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Structure spaces for rings of continuous functions with applications to realcompactifications

by

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Abstract. Let X be a completely regular space and let $A(X)$ be a ring of continuous real-valued functions on X which is closed under local bounded inversion. We show that the structure space of $A(X)$ is homeomorphic to a quotient of the Stone–Čech compactification of X. We use this result to show that any realcompactification of X is homeomorphic to a subspace of the structure space of some ring of continuous functions $A(X)$.

1. Introduction. Let X be a completely regular space and let $A(X)$ be a collection of continuous real-valued functions on X which form a ring under pointwise operations. Two special cases are $C(X)$, the ring of all continuous functions on X, and $C^*(X)$, the ring of bounded continuous functions on X. We study the class of rings of continuous functions which are closed under *local bounded inversion* (as defined in Section 2). This class includes any ring that contains $C^*(X)$, and any uniformly closed subring of $C^*(X)$, as well as others, including $C_0^1(X)$, the ring of continuously differentiable functions on a locally compact subset X of $\mathbb{R}$ which vanish at infinity (see [1]). Structure spaces for $C(X)$ and $C^*(X)$ have been studied extensively. (See for example [5], where it is shown, by different methods, that the structure space of each ring is isomorphic to βX, the Stone–Čech compactification of X.) We show that for any ring closed under local bounded inversion, the structure space is compact, and is homeomorphic with a quotient of the Stone–Čech compactification βX (Theorem 3.6). Our proofs use a map which assigns a z-filter to every noninvertible $f \in A(X)$; this map extends to one from ideals (maximal ideals) to z-filters (z-ultrafilters). For each $A(X)$ we identify a

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subspace $v_A X$ of $\mathfrak{M}(A)$ which we call the A-compactification of X. We show that $v_A X$ is a realcompactification of X and that every realcompactification arises in this way (Theorem 4.6). Thus every realcompactification of X is a quotient of a subspace of βX. We identify a class of rings which is in natural one-to-one correspondence with the realcompactifications of X (Theorem 4.7). As an application of our results, we prove an extension of the Banach–Stone theorem: Let X and Y be compact and let $A(X)$ and $B(Y)$ be closed under local bounded inversion; if $A(X)$ and $B(Y)$ are isomorphic, then X and Y are homeomorphic. (See the remark following Theorem 4.5.)

Rings of continuous functions other than $C(X)$ and $C^*(X)$ are also studied in [1], [6], [7], and [8].

2. Ideals and z-filters. Let X be a completely regular space and let $A(X)$ be a ring of continuous real-valued functions on X. A *zero set* in X is a set of the form $Z(f) = \{x \in X : f(x) = 0\}$ for some $f \in C(X)$; the complement of a zero set is called a *cozero set*. We define $Z[A(X)] = \{Z(f) : f \in A(X)\}$; the collection $Z[C(X)]$ of all zero sets is denoted by $Z[X]$. We always assume that the rings $A(X)$ that we consider contain the constants and separate points and closed sets in X. We make this assumption because of the following easily proved fact: $Z[A(X)]$ is a base for the closed sets in X iff $A(X)$ separates the points and closed sets of X.

If $f \in A(X)$ and E is a cozero set in X, then f is *E-regular* if there exists $g \in A(X)$ such that $fg|_E \equiv 1$; that is, f is *locally invertible* on E. To each $f \in A(X)$ we attach a collection $\mathcal{Z}_A(f)$ of subsets of X defined by

$$\mathcal{Z}_A(f) = \{E \in Z[X] : f \text{ is } E^{\mathrm{c}}\text{-regular}\}.$$

Clearly, $\mathcal{Z}_A(fg) \subset \mathcal{Z}_A(f) \cap \mathcal{Z}_A(g)$. It can be shown, as in [9], Theorem 1, that $\mathcal{Z}_A(f)$ is a z-filter on X iff f is not invertible in $A(X)$. For $S \subset A(X)$ we write $\mathcal{Z}_A[S] = \bigcup_{f\in S} \mathcal{Z}_A(f)$. It was shown in [9] and [3] that if $A(X)$ is a uniformly closed subring that contains or is contained in $C^*(X)$, then for an ideal I in $A(X)$, $\mathcal{Z}_A[I]$ is a z-filter on X. The proofs there depend on the assumption that $A(X)$ is uniformly closed. In Theorem 2.1 below we show that this is in fact true for any subring of $C(X)$. The inverse of the map $\mathcal{Z}_A$, considered as a set map, is denoted by $\mathcal{Z}_A^{\leftarrow}$ and defined by

$$\mathcal{Z}_A^{\leftarrow}[\mathcal{S}] = \{f \in A(X) : \mathcal{Z}_A(f) \subset \mathcal{S}\},$$

where $\mathcal{S}$ is a collection of zero sets in X. It follows immediately from the definition that $\mathcal{Z}_A^{\leftarrow}[\mathcal{Z}_A[S]] \supset S$ and $\mathcal{Z}_A[\mathcal{Z}_A^{\leftarrow}[\mathcal{S}]] \subset \mathcal{S}$ for all $S \subset A(X)$ and $\mathcal{S} \subset Z[X]$. For a z-filter $\mathcal{F}$ on X we define

$$I_A[\mathcal{F}] = \{f \in A(X) : \lim_{\mathcal{F}} fh = 0 \text{ for all } h \in A(X)\},$$

where $\lim_{\mathcal{F}} f$ denotes the limit of the filter base $f(\mathcal{F})$. Clearly, $I_A[\mathcal{F}]$ is an ideal of $A(X)$.

THEOREM 2.1. (a) *If I is an ideal in $A(X)$ then $\mathcal{Z}_A[I]$ is a z-filter on X.*
(b) *If $\mathcal{F}$ is a z-filter on X then $I_A[\mathcal{F}] \subset \mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$.*

Proof. (a) Clearly, $\emptyset \notin \mathcal{Z}_A[I]$, because I contains no invertible elements. If $F \in Z[X]$ and $F \supset E \in \mathcal{Z}_A[I]$, then $F \in \mathcal{Z}_A[I]$. Now let $E, F \in \mathcal{Z}_A[I]$, and choose $f, g \in I$ locally invertible on E^c and F^c respectively. Then there exist $h, k \in A(X)$ such that $fh|_{E^c} \equiv 1$ and $gk|_{F^c} \equiv 1$. Let $w = fh + gk - fhgk$. Then $w \in I$, and since $w|_{E^c \cup F^c} \equiv 1$, it follows that w is locally invertible on $E^c \cup F^c$. Thus $(E^c \cup F^c)^c = E \cap F \in \mathcal{Z}_A[I]$, and so $\mathcal{Z}_A[I]$ is a z-filter.

(b) For $f \in I_A[\mathcal{F}]$ we show that for every $E \in \mathcal{Z}_A(f)$ there exists $F \in \mathcal{F}$ such that $F \subset E$. If no such F exists, then $F \cap E^c \neq \emptyset$ for all $F \in \mathcal{F}$. Since $E \in \mathcal{Z}_A(f)$ there exists $h \in A(X)$ such that $hf|_{E^c} \equiv 1$. But then 1 is a cluster point of $\{fh(F) : F \in \mathcal{F}\}$, contradicting the hypothesis that $\lim_{\mathcal{F}} fh = 0$. Thus $\mathcal{Z}_A(f) \subset \mathcal{F}$; that is, $f \in \mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$. ∎

If $\mathcal{F}$ is a z-filter or even a z-ultrafilter on X, then $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$ is not necessarily an ideal. For example, in the ring $P(\mathbb{R})$ of polynomials on $\mathbb{R}$, for any z-filter $\mathcal{F}$, the set $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$ consists of all polynomials other than the nonzero constants. We now introduce a class of subrings for which $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$ is an ideal for every z-filter $\mathcal{F}$. A subring $A(X)$ of $C(X)$ is *closed under local bounded inversion* if every element of $A(X)$ that is bounded away from 0 on a cozero set E is locally invertible on E; that is, if $f(x) \geq c > 0$ for all $x \in E$, then f is E-regular in $A(X)$. Any subring of $C(X)$ that contains $C^*(X)$ is closed under local bounded inversion, and according to [3], Lemma 1.2(c), so also is any uniformly closed subring of $C^*(X)$. However, a subring of $C(X)$ that is closed under local bounded inversion need not be comparable to $C^*(X)$. (Consider, for example, the ring of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which $\lim_{x\to\infty} f(x)$ exists.)

In the study of $C(X)$ the zero sets $Z(f)$ play a central role. The following result gives a relationship between the z-filter $\mathcal{Z}_A(f)$ and the zero set $Z(f)$.

PROPOSITION 2.2. *If $A(X)$ is closed under local bounded inversion, then $Z(f) = \bigcap \mathcal{Z}_A(f)$.*

Proof. Suppose $y \notin Z(f)$ and without loss of generality assume that $f(y) > 0$. Choose a cozero set neighborhood G of y such that $f(x) \geq c > 0$ for all $x \in G$. By hypothesis f is locally invertible on G and so $G^c \in \mathcal{Z}_A(f)$. Thus $y \notin \bigcap \mathcal{Z}_A(f)$. This shows that $Z(f) \supset \bigcap \mathcal{Z}_A(f)$. The other inclusion is immediate. ∎

THEOREM 2.3. *Let $A(X)$ be closed under local bounded inversion. If $\mathcal{F}$ is a z-filter on X then $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}] = I_A[\mathcal{F}]$; in particular, $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$ is an ideal in $A(X)$.*

Proof. We claim that if $f \in A(X)$ and $\mathcal{F} \supset \mathcal{Z}_A(f)$, then $\lim_{\mathcal{F}} fh = 0$ for all $h \in A(X)$. To show this, let f be a noninvertible element of $A(X)$.

We show that $\lim_{\mathcal{Z}_A(f)} f = 0$. Let $[-\varepsilon, \varepsilon]$ be a neighborhood of 0 in $\mathbb{R}$ and let $E_\varepsilon = f^{-1}([-\varepsilon, \varepsilon])$. Set

$$F_1 = \{x \in X : f(x) > \varepsilon\} \quad \text{and} \quad F_2 = \{x \in X : f(x) < -\varepsilon\}.$$

Since $A(X)$ is closed under local bounded inversion, f is F_1-regular and F_2-regular, and hence $(F_1 \cup F_2)$-regular ([9], Lemma 1(b)). But $F_1 \cup F_2 = (E_\varepsilon)^{\mathrm{c}}$, so $E_\varepsilon \in \mathcal{Z}_A(f)$ for all $\varepsilon > 0$. Thus $\lim_{\mathcal{Z}_A(f)} f = 0$ for all $f \in A(X)$. In particular, for all $h \in A(X)$, $\lim_{\mathcal{Z}_A(fh)} fh = 0$, so since $\mathcal{Z}_A(fh) \subset \mathcal{Z}_A(f) \subset \mathcal{F}$, we have $\lim_{\mathcal{F}} fh = 0$. This proves the claim.

Now, let $\mathcal{F}$ be a z-filter and let $f \in \mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$; that is, $\mathcal{Z}_A(f) \subset \mathcal{F}$. Then by the claim $\lim_{\mathcal{F}} fh = 0$ for all $h \in A(X)$, so $f \in I_A[\mathcal{F}]$. Thus by Theorem 2.1(b), $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}] = I_A[\mathcal{F}]$. ∎

If M is a maximal ideal in $A(X)$, then $\mathcal{Z}_A[M]$ is not necessarily a z-ultrafilter. For example, let M_0 be the maximal ideal in $C(\mathbb{R})$ consisting of those functions which vanish at 0. Then $\mathcal{Z}_C[M_0]$ is the z-filter of zero-set neighborhoods of 0, which is properly contained in the z-ultrafilter of all zero sets that contain 0. On the other hand, if $A(X)$ is closed under local bounded inversion, then $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ is a maximal ideal whenever $\mathcal{U}$ is a z-ultrafilter, as we show in the following theorem.

THEOREM 2.4. *Let $A(X)$ be closed under local bounded inversion. If $\mathcal{U}$ is a z-ultrafilter on X, then $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ is a maximal ideal in $A(X)$.*

Proof. We first show that if $\mathcal{U}_1$ and $\mathcal{U}_2$ are z-ultrafilters on X, then the ideals $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}_1]$ and $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}_2]$ are either equal or not comparable. To this end, suppose that $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}_1]$ is properly contained in $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}_2]$, and choose $f \in \mathcal{Z}_A^{\leftarrow}[\mathcal{U}_2]$ such that $f \notin \mathcal{Z}_A^{\leftarrow}[\mathcal{U}_1]$. Then by Theorem 2.3, $\lim_{\mathcal{U}_2} fh = 0$ for all $h \in A(X)$, and $\lim_{\mathcal{U}_1} fk \neq 0$, or does not exist, for some $k \in A(X)$. In either case, there exists $c > 0$ and a zero set $E \in \mathcal{U}_1$ such that $f(x)k(x) > c$ for all $x \in E$. The set $G = \{x : f(x)k(x) > c\}$ is a cozero set containing E on which fk is bounded away from zero. Since $A(X)$ is closed under local bounded inversion there exists $g \in A(X)$ such that $fkg|_G \equiv 1$, and hence $fkg|_E \equiv 1$. Thus $1 - fkg|_E \equiv 0$, and so $\lim_{\mathcal{U}_1}(1 - fkg)h = 0$ for all $h \in A(X)$. But $\lim_{\mathcal{U}_2}(1 - fkg) = 1$, which means that $1 - fkg$ belongs to $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}_1]$ but not to $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}_2]$, contradicting the hypothesized containment.

Now, let $M = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ and $\mathcal{F} = \mathcal{Z}_A[\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]]$. Suppose $g \notin M$. We will show that the ideal generated by M and g is all of $A(X)$, which will prove that M is maximal. We claim that the collection $\mathcal{Z}_A(g) \cup \mathcal{F}$ does not have the finite intersection property. For if it did, then there would exist a z-ultrafilter $\mathcal{U}'$ containing both $\mathcal{Z}_A(g)$ and $\mathcal{F}$. In this case $\mathcal{U}' \supset \mathcal{Z}_A[\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]]$; hence $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}'] \supset \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$, and so by the first paragraph $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}'] = M$, which is impossible since $g \notin M$. This proves the claim. So there exist zero sets $E \in \mathcal{Z}_A(g)$ and $F \in \mathcal{F}$ such that $E \cap F = \emptyset$. By definition of $\mathcal{F}$ we can choose $f \in M$ such that $F \in \mathcal{Z}_A(f)$. Hence there exist $h, k \in A(X)$ such that $fh|_{F^c} \equiv 1$ and $gk|_{E^c} \equiv 1$. Now since $E^c \cup F^c = X$, we see that $fh + gk - fghk$ is identically 1 on all of X. Thus the ideal generated by M and g contains the function 1. ∎

A z-filter $\mathcal{F}$ is *fixed* if $\bigcap \mathcal{F} \neq \emptyset$, otherwise it is *free*. An ideal I is *fixed* if $\bigcap \mathcal{Z}_A[I] \neq \emptyset$, otherwise it is *free*. By Proposition 2.2, $\bigcap \mathcal{Z}_A[I] = \bigcap Z[I]$, so an ideal is fixed in this sense iff it is fixed in the usual sense ([5], page 54). Note that the fixed z-ultrafilters on X are precisely the z-ultrafilters $\mathcal{U}_p$, $p \in X$, where $\mathcal{U}_p = \{E \in Z[X] : p \in E\}$.

THEOREM 2.5. *Let $A(X)$ be closed under local bounded inversion and let M be a maximal ideal in $A(X)$.*

(a) *If M is fixed, then there exists $p \in X$ such that*

$$M = \{f \in A(X) : f(p) = 0\}.$$

Moreover, $\mathcal{U}_p$ is the unique z-ultrafilter containing $\mathcal{Z}_A[M]$.

(b) *If M is free, then there is a free z-ultrafilter $\mathcal{U}$ such that*

$$M = \{f \in A(X) : \mathcal{Z}_A(f) \subset \mathcal{U}\}.$$

Proof. (a) By Proposition 2.2 we have $Z(f) = \bigcap \mathcal{Z}_A(f)$ for every $f \in A(X)$, and so $\bigcap \mathcal{Z}_A[I] = \bigcap\{Z(f) : f \in M\}$. Since M is fixed, this intersection is not empty. So there exists $p \in X$ such that $f(p) = 0$ for all $f \in M$. Since M is maximal, we have $M = \{f \in A(X) : f(p) = 0\}$.

Now let $\mathcal{U}$ be any z-ultrafilter containing $\mathcal{Z}_A[M]$. We show that p belongs to every element of $\mathcal{U}$. For suppose that $p \notin E \in \mathcal{U}$. Then there exists $f \in A(X)$ such that $f(p) = 0$ and $f(E) \geq 1$ (and so $f \in M$). Let $F = \{x \in X : f(x) \geq 1/2\}$. Since $A(X)$ is closed under local bounded inversion, f is locally invertible on F^c, so $F \in \mathcal{Z}_A(f) \subset \mathcal{Z}_A[M] \subset \mathcal{U}$. But this is impossible because $E \cap F = \emptyset$. Thus $\mathcal{U} \supset \mathcal{U}_p$, and so $\mathcal{U} = \mathcal{U}_p$ is the unique z-ultrafilter containing $\mathcal{Z}_A[M]$.

(b) Since $\mathcal{Z}_A[M]$ is free, any z-ultrafilter $\mathcal{U} \supset \mathcal{Z}_A[M]$ is free. Clearly, $M = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$. ∎

COROLLARY 2.6. *Let $\mathcal{U}$ be a z-ultrafilter on X. Then $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ is a fixed (free) maximal ideal iff $\mathcal{U}$ is a fixed (free) z-ultrafilter.*

Proof. Suppose $M = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ is a fixed maximal ideal. Since $\mathcal{U} \supset \mathcal{Z}_A[M]$, $\mathcal{U}$ is fixed by Theorem 2.5(a). Conversely, if $\mathcal{U}$ is fixed, then $\bigcap \mathcal{U} \neq \emptyset$. Since $\mathcal{Z}_A[\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]] \subset \mathcal{U}$, we have $\bigcap \mathcal{Z}_A[\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]] \neq \emptyset$, and hence $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ is fixed. ∎

3. The structure space. In this section, $A(X)$ denotes a ring of continuous functions that is closed under local bounded inversion.

Let $\mathfrak{U}(X)$ denote the collection of all z-ultrafilters on X, and let $\mathfrak{M}(A)$ denote the collection of maximal ideals in $A(X)$. The map

$$\mathcal{Z}_A^{\leftarrow} : \mathfrak{U}(X) \to \mathfrak{M}(A)$$

is onto. This is because if M is a maximal ideal then the z-filter $\mathcal{Z}_A[M]$ is contained in a z-ultrafilter $\mathcal{U}$. Then $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}] \supset M$, and since M is maximal, $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}] = M$. We endow $\mathfrak{M}(A)$ with the hull-kernel topology. A base for the closed sets in this topology is given by the family of sets of the form

$$N_f = \{M \in \mathfrak{M}(A) : f \in M\}, \quad f \in A(X).$$

We denote by τ the Stone topology on $\mathfrak{U}(X)$; thus $(\mathfrak{U}(X), \tau)$ is βX, the Stone–Čech compactification of X. A base for the closed sets in τ is given by the family of sets of the form $S_E = \{\mathcal{U} \in \mathfrak{U}(X) : E \in \mathcal{U}\}$, $E \in Z[X]$, ([5], p. 87). We write $p, q, \ldots$ for the points of βX, but when we wish to emphasize that these are z-ultrafilters, we write $\mathcal{U}_p, \mathcal{U}_q, \ldots$; if $p \in X$ then $\mathcal{U}_p$ denotes the fixed z-ultrafilter defined before Theorem 2.5. We also endow $\mathfrak{U}(X)$ with a topology τ_A depending on $A(X)$. A base for the closed sets in this topology is given by the family of sets of the form

$$Z_f = \{\mathcal{U} \in \mathfrak{U}(X) : \mathcal{Z}_A(f) \subset \mathcal{U}\}, \quad f \in A(X).$$

By Proposition 2.2, $Z(f) = \bigcap \mathcal{Z}_A(f)$. Thus if $Z(f) \in \mathcal{U}$, then $\mathcal{Z}_A(f) \subset \mathcal{U}$, and so $S_{Z(f)} \subset Z_f$.

We now show that the collection $\{Z_f : f \in A(X)\}$ does indeed form a base for the closed sets in a topology on $\mathfrak{U}(X)$. First, note that X is naturally embedded in both $\mathfrak{U}(X)$ and $\mathfrak{M}(A)$: a point $p \in X$ is identified with the fixed z-ultrafilter $\mathcal{U}_p$ in $\mathfrak{U}(X)$, and with the maximal ideal $M = \{f \in A(X) : f(p) = 0\}$ in $\mathfrak{M}(A)$ (Theorem 2.5(a)).

THEOREM 3.1. (a) *τ_A is a topology on $\mathfrak{U}(X)$.*

(b) *The closure in $(\mathfrak{U}(X), \tau_A)$ of a zero set $E \in Z[X]$ is given by $\bigcap\{Z_f : f \in I_E\}$, where $I_E = \{f \in A(X) : f(E) = 0\}$.*

Proof. (a) First note that if f is invertible, then $Z_f = \emptyset$, so $\emptyset$ is in the base. We now show that for $f, g \in A(X)$ we have $Z_f \cup Z_g = Z_{fg}$. Observe that if $\mathcal{U}$ is a z-ultrafilter with $\mathcal{Z}_A(fg) \subset \mathcal{U}$, then $\mathcal{Z}_A(f) \subset \mathcal{U}$ or $\mathcal{Z}_A(g) \subset \mathcal{U}$; for if $\mathcal{Z}_A(fg) \subset \mathcal{U}$, then $fg \in \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$, which is a maximal ideal, and hence prime. Thus either $f \in \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ or $g \in \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$, and so either $\mathcal{Z}_A(f) \subset \mathcal{U}$ or $\mathcal{Z}_A(g) \subset \mathcal{U}$. Now if $\mathcal{U} \in Z_{fg}$ then $\mathcal{Z}_A(fg) \subset \mathcal{U}$, and hence $\mathcal{Z}_A(f) \subset \mathcal{U}$ or $\mathcal{Z}_A(g) \subset \mathcal{U}$; that is, $\mathcal{U} \in Z_f \cup Z_g$. Conversely, suppose $\mathcal{U} \in Z_f \cup Z_g$. Then either $\mathcal{Z}_A(f) \subset \mathcal{U}$ or $\mathcal{Z}_A(g) \subset \mathcal{U}$; since $\mathcal{Z}_A(fg) \subset \mathcal{Z}_A(f) \cap \mathcal{Z}_A(g)$, we have $\mathcal{Z}_A(fg) \subset \mathcal{U}$. Thus $\mathcal{U} \in Z_{fg}$.

(b) We show that $Z_f \supset E$ iff $f \in I_E$. The result follows from this since the collection $\{Z_f\}$ is a base for the closed sets in $\mathfrak{U}(X)$. So suppose $Z_f \supset E$. Then for every $p \in E$, we have $\mathcal{U}_p \in Z_f$, and so $\mathcal{Z}_A(f) \subset \mathcal{U}_p$ whenever $p \in E$.

Thus $f \in \mathcal{Z}_A^{\leftarrow}[\mathcal{U}_p]$, $p \in E$. By Theorem 2.5(a), this means that $f(p) = 0$ for all $p \in E$, and so $f \in I_E$. Conversely, if $f \in I_E$ then $f(p) = 0$ for all $p \in E$, and by Theorem 2.5(a), $f \in \mathcal{Z}_A^{\leftarrow}[\mathcal{U}_p]$ for all $p \in E$. Thus $\mathcal{Z}_A(f) \subset \mathcal{U}_p$, $p \in E$, and so $E = \{\mathcal{U}_p : p \in E\}$ is contained in $\{\mathcal{U} : \mathcal{Z}_A(f) \subset \mathcal{U}\} = Z_f$. ■

THEOREM 3.2. *X is a dense subspace of* $(\mathfrak{U}(X), \tau_A)$.

Proof. First we verify that the map $\iota : X \to \mathfrak{U}(X)$ taking p to $\mathcal{U}_p$ is a continuous embedding. (We do not distinguish between the points of X and $\iota(X)$; that is, we consider X to be a subset of $\mathfrak{U}(X)$.) Let $f \in A(X)$; then Z_f is a basic closed set in $\mathfrak{U}(X)$. We claim that $Z_f \cap X = Z(f)$, the zero set of f in X. To show this, let $p \in X$. If $f(p) = 0$, then $\mathcal{Z}_A(f) \subset \mathcal{U}_p$ and so $p \in Z_f$. Conversely, suppose $f(p) \neq 0$; we may assume without loss of generality that $f(p) = 1$. Let $E = \{x \in X : f(x) \leq 1/2\}$. Since $A(X)$ is closed under local bounded inversion, $E \in \mathcal{Z}_A(f)$, but since $p \notin E$, $E \notin \mathcal{U}_p$. Thus $\mathcal{Z}_A(f)$ is not contained in $\mathcal{U}_p$, and hence $p = \mathcal{U}_p \notin Z_f$. This verifies the claim, which in turn shows that the restrictions to X of the basic closed sets $\{Z_f : f \in A(X)\}$ in $\mathfrak{U}(X)$ are precisely the basic closed sets $\{Z(f) : f \in A(X)\}$ in X. Thus ι is a homeomorphism onto its image.

That X is dense in $(\mathfrak{U}(X), \tau_A)$ follows from Theorem 3.1(b), since the closure of X is $\bigcap\{Z_f : f \equiv 0\} = \mathfrak{U}(X)$. ■

THEOREM 3.3. $\tau_A \subset \tau$; *hence* $(\mathfrak{U}(X), \tau_A)$ *is compact.*

Proof. We show that every basic closed set Z_f in the topology τ_A is also closed in τ. Let $f \in A(X)$. Then $Z_f = \{\mathcal{U} \in \mathfrak{U}(X) : \mathcal{Z}_A(f) \subset \mathcal{U}\} = \bigcap\{S_E : E \in \mathcal{Z}_A(f)\}$, where $S_E = \{\mathcal{U} \in \mathfrak{U}(X) : E \in \mathcal{U}\}$ is a basic closed set in τ. ■

THEOREM 3.4. *The following are equivalent*:

(a) $\tau_A = \tau$.

(b) $A(X)$ *separates zero sets in X.*

(c) τ_A *is Hausdorff.*

Proof. (a)⇒(b). Suppose $\tau_A = \tau$ and let $E, F \in Z[X]$, with $E \cap F = \emptyset$. Then the closure of F in τ_A is the same as its closure $\mathrm{cl}_\tau F$ in τ. By Theorem 3.1(b) this means that

$$\bigcap\{Z_f : f \in I_F\} = \{\mathcal{U} : F \in \mathcal{U}\}. \tag{3.1}$$

Since E and F are disjoint, E does not belong to any $\mathcal{U}$ for which $F \in \mathcal{U}$. Suppose that for every $f \in I_F$, E meets every member of $\mathcal{Z}_A(f)$. Then E meets every element of the z-filter $\mathcal{Z}_A[I_F]$ (this is a z-filter since I_F is an ideal). Thus there is a z-ultrafilter $\mathcal{U}_1$ containing $\mathcal{Z}_A[I_F] \cup \{E\}$. This z-ultrafilter would belong to Z_f for every $f \in I_F$, and so $\mathcal{U}_1 \in \bigcap\{Z_f : f \in I_F\}$, the intersection on the left in (3.1). But as noted above, the fact that $E \in \mathcal{U}_1$

implies that $F \notin \mathcal{U}_1$, so $\mathcal{U}_1$ does not belong to the set on the right in (3.1), a contradiction. Thus there must exist $f \in I_F$ and $G \in \mathcal{Z}_A(f)$ such that $E \cap G = \emptyset$. Then there exists $g \in A(X)$ such that $fg = 1$ on $G^{\mathrm{c}} \supset E$. But $f \in I_F$, so $fg = 0$ on F; thus the function $fg \in A(X)$ separates the zero sets E and F.

(b)⇒(a). Suppose $A(X)$ separates zero sets, and let $E \in Z[X]$. We show that the closure of E in τ is equal to its closure in τ_A, namely $\bigcap\{Z_f : f \in I_E\}$ (Theorem 3.1(b)). This would mean that $\tau \subset \tau_A$, and so the desired result would follow from Theorem 3.3. If $\mathcal{U} \in \mathrm{cl}_\tau E$, then $E \in \mathcal{U}$. So for every $f \in I_E$, $\mathcal{Z}_A(f) \subset \mathcal{U}$, and hence $\mathcal{U} \in Z_f$. This shows that $\mathrm{cl}_\tau E \subset \bigcap\{Z_f : f \in I_E\}$. For the other containment, let $\mathcal{U}' \in \bigcap\{Z_f : f \in I_E\} = \{\mathcal{U} \in \mathfrak{U}(X) : \mathcal{Z}_A[I_E] \subset \mathcal{U}\}$. If $F \in \mathcal{U}'$, then F meets every element of $\mathcal{Z}_A[I_E]$. We claim that F meets E. For suppose that $E \cap F = \emptyset$. Then there exists $h \in A(X)$ such that $h(E) = 0$ and $h(F) = 1$. Clearly, $h \in I_E$. The set $G = \{x \in X : h(x) \le 1/2\}$ is a zero set that contains E; moreover, $G \in \mathcal{Z}_A(h)$ since h is bounded away from zero on G^{c}. But $F \cap G = \emptyset$, and this contradicts the fact that F meets every element of $\mathcal{Z}_A[I_E]$. Thus $E \cap F \neq \emptyset$. Since F is an arbitrary element of the z-ultrafilter $\mathcal{U}'$, this proves that $E \in \mathcal{U}'$. Thus $\mathcal{U}' \in \mathrm{cl}_\tau E$.

(a)⇔(c). Since τ is Hausdorff ([5], p. 87), it remains to show (c)⇒(a). If τ_A is Hausdorff, then the identity map $(\mathfrak{U}(X), \tau) \to (\mathfrak{U}(X), \tau_A)$ is a continuous function (Theorem 3.3) from a compact space to a Hausdorff space, and hence is a homeomorphism ([10], p. 123). ∎

The map $\mathcal{Z}_A^{\leftarrow} : \mathfrak{U}(X) \to \mathfrak{M}(A)$ is not necessarily one-to-one, but, as we have seen, it is onto. Thus we may define an equivalence relation $\underset{A}{\sim}$ on $\mathfrak{U}(X)$ by

$$\mathcal{U}_1 \underset{A}{\sim} \mathcal{U}_2 \quad \text{iff} \quad \mathcal{Z}_A^{\leftarrow}[\mathcal{U}_1] = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}_2].$$

Note that if $p \in X$, then $\mathcal{U}_p$ is the only element in its equivalence class modulo $\underset{A}{\sim}$. Thus $\underset{A}{\sim}$ does not identify points of X (understood to be embedded in $\mathfrak{U}(X)$ as described above Theorem 3.1).

We now prove a result which explains the choice of the topology τ_A on $\mathfrak{U}(X)$.

THEOREM 3.5. *$\mathfrak{M}(A)$ is homeomorphic to $(\mathfrak{U}(X), \tau_A)/\underset{A}{\sim}$.*

Proof. In this proof we denote $\mathcal{Z}_A^{\leftarrow}$ by Φ. We show that $\mathfrak{M}(A)$ has the quotient topology induced by Φ; that is, a set F is closed in $\mathfrak{M}(A)$ iff $\Phi^{-1}(F)$ is closed in $(\mathfrak{U}(X), \tau_A)$. We first observe that the inverse image under Φ of a basic closed set in $\mathfrak{M}(A)$ is a basic closed set in $(\mathfrak{U}(X), \tau_A)$. This follows from the following equation: $\Phi^{-1}(N_f) = \{\mathcal{U} : f \in \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]\} = \{\mathcal{U} : \mathcal{Z}_A(f) \subset \mathcal{U}\} = Z_f$. Since any closed set $F \subset \mathfrak{M}(A)$ is an intersection of basic closed sets, it follows that $\Phi^{-1}(F)$ is closed.

Conversely, if $\Phi^{-1}(F)$ is closed in $(\mathfrak{U}(X),\tau_A)$, then $\Phi^{-1}(F)=\bigcap\{Z_f : f\in\Lambda\}$ for some index set Λ. We claim that

$$\Phi\Big(\bigcap\{Z_f : f\in\Lambda\}\Big)=\bigcap\{\Phi(Z_f) : f\in\Lambda\}.$$

The result will then follow, since we would have $F=\Phi(\bigcap\{Z_f : f\in\Lambda\})=\bigcap\{\Phi(Z_f) : f\in\Lambda\}=\bigcap\{N_f : f\in\Lambda\}$, whence F is closed. So to prove the claim, let $M\in\Phi(\bigcap\{Z_f : f\in\Lambda\})$; then there exists $\mathcal{U}\in\bigcap\{Z_f : f\in\Lambda\}$ such that $M=\Phi(\mathcal{U})$. Now $\mathcal{U}\in Z_f$ for all $f\in\Lambda$, so $M\in\Phi(Z_f)$ for all $f\in\Lambda$, and hence $M\in\bigcap\{\Phi(Z_f) : f\in\Lambda\}$. For the other containment, suppose $M\in\bigcap\{\Phi(Z_f) : f\in\Lambda\}$. Then $M\in\Phi(Z_f)$ for all $f\in\Lambda$, so for each $f\in\Lambda$ there exists a z-ultrafilter $\mathcal{U}_f$ such that $M=\Phi(\mathcal{U}_f)$. This means that $f\in M$ for all $f\in\Lambda$, so $\mathcal{Z}_A[M]\supset\mathcal{Z}_A(f)$ for all $f\in\Lambda$. Thus if $\mathcal{U}$ is any z-ultrafilter containing $\mathcal{Z}_A[M]$ then $\mathcal{U}\in Z_f$ for all $f\in\Lambda$, and hence $\mathcal{U}\in\bigcap\{Z_f : f\in\Lambda\}$. So $\Phi(\mathcal{U})=M\in\Phi(\bigcap\{Z_f : f\in\Lambda\})$. ∎

THEOREM 3.6. *$\mathfrak{M}(A)$ is homeomorphic to a quotient of the Stone–Čech compactification βX; precisely, $\mathfrak{M}(A)\simeq(\mathfrak{U}(X),\tau)/\underset{A}{\sim}$.*

Proof. We first show that $\mathfrak{M}(A)$ is a compact Hausdorff space. By [5], p. 111, it suffices to show that given $M_1,M_2\in\mathfrak{M}(A)$, there exist $h_1,h_2\in A(X)$, $h_1\notin M_1$ and $h_2\notin M_2$, such that $h_1h_2\in\bigcap\mathfrak{M}(A)$. We claim that there exist $E\in\mathcal{Z}_A[M_1]$ and $F\in\mathcal{Z}_A[M_2]$ such that $E\cap F=\emptyset$. For otherwise there would exist a z-filter containing $\mathcal{Z}_A[M_1]\cup\mathcal{Z}_A[M_2]$, and so $\mathcal{Z}_A^{\leftarrow}[\mathcal{F}]$ would be a proper ideal containing $M_1\cup M_2$, which is impossible. So choose $f\in M_1$ and $g\in M_2$ such that $E\in\mathcal{Z}_A(f)$ and $F\in\mathcal{Z}_A(g)$; this means there exist $f_1,g_1\in A(X)$ such that $ff_1|_{E^c}\equiv 1$ and $gg_1|_{F^c}\equiv 1$. Since $ff_1\in M_1$ and $gg_1\in M_2$, we have $1-ff_1\notin M_1$ and $1-gg_1\notin M_2$. But $(1-ff_1)(1-gg_1)=0$ for all $x\in X$, and so $h_1=1-ff_1$ and $h_2=1-gg_1$ are the desired functions.

By Theorem 3.5, $\mathfrak{M}(A)$ is homeomorphic to $(\mathfrak{U}(X),\tau_A)/\underset{A}{\sim}$, and so the latter space is also compact Hausdorff. Now the identity map $\iota:(\mathfrak{U}(X),\tau)\to(\mathfrak{U}(X),\tau_A)$ is continuous by Theorem 3.3, and hence so is the induced map $\Psi:(\mathfrak{U}(X),\tau)/\underset{A}{\sim}\to(\mathfrak{U}(X),\tau_A)/\underset{A}{\sim}$. Thus Ψ is a homeomorphism, since it is a continuous bijection from a compact space to a Hausdorff space ([10], p. 123). ∎

4. The realcompactifications of a completely regular space. A *realcompactification* of a completely regular space X is a realcompact space in which X is densely embedded. (In particular, every compactification of X is a realcompactification.) In this section we show that every realcompactification of X is homeomorphic to a subspace of $\mathfrak{M}(A)$ for some ring of continuous functions $A(X)$.

If M is a maximal ideal in $A(X)$ then the field $A(X)/M$ contains a canonical copy of $\mathbb{R}$: the image of the constant functions under the quotient map. If $A(X)/M$ consists only of this copy of $\mathbb{R}$ then M is a *real maximal ideal*. We say that X is *A-compact* if every real maximal ideal in $A(X)$ is fixed. (Thus a compact space is C^*-compact and a realcompact space is C-compact.) A collection $\mathcal{C}$ of closed sets in X is called *A-stable* if every $f \in A(X)$ is bounded on some member of $\mathcal{C}$.

Throughout this section, $A(X)$ is a ring of continuous functions that is closed under local bounded inversion.

THEOREM 4.1. *Let $\mathcal{U}$ be a z-ultrafilter on X. Then $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ is a real maximal ideal iff $\mathcal{U}$ is A-stable.*

Proof. If $M = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ is real, then for every $f \in A(X)$ there exists a constant function $r \in \mathbb{R}$ such that $f - r \in M$. Thus by Theorem 2.3, $\lim_{\mathcal{U}}(f-r)h = 0$ for all $h \in A(X)$, and so, in particular, using $h \equiv 1$, we get $\lim_{\mathcal{U}} f = r$. Hence $\lim_{\mathcal{U}} f$ is a (finite) real number for every $f \in A(X)$, and so every f is bounded on some member of $\mathcal{U}$; that is, $\mathcal{U}$ is A-stable.

Conversely, if $\mathcal{U}$ is A-stable then $\lim_{\mathcal{U}} f$ is a real number for every $f \in A(X)$. So it follows from Theorem 2.3 that $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}] = \{f \in A(X) : \lim_{\mathcal{U}} f = 0\}$. Thus the map

$$\psi : A(X) \to \mathbb{R}, \quad f \mapsto \lim_{\mathcal{U}} f,$$

is a well-defined ring homomorphism with kernel $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$, which is therefore real. ∎

THEOREM 4.2. *The space X is A-compact iff every A-stable z-ultrafilter on X converges.*

Proof. Suppose X is A-compact. If $\mathcal{U}$ is an A-stable z-ultrafilter then the ideal $M = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ is real by Theorem 4.1, and since X is A-compact, M is fixed. Thus by Corollary 2.6, $\mathcal{U}$ is fixed and hence converges. Conversely, suppose X is not A-compact. Then there exists a real maximal ideal M that is not fixed. Let $\mathcal{U}$ be a z-ultrafilter such that $M = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$. By Theorem 4.1, $\mathcal{U}$ is A-stable, and by Corollary 2.6, $\mathcal{U}$ is not fixed. Thus $\mathcal{U}$ is an A-stable z-ultrafilter that does not converge in X. ∎

COROLLARY 4.3. *If $B(X) \subset A(X)$ and X is B-compact, then X is A-compact.*

Proof. Let $\mathcal{U}$ be an A-stable z-ultrafilter on X. Then $\mathcal{U}$ is also B-stable, so $\mathcal{U}$ converges, since X is B-compact. Thus X is A-compact. ∎

Since all the rings that we consider are contained in $C(X)$, it follows that an A-compact space is realcompact.

Let $\mathfrak{M}_{\mathrm{r}}(A)$ denote the subspace of $\mathfrak{M}(A)$ consisting of the real maximal ideals in $A(X)$, and let $\mathfrak{U}_{\mathrm{s}}(X)$ denote the subspace of $(\mathfrak{U}(X), \tau_A)$ consisting of the A-stable z-ultrafilters on X. We define

$$\upsilon_A X = (\mathfrak{U}_{\mathrm{s}}(X), \tau_A)/\underset{A}{\sim}.$$

This space will serve as the A-compactification of X. If $f \in A(X)$ and if $\mathcal{U}_p$ is an A-stable z-ultrafilter, then we define

$$f^A(p) = \lim_{\mathcal{U}_p} f.$$

Then f^A is a continuous extension of f to $\mathfrak{U}_{\mathrm{s}}(X)$. If $\mathcal{U}_p$ and $\mathcal{U}_q$ are A-stable, then $\mathcal{U}_p \underset{A}{\sim} \mathcal{U}_q$ if and only if $f^A(p) = f^A(q)$ for all $f \in A(X)$ (by Theorem 4.1 and an argument as in [3], Lemma 2.1). Thus f^A composed with the quotient map modulo $\underset{A}{\sim}$ is a continuous extension of f to $\upsilon_A X$, which we shall also call f^A. Let $A(\upsilon_A X) = \{f^A : f \in A(X)\}$. Clearly, $A(\upsilon_A X) \simeq A(X)$.

THEOREM 4.4. *$\upsilon_A X$ is homeomorphic to $\mathfrak{M}_{\mathrm{r}}(A)$ by a homeomorphism that keeps X pointwise fixed.*

Proof. By Theorem 4.1, $\mathcal{Z}_A^{\leftarrow}$ maps $\mathfrak{U}_{\mathrm{s}}(X)$ onto $\mathfrak{M}_{\mathrm{r}}(A)$. The result now follows from Theorem 3.5. ∎

THEOREM 4.5. *If $A(X)$ and $B(Y)$ are isomorphic, then $\upsilon_A X$ and $\upsilon_B Y$ are homeomorphic.*

Proof. It follows from Theorem 4.4 that the topological spaces $\upsilon_A X$ and $\upsilon_B Y$ are determined entirely by the algebraic structures of $A(X)$ and $B(Y)$. ∎

In Theorem 4.5, if we assume in addition that X is A-compact and Y is B-compact, then we can conclude that X is homeomorphic to Y, because in this case $\upsilon_A X \simeq X$ and $\upsilon_B Y \simeq Y$.

Remark. If X is compact, then by Theorem 4.2, X is A-compact for any $A(X)$. In this case $\upsilon_A X \simeq X$. Thus for compact X and Y, it follows from Theorem 4.5 that if $A(X)$ and $B(Y)$ are isomorphic, then X and Y are homeomorphic.

THEOREM 4.6. (a) *$\upsilon_A X$ is an A-compactification of X (and hence a realcompactification of X).*

(b) *If αX is a realcompactification of X, then there exists a ring $A(X)$ of continuous functions such that $\alpha X \simeq \upsilon_A X$ by a homeomorphism that keeps X pointwise fixed.*

Proof. (a) Let M be a real maximal ideal in $A(X)$. Then every z-ultrafilter $\mathcal{U}_p$ containing $\mathcal{Z}_A[M]$ is A-stable, and so $M = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}_p] = \{f \in A(X) : \lim_{\mathcal{U}_p} f = 0\}$, as in the proof Theorem 4.1. Thus $M = \{f \in A(X) : f^A(p) = 0\}$, so the ideal M is fixed (considered as an ideal in $A(\upsilon_A X)$). Thus $\upsilon_A X$ is A-compact; since X is dense in $\upsilon_A X$ (Theorem 3.2), $\upsilon_A X$ is an A-compactification of X. Since $A(X) \subset C(X)$, $\upsilon_A X$ is also C-compact by Corollary 4.3; that is, it is realcompact.

(b) We first observe that if Y is realcompact, then $A(X) \simeq C(Y)$ implies $\upsilon_A X \simeq Y$ (Theorem 4.5). Now let $A(X) = \{f \in C(X) : f$ has a continuous extension f^α to $\alpha X\}$. Then $A(X) \simeq C(\alpha X)$. Since αX is realcompact, every real maximal ideal in $C(\alpha X)$ is fixed. But $C(\alpha X) \simeq A(X) \simeq A(\upsilon_A X)$, and so $\upsilon_A X \simeq \alpha X$. ∎

It follows immediately from Theorem 4.6(b) that every realcompactification of X is a quotient of a subspace of βX.

Two different rings of continuous functions $A(X)$ and $B(X)$ may give equivalent realcompactifications $\upsilon_A X$ and $\upsilon_B X$ (that is, they are homeomorphic by a homeomorphism that leaves X pointwise fixed). For example, if $H(\mathbb{N})$ denotes the ring of all sequences on $\mathbb{N}$ which are coefficients in the Taylor series of an analytic function on the disc, then $\mathbb{N}$ is H-compact and C-compact [2]. This situation cannot occur for two different "C-rings", defined as follows: A ring of continuous functions $A(X)$ is a *C-ring* if there exists a completely regular space Y such that $A(X)$ is ring isomorphic to $C(Y)$. (For example, $C^*(X)$ is a C-ring, since $C^*(X)$ is isomorphic to $C(\beta X)$.)

We now show that every realcompactification of X is equivalent to $\upsilon_A X$ for some C-ring $A(X)$.

THEOREM 4.7. *Let X be a completely regular space. There is a one-to-one correspondence between realcompactifications of X and C-rings on X.*

Proof. We first observe that $A(X)$ is a C-ring iff $A(X) \simeq C(\upsilon_A X)$. This is because if $A(X) \simeq C(Y)$ then $\upsilon_A X \simeq \upsilon_C Y$ (Theorem 4.5). Thus $C(\upsilon_A X) \simeq C(\upsilon_C Y) \simeq C(Y)$, and so $A(X) \simeq C(\upsilon_A X)$.

Now, suppose αX is a realcompactification of X, and let $A(X) = \{f|_X : f \in C(\alpha X)\}$. Then $A(X) \simeq C(\alpha X)$ since X is dense in αX. Thus $A(X)$ is a C-ring, so $A(X) \simeq C(\upsilon_A X)$, and hence $\alpha X \simeq \upsilon_A X$ (because αX and $\upsilon_A X$ are realcompact). Conversely, if $A(X)$ is a C-ring then $A(X) \simeq C(Y)$, where Y can be chosen to be realcompact. Then $\upsilon_A X \simeq Y$. This proves that the correspondence between C-rings $A(X)$ and the realcompactifications $\upsilon_A X$ is one-to-one. ∎

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