where $n < q < 2n$. Thus $X\langle n, 2n\rangle$ is a Postnikov section which is rationally a product of r spheres each of dimension $2n-1$. It follows that there is a homology representation

$$\mathrm{Aut}(X\langle n, 2n\rangle) \to \mathrm{GL}(r, \mathbb{Z})$$

given by

$$f \mapsto H_{2n-1}(f; \mathbb{Z})/\mathrm{torsion}$$

with finite kernel and finite cokernel. Since $r \geq 2$, this implies $\mathrm{Aut}(X\langle n, 2n\rangle)$ is infinite. We will show that the image of $\mathrm{Aut}(X\langle n\rangle)$ in this group is finite; to this end consider the two representations

$$\mathrm{Aut}(X) \to \mathrm{GL}(r, \mathbb{Z})$$

given by

$$f \mapsto H_n(f; \mathbb{Z}) \quad \text{and} \quad f \mapsto \pi_{2n-1}(f)/\mathrm{torsion}.$$

When n is even there are no maps $S^n \times S^n \to S^n$ which restrict to rational equivalences on both factors. It follows that one can choose a basis for $H_n(X; \mathbb{Z})$ such that in both representations, no matrix has two or more nonzero entries in any row or column. Indeed, the second representation consists solely of the permutation matrices whose entries are zeros and ones. This follows from basic properties of the Whitehead product. The image of each representation is clearly finite. Using Neisendorfer's localization functor it then follows that the image of

$$\mathrm{Aut}(X\langle n\rangle) \to \mathrm{Aut}(X\langle n, 2n\rangle)$$

is finite as well. Its cokernel is thus infinite and so by Theorem 3 of [19], the set $\mathrm{SNT}(X\langle n\rangle)$ is uncountably large.

A glance back at Example 6.1 reveals it to be a special case of the following result.

Proposition 6.0. *Let X be a finite complex which satisfies the hypothesis of Theorem* 1 *and assume that Y is a nilpotent finite complex such that $[X, Y] = [\Sigma X, Y] = *$. Then for each natural number n,*

$$[X\langle n\rangle, Y] \approx \mathrm{Ph}(X\langle n\rangle, Y) \approx \prod_k H^k(X\langle n\rangle; \pi_{k+1}Y \otimes \mathbb{R}).$$

The bijections here are those of pointed sets. The first one says that every map from $X\langle n\rangle$ to Y is a phantom map; the second reduces the computation of such homotopy classes to a rational calculation. Here $\mathbb{R}$ denotes the real numbers regarded only as a rational vector space. The result stated here is not the most general one; for example, Y could be replaced by a localization of itself. We leave to the reader the task of further generalizations from finite complexes to B-null spaces.

P r o o f. Consider maps from the spaces in the principal fibration

$$\Omega X^{(n)} \to X\langle n\rangle \xrightarrow{j} X$$

to the profinite completion $\widehat{Y}$. According to a theorem of Zabrodsky (Theorem 5.6 of [17]), the function space $\mathrm{map}_*(\Omega X^{(n)}, \widehat{Y})$ is weakly contractible and so, by the Zabrodsky Lemma (Lemma 5.5 *ibid.*), the map j induces a weak equivalence

$$\mathrm{map}_*(X, \widehat{Y}) \approx \mathrm{map}_*(X\langle n\rangle, \widehat{Y}).$$

In particular, this means that

$$[X\langle n\rangle, \widehat{Y}] = [X, \widehat{Y}] = *.$$

It follows that the only maps from $X\langle n\rangle$ to Y are phantom maps because between spaces of finite type these are the only maps which vanish when completed (Theorem 5.1 *ibid.*). The first bijection is thus established.

Given a connected, nilpotent space W, there is a well-known sequence

$$W_\tau \to W \xrightarrow{r} W_0,$$

which is both a fiber sequence and a cofiber sequence. As usual, W_0 denotes the rationalization of W. By Theorem 5.1 *ibid.*, phantom maps are precisely those maps which factor through the rationalization of their domain; that is,

$$\mathrm{Ph}(W, Y) = r^*[W_0, Y].$$

Another result of Zabrodsky (Theorem 5.2 *ibid.*) is the bijection

$$[W_0, Y] \approx \prod_k H^k(W; \pi_{k+1} Y \otimes \mathbb{R}).$$

Therefore to complete the proof of 6.0, it suffices to show that the induced function $r^* : [(X\langle n\rangle)_0, Y] \to [X\langle n\rangle, Y]$ is injective. This will follow by exactness once we show that $[\Sigma(X\langle n\rangle)_\tau, Y] = *$. To this end, we use the bijections

$$* = [X, \widehat{\Omega Y}] = [X\langle n\rangle, \widehat{\Omega Y}] = [(X\langle n\rangle)_\tau, \widehat{\Omega Y}].$$

The first follows since $[\Sigma X, Y] = *$, by hypothesis. The next is an application of the Zabrodsky Lemma, as at the beginning of the proof with Y replaced by ΩY. The last is another application of the Zabrodsky Lemma, namely to the principal fibration

$$\Omega(X\langle n\rangle)_0 \to (X\langle n\rangle)_\tau \to X\langle n\rangle,$$

together with the weak contractibility of $\operatorname{map}_*(\Omega(X\langle n\rangle)_0, \widehat{\Omega Y})$. Therefore, it follows that every map from $(X\langle n\rangle)_\tau$ to ΩY vanishes upon completion. Since the domain here does not have finite type in general, this means that every map from $(X\langle n\rangle)_\tau$ to ΩY is a phantom map of the second kind; that is, its restriction to any finite subcomplex of the domain is null-homotopic. But since the domain here is a torsion space (that is, its integral homology groups have only torsion in positive degrees), it follows from ([11], Example 4.1) that there are no essential phantom maps in this case. Thus $[\Sigma(X\langle n\rangle)_\tau, Y] = *$ and the proof of Proposition 6.0 is complete.

Proof of Example 6.2. Given a connected nilpotent space Y of finite type, the members of $\mathrm{WI}(Y)$ are easily seen to be those classes in $\pi_0 \operatorname{map}(Y,Y)$ which rationalize to the identity and whose profinite completion is the identity. So consider the pullback square of mapping spaces induced by rationalization and profinite completion

$$\begin{array}{ccc} \operatorname{map}(Y,Y) & \longrightarrow & \operatorname{map}(Y,\widehat{Y}) \\ \downarrow & & \downarrow \\ \operatorname{map}(Y,Y_0) & \longrightarrow & \operatorname{map}(Y,\overline{Y}) \end{array}$$

Again $\overline{Y}$ denotes the formal completion of Y; it is homotopy equivalent to $(\widehat{Y})_0$. Now take the corresponding Mayer–Vietoris sequence [10]. For our purposes the relevant portion of this sequence is

$$\pi_1 \operatorname{map}(Y,Y_0) \times \pi_1 \operatorname{map}(Y,\widehat{Y}) \to \pi_1 \operatorname{map}(Y,\overline{Y}) \to \pi_0 \operatorname{map}(Y,Y)$$

From this one obtains the double coset presentation

$$\mathrm{WI}(Y) \approx \pi_1 \operatorname{map}(Y,Y_0)\backslash \pi_1 \operatorname{map}(Y,\overline{Y})/\pi_1 \operatorname{map}(Y,\widehat{Y}),$$

where the left and right subgroups are embedded by the formal completion and rationalization functors respectively, and the basepoints are the obvious choices. Now let $Y = X\langle 2n\rangle$, where $X = S^{2n} \vee S^{4n}$. Thus

$$Y = (S^{2n} \vee S^{4n})\langle 2n\rangle \simeq S^{2n}\langle 2n\rangle \vee S^{4n}.$$

Then

$$\pi_1 \operatorname{map}(Y,Y_0) \approx \pi_1 \operatorname{map}(S^{4n-1} \vee S^{4n}, Y_0) \approx \pi_{4n} Y_0 \oplus \pi_{4n+1} Y_0 \approx \mathbb{Q}.$$

Similarly, $\pi_1 \operatorname{map}(Y,\overline{Y}) \approx \widehat{\mathbb{Q}}$. However, using Neisendorfer's theorem,

$$\begin{aligned} \pi_1 \operatorname{map}(Y,\widehat{Y}) &\approx \pi_1 \operatorname{map}(\widehat{Y},\widehat{Y}) \approx \pi_1 \operatorname{map}(\widehat{X},\widehat{X}) \\ &\approx \pi_1 \operatorname{map}(S^{2n} \vee S^{4n}, \widehat{X}) \approx \pi_{2n+1}\widehat{X} \oplus \pi_{4n+1}\widehat{X}. \end{aligned}$$

As $\pi_1 \operatorname{map}(Y,\widehat{Y})$ is evidently a finite group, its image in $\pi_1 \operatorname{map}(Y,\overline{Y})$ is trivial. Therefore $\mathrm{WI}(Y) \approx \widehat{\mathbb{Q}}/\mathbb{Q}$, which is uncountably large. Thus Example

6.2 is verified. The reader may have noticed that Example 6.1 could also have been verified directly with this sort of analysis.

The following fibration lemma is a special case of results of Dror–Farjoun, [9], or of Bousfield ([2], §4). It is a crucial tool in proving the results described in §2.

LEMMA 7.0. *Let* $F \xrightarrow{i} W \xrightarrow{\pi} Z$ *be a homotopy fiber sequence. Then*

$$L_\varphi(F) \xrightarrow{L_\varphi(i)} L_\varphi(W) \xrightarrow{L_\varphi(\pi)} L_\varphi(Z)$$

is a homotopy fiber sequence provided either (a) $L_\varphi(F) \simeq \bullet$, *or* (b) $L_\varphi(Z) \simeq Z$.

Remarks. An immediate consequence of part (a) is that the localizations $L_\varphi(W)$ and $L_\varphi(Z)$ are homotopy equivalent under the conditions stated. The hypothesis in part (b) is equivalent to saying that the space Z is B-null.

Proof of Theorem 7.1. Take the principal fibration

$$\Omega Y \to F \to X$$

induced by the map f. Since X is B-null, the localized fiber sequence

$$L_\varphi(\Omega Y) \to L_\varphi(F) \to L_\varphi(X)$$

is also a fibration by part (b) of Lemma 7.0. In this new fibration, the base space $L_\varphi(X) \simeq X$, since X is B-null; the other two spaces are easily seen to be simple. Hence the p-completion of this fibration is again a fibration. Now $\mathcal{L}_p(K(\pi, m)) \simeq \bullet$ for any abelian group when $m \geq 2$ and any torsion abelian group when $m = 1$, by ([4], §7). Thus a finite induction, going up the Postnikov tower, shows $\mathcal{L}_p(\Omega Y) \simeq \bullet$. The above fibration, completed at p, thus yields

$$\bullet \to \mathcal{L}_p(F) \to X_p$$

and hence a homotopy equivalence between the new total space and base.

Proof of Theorem 7.4. The proof involves the following commutative diagram:

$$\begin{array}{ccccc}
\Omega G & \longrightarrow & \bullet & \longrightarrow & G \\
\downarrow & & \downarrow & & \downarrow \\
X\langle n\rangle & \longrightarrow & X & \longrightarrow & X^{(n)} \\
\downarrow & & \downarrow & & \downarrow \\
F & \longrightarrow & X & \longrightarrow & X_0^{(n)}
\end{array}$$

in which all the rows and columns are homotopy fiber sequences. The fiber G has only torsion homotopy groups and at most $n-1$ of them are nonzero. Thus $L_\varphi(\Omega G) \simeq \bullet$ as in the proof of 7.1. Next apply L_φ to the vertical fiber sequence on the left. It follows that $L_\varphi(X\langle n\rangle) \simeq L_\varphi(F)$, by part (a) of Lemma 7.0. In the fiber sequence along the bottom, the functor L_φ fixes both the base and the total space. Therefore it also fixes the fiber, that is, $L_\varphi(F) \simeq F$, again by Lemma 7.0, part (b). Thus $L_\varphi(X\langle n\rangle) \simeq F$, as claimed.

Proof of Theorem 7.5. Let $\pi = \pi_2 X$ and let T denote its torsion subgroup. Let E denote the homotopy fiber of the composition

$$X \to X^{(2)} = K(\pi, 2) \to K(\pi/T, 2).$$

The first map is the usual inclusion and the last is induced by the quotient homomorphism $\pi \to \pi/T$. We thus have a fiber sequence

$$K(\pi/T, 1) \to E \xrightarrow{j} X.$$

There are no essential maps from $B\mathbb{Z}/p$ into the fiber here. This follows from the universal coefficient sequence for cohomology with coefficients in the torsion-free group π/T. Consequently, the fiber is B-null. The base X is B-null by assumption. It then follows from Lemma 7.0, part (b), that the total space E is also B-null.

Notice that the map $j : E \to X$ induces a homotopy equivalence $E\langle n\rangle \simeq X\langle n\rangle$ for each $n \ge 2$. Thus for these values of n we have

$$\mathcal{L}_p(X\langle n\rangle) \simeq \mathcal{L}_p(E\langle n\rangle) \simeq E_p$$

by Theorem 7.1.

Proof of Proposition 8.1. Fix a prime p, assume that $t > 3$, and let Y denote the p-completion of $\Omega^{t-3}X$. It suffices to show that the loop space $Y\langle m\rangle$ has no geometric exponent for any integer $m \ge 1$. Take the principal fibration

$$\Omega Y^{(m)} \to Y\langle m\rangle \to Y$$

and apply $\mathcal{L}_p$ to it. Since X is B-null, so is Y, and thus

$$\mathcal{L}_p(\Omega Y^{(m)}) \to \mathcal{L}_p(Y\langle m\rangle) \to Y$$

is a homotopy fiber sequence. Assume for the moment that the homotopy groups of the new fiber vanish in dimensions greater than 1. It then follows by exactness that $\pi_3\mathcal{L}_p(Y\langle m\rangle) \approx \pi_3 Y$, which is not a torsion group, by hypothesis.

In general, there is a natural equivalence of loop spaces, $L_f(\Omega W) \simeq \Omega L_{\Sigma f}(W)$ (cf. [9]), and it evidently takes power maps to power maps. Thus the existence of a geometric exponent on $Y\langle m\rangle$ would imply the same for $\mathcal{L}_p(Y\langle m\rangle)$. But the power map $x \mapsto x^\lambda$ induces multiplication by λ on homotopy groups, and this is not the zero endomorphism of $\pi_3\mathcal{L}_p(Y\langle m\rangle)$

in particular, unless $\lambda = 0$. Thus $Y\langle m\rangle$ has no geometric exponent at any prime p.

To finish the proof, let π_i stand for $\pi_i \Omega Y^{(m)}$. We will show that

$$\mathcal{L}_p(\Omega Y^{(m)}) \simeq \pi_0 \times K(\pi_1/\text{torsion}, 1).$$

Since the path components of $\Omega Y^{(m)}$ all have the same homotopy type it suffices to determine the localization of any one of them. So let P denote a path component of $\Omega Y^{(m)}$. There is a fibration

$$F \to P \to K(\pi_1/\text{torsion}, 1)$$

in which $\pi_1 P$ maps onto $\pi_1/\text{torsion}$. Since the base space of this fibration is B-null (as was noted in the proof of 7.5), the application of $\mathcal{L}_p$ yields the homotopy fiber sequence

$$\mathcal{L}_p(F) \to \mathcal{L}_p(P) \to K(\pi_1/\text{torsion}, 1).$$

Since the fiber F is a Postnikov space with a torsion fundamental group, $\mathcal{L}_p(F) \simeq \bullet$ as in the proof of 7.1. Consequently, $\mathcal{L}_p(P)$ is a $K(G, 1)$ as claimed. This completes the proof of 8.1.

Proof of Proposition 9.1. Suppose that μ is a multiplication on $X\langle n\rangle$. Dwyer has shown that there is a natural equivalence between $L_f(Y \times Z)$ and $L_f(Y) \times L_f(Z)$ (see [9]). Completion at p is also known to respect products [3]. It follows easily from these facts that $\mathcal{L}_p(\mu)$ is a multiplication on X_p. Suppose that μ is homotopy commutative. Thus $\mu \simeq \mu T$, where T is the twist map on $X\langle n\rangle \times X\langle n\rangle$. It is easy to check that $\mathcal{L}_p(T)$ is homotopic to the twist map on X_p. Thus it follows by functoriality that X_p has a homotopy commutative multiplication, namely $\mathcal{L}_p(\mu)$. The proof for homotopy associativity amounts to applying $\mathcal{L}_p$ to the usual diagram.

Proof of Proposition 9.2. A p-complete space Y is irreducible if and only if $[Y, Y]$ contains no nontrivial idempotents [1]. Clearly, the n-connected cover functor and its inverse $\mathcal{L}_p$ take idempotents to idempotents, and so the result follows.

Proof of Theorem 10.1. If $X\langle n\rangle = Y\langle n\rangle$, then $X_p \simeq Y_p$ for each prime p, by Theorem 1. Thus X and Y are in the same completion genus. Wilkerson has shown that the completion genus of X is a finite set of homotopy types when X is a 1-connected finite CW-complex [30]. Thus given $X\langle n\rangle$ there are at most a finite number of possibilities for Y in $\mathcal{C}$ with $Y\langle n\rangle = X\langle n\rangle$. For the second statement, note that if X and Y are in the same completion genus, then clearly $X\langle n\rangle_p \simeq Y\langle n\rangle_p$ for each prime. If X is rationally elliptic, then $X\langle n\rangle$ is rationally trivial for n sufficiently large and thus $X\langle n\rangle \simeq \prod_p X\langle n\rangle_p$. The result follows.

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