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On the $*$-product in kneading theory

by

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Abstract. We discuss a generalization of the $*$-product in kneading theory to maps with an arbitrary finite number of turning points. This is based on an investigation of the factorization of permutations into products of permutations with some special properties relevant for dynamics on the unit interval.

1. Introduction. Kneading theory is a powerful symbolic method used to study the combinatorial aspects of the dynamics of one-dimensional maps. The foundations of this theory go back to [MTh] (which was circulated as a preprint since 1977), with early precursors in [My] and [MSS]. In the special case of continuous maps on the interval with a single turning point, or *unimodal maps*, a particular operation on symbolic sequences, the $*$-product (cf. [DGP], see also the monograph [CEc]), is of paramount importance for its utility in the description of renormalization (for a recent review of renormalization, see the monograph [MSt] which contains a huge bibliography, and [CTr], [TCo], [Fe1], [Fe2] for early references). It is thus natural to hope for an appropriate generalization of the $*$-product to maps with an arbitrary number of turning points which would play an analogous role in a more general theory of renormalization. Previous special cases of this program appear in [Mu], [LMu1], and [LMu2] for maps with two turning points. The aim of this paper is to extend the $*$-product to a large class of renormalizable maps with an arbitrary finite number of turning points. Extensions of the $*$-product to some kinds of piecewise-continuous one-dimensional maps appear in [PTT] and [LMu2].

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Some basic material is assembled in Section 2. Most of the discussion in this paper involves an interplay between kneading theory and the theory of permutations induced on finite orbits of n points embedded in the real line; the basic material in Section 2 is presented accordingly. In Section 3 we quickly review the classical theory of the $*$-product in a form suitable for the rest of this exposition. Section 4 contains a brief discussion of maps which can occur as a composition of unimodal maps. This is the prelude to our first main result in Section 5, which explains how to extract unimodal factors from products. Section 6 then combines the procedure of recovering unimodal factors in a product with known results in kneading theory to define a $*$-product for a large class of finite symbolic sequences. We indicate how this definition can be extended to infinite sequences in Section 7.

The description of the $*$-product is quite explicit in the case of unimodal maps, both for finite or infinite sequences. In this case we can write down the formula for the product from the sole knowledge of the symbols of the factors. The case of a greater number of turning points is different in nature. Here we must first factor one of the terms entering the $*$-product, so that the $*$-product cannot be described explicitly in terms of the symbols of its factors. If one factor has infinite length, its factorization cannot be obtained directly in a finite number of steps and we offer instead a method to get successive approximations to the $*$-product.

2. Basics

2.1. A pair (E, f), where E is a non-empty topological space and $f : E \to E$ is a continuous map, constitutes a simple example of a *dynamical system*. The semigroup $\mathbb{Z}^+$ acts on E by iteration of the map f. We use Milnor's notation:

$$f^{\circ 0}(x) = x, \quad f^{\circ 1}(x) = f(x), \ \ldots, \ f^{\circ n}(x) = f(f^{\circ (n-1)}(x)). \tag{1}$$

The *orbit* of a point $x \in E$ is the set

$$O(x) = \{x, f(x), f^{\circ 2}(x), \ldots\}. \tag{2}$$

We say that $f^{\circ n}$ is the *nth iterate of* f, and that $f^{\circ n}(x)$ is the *nth iterate of* x. The point x is *periodic* if $O(x) = O(f(x))$; the *period* of x (or of $O(x)$) is then the number of elements $|O(x)|$ of the set $O(x)$ (in general, we denote by $|F|$ the cardinal of the set F). If the period of x is p, we also say that x or $O(x)$ is *p-periodic*.

A *morphism* (or *semi-conjugacy*) from (E, f) to (E', f') is a continuous onto map $h : E \to E'$ such that $f' \circ h = h \circ f$. If h is invertible, we call it a *conjugacy* and in this case say that the dynamical systems (E, f) and (E', f') are *topologically conjugate*, and write this as $f \cong f'$.

2.2. Let I be an interval in the real line, and assume that the points $x_1 < \ldots < x_p$ comprise a union of periodic orbits of $f : I \to I$. Then there exists a permutation $\sigma(\{x_1, \ldots, x_p\}) = \sigma \in S_p$, the symmetric group on p elements, such that

$$f(x_i) = x_{\sigma(i)}. \tag{3}$$

For instance, if $E = \{x_1, \ldots, x_p\}$ forms a single periodic orbit, then σ is a p-cycle in S_p. Let h be any orientation preserving homeomorphism from I to some interval I' containing the integers $1, \ldots, p$, and such that $h(E) = \{1, \ldots, p\}$. If we let $g = h \circ f \circ h^{-1}$, then $g|_{\{1,\ldots,p\}} = \sigma$; we write $f|_E \sim \sigma$ and say that σ *represents* $f|_E$ and that f *extends* σ.

Let $k : J \to J$ be another dynamical system for a real interval J and let $F \subset J$ be a finite union of finite orbits. If both $f|_E$ and $k|_F$ are represented by the same permutation, then we write $f|_E \sim k|_F$ and say that $f|_E$ and $k|_F$ are *combinatorially equivalent*. Define Δ_n to be the permutation in S_n which sends $n-j+1$ to j, for each $j = 1, \ldots, n$; if $|E| = |F| = n$, we say that $f|_E$ and $k|_F$ are *combinatorially similar* if both $f|_E$ and $k|_F$ are represented by the same permutation $\sigma \in S_n$, up to conjugacy of σ by a power of Δ_n (notice that $(\Delta_n)^2$ is the identity).

For early occurrences of explicit relationship between permutations and interval dynamics, see, e.g., [Bl] and [Be]. See also [MNi] and references therein.

2.3. Let I be the interval $[\alpha, \beta]$ and $f : I \to I$. If there is a sequence of *extremal points*

$$\alpha = c_{-1} < c_0 < \ldots < c_m = \beta \tag{4}$$

such that

$$\begin{aligned} &\text{for } -1 \le i \le m-1,\ f|_{[c_i, c_{i+1}]} \text{ is strictly monotone,} \\ &\text{for } -1 \le j \le m-2,\ f|_{[c_j, c_{j+2}]} \text{ is not monotone,} \end{aligned} \tag{5}$$

we say that f is *m-modal*. The points $c_0, c_1, \ldots, c_{m-1}$ are called the *turning points* of f, and the intervals $l_0 = [c_{-1}, c_0)$, $l_1 = (c_0, c_1)$, $\ldots$, $l_{m-1} = (c_{m-2}, c_{m-1})$, $l_m = (c_{m-1}, c_m]$ the *laps* of f. Thus f is m-modal if and only if f has m turning points. If f is m-modal for some $m \ge 0$, then we say that f is *multimodal* or *piecewise monotone*. The words *amodal*, *unimodal*, *bimodal*, etc. are used for $m = 0, 1, 2$, etc. If f is m-modal, we call $m = m(f)$ the *modality* of f. The numbers $f(c_i)$ for $i = -1, 0, \ldots, m$ are called the *extremal values* of f, and also the *turning values* if $i \in \{0, 1, \ldots, m-1\}$. A piecewise monotone map is *full* if it maps each extremal point to a boundary point of I.

Any multimodal map has a *shape* $s(f) = (s_0, s_1, \ldots, s_m)$, where $s_i \in \{+, -\}$ and where we write $s_i = +$ if f is increasing on (c_{i-1}, c_i) for $0 \le$

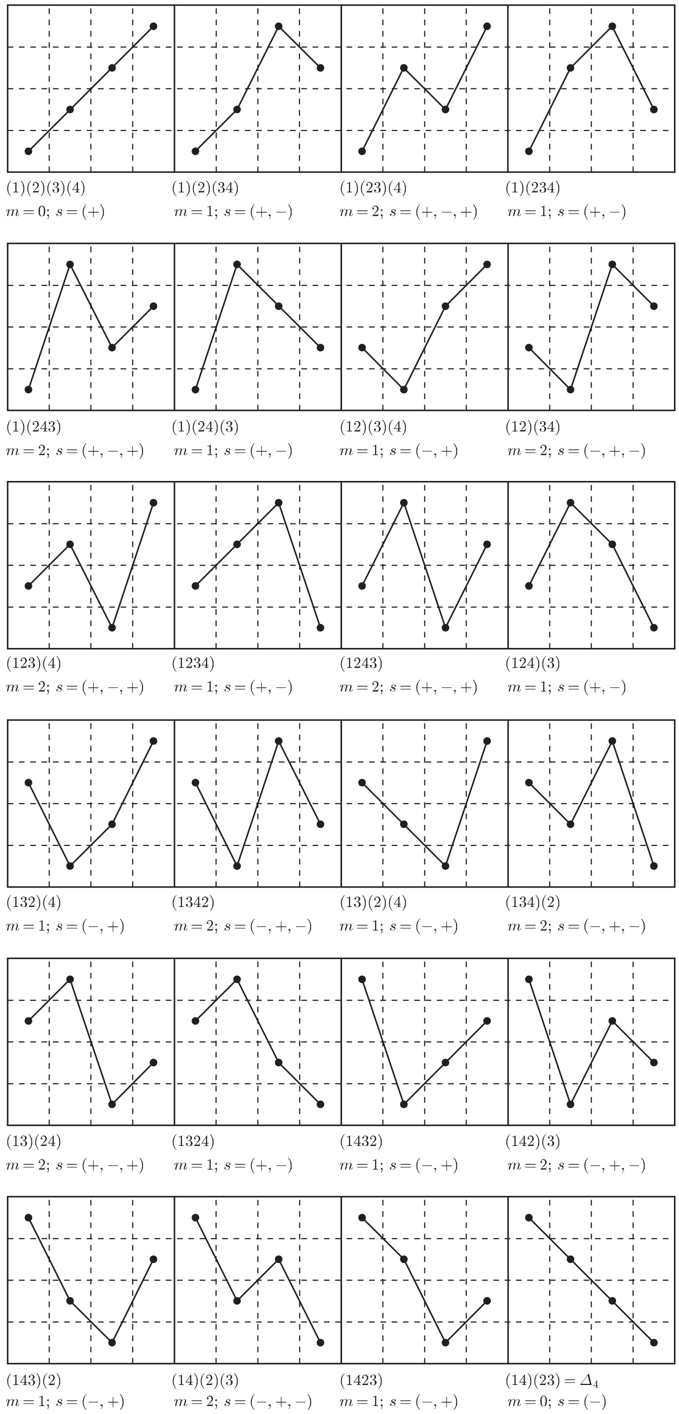
(1)(2)(3)(4)
$m=0$; $s=(+)$
(1)(2)(34)
$m=1$; $s=(+,-)$
(1)(23)(4)
$m=2$; $s=(+,-,+)$
(1)(234)
$m=1$; $s=(+,-)$
(1)(243)
$m=2$; $s=(+,-,+)$
(1)(24)(3)
$m=1$; $s=(+,-)$
(12)(3)(4)
$m=1$; $s=(-,+)$
(12)(34)
$m=2$; $s=(-,+,-)$
(123)(4)
$m=2$; $s=(+,-,+)$
(1234)
$m=1$; $s=(+,-)$
(1243)
$m=2$; $s=(+,-,+)$
(124)(3)
$m=1$; $s=(+,-)$
(132)(4)
$m=1$; $s=(-,+)$
(1342)
$m=2$; $s=(-,+,-)$
(13)(2)(4)
$m=1$; $s=(-,+)$
(134)(2)
$m=2$; $s=(-,+,-)$
(13)(24)
$m=2$; $s=(+,-,+)$
(1324)
$m=1$; $s=(+,-)$
(1432)
$m=1$; $s=(-,+)$
(142)(3)
$m=2$; $s=(-,+,-)$
(143)(2)
$m=1$; $s=(-,+)$
(14)(2)(3)
$m=2$; $s=(-,+,-)$
(1423)
$m=1$; $s=(-,+)$
$(14)(23)=\Delta_4$
$m=0$; $s=(-)$

Fig. 1

$i \leq m$, and $s_i = -$ if f is decreasing on this subinterval. Define $s_{i+1} = \overline{s}_i$ for $\overline{+} = -$ and $\overline{-} = +$. If $s(f) = (s_0, s_1, \ldots, s_m)$ we can say that f is (s_0)-*m*-*modal* with no loss of information regarding the shape of f. The first sign $s_0(f)$ in $s(f)$ is called the *type* of f.

2.4. We define the *modality of* σ (in S_n), written $m(\sigma)$, as the smallest modality of all maps on a real interval which extend σ. For instance, we have $m(\Delta_n) = 0$.

We identify the *graph* of a permutation $\sigma \in S_n$ with the set $\{(1, \sigma(1)), \ldots \ldots, (n, \sigma(n))\}$ of integer lattice points in $\mathbb{R}^2$. If σ represents $f|_E$, then the graph of σ will also be called the *graph of* $f|_E$, and will be denoted by $\mathrm{Graph}(f|_E)$. We define the *linear interpolation* of a permutation $\sigma \in S_n$, denoted by $\mathrm{lin}(\sigma)$, as the map from the interval $[1, n]$ to itself whose graph is obtained by linear interpolation between the successive points in the graph of σ; it is easy to check that $m(\sigma) = m(\mathrm{lin}(\sigma))$. We also set $s(\sigma) = s(\mathrm{lin}(\sigma))$, and define a *turning point of* σ as a turning point of $\mathrm{lin}(\sigma)$. As an example, the permutations in S_4 are represented in Figure 1 by their respective graphs; we have also drawn their linear interpolations and indicated their modalities and their shapes.

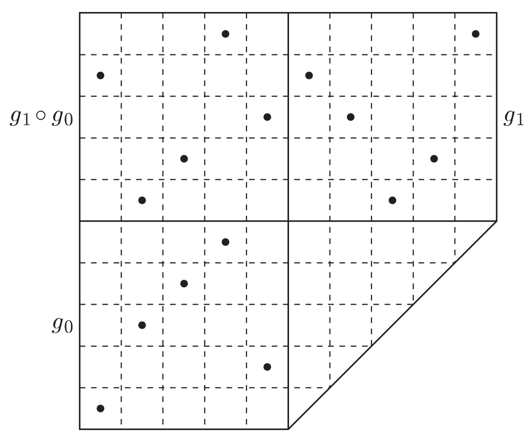

Fig. 2

The graphs of two permutations in S_n can be combined pictorially, as displayed in Figure 2 for $n = 5$, to represent the composition (i.e., multiplication) of the two permutations. We call such a representation a *composition machine in* S_n. The same picture also provides a means of performing the composition of extensions of the permutations. In particular, this immediately shows that the composition of the linear interpolations of two permutations is in general different from the linear interpolation of the composition of the permutations; in general, the former is not a linear interpolation.

When used to compose graphs of continuous maps which extend permutations in S_n, the composition machine will be called an *extended composition machine in* S_n.

Remark. The study of permutations from the point of view of modality and turning points is reminiscent of, but different from, the notions of *descents* and *rising sequences* of a permutation, which have been studied in the context of understanding the structure of the group algebra of the symmetric group as well as other Coxeter groups [So]. This has led to many interesting results in algebraic combinatorics, among which is a deeper understanding of the mathematics of card shuffling [BDi]. Similar results have recently been obtained for the analogously defined *turning point algebra* [DRSS].

2.5. With the notation of 2.3, we define the *address* of $x \in I$, written $a(x)$, to be one of the symbols L_i, $0 \le i \le m$, or C_j, $0 \le j < m$, according to:

$$a(x) = \begin{cases} L_i & \text{if } x \in l_i, \\ C_j & \text{if } x = c_j. \end{cases} \tag{6}$$

The *itinerary* of a point $x \in I$ is the sequence of addresses

$$a^*(x) = a(x), \quad a(f(x)), \quad a(f^{\circ 2}(x)), \ \ldots, \tag{7}$$

or equivalently, the (possibly infinite) word in the symbols L_i and C_j formed by concatenating the elements of the sequence (7).

Remark. If we are discussing two maps at the same time, or plan to use one map in conjunction with another, we may use the symbol K instead of C, and M instead of L.

We say that an orbit $O(x)$ is *critical* if it contains at least one turning point and *acritical* otherwise. The *criticality index* of an orbit is the number of turning points it contains.

2.6. Let $\mathcal{A}$ denote the set of possible addresses under f and $\mathcal{A}'$ the set of lap symbols. We endow $\mathcal{A}$ with the order $\ldots < C_k < L_{k+1} < C_{k+1} < \ldots$, so that $a(x) < a(y) \Rightarrow x < y$.

We set $P(L_i) = +$ if f is increasing on l_i, $P(L_i) = -$ if f is decreasing on l_i, and adopt the obvious multiplication rule for signs. For any finite word in the symbols L_i, $W = L_{i_1} \ldots L_{i_n}$, we naturally define the *parity* of W, denoted by $P(W)$, as

$$P(W) = P(L_{i_1}) \cdot \ldots \cdot P(L_{i_n}), \tag{8}$$

so that, if W represents the beginning of the itinerary of x, then

- W is *even*, i.e., $P(W) = +$, if $f^{\circ n}$ is sense-preserving near x,
- W is *odd*, i.e., $P(W) = -$, if $f^{\circ n}$ is sense-reversing near x.

Let $W_0 \neq W_1$ be words such that W_i, $i \in \{0,1\}$, begins AS_i..., where A is a finite word in $\mathcal{A}'$ and $S_i \in \mathcal{A}$. We set

$$W_0 < W_1 \Leftrightarrow \begin{cases} \text{either } A \text{ is even and } S_0 < S_1, \\ \text{or } A \text{ is odd and } S_0 > S_1. \end{cases} \tag{9}$$

Hence $a^*(x) < a^*(y) \Rightarrow x < y$.

For $i \in \{0, 1, \ldots, m-1\}$, we set

$$C_i^+ = L_{i+1} \quad \text{and} \quad C_i^- = L_i. \tag{10}$$

To shorten notation, for a finite word W in the L_i's, we shall write C_i^W for $C_i^{P(W)}$ and C_i^{-W} for $C_i^{-P(W)}$. It is easy to check that

$$WC_i^{-W} < WC_i < WC_i^W. \tag{11}$$

2.7. The itinerary of any point x of a p-periodic orbit is a periodic word whose period divides p. Let $W(x)$ be the word of length p that coincides with the beginning of $a^*(x)$. If $y \in O(x)$, then $W(y)$ is obtained from $W(x)$ by a cyclic permutation of the letters. For any such word $W(x)$, the *cyclic word* $\mathcal{S}(O(x))$ is defined to be the set of the images of all such cyclic permutations of the symbols; we sometimes identify this set with any of its elements and conversely. We also say that $\mathcal{S}(O(x))$ is the *symbol* of the p-periodic orbit $O(x)$. More generally, the *symbol of a set of periodic orbits* is the corresponding set of symbols of individual periodic orbits.

2.8. The *kneading data* of the m-modal map f consists of $s(f)$ together with the $(m+2)$-tuple

$$\{a^*(f(c_{-1})), a^*(f(c_0)), \ldots, a^*(f(c_m))\}$$

of itineraries of the extremal points. Two maps are called *monotone equivalent* if they have the same kneading data. We denote by $\mathrm{Mon}(f)$ the monotone equivalence class of f.

3. Renormalization and the ∗-product

3.1. Renormalization group theory was first explicitly adapted from statistical mechanics to dynamical systems theory in [CTr] and [TCo] (see also [Fe1], [Fe2] for similar material). Since then, various formulations have arisen, their precise definitions often depending on the intended application. Here, we reproduce the notion of renormalizability following [BORT] but only at a level of generality necessary for this discussion. The more "conventional" approach to renormalization group theory, adapted to continuous maps acting on an interval or some subinterval, is reviewed at length in [MSt], for instance. This conventional approach merely corresponds to what we call C-renormalizability below. Our more general framework is needed for applying combinatorial analysis to restrictions of a map on the interval to finite or Cantor subsets.

For any positive integer $n \geq 1$, let $(\mathbb{Z}/n\mathbb{Z}, +1)$ denote the dynamical system where $+1$ stands for the translation by 1 on the group $\mathbb{Z}/n\mathbb{Z}$ with the discrete topology, i.e.,

$$+1 : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}, \quad j \mapsto j+1 \pmod{n}. \tag{12}$$

Let I be an interval, $f : I \to I$, and E be a closed, f-invariant subset of I. We say that $(E, f|_E)$ (or $f|_E$) is *n-renormalizable* if there is a morphism ϕ from (E,f) to $(\mathbb{Z}/n\mathbb{Z}, +1)$ such that the fibers of ϕ are contained in n disjoint intervals, which we can denote as $I_0 < I_1 < \ldots < I_{n-1}$. Note that this implies that each $E_j = I_j \cap E$ is $f^{\circ n}$-invariant. Then we say that $f^{\circ n}|_{E_j}$ is the *jth renormalization* of f. We say that $f|_E$ is *renormalizable* if it is n-renormalizable for some $n \notin \{1, |E|\}$. Set $E' = I_0 \cup I_1 \cup \ldots \cup I_{n-1}$. If E' is f-invariant and $(E', f|_{E'})$ is also renormalizable, we say $f|_E$ is *C-renormalizable*. Finally, if $(E, f|_E)$ is both n-renormalizable and C-renormalizable we say that it is *n-C-renormalizable*.

Remarks. 1. The subscript i of I_i is understood as belonging to $\mathbb{Z}/n\mathbb{Z}$, so that any arithmetic involving indices is to be performed mod n.

2. Instead of translation by 1, any transitive permutation of the n elements of $\mathbb{Z}/n\mathbb{Z}$ could have been used in the definition of renormalization: the rationale for this particular choice will appear in Section 7 when we define infinitely renormalizable maps (see also [BORT]).

3.2. Here we describe an approach to the $*$-product for unimodal maps. The following proposition, formulated with the generality we need, is well known:

PROPOSITION 1. *Assume O is a critical p-periodic orbit of the unimodal map f. Assume furthermore that $f|_O$ is q-C-renormalizable for some $q \notin \{1,p\}$. Then the maps $f_j = f^{\circ q}|_{I_j}$ are topologically conjugate to each other.*

Proof. Notice that all but at most one of the $f|_{I_i}$'s are homeomorphisms. The proof now follows by comparing the formulas for the f_j's written as products of the $f|_{I_i}$'s. More precisely, assume that $C_0 \in I_k$ and let σ represent $f|_O$. Then we have the list:

$$\begin{aligned} f_k &= f|_{I_{\sigma^{(n-1)}(k)}} \circ \ldots \circ f|_{I_{\sigma^2(k)}} \circ f|_{I_{\sigma(k)}} \circ f|_{I_k}, \\ f_{\sigma^{-1}(k)} &= f|_{I_{\sigma^{(n-2)}(k)}} \circ \ldots \circ f|_{I_{\sigma^2(k)}} \circ f|_{I_{\sigma(k)}} \circ f|_{I_k} \circ f|_{I_{\sigma^{(n-1)}(k)}}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_{\sigma^{n-1}(k)} &= f|_{I_k} \circ f|_{I_{\sigma^{(n-1)}(k)}} \circ \ldots \circ f|_{I_{\sigma^2(k)}} \circ f|_{I_{\sigma(k)}}. \end{aligned}$$

Since $f|_{I_k}$ is the only non-homeomorphic restriction, each map in the list, except for the first, is obtained from the previous one by a conjugacy. ■

Using the notation of Proposition 1, and assuming that f is $(+)$-unimodal, let g denote a $(+)$-unimodal map, conjugate to a fixed $f^{\circ q}|_{I_j}$. Then g has a critical (p/q)-periodic orbit O'' and we let $\sigma_{\mathcal{R}} = g|_{O''} \in S_{(p/q)}$. We also write σ for $\sigma(O)$ (cf. Section 2.3) and $\sigma_{\mathcal{G}}$ for the element of S_q describing the way in which the E_i's are permuted by f, i.e.,

(13) $$f(E_j) = E_{\sigma_{\mathcal{G}}(j)}.$$

Let h be a $(+)$-unimodal map with a critical periodic orbit O' such that $\sigma(O') = \sigma_{\mathcal{G}}$.

Renormalizability implies that the symbols (cf. Section 2.7) of the various orbits O, O', and O'' are related. So let

(14) $$A = \mathcal{S}(O') = A'C_0, \qquad B = \mathcal{S}(O''), \qquad W = \mathcal{S}(O).$$

With Σ_A standing for the symbolic substitution

(15) $$L_0 \mapsto A'C_0^{-A'}, \qquad L_1 \mapsto A'C_0^{A'}, \qquad C_0 \mapsto A'C_0,$$

we define

(16) $$A' * B = \Sigma_A(B).$$

If f is $(-)$-unimodal, we assume the same for g and h. Then, with Σ_A standing for the symbolic substitution

(17) $$L_0 \mapsto A'C_0^{A'}, \qquad L_1 \mapsto A'C_0^{-A'}, \qquad C_0 \mapsto A'C_0,$$

we define, as in the $(+)$-unimodal case,

(18) $$A' * B = \Sigma_A(B).$$

Then we have

PROPOSITION 2. *$W = \Sigma_A(B)$.*

Proof. By inspection; see, e.g., [CEc], [BORT]. ■

Notice that σ is 1-modal while $\sigma_{\mathcal{G}}$ and $\sigma_{\mathcal{R}}$ are either 0- or 1-modal. Accordingly, we can extend the $*$-product to cyclic permutations of modality at most one by writing

(19) $$\sigma = \sigma_{\mathcal{G}} * \sigma_{\mathcal{R}}.$$

Note that a cyclic permutation can be 0-modal only in the case of $n \in \{1, 2\}$.

The $*$-product for unimodal maps appeared in [DGP]. See also [Mi] and [JR] for related ideas, and, e.g., [CEc], [BORT] for detailed expositions. The reinterpretation of the $*$-product as a substitution is from [PTT] (see also [BORT]).

3.3. Assume now that h is m-modal with $m \geq 1$ and that O' is a critical n-periodic orbit of h with criticality index 1, so that $\mathcal{S}(O') = A = A'C_i$ for a single turning point c_i such that $O' = O(c_i)$. Let O'' be a critical p-periodic orbit of some (s)-unimodal map g, where $s = s_i$, and let $\mathcal{S}(O'') = B$. If we define the substitution Σ_A by

$$L_0 \mapsto A'C_i^{\bar{s}_i A'}, \quad L_1 \mapsto A'C_i^{s_i A'}, \quad C_0 \mapsto A'C_i, \tag{20}$$

it is easy to check that $\Sigma_A(B)$ is the symbol of an n-renormalizable critical np-periodic orbit O with criticality index 1 of an m-modal map f. Consequently, we define

$$A' * B := \Sigma_A(B). \tag{21}$$

This solves the $*$-product construction for this special, essentially unimodal case. We leave the parallel discussion in terms of permutations to the reader. We will consider more general cases starting in Section 5, after we present some preparatory material in Section 4.

4. Topologically even maps

4.1. Let $f : I \to I$ be a continuous map. We say f is *topologically even* if there exists an even map $g : [-1,1] \to [-1,1]$ such that $f \cong g$. The following result is elementary.

PROPOSITION 3. *Let $f : I \to I$ be a piecewise monotone map (hence not constant on any non-trivial subinterval $J \subseteq I$). Then f is topologically even if and only if the collection of turning points and the boundary points of I can be written as*

$$\{c_{-r}, c_{-(r-1)}, \ldots, c_0, c_1, \ldots, c_r\} \quad \textit{with} \quad f(c_i) = f(c_{-i}), \quad i = 1, \ldots, r.$$

Proof. Necessity is obvious, so we only prove sufficiency. This is obtained by using the metric d which is the usual one on the right of c_0, and defined on the left of c_0 so that $d(c_i, c_0) = d(c_{-i}, c_0)$, and, for $y \in (c_{-(i+1)}, c_{-i})$ and $x \in (c_i, c_{i+1})$ with $f(y) = f(x)$, $d(y, c_0) = d(c_0, x)$. It is also possible to use the metric defined in 4.2 below. ■

In particular, every unimodal endomorphism f of $I = [\alpha, \beta]$ such that $f(\alpha) = f(\beta) \in \{\alpha, \beta\}$ is topologically even. The turning point c_0 is called *central*.

4.2. Given a topologically even map f, conjugated by h to the even map g, we will need to measure the distance from a point in the interval to the central point c_0, in such a way that x is closer to c_0 than y (in this distance) if and only if $h(x)$ is closer to 0 than $h(y)$ in the usual sense. Thus, let f be topologically even with extremal points $c_{-r} < \ldots < c_0 < \ldots < c_r$ and laps $l_{-r} < \ldots < l_{-1} < l_1 < \ldots < l_r$, where $l_k = (\min(c_k, c_{k-|k|/k}), \max(c_k, c_{k-|k|/k}))$. For x in the closure of l_k, we define

$$d_f(x, c_0) = |k| - 1 + \frac{f(x) - f(c_{k-|k|/k})}{f(c_k) - f(c_{k-|k|/k})}. \tag{22}$$

The associated metric for such a distance is simply defined by $\|x - y\|_f = |d_f(x, c_0) - d_f(y, c_0)|$.

It is now easy to check the following.

PROPOSITION 4. $d_f(x, c_0) < d_f(y, c_0) \Leftrightarrow |h(x)| < |h(y)|$.

4.3. We also have

PROPOSITION 5. *The composition of n topologically even unimodal maps is topologically even, has at most $2^n - 1$ turning points and at most n turning values.*

Proof. Assume f is topologically even. From the proof of Proposition 3 or from Proposition 4, f can be rewritten as an even map in the proper metric (e.g. $\|\cdot\|_f$ defined above). In such a metric, for any map g we have

$$f(-x) = f(x) \Rightarrow g(f(-x)) = g(f(x)).$$

Hence, if f is topologically even, so is $g \circ f$ for any g. The first statement in the proposition follows. The two other statements are obtained by induction, using the chain rule. ∎

Remark. Whenever we want to exploit the fact that a map is topologically even, we use the notation of this section for turning points and laps.

5. Peeling

5.1. A multimodal map $f : I \to I$ is a *captive map* if it maps the boundary ∂I to itself. If the modality of f is odd, f maps the boundary to one of the endpoints, while if the modality of f is even, f either fixes or exchanges the endpoints. It is clear that captive maps form an invariant class under composition, and that $f \circ g$ and f are of the same type (in the sense of 2.3) whenever the modality of f is odd.

Assume $f : I \to I$ is a captive map, and that $\mathbf{S} = \{O_1, \ldots, O_M\}$ is a set of periodic orbits of f such that each turning value of f belongs to some O_i in $\mathbf{S}$. Let $E = O_1 \cup \ldots \cup O_M$, and assume furthermore that f has no turning points outside the interval $[\min E, \max E]$. In this case we call (f, E) a *fundamental pair* (*based on* E).

5.2. Assume f, g_0 and g_1 are captive maps on I, and that (f, E), (g_0, E) and (g_1, E) are fundamental pairs based on the same set E. Assume furthermore $f = g_1 \circ g_0$, where g_0 is a map with modality at most one and g_1 is amodal or has odd modality.

Lemma 1. *The monotone equivalence classes of g_0 and g_1 are uniquely determined and can be computed from the knowledge of the kneading data of f, the set E, and the type of g_0.*

Proof. The cases when g_0 or g_1 is amodal are trivial, hence in the rest of the proof we assume that g_0 is unimodal and that g_1 is not amodal. We set $N = |E|$. With no loss of generality, we can assume that $I = [0, N+1]$ and $E = \{1, \ldots, N\}$. Using the extended composition machine in S_n (see 2.4) as a factoring machine, we are going to present an algorithm to compute $\mathrm{Graph}(g_0|_E)$. Notice that this algorithm does not require us to know the modality of g_1 beforehand, but that the process yields this modality. We first collect three simple observations:

(a) The map f is topologically even with central point $c_0 \in E$.

(b) We assume $s_0(f) = s_0(g_0) = +$, the other cases being treated similarly (the type of f is also the type of g_1; see 5.1).

(c) If we think of $E \times E$ as an N by N checkerboard, the graph of $g_0|_E$ will have exactly one point in each row and on each column; we take the rows parallel to the source and the columns parallel to the target.

From (a), (b), (c), the construction of $\Gamma = \mathrm{Graph}(g_0|_E)$ starts with

$$(c_0, N) \in \Gamma. \tag{23}$$

From (c), we know that one of the points $(c_0 - 1, N - 1)$, $(c_0 + 1, N - 1)$ is in Γ; clearly, the former case corresponds to $d_f(c_0 - 1, c_0) < d_f(c_0 + 1, c_0)$ and the latter to the reverse inequality.

Now assume that the first Q points of Γ have been recognized; their abscissae are consecutive, say $R, R+1, \ldots, R+Q-1$, and the ordinates are $N, N-1, \ldots, N-Q+1$. One of the points $(R-1, N-Q)$, $(R+Q, N-Q)$ is in Γ with the former case corresponding to

$$d_f(R-1, c_0) < d_f(R+Q, c_0) \tag{24}$$

and the latter to the reverse inequality.

Knowledge of Γ entails the knowledge of the critical point and critical value of g_0, thus the extended composition machine setting allows us next to locate as well the critical points and critical values of g_1. Since the modality of g_0 is one, the behavior of g_0 out of $[\min E, \max E]$ just depends on its type. Similarly, since the modality of g_1 is odd, the behavior of g_1 out of $[\min E, \max E]$ just depends on its type. ∎

5.3. With the notation of 5.2, we say that $\mathrm{Mon}(g_0)$ and $\mathrm{Mon}(g_1)$ have been obtained from $\mathrm{Mon}(f)$ by *peeling off* $\mathrm{Mon}(g_0)$. Iterating the peeling procedure, we have the following theorem.

Theorem 1. *We can completely recover the factors of any product of captive maps of given types and given modalities in* $\{0,1\}$, *assuming the maps are part of fundamental pairs based on the same set.*

Remark. The necessity of going beyond the case of a single periodic orbit in $\mathbf{S}$ comes from the fact that the condition that the permutation $f|_E$ is cyclic is generally not inherited by $g_1|_E$ (nor by $g_0|_E$). Thus, to iterate the peeling algorithm as formulated in the proof of Lemma 1, we needed to formulate it for permutations.

5.4. Example. Consider the 6-cycle $\sigma_B = (156342)$, with symbol $M_{-3}K_2M_4K_0M_1K_{-1}$ (see the Remarks in 2.5 and 4.2). Figure 3 shows how to successively extract the permutations

- σ_1, unimodal of type $-$,
- σ_2, unimodal of type $+$,
- σ_3, unimodal of type $+$,

such that

$$\sigma_B = \varDelta_6 \circ \sigma_3 \circ \sigma_2 \circ \sigma_1. \tag{25}$$

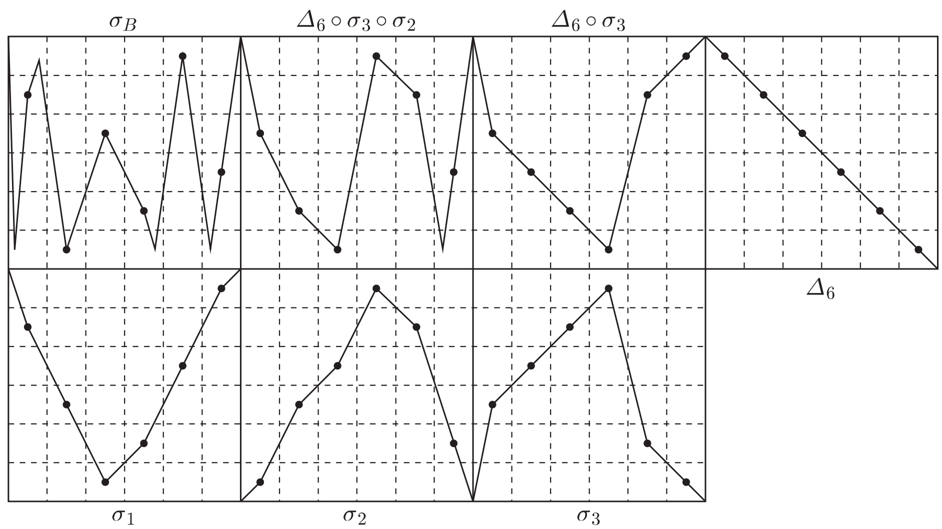

Fig. 3

We have (in classical cycle notation)

$$\sigma_1 = (15423)(6), \qquad \sigma_2 = (1)(2346)(5), \qquad \sigma_3 = (135246). \tag{26}$$

6. Generalized $*$-products of finite sequences

6.1. Hypothesis H1. Let $f : I \to I$ have a periodic orbit O of period p and criticality index n, and assume $f|_O$ is q-C-renormalizable for some proper divisor q of p. With the notation of 3.1, assume furthermore that each of the intervals $I_0, I_1, \ldots, I_{q-1}$ contains at most one turning point of f, and that any turning point in an I_i belongs to O.

We propose to generalize the unimodal $*$-product to the cases covered by H1, by describing how to compute the symbol $\mathcal{S}(O)$ of O from the symbolic description of

(i) the way f permutes the O_i's, with $O_i = I_i \cap O$

and of

(ii) any of the renormalized maps $f^{\circ q}|_{I_j}$.

The next two subsections are devoted to making precise the notion of "symbolic descriptions of (i) and (ii)".

6.2. Let $\sigma_{\mathcal{G}}$ be the cyclic permutation describing the way f permutes the O_i's, i.e., $f(O_j) = O_{\sigma_{\mathcal{G}}(j)}$. Let f' be a map with the same modality as f, and assume f' has a periodic orbit O' such that $f'|_{O'} \sim \sigma_{\mathcal{G}}$. Write $O' = \{x_0, x_1, \ldots, x_{q-1}\}$ with $x_0 < x_1 < \ldots < x_{q-1}$ and assume furthermore that O' is chosen so that

$$a(x_i) = \begin{cases} L_j & \text{if } I_i \subset l_j, \\ C_j & \text{if } c_j \in I_i. \end{cases} \tag{27}$$

We write

$$A = \mathcal{S}(O') = W_0 C_{k_0} W_1 C_{k_1} \ldots W_{n-1} C_{k_{n-1}}, \tag{28}$$

where the $W_i = W_{i,1} W_{i,2} \ldots W_{i,u(i)-1}$'s are words in $\mathcal{A}'$.

6.3. Now choose $j \in \{0, 1, \ldots, q-1\}$, and let B be the symbol of O_j, considered as a periodic orbit of $f^{\circ q}|_{I_j}$. Since in general the $f^{\circ q}|_{I_i}$'s are not topologically conjugate to each other, we must keep track of which j is chosen.

A key point in our discussion is that only symbols are relevant, and that the maps f and f' can be deformed as long as such deformations do not affect symbols; this is exploited in Lemma 2 to choose a deformation F of f with some specific properties. Furthermore, we need to check that, for the I_i's well chosen, the $F^{\circ q}|_{I_i}$'s are endomorphisms of the I_i's, and more precisely captive maps (see Lemma 3). Once such I_i's are selected, the peeling technique allows us to extract the $f|_{O_i}$'s from B, hence $\mathcal{S}(O)$ from A, B and j (see Theorem 2).

6.4. With the notation of 6.1, we say the map F is an *O-compatible* deformation of f if F has the same shape as f and gives the same symbol to O. It is plain that O-compatibility is an equivalence relation, and that the star product formulae only depend on the equivalence classes of this relation. This motivates the next two lemmata.

LEMMA 2. *Assuming the notation of* 6.2, *we can find an O-compatible deformation of f which has a periodic orbit O'_- with symbol*

$$A_- = \mathcal{S}(O'_-) = W_0 C_{k_0}^{-s_{k_{n-1}}{}^{W_0}} W_1 C_{k_1}^{-s_{k_0}{}^{W_1}} \ldots W_{n-1} C_{k_{n-1}}^{-s_{k_{n-2}}{}^{W_{n-1}}},$$

and for each $v \in \{0, 1, \ldots, n-1\}$, $u(v)$ the length of the block $W_v C_{k_v}$, and $j \in \{1, 2, \ldots, u(v)-1\}$, find points with itineraries

$$\begin{aligned} a^*(P^+_{v,u(v)}) = C_{k_v}^{s_{k_{v-1}}{}^{W_v}} & W_{v+1} C_{k_{v+1}}^{-s_{k_v}{}^{W_{v+1}}} \ldots \\ & \ldots W_{v-1} C_{k_{v-1}}^{-s_{k_{v-2}}{}^{W_{v-1}}} W_v (C_{k_v}^{-s_{k_{v-1}}{}^{W_v}} W_{v+1} C_{k_{v+1}}^{-s_{k_v}{}^{W_{v+1}}} \ldots \\ & \ldots W_{v-1} C_{k_{v-1}}^{-s_{k_{v-2}}{}^{W_{v+1}}} W_v)^\infty, \end{aligned}$$

and

$$\begin{aligned} a^*(P^+_{v,j}) = W_{v,j} W_{v,j+1} \ldots & W_{v,u(v)-1} C_{k_v}^{s_{k_{v-1}}{}^{W_v}} W_{v+1} C_{k_{v+1}}^{-s_{k_v}{}^{W_{v+1}}} \ldots \\ & \ldots W_{v-1} C_{k_{v-1}}^{-s_{k_{v-2}}{}^{W_{v-1}}} W_{v,1} W_{v,2} \ldots W_{v,j-1} (W_{v,j} W_{v,j+1} \ldots \\ & \ldots W_{v,u(v)-1} C_{k_v}^{-s_{k_{v-1}}{}^{W_v}} W_{v+1} C_{k_{v+1}}^{-s_{k_v}{}^{W_{v+1}}} \ldots \\ & \ldots W_{v-1} C_{k_{v-1}}^{-s_{k_{v-2}}{}^{W_{v+1}}} W_{v,1} W_{v,2} \ldots W_{v,j-1})^\infty. \end{aligned}$$

Proof. The proof is a simple but tedious kneading theory argument, so we merely sketch it, leaving details to the reader.

Start with a map G with the same shape as f, but full (see 2.3) and with slopes greater than one in absolute value in all laps. We can construct from G stunted maps as described in Figure 4 (see [DGMT] and references

Fig. 4

therein). We stunt G only at the turning points whose symbolic name appears in $\mathcal{S}(O)$, and just enough so that $\mathcal{S}(O)$ is the symbol of an orbit. Call this map F. We assume the convention in which we consider the middle point of each plateau to be a critical point. It remains to check that the map F satisfies the properties we seek. This check could be done by brute force computations, using the order on symbolic sequences defined in 2.6. Instead, we offer a more geometrical approach, based on elementary bifurcation theory considerations. To simplify this discussion, we leave unchanged the names of orbits which are deformed as a result of deforming maps, as long as their symbols do not change.

First, continue to stunt F to a map F_1, until O is deformed to an orbit O_1 of F_1 characterized as follows: O_1 contains a boundary of each of the plateaus retracted from those plateaus of F which contain a (necessarily critical) point of O.

Deforming F_1 back to F, O_1 splits into two orbits, O' and O'_-: O'_- has the same symbol as O_1 and O' contains a point of each plateau retracted from where O contains a critical point.

Clearly, to write the symbol of O'_-, we replace each C_{k_i} by the neighbor of C_{k_i} in $\mathcal{A}'$ chosen so that the image of the corresponding point in the orbit is:

(1) smaller if the plateau is a maximum, and
(2) bigger if the plateau is a minimum.

Consider now the qth iterate of F near each plateau containing a point of O. The locally unimodal graph cuts the diagonal at a point $P^-_{v,u(v)}$ or $P^-_{v,j}$, according to whether or not the plateau we consider contains a critical point of F. We denote accordingly by $P^+_{v,u(v)}$ or by $P^+_{v,j}$ the point on the other side from $P^-_{v,u(v)}$ or $P^-_{v,j}$ of the plateau of $F^{\circ q}$ which has the same image under $F^{\circ q}$ as does $P^-_{v,u(v)}$ or $P^-_{v,j}$.

An easy check now confirms that these geometrical considerations lead to the formulas stated in the lemma. ■

Notice that the map F constructed in the proof is such that to each point $P^+_{v,l}$ there corresponds a single point $P^-_{v,l}$ in O'_- whose itinerary is the same as that of $P^+_{v,l}$, except for the $(u(v)-l+1)$st letter, which reads $C_{k_v}^{-s_{k_v-1}W_v}$ instead of $C_{k_v}^{s_{k_v-1}W_v}$. Another simple but tedious kneading theory argument then shows that the sets $\{P^-_{v,l}, P^+_{v,l}\}$ furnish the bounds of the intervals I_i which we seek. This is formulated in the following lemma, which is easily confirmed using the geometrical discussion in the proof of Lemma 2.

LEMMA 3. *The sets* $\{P^-_{v,l}, P^+_{v,l}\}$ *are boundaries of a set of* I_i*'s such that* $f(I_i) \subset I_{\sigma_{\mathcal{G}}(i)}$, *and such that the* $F^{\circ q}|_{I_i}$*'s are captive maps.*

6.5. Putting together the peeling results and the previous lemma, we get the following.

THEOREM 2. *Under the hypothesis* H1, $\mathcal{S}(O)$ *can be computed from* A *and* B.

6.6. EXAMPLE. We consider the case of a (+)-trimodal map, with $A = C_1C_0C_2L_3$ and $B = M_{-3}K_2M_4K_0M_1K_{-1}$ at c_1. The computations in 5.4 give the desired peeling of the renormalized map near the turning point c_1.

We can then compute the following products:

$$\begin{aligned}\sigma_{B_1} &= \sigma_1 \circ \Delta_6 \circ \sigma_3 \circ \sigma_2 = (123546),\\ \sigma_{B_2} &= \sigma_2 \circ \sigma_1 \circ \Delta_6 \circ \sigma_3 = (134562), \qquad (29)\\ \sigma_{B_3} &= \sigma_3 \circ \sigma_2 \circ \sigma_1 \circ \Delta_6 = (143562).\end{aligned}$$

From the knowledge of the σ_i's, $i = 1, 2, 3$, we also extract the following facts:

F1: the turning point of σ_1 is at 3,
F2: the turning point of σ_2 is at 4,
F3: the turning point of σ_3 is at 4, and
F4: $\sigma_1(3) = 1$, $\sigma_2(1) = 1$, $\sigma_3(1) = 3$.

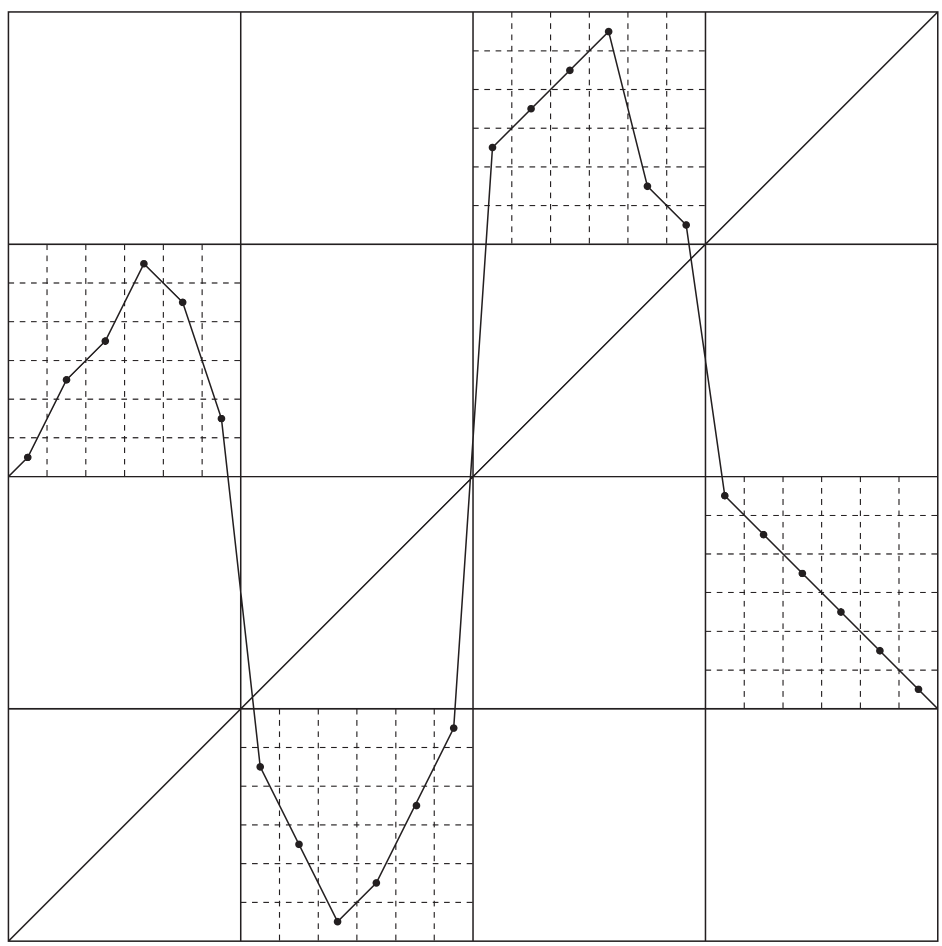

Fig. 5

As we now explain, the facts F1–F4 are all we need to compute $\mathcal{S}(O) = S_1 \dots S_{24}$.

First, in relation to the information we chose to put in F4, we take the symbol for c_1 as S_1.

Using the formula for B and F1 we have:

$$S_1 = C_1;\quad S_5 = L_2;\quad S_9 = L_1;\quad S_{13} = L_1;\quad S_{17} = L_2;\quad S_{21} = L_2. \tag{30}$$

Using the formula for σ_{B_1} in (29), and facts F2 and F4 we have:

$$\begin{aligned} S_2 = L_0;\quad S_6 = L_0;\quad S_{10} = L_0;\\ S_{14} = L_1;\quad S_{18} = C_0;\quad S_{22} = L_1. \end{aligned} \tag{31}$$

Using the formula of σ_{B_2} in (29), F3 and F4 we have:

$$\begin{aligned} S_3 = L_2;\quad S_7 = L_2;\quad S_{11} = C_2;\\ S_{15} = L_3;\quad S_{19} = L_3;\quad S_{23} = L_2. \end{aligned} \tag{32}$$

It is straightforward that:

$$\begin{aligned} S_4 = L_3;\quad S_8 = L_3;\quad S_{12} = L_3;\\ S_{16} = L_3;\quad S_{20} = L_3;\quad S_{24} = L_3. \end{aligned} \tag{33}$$

Collecting (30)–(33), we obtain (see Figure 5)

$$\begin{aligned} &\mathcal{S}(O)\\ &= C_1L_0L_2L_3L_2L_0L_2L_3L_1L_0C_2L_3L_1L_1L_3L_3L_2C_0L_3L_3L_2L_1L_2L_3. \end{aligned} \tag{34}$$

7. Generalized $*$-products and infinitely renormalizable maps

7.1. Up to this point, we have considered exclusively the case of finite orbits. In the unimodal case, the substitution also works when B is an infinite string. In any modality, results for the case when B is an infinite aperiodic sequence representing the itinerary of some point P can in general be obtained from the finite case by considering, as usual, longer and longer periodic approximations of the orbit of P, obtained by closing the orbit at the successive closest returns to P. The next subsection is devoted to the class of infinite sequences which is most important in the context of $*$-products considerations.

7.2. With the notation of 3.1, assume that $f|_E$ is n_0-renormalizable with $n_0 \notin \{1, |E|\}$ and that the $f^{\circ n_0}|_{E_j}$'s are all n_1-renormalizable with

$n_1 \notin \{1, |E_j|\}$, in such a way that $f|_E$ is $n_1 n_0$-renormalizable with $n_1 n_0 \notin \{1, |E|\}$. We then say that $f|_E$ is *twice renormalizable*. We say that $f|_E$ is *m-fold renormalizable* if the $f^{\circ n_0}|_{E_j}$'s are all $(m-1)$-fold renormalizable, and k-renormalizable with $k = n_{m-1} n_{m-2} \ldots n_1$ in such a way that $f|_E$ is kn_0-renormalizable with $kn_0 \notin \{1, |E|\}$. If $f|_E$ is n-fold renormalizable for each $n > 0$, we call it *infinitely renormalizable*. In this case there is a morphism ϕ from $(E, f|_E)$ to (G, T), where

(i) G is a group given as an inverse limit

$$\widehat{\mathbb{Z}}_Q = \varprojlim_i \mathbb{Z}/q_i\mathbb{Z},$$

where Q stands for a supernatural number

$$Q = \prod_p p^{k_p} \qquad \text{where, for all } p \text{ prime, } 0 \le k_p \le \infty,$$

and the $q_i = n_i n_{i-1} \ldots n_0$'s form a sequence of divisors of Q ordered by divisibility, and

(ii) the map T is addition of 1 on $G = \widehat{\mathbb{Z}}_Q$, i.e., a generalized adding machine, where the usual adding machine corresponds to the case when $n_i \equiv 2$.

The set E is then a Cantor set, and $f|_E$ is a homeomorphism if the fibers of ϕ are points (for a thorough discussion see [BORT]).

Notice that in this most important case when $f|_E$ is infinitely renormalizable while all the fibers of ϕ are points, m-fold renormalized maps from $f|_E$ for m large enough are all supported in intervals containing at most one turning point of f, so that Hypothesis H1 is automatically satisfied. Then the symbol of an orbit in E can be obtained by iterating the operation on finite symbols described in Section 6, i.e., like in the unimodal case (see [DGP, CEc, BORT]), no further approximation method is necessary.

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