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A Nielsen theory for intersection numbers

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Abstract. Nielsen theory, originally developed as a homotopy-theoretic approach to fixed point theory, has been translated and extended to various other problems, such as the study of periodic points, coincidence points and roots. In this paper, the techniques of Nielsen theory are applied to the study of intersections of maps. A Nielsen-type number, the Nielsen intersection number $\mathrm{NI}(f,g)$, is introduced, and shown to have many of the properties analogous to those of the Nielsen fixed point number. In particular, $\mathrm{NI}(f,g)$ gives a lower bound for the number of points of intersection for all maps homotopic to f and g.

1. Introduction. Nielsen fixed point theory, a homotopy-theoretic approach to fixed-point theory, grew out of Nielsen's work in the 1920's on surface homeomorphisms. From those origins, Nielsen fixed point theory has grown into a richly developed theory for fixed points. Moreover, the methods of Nielsen theory have been translated from fixed point problems into other domains, such as the study of periodic points, coincidence points, roots, etc. That is, there are Nielsen coincidence numbers, Nielsen root numbers, etc., defined in the similar fashion, and with similar properties, to the original Nielsen fixed point number. In this paper, we consider another translation of the Nielsen machinery into a new setting: intersections of maps.

Given $f : X \to Z$ and $g : Y \to Z$, the *intersection set* of f and g is

$$\mathrm{Int}(f,g) = \{(x,y) \in X \times Y \mid f(x) = g(y)\}.$$

In general, the intersection set can be quite complicated, and the problem of "describing $\mathrm{Int}(f,g)$" correspondingly intractable. In some settings, such as the study in algebraic geometry of the intersection of algebraic varieties, there are highly developed theories which are tailored to that setting. Here, we will focus on the topological setting, and will assume that all of the spaces involved are compact, Hausdorff, path-connected and admit universal

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covering spaces. In this setting, the intersection set is clearly compact, and our attempt to describe it will be limited to counting its components.

Actually, it is not exactly "counting components" for a single pair of maps f and g that we are interested in. If either f or g is deformed by a homotopy, the intersection set will change. In particular, we can create arbitrarily large intersection sets via homotopies of f and g. To discount the contribution of such spurious intersection points, we instead study

$$\mathrm{MI}(f,g) = \min\{|\pi_0(I(f',g'))| \mid f' \simeq f,\ g' \simeq g\}.$$

Note that we count path components, rather than the cardinality of the set, so that $\mathrm{MI}(f,g)$ does not become "trivially infinite" when the intersection set has positive dimension. Even so, and even if X, Y and Z are all compact, it is not clear that $\mathrm{MI}(f,g)$ must be finite. However, in one important setting, it will be. Namely, if X, Y and Z are manifolds, then an arbitrarily small perturbation of f and/or g makes the two maps transverse at all intersection points. In particular, if $\dim(X)+\dim(Y)=\dim(Z)$, then $\mathrm{Int}(f,g)$ will be discrete, and hence finite.

Now, just as the Nielsen fixed point number provides a lower bound for $\mathrm{MF}(f)$, we will develop a Nielsen intersection number $\mathrm{NI}(f,g)$ that will provide a lower bound for $\mathrm{MI}(f,g)$. In broad terms, the development of a Nielsen theory for intersections parallels that of fixed point theory. It is patterned even more strongly after the Nielsen theory for coincidences [1], and the Nielsen theory for roots and generalized roots [2, 4]. The reason for this is that there is an important property that distinguishes fixed point theory from both coincidence theory and intersection theory: the ability to define a fixed point index in great generality. A fixed point index can be defined for any self-map on a polyhedron. In contrast, a coincidence index is only defined when the domain and range are compact orientable manifolds of the same dimension; and an intersection index is only defined when X, Y and Z are compact orientable manifolds with $\dim(X)+\dim(Y)=\dim(Z)$. Since the fixed point index plays an essential role (pun intended) in the development of Nielsen fixed point theory, the lack of an index forces the development of coincidence theory down a different path, and suggests that intersection theory should follow that path.

We begin with a quick survey of the intersection index for manifolds. Even though this index can only be used in a special case, it is an important special case, and will be a valuable tool when available. The Nielsen intersection number is defined in §3, and its basic properties are established in the remaining sections. §4 explores the Wecken theorem and §5 establishes the Jiang condition for intersection numbers. The next three sections deal with the functoriality of Nielsen intersection numbers, and the ability to move the computation of Nielsen intersection numbers from one triple of spaces to another. §6 provides some of the general functoriality results, while §§7 and 8 apply these results to intersection number formulas for covering spaces and fibrations respectively. For all of these topics, the treatment here should be considered a preliminary effort. More can be proven about all of these topics, and there are many other aspects of Nielsen theory that are not considered at all in this paper. The last section surveys some of the open questions and directions for development that lie ahead for the theory.

2. The intersection index. The intersection index, like the fixed point and coincidence indices, can be constructed from either a differential [7] or homological [5] point of view. In either approach, we require that X, Y and Z are compact, orientable manifolds of dimensions p, q and $n = p + q$ respectively. Fix orientations for X, Y and Z. A set $J \subset \mathrm{Int}(f, g)$ is an *isolated* set of intersections if there exists a neighborhood $U \subset X \times Y$ of J such that $\mathrm{Int}(f, g) \cap U = J$.

In the differential definition of the index, we can, by an arbitrarily small perturbation of f and g, assume that the maps are smooth and transverse at every point of intersection. For every $(x, y) \in J$, take a basis $\{v_1, \ldots, v_p\}$ for $T_x X$ and a basis $\{w_1, \ldots, w_q\}$ for $T_y Y$. Then

$$\{Df_x(v_1), \ldots, Df_x(v_p), Dg_y(w_1), \ldots, Dg_y(w_q)\}$$

forms a basis for $T_{f(x)}Z$. If this basis has the same orientation as that fixed for Z, we define $\mathrm{Ind}(f, g; (x, y)) = 1$; if it has the opposite orientation, we define $\mathrm{Ind}(f, g; (x, y)) = -1$. We then define

$$\mathrm{Ind}(f, g; J) = \sum_{(x,y)\in J} \mathrm{Ind}(f, g; (x, y)).$$

Clearly, $\mathrm{Ind}(f, g; J) = (-1)^{pq} \mathrm{Ind}(g, f; J)$. Of course, to show that this is well-defined, it must be shown that the quantity $\mathrm{Ind}(f, g; J)$ is independent of the transverse approximations of f and g chosen.

For the homological definition of $\mathrm{Ind}(f, g; J)$, choose neighborhoods $J \subset U \subset V$ in $X \times Y$ such that $\overline{U} \subset V^\circ$ and such that $V \cap \mathrm{Int}(f, g) = J$. Then consider the composition

$$\begin{aligned} &H_n(X \times Y) \to H_n(X \times Y, X \times Y \setminus U) \\ &\qquad \xleftarrow{\cong} H_n(V, V \setminus U) \xrightarrow{(f\times g)_*} H_n(Z \times Z, Z \times Z \setminus \Delta(Z)) \end{aligned}$$

in integer homology. Since $X \times Y$ and Z are compact orientable n-manifolds,

$$H_n(X \times Y) \cong H_n(Z \times Z, Z \times Z \setminus \Delta(Z)) \cong \mathbb{Z},$$

so the image of the generator of $H_n(X \times Y)$ under this composition gives an integer quantity $\mathrm{Ind}(f, g; J)$. Of course, here the index must be shown to be independent of the neighborhoods U and V chosen.

Omitting a number of proofs (including the proof that the two definitions coincide), we have an integer index with the following properties:

1. Given $F : X \times [0,1] \to Z$ and $G : Y \to Z$, consider the "fat homotopies" $\mathbb{F} : X \times [0,1] \to Z \times [0,1]$ and $\mathbb{G} : Y \times [0,1] \to Z \times [0,1]$, defined by $\mathbb{F}(x,t) = (F(x,t),t)$, $\mathbb{G}(y,t) = (G(y,t),t)$. If $\mathcal{J} \subset \mathrm{Int}(\mathbb{F},\mathbb{G})$ is an isolated set of intersections, then so is every $J_{t_0} = \mathcal{J} \cap \{t = t_0\}$, and $\mathrm{Ind}(F_t, G_t; J_t)$ is independent of t.
2. If $J = \emptyset$, then $\mathrm{Ind}(f,g;J) = 0$.
3. If $J_1,\ldots,J_n$ are disjoint isolated intersection sets, then

$$\mathrm{Ind}\Big(f,g;\bigcup_{k=1}^{n} J_k\Big) = \sum_{k=1}^{n} \mathrm{Ind}(f,g;J_k).$$

4. Suppose $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$, and $Z = Z_1 \times Z_2$, with $\dim(X_i) + \dim(Y_i) = \dim(Z_i)$. Given $f_i : X_i \to Z_i$ and $g_i : Y_i \to Z_i$ and isolated isolated intersection sets $J_i \subset \mathrm{Int}(f_i, g_i)$, define $f = f_1 \times f_2$, $g = g_1 \times g_2$ and $J = J_1 \times J_2$. Then

$$\mathrm{Ind}(f,g;J) = \mathrm{Ind}(f_1,g_1;J_1)\,\mathrm{Ind}(f_2,g_2;J_2).$$

5. If (x,y) is an isolated intersection of f and g with $\mathrm{Ind}(f,g;(x,y)) = 0$ and $U \times V$ is a neighborhood of (x,y) in $X \times Y$, then there are homotopies F and G such that $F_0 = f$, and $G_0 = g$; $\mathrm{Int}(F_1,G_1) = \emptyset$; and $F_t = F_0$ and $G_t = G_0$ on $X \setminus U$ and $Y \setminus V$ respectively. Moreover, these homotopies can be chosen arbitrarily close to f and g.

The index of the total intersection set $\mathrm{LI}(f,g) = \mathrm{Ind}(f,g;\mathrm{Int}(f,g))$ ($\#(f,g)$ in [7]) is the (*Lefschetz*) *intersection number* of f and g. It should be thought of as an analogue to the Lefschetz number for fixed points or coincidences. From the properties above, it is clear that

Theorem 2.1. $\mathrm{LI}(f,g)$ *is a homotopy invariant*, *and if* $\mathrm{LI}(f,g) \neq 0$, *then for every* $f' \simeq f$, $g' \simeq g$, $\mathrm{Int}(f',g')$ *is nonempty.*

However, $\mathrm{LI}(f,g)$ also suffers from the same limitations as the Lefschetz number: when nonzero, it does not estimate the number of intersections; and when zero, it does not guarantee that f and g are intersection-free. Consider the example from [7, Figure 5-3] shown in Figure 1. Let $X = Y = S^1$, $Z = T^2 \# T^2$, and consider the submanifolds M and N to be the images of embeddings $f, g : S^1 \to T^2 \# T^2$. There are two intersections of opposite orientation, so $\mathrm{LI}(f,g) = 0$. That is, $\mathrm{LI}(f,g) = 0$ does not imply that f and g are intersection-free. Moreover, it appears that no deformation of f or g will remove the intersections, so $\mathrm{LI}(f,g) = 0$ does not even imply that f and g can be deformed to be intersection-free. This picture is, in some sense, a motivating example: neither intersection point can be removed individually by a deformation of f or g; and the two intersection points cannot

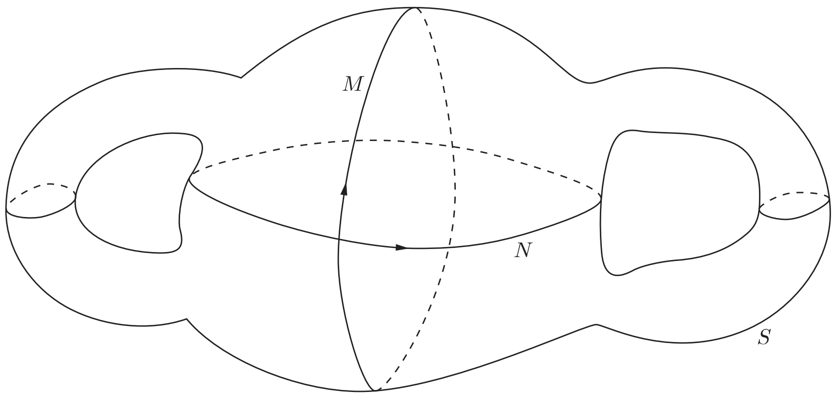

Fig. 1. Essential intersections in the double torus

be "cancelled" because we cannot deform the image of f into the image of g. The essence of the definition of the Nielsen intersection number lies in making these ideas precise and general.

3. Nielsen intersection numbers. As with any Nielsen-type number, the basic steps in the definition of the Nielsen intersection number are to partition the intersection set into intersection classes, define a notion of an essential class, and count the number of essential classes. The data required consists of compact, path-connected spaces X, Y and Z, and continuous functions $f : X \to Z$ and $g : Y \to Z$. We define an equivalence relation on $\mathrm{Int}(f,g)$ by $(x_0, y_0) \sim_{\mathrm{N}} (x_1, y_1)$ if there exist paths α in X from x_0 to x_1 and β in Y from y_0 to y_1 in Y such that $f\alpha \simeq g\beta \text{ rel}\{0,1\}$. Alternatively, we can think of this as requiring a path ω in $X \times Y$ from (x_0, y_0) to (x_1, y_1) such that $(f \times g)\omega$ is homotopic $\text{rel}\{0,1\}$ to a path in the diagonal $\Delta(Z)$ in $Z \times Z$. Equivalence classes will be called *intersection classes* and the set of intersection classes will be denoted by $\mathcal{I}(f,g)$.

Proposition 3.1. *$\mathcal{I}(f,g)$ is finite, and each intersection class J is a union of components of* $\mathrm{Int}(f,g)$.

There is also a covering space approach to this partitioning of $\mathrm{Int}(f,g)$ into intersection classes. If $\widetilde{X}$, $\widetilde{Y}$ and $\widetilde{Z}$ are the universal covers of X, Y and Z, and $(x,y) \in \mathrm{Int}(f,g)$, choose $\widetilde{x} \in p_X^{-1}(x)$ and $\widetilde{y} \in p_Y^{-1}(y)$. Then there exists a covering transformation $\gamma \in \mathcal{D}(Z)$ such that $\widetilde{f}(\widetilde{x}) = \gamma\widetilde{g}(\widetilde{y})$.

Proposition 3.2. *For every $\gamma \in \mathcal{D}(Z)$, $(p_X \times p_Y)(\mathrm{Int}(\widetilde{f}, \gamma\widetilde{g}))$ is an intersection class. Further, $(p_X \times p_Y)(\mathrm{Int}(\widetilde{f}, \gamma\widetilde{g})) = (p_X \times p_Y)(\mathrm{Int}(\widetilde{f}, \gamma'\widetilde{g}))$ if and only if there exist $\alpha \in \mathcal{D}(X)$, $\beta \in \mathcal{D}(Y)$ such that $(f_\#\alpha)\gamma = \gamma'(g_\#\beta)$. Otherwise, $(p_X \times p_Y)(\mathrm{Int}(\widetilde{f}, \gamma\widetilde{g}))$ and $(p_X \times p_Y)(\mathrm{Int}(\widetilde{f}, \gamma'\widetilde{g}))$ are disjoint.*

P r o o f. First, suppose $(\widetilde{x},\widetilde{y}),(\widetilde{x}',\widetilde{y}')\in \mathrm{Int}(\widetilde{f},\gamma\widetilde{g})$. Choose a path $\widetilde{\omega}$ from $(\widetilde{x},\widetilde{y})$ to $(\widetilde{x}',\widetilde{y}')$ in $\widetilde{X}\times\widetilde{Y}$. Then $\omega=(p_X\times p_Y)\widetilde{\omega}$ is a path from $(x,y)=(p_X\times p_Y)(\widetilde{x},\widetilde{y})$ to $(x',y')=(p_X\times p_Y)(\widetilde{x}',\widetilde{y}')$. Since $(\widetilde{f}\times\gamma\widetilde{g})\widetilde{\omega}$ has its endpoints in $\Delta(\widetilde{Z})$, it is endpoint-homotopic to a path in $\Delta(\widetilde{Z})$. Clearly, its image $(f\times g)\omega$ is then endpoint-homotopic to a path in $\Delta(Z)$.

On the other hand, suppose $(x,y)\in(p_X\times p_Y)(\mathrm{Int}(\widetilde{f},\gamma\widetilde{g}))$, and (x,y) and (x',y') are in the same intersection class, with a path ω connecting them. Let $\widetilde{\omega}$ be the lift of ω in $\widetilde{X}\times\widetilde{Y}$ based at $(\widetilde{x}',\widetilde{y}')\in(p_X\times p_Y)(\mathrm{Int}(\widetilde{f},\gamma\widetilde{g}))$. Let $(\widetilde{x}',\widetilde{y}')=\widetilde{\omega}(1)$. Since $(f\times g)\omega$ is endpoint-homotopic to a path in $\Delta(Z)$, its lift based at $(\widetilde{f}(\widetilde{x}),\gamma\widetilde{g}(\widetilde{y}))$ is endpoint-homotopic to a path in $\Delta(\widetilde{Z})$. But since its endpoint is $(\widetilde{f}(\widetilde{x}'),\gamma\widetilde{g}(\widetilde{y}'))$, the point $(\widetilde{f}(\widetilde{x}'),\gamma\widetilde{g}(\widetilde{y}'))$ lies in $\Delta(\widetilde{Z})$, and $(\widetilde{x}',\widetilde{y}')\in\mathrm{Int}(\widetilde{f},\gamma\widetilde{g})$.

This establishes that $(p_X\times p_Y)(\mathrm{Int}(\widetilde{f},\gamma\widetilde{g}))$ is an intersection class. Clearly, any two such sets either coincide or are disjoint. If $(p_X\times p_Y)(\mathrm{Int}(\widetilde{f},\gamma\widetilde{g}))=(p_X\times p_Y)(\mathrm{Int}(\widetilde{f},\gamma'\widetilde{g}))$, then there are points $(\widetilde{x},\widetilde{y})\in\mathrm{Int}(\widetilde{f},\gamma\widetilde{g})$ and $(\widetilde{x}',\widetilde{y}')\in\mathrm{Int}(\widetilde{f},\gamma'\widetilde{g})$ with $(p_X\times p_Y)(\widetilde{x},\widetilde{y})=(p_X\times p_Y)(\widetilde{x}',\widetilde{y}')$. That is,

$$\widetilde{f}(\widetilde{x})=\gamma\widetilde{g}(\widetilde{y}),\qquad \widetilde{f}(\widetilde{x}')=\gamma'\widetilde{g}(\widetilde{y}'),\qquad \widetilde{x}'=\alpha(\widetilde{x}),\qquad \widetilde{y}'=\beta(\widetilde{y})$$

for some $\alpha\in\mathcal{D}(X)$, $\beta\in\mathcal{D}(Y)$, $\gamma,\gamma'\in\mathcal{D}(Z)$. Then

$$\begin{aligned}\widetilde{f}(\widetilde{x}')&=\gamma'\widetilde{g}(\widetilde{y}'), &\qquad (f_\#\alpha)\widetilde{f}(\widetilde{x})&=(\gamma' g_\#\beta)\widetilde{g}(\widetilde{y}),\\ \widetilde{f}\alpha(\widetilde{x})&=\gamma'\widetilde{g}\beta(\widetilde{y}), &\qquad (f_\#\alpha)\gamma\widetilde{g}(\widetilde{y})&=\gamma'(g_\#\beta)\widetilde{g}(\widetilde{y}),\end{aligned}$$

which implies that $(f_\#\alpha)\gamma=\gamma'(g_\#\beta)$. ∎

We therefore define *Reidemeister intersection classes* in $\mathcal{D}(Z)$ by the equivalence relation $\gamma\sim\gamma'$ if and only if there exist $\alpha\in\mathcal{D}(X)$, $\beta\in\mathcal{D}(Y)$ such that $(f_\#\alpha)\gamma=\gamma(g_\#\beta)$. The set of Reidemeister intersection classes will be denoted by $\mathrm{RI}(f,g)$. Of course, if $f(x)=z=g(y)$, then we can also define the Reidemeister relation in terms of the fundamental group homomorphisms

$$\pi_1(X,x)\xrightarrow{f_\#}\pi_1(Z,z)\xleftarrow{g_\#}\pi_1(Y,y).$$

Note that $\mathrm{RI}(f,g)$ can be viewed as a double coset:

$$\mathrm{RI}(f,g)=f_\#(\pi_1(X))\backslash\pi_1(Z)/g_\#(\pi_1(Y)).$$

This can be approximated by abelianizing the problem (i.e. passing from homotopy to homology). Let $\mathrm{RI}_*(f,g)=H_1(Z)/\langle\mathrm{im}(f_{1*}),\mathrm{im}(g_{1*})\rangle$. Clearly, $\mathrm{RI}(f,g)$ maps onto $\mathrm{RI}_*(f,g)$, so we have

PROPOSITION 3.3. 1. $|\mathcal{I}(f,g)|\le|\mathrm{RI}(f,g)|$.

2. $|\mathrm{RI}_*(f,g)|\le|\mathrm{RI}(f,g)|$, *with equality if* $\pi_1(Z)$ *is abelian.*

The inequality $|\mathcal{I}(f,g)| \le |\mathrm{RI}(f,g)|$ occurs because there may be Reidemeister classes for which the corresponding intersection class is empty. It will also be useful in later sections to consider intersections on the algebraic level. That is, given $(x,y) \in \mathrm{Int}(f,g)$, we define

$$I(f_\#, g_\#; (x,y)) = \{\alpha \times \beta \in \pi_1(X,x) \times \pi_1(Y,y) \mid f_\#(\alpha) = g_\#(\beta)\}.$$

It is a simple matter to check that

PROPOSITION 3.4. *For every $(x,y) \in \mathrm{Int}(f,g)$, $I(f_\#, g_\#; (x,y))$ is a subgroup of $\pi_1(X,x) \times \pi_1(Y,y)$. If $(x,y) \sim_{\mathrm{N}} (x',y')$, with paths α in X and β in Y relating them, then the isomorphism*

$$\alpha_\# \times \beta_\# : \pi_1(X,x) \times \pi_1(Y,y) \to \pi_1(X,x') \times \pi_1(Y,y')$$

maps $I(f_\#, g_\#; (x,y))$ to $I(f_\#, g_\#; (x',y'))$.

Proof. It is a simple matter to check that $I(f_\#, g_\#; (x,y))$ is a subgroup of $\pi_1(X,x) \times \pi_1(Y,y)$.

Now, given paths α from x to x' and β from y to y' with $f\alpha \simeq g\beta$, there is a commutative diagram

$$\begin{tikzcd}
\pi_1(X,x) \arrow[r, "f_\#"] \arrow[d, "\alpha_\#"'] & \pi_1(Z,z) \arrow[d, shift right, "f\alpha_\#"'] \arrow[d, shift left, "g\beta_\#"] & \pi_1(Y,y) \arrow[l, "g_\#"'] \arrow[d, "\beta_\#"] \\
\pi_1(X,x') \arrow[r, "f_\#"] & \pi_1(Z,z') & \pi_1(Y,y') \arrow[l, "g_\#"']
\end{tikzcd}$$

with $(f\alpha)_\# = (g\beta)_\#$. Now, suppose $\omega \in \pi_1(X,x)$, $\omega' \in \pi_1(Y,y)$ such that $f_\#(\omega) = g_\#(\omega')$. Then

$$f_\#\alpha_\#(\omega) = (f\alpha)_\# f_\#(\omega) = (g\beta)_\# g_\#(\omega') = g_\#\beta_\#(\omega'),$$

so $\alpha_\# \times \beta_\#$ maps $I(f_\#, g_\#; (x,y))$ into $I(f_\#, g_\#; (x',y'))$. Clearly, reversing the path gives the reverse inclusion. ∎

We can then unambiguously define $I(f_\#, g_\#; J)$ for an intersection class J. This quantity will be useful in later sections, when we investigate the functorial properties of the Nielsen numbers.

But first we must complete the definition of the Nielsen intersection number. To do so, we first need the concept of an essential intersection class. When an intersection index is defined, we say that an intersection class J is *algebraically essential* if $\mathrm{Ind}(f,g;J) \neq 0$. However, intersection indices are not defined in general, so we need a more general notion of an essential class. Following Brooks [1], we can define in full generality a concept of a *topologically essential* interesection class, which explicitly captures the meaning of "essential" as "unable to be removed by a homotopy".

To do so, we return to the fat homotopies introduced in the previous section. There, they were used (in the setting of orientable manifolds of the appropriate dimensions) to describe the homotopy invariance of the intersection index. It followed then that intersection classes with nonzero intersection index cannot be removed by a homotopy. We now use the fat homotopy construction to describe directly an essential intersection class. Consider homotopies $F : X \times [0,1] \to Z$ and $G : Y \times [0,1] \to Z$. Define $\mathbb{F} : X \times [0,1] \to Z \times [0,1]$ and $\mathbb{G} : Y \times [0,1] \to Z \times [0,1]$ by $\mathbb{F}(x,t) = (F(x,t),t)$ and $\mathbb{G}(y,t) = (G(y,t),t)$. Then $\mathrm{Int}(\mathbb{F},\mathbb{G}) = \bigcup_{t\in[0,1]} \mathrm{Int}(F_t,G_t)$. Further, the partitioning of $\mathrm{Int}(\mathbb{F},\mathbb{G})$ into intersection classes preserves this decomposition:

LEMMA 3.5. *Supppose J is an intersection class in $\mathcal{I}(\mathbb{F},\mathbb{G})$. Then, for every $t \in [0,1]$, $J_t = J \cap (X \times Y \times \{t\})$ is an intersection class for F_t and G_t.*

Proof. If J is an intersection class in $\mathcal{I}(\mathbb{F},\mathbb{G})$, then there are lifts $\widetilde{\mathbb{F}} : \widetilde{X} \times [0,1] \to \widetilde{Z} \times [0,1]$ and $\widetilde{\mathbb{G}} : \widetilde{Y} \times [0,1] \to \widetilde{Z} \times [0,1]$ such that $J = (p_X \times \mathrm{id} \times p_Y \times \mathrm{id})(\mathrm{Int}(\widetilde{\mathbb{F}},\widetilde{\mathbb{G}}))$. Clearly then, $J_t = (p_X \times p_Y)(\mathrm{Int}(\widetilde{F}_t,\widetilde{G}_t)) \times \{t\}$, so J_t is an intersection class of (F_t,G_t).

If $(x,y),(x',y') \in J_{t_0}$, then there exists a path $\omega \subset X \times [0,1] \times Y \times [0,1]$ such that $\omega(0) = (x,t_0,y,t_0)$, $\omega(1) = (x',t_0,y',t_0)$ and $\mathbb{F}\omega \simeq \mathbb{G}\omega$ in $Z \times [0,1] \times Z \times [0,1]$. If $\omega(s) = (x(s),t_1(s),y(s),t_2(s))$, then define $\Psi(s) = (x(s),t_0,y(s),t_0)$. Clearly, $\Psi \simeq \omega$, so $\overline{F}\Psi \simeq \overline{F}\omega \simeq \overline{G}\omega \simeq \overline{G}\Psi$. Moreover, while the homotopy from $\overline{F}\Psi$ to $\overline{G}\Psi$ may run through all of $Z \times [0,1] \times Z \times [0,1]$, it clearly projects to a homotopy contained in $Z \times \{t_0\} \times Z \times \{t_0\}$. Dropping the "$\times\{t_0\}$", we have a path ψ in $X \times Y$ from (x,y) to $(x'y')$ such that $F_{t_0}\psi \simeq G_{t_0}\psi$.

The converse is trivial. ∎

We say that J_0 and J_1 are (F,G)-*related*, or more informally, that J_0 can be *deformed* to J_1. This defines an equivalence relation on the union $\bigcup_{f'\in[f],g'\in[g]} \mathcal{I}(f',g')$ by declaring $J_0 \in \mathcal{I}(f',g')$ and $J_1 \in \mathcal{I}(f'',g'')$ to be related if there exist homotopies F,G such that J_0 and J_1 are (F,G)-related. Now, given $J \in \mathcal{I}(f,g)$, we declare J to be *inessential* if it can be deformed to the empty set. J is *essential* if, for every $F : X \times [0,1] \to Z$ and $G : Y \times [0,1] \to Z$ with $F_0 = f$, $G_0 = g$, the set $\mathrm{Int}(F_1,G_1)$ is nonempty.

The *Nielsen intersection number* $\mathrm{NI}(f,g)$ is the number of essential intersection classes in $\mathcal{I}(f,g)$. Clearly, from its construction, $\mathrm{NI}(f,g) \le |\mathcal{I}(f,g)|$. Moreover, for every $f' \simeq f$, $g' \simeq g$, $\mathrm{NI}(f,g) \le \mathrm{NI}(f',g')$. That is,

THEOREM 3.6. $\mathrm{NI}(f,g)$ *is a homotopy invariant, and* $\mathrm{NI}(f,g) \le \mathrm{MI}(f,g)$.

Note that $\mathrm{NI}(f, g)$ is not a homotopy-type invariant. For example, take $X = Y = Z = S^1$ and $f = g = \mathrm{id}$. There is only one Reidemeister class, so there is only one intersection class. And, since any two maps f', g' homotopic to the identity must map onto Z, there is for every $z \in Z$ an $x \in X$ and $y \in Y$ such that $f'(x) = g'(y) = z$. In particular, $\mathrm{Int}(f', g') \neq \emptyset$, so the intersection class can never be removed by a homotopy, and so is essential. That is, $\mathrm{NI}(\mathrm{id}, \mathrm{id}) = 1$. But, if we replace Z by $S^1 \times [0, 1]$, and define $f(x) = (x, 0)$, $g(y) = (y, 1)$, then $\mathrm{Int}(f, g) = \emptyset$, and $\mathrm{NI}(f, g) = 0$.

On the other hand, NI is a topological invariant, in the sense that pre-composition or post-composition of f and g with homeomorphisms does not change $\mathrm{NI}(f, g)$. More precisely,

Proposition 3.7. *If $\alpha : X' \to X$, $\beta : Y' \to Y$ and $\gamma : Z \to Z'$ are homeomorphisms, then for any $f : X \to Z \leftarrow Y : g$,*

$$\mathrm{NI}(f, g) = \mathrm{NI}(\gamma f \alpha, \gamma g \beta).$$

Returning to the example at the end of §2, it is clear that the two intersection points lie in different intersection classes. To see this, take the representation

$$\langle \alpha_1, \beta_1, \alpha_2, \beta_2 \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} = \alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1} \rangle$$

of $\pi_1(S)$. Then $\pi_1(M) = \langle \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \rangle$ and $\pi_1(N) = \langle \beta_1\beta_2^{-1} \rangle$. Thus, if we take any choice of paths α and β in S^1 between the intersection points, then

$$[f\alpha\overline{g\beta}] = (\alpha_1\beta_1\alpha_1^{-1}\beta^{-1})^m * \beta_1(\beta_1\beta_2^{-1})^n$$

for some choice of m and n. No choice of m and n ever renders the trivial element, so this loop is essential and the two intersection points are in different intersection classes. Since both classes consist of a single intersection point with index ± 1, both classes are essential, and $\mathrm{NI}(f, g) = 2$. Further, since

$$\mathrm{NI}(f, g) \leq \mathrm{MI}(f, g) \leq |\mathrm{Int}(f, g)|,$$

it follows that $\mathrm{MI}(f, g) = 2$ as well. That is, the Nielsen intersection number correctly detects the fact that every deformation of f and g has at least two intersection points.

4. The Wecken theorem. Having defined the Nielsen intersection number, we are faced immediately with two natural and interrelated questions. First, note that if X, Y and Z are compact orientable manifolds with $\dim(X) + \dim(Y) = \dim(Z)$, we have two alternatives for defining an essential intersection class, and hence for defining the Nielsen intersection number. On the one hand, we can count algebraically essential classes (i.e. those with nonzero index); or we can count topologically essential classes (i.e. those which cannot be removed by a homotopy). Clearly, we would like to know if those two definitions coincide. Second, having established that $\mathrm{NI}(f,g)$ is a lower bound for $\mathrm{MI}(f,g)$, we would like to know if it is a sharp lower bound. What conditions on X,Y,Z and f,g are needed to guarantee that there exist $f' \simeq f$ and $g' \simeq g$ such that $\mathrm{Int}(f',g')$ has cardinality $\mathrm{NI}(f,g)$? Our exploration of both of these questions closely parallels the work in [3] and [4].

The two are actually closely related. The one setting in which we will show $\mathrm{NI}(f,g) = \mathrm{MI}(f,g)$ is that of compact orientable manifolds, and the knowledge that algebraically essential classes coincide with topologically essential classes will be an important step in that proof. We begin, then, with the question of the relation between topologically essential and algebraically essential classes.

Suppose X, Y and Z are compact smooth orientable manifolds with $\dim(X) + \dim(Y) = \dim(Z)$. We can assume without loss that $f : X \to Z$ and $g : Y \to Z$ are smooth maps which intersect each other transversely. Suppose $J \in \mathcal{I}(f,g)$ is an intersection class. We will show that J is algebraically inessential if and only if it is topologically inessential. One direction is clear: if J is topologically inessential, then it can be deformed to the empty set, and so has index 0. The other implication requires more work and some hypotheses. The starting point is the Whitney lemma [11]:

Lemma 4.1. *Consider embeddings $f : D^p \to M$ and $g : D^q \to M$, where M is a compact orientable manifold of dimension $p+q$ and $p,q > 2$. Suppose $\mathrm{Int}(f,g)$ consists of exactly two points (x_1,y_1), (x_2,y_2) with $\mathrm{Ind}(f,g;(x_1,y_1)) = -\mathrm{Ind}(f,g;(x_2,y_2))$. If there are paths α from x_1 to x_2 and β from y_1 to y_2 such that $f\alpha$ is smoothly homotopic to $g\beta$, then there exists a smooth homotopy $F : D^p \times [0,1] \to M$ such that $F_0 = f$ and $\mathrm{Int}(F_1,g) = \emptyset$.*

Using this, we can show that

Lemma 4.2. *Suppose X, Y, Z are compact orientable manifolds with $\dim(X) + \dim(Y) = \dim(Z)$ and $\max\{\dim(X),\dim(Y)\} > 2$. If $J \in \mathcal{I}(f,g)$ has $\mathrm{Ind}(f,g;J) = 0$, then there exist homotopies $F : X \times [0,1] \to Z$ and $G : Y \times [0,1] \to Z$ such that J is (F,G)-related to the empty set.*

Proof. Since f and g intersect transversely, the only way J can have index 0 is for J to consist of $2m$ points, half of which have index $+1$, the other half having index -1. To remove J, it suffices to show that we can cancel a pair of points (x,y), (x',y'). If we assume without loss that $\dim(X) \le \dim(Y)$, we must consider separately the two cases $\dim(X) < \dim(Y)$ and $\dim(X) = \dim(Y)$. In the former, $\dim(Z) > 2\dim(X)$, and an arbitrarily small perturbation will make f an embedding [7, Theorem 2.2.13]. In the latter, $\dim(Z) = 2\dim(X)$, and an arbitrarily small perturbation will make

f an immersion with *clean double points* [7, Theorem 2.2.12 & Ex. 3.2.1]. That is, there is a finite set $S_f \subset X$ such that

1. f is injective on $X \setminus S_f$;
2. for each $x \in S_f$ there is a neighborhood U such that $f|_U$ is injective;
3. if $f(x_i) = f(x_j)$, then $f(U_i) \cap f(U_j) = f(x_i)$, and $f|_{U_i}$ and $f|_{U_j}$ are transverse at that point.

Since $\dim(Z) = 2\dim(Y)$ as well, a similar deformation of g can be made, and a similar set S_g defined.

Case I: $\dim(X) < \dim(Y)$. To apply the Whitney lemma, both of the maps involved must be embeddings. Since embeddings are not dense in $C^0(Y, Z)$, we must "manufacture" an embedding. To do so, consider $f \times \mathrm{id} : X \times Y \to Z \times Y$, and $\widehat{g} : Y \to Z \times Y$ defined by $\widehat{g}(y) = (g(y), y)$. These are both embeddings of manifolds with dimension at least 3. If $\delta : Y \to Y \times Y$ is the diagonal embedding, then $\mathrm{id} \times \delta : X \times Y \to X \times Y \times Y$ maps $\mathrm{Int}(f, g)$ homeomorphically to $\mathrm{Int}(f \times \mathrm{id}, \widehat{g})$. There are paths $\alpha \times \beta$ from (x, y) to (x', y') in $X \times Y$ and β from y to y' in Y such that $f\alpha \times \beta \simeq g\beta \times \beta$ in $Z \times Y$. To apply the Whitney lemma, we must verify that $\mathrm{Ind}(f \times \mathrm{id}, \widehat{g}; (x, y, y)) = \mathrm{Ind}(f, g; (x, y))$ and $\mathrm{Ind}(f \times \mathrm{id}, \widehat{g}; (x', y', y')) = \mathrm{Ind}(f, g; (x', y'))$.

If $\{v_1, \ldots, v_p\}$ is a basis for $T_x X$ and $\{w_1, \ldots, w_q\}$ is a basis for $T_y Y$, then

- $\{Df_x(v_1), \ldots, Df_x(v_p), Dg_y(w_1), \ldots, Dg_y(w_q)\}$ is a basis for $T_{f(x)}Z$;
- $\{v_1 \times 0, \ldots, v_p \times 0, 0 \times w_1, \ldots, 0 \times w_q\}$ is a basis for $T_{(x,y)}(X \times Y)$;
- $\{Df_x(v_1) \times 0, \ldots, Df_x(v_p) \times 0, Dg_y(w_1) \times 0, \ldots, Dg_y(w_q) \times 0, 0 \times w_1, \ldots$ $\ldots, 0 \times w_q\}$ is a basis for $T_{(f(x),y)}(Z \times Y)$.

If we use these bases for Z, $X \times Y$ and $Z \times Y$ (which we are free to do), then

$$\mathrm{Ind}(f, g; (x, y)) = \mathrm{Ind}(f \times \mathrm{id}, \widehat{g}; (x, y, y)) = 1.$$

We now have to consider $\mathrm{Ind}(f \times \mathrm{id}, \widehat{g}; (x', y', y'))$ and $\mathrm{Ind}(f, g; (x', y'))$.

Since (x, y) and (x', y') lie in the same intersection class, there are paths α from x to x' and β from y to y' such that $f\alpha \simeq g\beta$. Let $\{v'_1, \ldots, v'_p\}$ and $\{w'_1, \ldots, w'_q\}$ denote the translates along α and β of $\{v_1, \ldots, v_p\}$ and $\{w_1, \ldots, w_q\}$ to $T_{x'}X$ and $T_{y'}Y$. For $T_{f(x')}Z$, there are two bases to consider:

$$\{Df_{x'}(v'_1), \ldots, Df_{x'}(v'_p), Dg_{y'}(w'_1), \ldots, Dg_{y'}(w'_q)\}$$

and the basis formed by translating

$$\{Df_x(v_1), \ldots, Df_x(v_p), Dg_y(w_1), \ldots, Dg_y(w_q)\}$$

along $f\alpha$ from $T_{f(x)}Z$ to $T_{f(x')}Z$. If C is the change of basis matrix relating these two bases, then $\mathrm{Ind}(f, g; (x', y')) = \det(C)$. Similarly, we can translate the basis of $T_{(x,y)}(X \times Y)$ to the basis

$$\{v'_1 \times 0, \ldots, v'_p \times 0, 0 \times w'_1, \ldots, 0 \times w'_q\}$$

for $T_{(x',y')}(X \times Y)$. Then $T_{(f(x'),y')}(Z \times Y)$ has the basis

$$\{Df_{x'}(v'_1) \times 0, \ldots, Df_{x'}(v'_p) \times 0,\\ Dg_{y'}(w'_1) \times 0, \ldots, Dg_{y'}(w'_q) \times 0, 0 \times w'_1, \ldots, 0 \times w'_q\}$$

and the basis formed by translating

$$\{Df_x(v_1) \times 0, \ldots, Df_x(v_p) \times 0,\\ Dg_y(w_1) \times 0, \ldots, Dg_y(w_q) \times 0, 0 \times w_1, \ldots, 0 \times w_q\}$$

from $T_{(f(x),y)}(Z \times Y)$ along $g\beta \times \beta$. The change of basis matrix relating these two bases is

$$D = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix},$$

so

$$\mathrm{Ind}(f \times \mathrm{id}, \widehat{g}; (x', y', y')) = \det(D) = \det(C) = \mathrm{Ind}(f, g; (x', y')).$$

The Whitney lemma then implies that there exists a $\widehat{G} : Y \times [0,1] \to Z \times Y$ such that $\widehat{G}_0 = \widehat{g}$ and $\mathrm{im}(\widehat{G}_1) \cap (f \times \mathrm{id}) = \emptyset$. Let G be the projection $\varrho_1 \circ \widehat{G} : Y \times [0,1] \to Z$ of $\widehat{G}$ onto Z. It is a simple calculation to see that $f \times \mathrm{id}$ and $\widehat{G}_1$ are intersection-free if and only if f and G_1 are. This, then, is the required homotopy. Moreover, since the Whitney lemma is really a local result, $\widehat{G}$ and G can be constructed so that g is deformed only in a neighborhood of β. In particular, no new intersection points are formed. Applying this process iteratively removes all of the intersection points in J.

Case II: $\dim(X) = \dim(Y)$. If $\dim(Z) = 2\dim(X) = 2\dim(Y)$, then arbitrarily small perturbations make f and g immersions with clean double points and with transverse intersections. Moreover, since f and g each have only a finite number of self-intersections, we may assume that $\mathrm{Int}(f,g)$ is disjoint from $X \times S_g$ and $S_f \times Y$; and that if α and β are paths connecting points in J, then the homotopy $H : f\alpha \simeq g\beta$ is likewise disjoint from $f(S_f)$ and $g(S_g)$. Now, take a neighborhood V of $\alpha([0,1])$ which avoids S_f, and whose only intersections with J are x_1 and x_2. The Whitney lemma implies that there is a homotopy $F : X \times [0,1] \to Z$ such that $F_0 = f$, $F_t = f$ outside of V for every t, and $F_1(V) \cap g(Y) = \emptyset$. Applying this process successively to pairs of points in J eventually eliminates all intersections in J. ∎

Thus we know that

THEOREM 4.3. *If X, Y and Z are compact orientable manifolds with* $\dim(Z) \geq \dim(X) + \dim(Y)$ *and* $\max\{\dim(X), \dim(Y)\} > 2$, *then for any*

$f : X \to Z$ and $g : Y \to Z$, an intersection class J is topologically essential if and only if it is algebraically essential.

Now, apart from its own intrinsic importance, this result is also a significant step towards establishing the Wecken theorem for intersection numbers. For fixed points, it is true for almost all spaces that, given $f : X \to X$, there is an $f' \simeq f$ such that f' has exactly $N(f)$ fixed points. For coincidences and roots, the results are more restrictive, requiring that the spaces be manifolds of sufficiently high dimension. Our results for intersection numbers parallel these, requiring the same restrictions for the same reason—to allow use of the Whitney lemma. It is not clear, however, that such restrictions are truly necessary. While certainly not the last word on the subject, we do have the following

THEOREM 4.4. *If X, Y and Z are compact orientable manifolds such that* $\dim(Z) \geq \dim(X) + \dim(Y)$ *and* $\max\{\dim(X), \dim(Y)\} > 2$, *then for every $f : X \to Z$ and $g : Y \to Z$, there exists $f' \simeq f$ and $g' \simeq g$ such that f' and g' have exactly* $\mathrm{NI}(f, g)$ *points of intersection.*

Proof. The case $\dim(Z) > \dim(X) + \dim(Y)$ is trivial: in this case, transversality allows us to perturb f and g to be intersection free. The case we are really interested in is $\dim(Z) = \dim(X) + \dim(Y)$. As with Lemma 4.2, we assume $\dim(Y) \geq \dim(X)$. We follow the canonical construction to obtain $f' \simeq f$, $g' \simeq g$ such that $|\mathrm{Int}(f', g')| = \mathrm{NI}(f, g)$:

1. If an intersection class J has index 0, deform f and g to remove J.
2. If an intersection class has nonzero index and contains more than one point, deform f and g to consolidate the class into a single point.

Lemma 4.2 takes care of the first step. For the second step, suppose J is an intersection class with $\mathrm{Ind}(f, g; J) = \nu \neq 0$. We may apply the same process to cancel all intersections in J with indices of opposite sign, so we may assume that J consists of $|\nu|$ intersection points, all having the same index. We may also assume that either f is an embedding (if $\dim(X) < \dim(Y)$) or both f and g are immersions with clean double points (if $\dim(X) = \dim(Y)$). Choose a point $(x_0, y_0) \in J$, and euclidean neighborhoods V of y_0 and W of $g(y_0)$ such that $V \cap \varrho_2(\mathrm{Int}(f, g)) = y_0$; $V \cap S_g = \emptyset$ and $W \cap (f(S_f) \cup g(S_g)) = \emptyset$ (if S_f is nonempty); and $f(V) \subset W$. Now, $f(X) \cap W$ is a submanifold of W, so we are in the same setting as [4, §3.3]. The argument used there may be employed to produce points $y_1, \ldots, y_{|\nu|}, \overline{y} \in V$ and $g' \simeq g$ such that

1. $g' = g$ outside of V.
2. $\varrho_2(\mathrm{Int}(f, g')) \cap V = \{y_0, y_1, \ldots, y_{|\nu|}, \overline{y}\}$.
3. All intersection points in V lie in a single intersection class, which is related to J.

4. $\mathrm{Ind}(f, g'; (x_0, y_0)) = \mathrm{Ind}(f, g; (x_0, y_0))$, $\mathrm{Ind}(f', g; (\overline{x}, \overline{y})) = \nu$ and, for $i = 1, \ldots, |\nu|$, $\mathrm{Ind}(f', g; (x_i, y_i)) = -\mathrm{Ind}(f, g; (x_0, y_0))$.

Thus J is deformed to an intersection class with $2|\nu| + 1$ intersection points. Of these, $|\nu|$ have index $+1$, $|\nu|$ have index -1 and one has index ν. Applying the arguments used in Lemma 4.2 again, we can cancel the $2|\nu|$ points with index ± 1, and deform J to the single point $(\overline{x}, \overline{y})$ with index ν. ∎

It is worth noting in this result that, in all cases, we realized the lower bound $\mathrm{NI}(f, g)$ by first deforming one map to an immersion with at most finitely many self-intersections, then deforming the other to eliminate all extra intersection points. We can interpret this as saying that the Nielsen intersection number is "almost" one-sided. That is, if we allow one arbitrarily small perturbation of one of the maps, we can eliminate all inessential classes by then holding that map fixed, and performing all deformations on the other map.

5. The Jiang condition. At this point, we have established some of the properties that make Nielsen intersection numbers useful. However, we have not established any properties that help us to actually compute them. In the remaining sections, we turn to computational issues. In this section, we consider the Jiang condition as an aid in directly computing $\mathrm{NI}(f, g)$. In the remaining sections, we consider methods by which Nielsen intersection numbers for one intersection problem can be related to intersection numbers for other (presumably solved) intersection problems.

The Jiang condition is a well-known computational technique for Nielsen fixed point numbers [8]. It has been generalized to coincidence numbers [1]. Its generalization to intersection theory is similar, and arises quite naturally in the process of defining Nielsen intersection numbers. To determine if an intersection class $J \in \mathcal{I}(f, g)$ is essential, all homotopies F and G based at f and g are considered. If J cannot be deformed to the empty set, it is essential—and so is any class that J can be deformed to. Now, among all of the homotopies based at f, there may be *cyclic homotopies*: maps $F : X \times [0, 1] \to Z$ such that $F_0 = F_1 = f$. Then J_0 and J_1 are both intersection classes in $\mathcal{I}(f, g)$ and they are either both essential, or both inessential. This suggests that there should be an equivalence relation on $\mathcal{I}(f, g)$, generated by examining cyclic homotopies, and that given a "Jiang equivalence class", either all intersection classes in that Jiang class are essential, or all are inessential. If this equivalence relation can be clearly understood, it can greatly simplify the computation of $\mathrm{NI}(f, g)$.

To make this idea precise, we begin by considering the Jiang subgroup for a single map $f : X \to Z$. If we fix a "reference lift" $\widetilde{f} : \widetilde{X} \to \widetilde{Z}$ of f, then any other lift of f has the form $\gamma\widetilde{f}$, $\gamma \in \mathcal{D}(Z)$. If we take the lift $\widetilde{F}$ of a cyclic homotopy F with $\widetilde{F}_0 = \gamma\widetilde{f}$, then $\widetilde{F}_1 = \delta\widetilde{f}$ for some $\delta \in \mathcal{D}(Z)$. We then declare γ and δ to be *Jiang equivalent*, written $\gamma \sim_{\mathrm{J}} \delta$. This is easily seen to be an equivalence relation. In fact, equivalence classes are actually cosets of $\mathcal{D}(Z)$. To see this, define

$$J(\widetilde{f}\,) = \{\gamma \in \mathcal{D}(Z) \mid \exists \text{ a cyclic homotopy } F : f \simeq f \text{ with lift } \widetilde{F} \\ \text{such that } \widetilde{F}_0 = \widetilde{f} \text{ and } \widetilde{F}_1 = \gamma\widetilde{f}\}.$$

LEMMA 5.1. *$J(\widetilde{f}\,)$ is a subgroup of $\mathcal{D}(Z)$. If $\delta \in \mathcal{D}(Z)$, then $J(\delta\widetilde{f}\,) = \delta^{-1}J(\widetilde{f}\,)\delta$.*

Proof. If $F : f \simeq f$ and $F' : f \simeq f$ are cyclic homotopies that lift to $\widetilde{F} : \widetilde{f} \simeq \gamma\widetilde{f}$ and $\widetilde{F}' : \widetilde{f} \simeq \delta\widetilde{f}$, then

$$F''(x,t) = \begin{cases} F(x,2t), & t \le 1/2, \\ F'(x,2t-1) & t \ge 1/2, \end{cases}$$

is a cyclic homotopy that lifts to

$$\widetilde{F}''(x,t) = \begin{cases} \widetilde{F}(x,2t), & t \le 1/2, \\ \gamma\widetilde{F}'(x,2t-1), & t \ge 1/2. \end{cases}$$

That is, $\widetilde{F}'' : \widetilde{f} \simeq \gamma\delta f$, so $\gamma\delta \in J(\widetilde{f}\,)$. Similarly, if $\overline{F}(x,t) = F(x,1-t)$, then $\overline{F}$ lifts to $\widetilde{\overline{F}} : \gamma\widetilde{f} \simeq \widetilde{f}$, or $\gamma^{-1}\widetilde{\overline{F}} : \widetilde{f} \simeq \gamma^{-1}\widetilde{f}$. That is, $\gamma^{-1} \in J(\widetilde{f}\,)$.

If $\gamma \in J(\delta\widetilde{f}\,)$, then there is a cyclic homotopy $F : f \simeq f$ with lift $\widetilde{F}$ such that $\widetilde{F}_0 = \delta\widetilde{f}$ and $\widetilde{F}_1 = \gamma\delta\widetilde{f}$. Then $\delta^{-1}\widetilde{F}$ is a homotopy from $\widetilde{f}$ to $\delta^{-1}\gamma\delta\widetilde{f}$, so $\delta^{-1}\gamma\delta \in J(\widetilde{f}\,)$. ∎

We refer to $J(\widetilde{f}\,)$ as the *Jiang subgroup* of f based at $\widetilde{f}$. Clearly, the Jiang equivalence classes are just the cosets of $\mathcal{D}(Z)/J(\widetilde{f}\,)$.

In its covering space formulation, the Jiang subgroup depends on the lift $\widetilde{f}$ chosen as the reference lift. There is a fundamental group formulation, which replaces this dependence on the lift with a dependence on a base point. If $f(x) = z$, define

$$J(f,x) = \{[\omega] \in \pi_1(Z,z) \mid \omega(t) = F(x,t) \text{ for some } F : f \simeq f\}.$$

It is easy to check that $J(f,x)$ is a subgroup of $\pi_1(Z,z)$. To determine the dependence on x, take $x' \in X$, and choose a path α from x to x'. Then $F(x,-) * f\alpha$ is endpoint-homotopic to $f\alpha * F(x',-)$, with homotopy $H(s,t) = F(\alpha(s),t)$, so the isomorphism $(f\alpha)_{\#} : \pi_1(Z,z) \to \pi_1(Z,z')$ maps $J(f,x)$ isomorphically to $J(f,x')$.

Let $\Phi_{\tilde{f}} : \mathcal{D}(Z) \to \pi_1(Z,z)$ denote the isomorphism that identifies γ with the projection of a path from $\widetilde{f}(\widetilde{x})$ to $\gamma\widetilde{f}(\widetilde{x})$. Then the two formulations of the Jiang subgroup coincide:

PROPOSITION 5.2. *$\Phi_{\tilde{f}}$ maps $J(\widetilde{f}\,)$ isomorphically to $J(f,x)$.*

We will find it more convenient to work with the fundamental group formulation, but we will move back and forth between the two whenever it is convenient to do so.

The two most important properties of the Jiang subgroup (cf. [8, Lemmas II.3.3, II.3.4]) are:

Proposition 5.3. *Given* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *with* $f(x) = y$, $g(y) = z$, *we have*

1. $g_\# J(f,x) \subset J(gf,x)$ *and* $J(g,y) \subset J(gf,x)$;
2. $J(f,x) \subset Z_{\pi_1(Y,y)}(f_\#(\pi_1(X,x)))$.

Proof. If $F : f \simeq f$ has $F(x,t) = \omega(t)$, then $gF : gf \simeq gf$ has $gF(x,t) = g\omega(t)$, so $g_\#[\omega] \in J(gf,x)$. Similarly, given $G : g \simeq g$, $Gf : gf \simeq gf$ has $Gf(x,t) = G(y,t)$, so $[G(y,t)] \in J(gf,x)$.

Finally, given $[\alpha] \in \pi_1(X,x)$ and $[F(x,t)] \in \pi_1(Y,y)$, define $H : [0,1] \times [0,1] \to Y$ by $H(s,t) = F(\alpha(s),t)$. Then the two "edges" of the square are $f\alpha * F(x,-)$ and $F(x,-) * f\alpha$, and $F(x,-)$ centralizes $[f\alpha]$. ■

Corollary 5.4. *The isomorphism* $(f\alpha)_\# : J(f,x) \to J(f,x')$ *is independent of the path* α *from* x *to* x'.

In particular, if we take $Y = Z$ and $g = \mathrm{id}$ then the Jiang group $J(Y) = J(\mathrm{id},y)$ is an abelian subgroup (since $J(\mathrm{id},y) \subset Z(\pi_1(Y,y))$). Then composing id_Y with any $f : X \to Y$, we see that $J(Y) \subset J(f,x) \subset Z_{\pi_1(Y,y)}(f_\#\pi_1(X,x))$. If Y is a Jiang space (i.e. $J(Y) = \pi_1(Y,y)$), then $\pi_1(Y,y)$ must be abelian, and $J(f,x) = \pi_1(Y,y)$ for every f.

It is also worth noting that, since $J(f,x)$ centralizes $f_\#(\pi_1(X,x))$, their product

$$J(f,x)f_\#(\pi_1(X,x)) = f_\#(\pi_1(X,x))J(f,x)$$

is a subgroup of $\pi_1(Z,z)$. We will denote this subgroup by $J_*(f,x)$. Like $J(f,x)$, $J_*(f,x)$ is in some sense independent of the basepoint chosen, in that we can canonically identify the subgroups defined at different points.

Proposition 5.5. *If* α *is a path in* X *from* x *to* x', *then* $(f\alpha)_\#$ *maps* $J_*(f,x)$ *isomorphically to* $J_*(f,x')$.

To make contact with intersection theory, consider cyclic homotopies $F : X \times [0,1] \to Z$ and $G : Y \times [0,1] \to Z$. Suppose that J is an intersection class of $f = F_0$ and $g = G_0$. Choose lifts $\widetilde{f}$ and $\widetilde{g}$ such that J has the form $(p_X \times p_Y)(\mathrm{Int}(\widetilde{f},\widetilde{g}))$, and lift F and G to $\widetilde{F}$ and $\widetilde{G}$ with $\widetilde{F}_0 = \widetilde{f}$ and $\widetilde{G}_0 = \widetilde{g}$. Then $\widetilde{F}_1 = \gamma_1\widetilde{f}$, $\widetilde{G}_1 = \gamma_2\widetilde{g}$ for some $\gamma_1 \in J(\widetilde{f})$, $\gamma_2 \in J(\widetilde{g})$, and J is (F,G)-related to

$$(p_X \times p_Y)(\mathrm{Int}(\gamma_1\widetilde{f},\gamma_2\widetilde{g})) = (p_X \times p_Y)(\mathrm{Int}(\widetilde{f},\gamma_1^{-1}\gamma_2\widetilde{g})),$$

which we label as $J_{\gamma_1^{-1}\gamma_2}$. Clearly, J is essential if and only if $J_{\gamma_1^{-1}\gamma_2}$ is; and if an intersection index is defined, $\mathrm{Ind}(f,g;J)=\mathrm{Ind}(f,g;J_{\gamma_1^{-1}\gamma_2})$.

This motivates the definition of a Jiang equivalence relation: $J\sim_{\mathrm{J}} J'$ if $J'=J_{\gamma_1\gamma_2}$ for some $\gamma_1\in J(\widetilde{f})$, $\gamma_2\in J(\widetilde{g})$. Similarly, we can define an equivalence relation on the set of Reidemeister classes $\mathrm{RI}(f,g)=f_\#(\pi_1(X,x))\backslash\pi_1(Z,z)/g_\#(\pi_1(Y,y))$ by taking a further double quotient by $J(f,x)$ and $J(Y,y)$. That is, $\gamma\sim_{\mathrm{J}}\gamma'$ in $\pi_1(Z,z)$ if

$$\gamma'=f_\#(\alpha)\gamma_1\gamma\gamma_2 g_\#(\beta)$$

with $\alpha\in\pi_1(X)$, $\beta\in\pi_1(Y)$, $\gamma_1\in J(f)$ and $\gamma_2\in J(g)$. The set of equivalence classes is the double coset $J_*(f,x)\backslash\pi_1(Z,z)/J_*(g,y)$.

Proposition 5.6. *If $J\sim_{\mathrm{J}} J'$ in $\mathcal{I}(f,g)$, then J is essential if and only if J' is. If there is an intersection index defined, then* $\mathrm{Ind}(f,g;J)=\mathrm{Ind}(f,g;J')$.

This analysis is most valuable when all intersection classes lie in a single Jiang class. In fixed point theory, the Jiang condition is a condition on the fundamental group level which guarantees that there is a single Jiang classs. The efficacy of that condition depends heavily on the ability to iterate self-maps. Since there is no such ability in the intersection problem, the "Jiang condition" takes a rather different form, but the significance of the condition remains the same.

Theorem 5.7. *If J is an intersection class in* $\mathrm{Int}(f,g)$ *and*

$$J_*(f,x)J_*(g,y)=\pi_1(Z,z)$$

for some $(x,y)\in J$, then every intersection class is Jiang equivalent to J. That is, either all intersection classes are essential, or all are inessential; and either $\mathrm{NI}(f,g)=0$ *or* $\mathrm{NI}(f,g)=|\mathrm{RI}(f,g)|$. *If the intersection index* $\mathrm{Ind}(f,g)$ *is defined, then all intersection classes have the same index, and*

$$\mathrm{NI}(f,g)=\begin{cases}0, & \mathrm{LI}(f,g)=0,\\ |\mathrm{RI}(f,g)|, & \mathrm{LI}(f,g)\neq 0.\end{cases}$$

Some comments on the quantity $J_*(f,x)J_*(g,y)$ are in order. First, since the two subgroups $J_*(f,x)$ need not normalize each other, their product is *not* a subgroup of $\pi_1(Z,z)$ in general. Second, while $J_*(f,x)J_*(g,y)$ is constant (or canonically isomorphic) as (x,y) varies across an intersection class, it can vary as (x,y) moves from one intersection class to another. On the other hand, the Jiang condition $J_*(f,x)J_*(g,y)=\pi_1(Z,z)$ is independent of the intersection class (x,y) lies in. That is,

Proposition 5.8. *If* $J_*(f,x)J_*(g,y) = \pi_1(Z,z)$ *for some* $(x,y) \in \mathrm{Int}(f,g)$, *then*
$$J_*(f,x')J_*(g,y') = \pi_1(Z,z')$$
for all $(x',y') \in \mathrm{Int}(f,g)$.

Proof. Choose paths α from x to x' and β from y to y'. Then
$$J_*(f,x')J_*(g,y') = (f\alpha)^{-1} * J_*(f,x) * (f\alpha) * (g\beta)^{-1} * J_*(g,y) * (g\beta).$$
Now $(f\alpha) * (g\beta)^{-1}$ is a loop at z, and so is an element of $J_*(f,x)J_*(g,y)$. That is,
$$\begin{aligned} J_*(f,x')J_*(g,y') &= (f\alpha)^{-1} * J_*(f,x)J_*(g,y) * (g\beta) \\ &= (f\alpha)^{-1} * \pi_1(Z,z) * (g\beta) = \pi_1(Z,z). \ \blacksquare \end{aligned}$$

Thus, the Jiang condition is satisfied at every point in $\mathrm{Int}(f,g)$, or at none.

Finally, if Z is a Jiang space, then $J(Z) = \pi_1(Z,z)$, so $J(f,x) = J(f) = \pi_1(Z,z)$ for all $f : X \to Z$, and the Jiang condition is satisfied for all intersection problems. Of course, if Z is a Jiang space, then $\pi_1(Z,z)$ is abelian, and Proposition 3.3 allows us to replace $\mathrm{RI}(f,g)$ with the more computable $\mathrm{RI}_*(f,g)$.

Theorem 5.9. *If Z is a Jiang space, then for every X and Y and every $f : X \to Z$ and $g : Y \to Z$, either all intersection classes in $\mathcal{I}(f,g)$ are essential, or all are inessential. Thus, either* $\mathrm{NI}(f,g) = 0$ *or* $\mathrm{NI}(f,g) = |\mathrm{RI}_*(f,g)|$. *If the intersection index* $\mathrm{Ind}(f,g)$ *is defined, then all intersection classes have the same index, and*
$$\mathrm{NI}(f,g) = \begin{cases} 0, & \mathrm{LI}(f,g) = 0, \\ |\mathrm{RI}_*(f,g)|, & \mathrm{LI}(f,g) \neq 0. \end{cases}$$

Let Z be the n-torus T^n, and let X, Y be tori of dimension p and $q = n-p$ respectively. Z is a Jiang space, so
$$\mathrm{NI}(f,g) = \begin{cases} 0, & \mathrm{LI}(f,g) = 0, \\ |\mathrm{RI}_*(f,g)|, & \mathrm{LI}(f,g) \neq 0. \end{cases}$$
If $\mathrm{LI}(f,g) \neq 0$, then $\mathrm{RI}_*(f,g) = H_1(T^n)/\langle \mathrm{im}(f_{1*}), \mathrm{im}(g_{1*})\rangle$ is just the cokernel of the $n \times n$ matrix $A = [f_{1*}\ g_{1*}] : H_1(T^p;\mathbb{Q}) \oplus H_1(T^q;\mathbb{Q}) \to H_1(T^n;\mathbb{Q})$, so $\mathrm{NI}(f,g) = |\det(A)|$. In fact, by exploiting the relationship between Nielsen intersection numbers and root numbers, we will show [10] that
$$\mathrm{NI}(f,g) = |\mathrm{LI}(f,g)| = |\det[f_{1*}\ g_{1*}]|$$
for all maps from tori to tori.

6. Functoriality. As with any topological invariant, the range of computability in Nielsen theory is expanded by the ability to relate Nielsen numbers from one problem to those from another. In particular, computational formulas for fibrations and finite covering spaces have been extensively developed. An important preliminary to developing these formulas for intersection theory is the establishment of the basic functorial properties of Nielsen intersection numbers. We consider then the following diagram:

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "a"] & Z_1 \arrow[d, "c"] & Y_1 \arrow[l, "g_1"'] \arrow[d, "b"] \\ X_2 \arrow[r, "f_2"] & Z_2 & Y_2 \arrow[l, "g_2"'] \end{tikzcd}$$

That is, if we wanted to define a category of intersection problems, we could consider $X_1 \xrightarrow{f_1} Z_1 \xleftarrow{g_1} Y_1$ and $X_2 \xrightarrow{f_2} Z_2 \xleftarrow{g_2} Y_2$ to be objects in the category and the trio of maps (a, b, c) to be a morphism.

Some of the basic questions to be asked here are:

- Do intersection points map to intersection points, and intersection classes to intersection classes?
- Given a class $J_2 \in \mathcal{I}(f_2, g_2)$, how many classes in $\mathcal{I}(f_1, g_1)$ map to it?
- Do essential classes map to essential classes? Do inessential classes map to inessential classes?

If we could answer all of these questions, we could relate $\mathrm{NI}(f_1, g_1)$ and $\mathrm{NI}(f_2, g_2)$. Unfortunately, we will not be able to answer these questions in general. However, we will establish some partial results, which will be useful in the examination of finite covers and fibrations.

PROPOSITION 6.1. *The map* $(a \times b) : X_1 \times Y_1 \to X_2 \times Y_2$ *maps* $\mathrm{Int}(f_1, g_1)$ *to* $\mathrm{Int}(f_2, g_2)$. *Moreover, if* J_1 *is an intersection class in* $\mathrm{Int}(f_1, g_1)$, *then* $(a \times b)(J_1)$ *is contained in a single intersection class in* $\mathrm{Int}(f_2, g_2)$.

Proof. It is trivial to check that $a \times b$ maps $\mathrm{Int}(f_1, g_1)$ into $\mathrm{Int}(f_2, g_2)$. If $(x, y) \sim_{\mathrm{N}} (x', y')$ in $\mathrm{Int}(f_1, g_1)$, with paths α in X_1 and β in Y_1 relating them, then $a\alpha$ and $b\beta$ are paths in X_2 and Y_2 with $a\alpha : a(x) \simeq a(x')$, $b\beta : b(y) \simeq b(y')$ and $f_2 a\alpha \simeq g_2 b\beta$. ∎

There is then a well-defined map $\mathcal{I}(a \times b) : \mathcal{I}(f_1, g_1) \to \mathcal{I}(f_2, g_2)$. Similarly, $c_{\#} : \pi_1(Z_1, z_1) \to \pi_1(Z_2, z_2)$ maps Reidemeister classes to Reidemeister classes, so there is a well-defined map $\mathrm{RI}(c) : \mathrm{RI}(f_1, g_1) \to \mathrm{RI}(f_2, g_2)$. Moreover, these maps are intertwined.

PROPOSITION 6.2. *There is a commutative diagram*

$$\begin{tikzcd} \mathcal{I}(f_1, g_1) \arrow[r] \arrow[d, "\mathcal{I}(a\times b)"] & \mathrm{RI}(f_1, g_1) \arrow[d, "\mathrm{RI}(c)"] \\ \mathcal{I}(f_2, g_2) \arrow[r] & \mathrm{RI}(f_2, g_2). \end{tikzcd}$$

Now, given $J, J' \in \mathcal{I}(f_1, g_1)$, we would like to know when $\mathcal{I}(a \times b)(J) = \mathcal{I}(a \times b)(J')$. As approximation of this, we would like to be able to compute the cardinality of $\mathrm{RI}(c)^{-1}[\gamma_2]$ for any $[\gamma_2] \in \mathrm{RI}(f_2, g_2)$. This is slightly less useful, because we may not know whether or not $\mathrm{Int}(\widetilde{f}_2, \gamma_2\widetilde{g}_2)$ (or its preimages) is essential, or even non-empty. That is, we do not know if we are obtaining information about intersection classes when we get information about Reidemeister classes. On the other hand, since $\mathrm{RI}(c)$ is defined algebraically, we might expect it to be more easily understood.

Unfortunately, even calculations with $\mathrm{RI}(c)$ are difficult. Following [8, Proposition III.1.15], we can determine the cardinality of $\mathrm{RI}(c)^{-1}$ in one important case. Suppose that

$$\mathrm{im}(a_{\#}) \lhd \pi_1(X_2, x_2), \qquad \mathrm{im}(b_{\#}) \lhd \pi_1(Y_2, y_2), \qquad \mathrm{im}(c_{\#}) \lhd \pi_1(Z_2, z_2).$$

There is then a commutative diagram of groups and homomorphisms

$$\begin{tikzcd}
\pi_1(X_1, x_1) \arrow[r, "f_{1\#}"] \arrow[d, "a_{\#}"] & \pi_1(Z_1, z_1) \arrow[d, "c_{\#}"] & \pi_1(Y_1, y_1) \arrow[l, "g_{1\#}"'] \arrow[d, "b_{\#}"] \\
\pi_1(X_2, x_2) \arrow[r, "f_{2\#}"] \arrow[d, "\varrho_a"] & \pi_1(Z_2, z_2) \arrow[d, "\varrho_c"] & \pi_1(Y_2, y_2) \arrow[l, "g_{2\#}"'] \arrow[d, "\varrho_b"] \\
\mathrm{coker}(a_{\#}) \arrow[r, "\bar{f}"] & \mathrm{coker}(c_{\#}) & \mathrm{coker}(b_{\#}) \arrow[l, "\bar{g}"']
\end{tikzcd}$$

Just as $\mathrm{Int}(f_{2\#}, g_{2\#}; (x_2, y_2))$ is a subgroup of $\pi_1(X_2, x_2) \times \pi_1(Y_2, y_2)$, $\mathrm{Int}(\bar{f}, \bar{g})$ is a subgroup of $\mathrm{coker}(a_{\#}) \times \mathrm{coker}(b_{\#})$. Further, $(\varrho_a \times \varrho_b)$ maps $\mathrm{Int}(f_{2\#}, g_{2\#}; (x_2, y_2))$ into $\mathrm{Int}(\bar{f}, \bar{g})$. Indeed, while we will not make use of the fact, $(\varrho_a \times \varrho_b)(\mathrm{Int}(f_{2\#}, g_{2\#}; (x_2, y_2)))$ is a normal subgroup of $\mathrm{Int}(\bar{f}, \bar{g})$. In any event, we define

$$P(\bar{f}, \bar{g}; (x_2, y_2)) = |\mathrm{Int}(\bar{f}, \bar{g}) : (\varrho_a \times \varrho_b)\, \mathrm{Int}(f_{2\#}, g_{2\#}; (x_2, y_2))|.$$

Proposition 6.3. *$P(\bar{f}, \bar{g}; (x_2, y_2))$ depends only on the intersection class of (x_2, y_2).*

Proof. Take $(x_1, y_1), (x_1', y_1') \in \mathrm{Int}(f_1, g_1)$ such that $(x_2, y_2) = (a(x_1), b(y_1))$ and $(x_2', y_2') = (a(x_1'), b(y_1'))$ are in the same intersection class in $\mathrm{Int}(f_2, g_2)$. We must show that the diagrams

$$\begin{tikzcd}
\pi_1(X_2, x_2) \arrow[r, "f_{2\#}"] \arrow[d, "\varrho_a"] & \pi_1(Z_2, z_2) \arrow[d, "\varrho_c"] & \pi_1(Y_2, y_2) \arrow[l, "g_{2\#}"'] \arrow[d, "\varrho_b"] \\
\mathrm{coker}(a_{\#}) \arrow[r, "\bar{f}"] & \mathrm{coker}(c_{\#}) & \mathrm{coker}(b_{\#}) \arrow[l, "\bar{g}"']
\end{tikzcd}$$

and

$$\begin{array}{ccccc}
\pi_1(X_2,x_2') & \xrightarrow{f_{2\#}} & \pi_1(Z_2,z_2') & \xleftarrow{g_{2\#}} & \pi_1(Y_2,y_2') \\
\downarrow{\scriptstyle \varrho_a'} & & \downarrow{\scriptstyle \varrho_c'} & & \downarrow{\scriptstyle \varrho_b'} \\
\operatorname{coker}(a'_{\#}) & \xrightarrow{\bar f'} & \operatorname{coker}(c'_{\#}) & \xleftarrow{\bar g'} & \operatorname{coker}(b'_{\#})
\end{array}$$

are conjugate. Now, there are paths α from x_2 to x_2' and β from y_2 to y_2' such that $\gamma = f_2\alpha \simeq g_2\beta$. There is then a commutative diagram

$$\begin{array}{ccccc}
\pi_1(X_2,x_2) & \xrightarrow{f_{2\#}} & \pi_1(Z_2,z_2) & \xleftarrow{g_{2\#}} & \pi_1(Y_2,y_2) \\
\downarrow{\scriptstyle \alpha_{\#}} & & \downarrow{\scriptstyle \gamma_{\#}} & & \downarrow{\scriptstyle \beta_{\#}} \\
\pi_1(X_2,x_2') & \xrightarrow{f_{2\#}} & \pi_1(Z_2,z_2') & \xleftarrow{g_{2\#}} & \pi_1(Y_2,y_2')
\end{array}$$

Since $\operatorname{im}(a_{\#})$ is normal in $\pi_1(X_2,x_2)$, $\alpha_{\#}$ maps $\operatorname{im}(a_{\#})$ to $\operatorname{im}(a'_{\#})$. Thus $\alpha_{\#}$ defines an isomorphism $\overline{\alpha}_{\#} : \operatorname{coker}(a_{\#}) \to \operatorname{coker}(a'_{\#})$. Likewise, $\beta_{\#}$ and $\gamma_{\#}$ define isomorphisms $\overline{\beta}_{\#}$ and $\overline{\gamma}_{\#}$. These isomorphisms conjugate the diagram at (x_2,y_2) to the diagram at (x_2',y_2'), and so map $(\varrho_a \times \varrho_b)(\operatorname{Int}(f_{2\#},g_{2\#};(x_2,y_2)))$ and $\operatorname{Int}(\bar f,\bar g)$ to $(\varrho_a' \times \varrho_b')(\operatorname{Int}(f_{2\#},g_{2\#};(x_2',y_2')))$ and $\operatorname{Int}(\bar f',\bar g')$. ∎

We can then speak of $P(\bar f,\bar g;J_2)$ for $J_2 \in \mathcal{I}(f_2,g_2)$. This quantity is of interest in determining how intersection classes in $\operatorname{Int}(f_1,g_1)$ map to intersection classes in $\operatorname{Int}(f_2,g_2)$.

PROPOSITION 6.4. *Suppose $J_i \in \mathcal{I}(f_i,g_i)$ are intersection classes such that $\mathcal{I}(a\times b)(J_1) = J_2$. Fix basepoints $(x_1,y_1) \in J_1$ and let $x_2 = a(x_1)$, $y_2 = b(y_1)$, $z_i = f_i(x_i) = g_i(y_i)$. If $c_{\#} : \pi_1(Z_1,z_1) \to \pi_1(Z_2,z_2)$ is injective and*

$$\operatorname{im}(a_{\#}) \lhd \pi_1(X_2,x_2), \qquad \operatorname{im}(b_{\#}) \lhd \pi_1(Y_2,y_2), \qquad \operatorname{im}(c_{\#}) \lhd \pi_1(Z_2,z_2),$$

then $\operatorname{RI}(c)^{-1}(J_2)$ *has cardinality* $P(\bar f,\bar g;J_2)$.

Proof. We will work with the diagram

$$\begin{array}{ccccc}
\pi_1(X_1,x_1) & \xrightarrow{f_{1\#}} & \pi_1(Z_1,z_1) & \xleftarrow{g_{1\#}} & \pi_1(Y_1,y_1) \\
\downarrow{\scriptstyle a_{\#}} & & \downarrow{\scriptstyle c_{\#}} & & \downarrow{\scriptstyle b_{\#}} \\
\pi_1(X_2,x_2) & \xrightarrow{f_{2\#}} & \pi_1(Z_2,z_2) & \xleftarrow{g_{2\#}} & \pi_1(Y_2,y_2)
\end{array}$$

In these coordinates,

$$\operatorname{RI}(c)^{-1}(J_2) = \{[\gamma_1] \in \operatorname{RI}(f_1,g_1) \mid \gamma_1 = f_{2\#}(\alpha_2)g_{2\#}(\beta_2^{-1}) \text{ for some } \alpha_2 \in \pi_1(X_2),\ \beta_2 \in \pi_1(Y_2)\}$$

$$= \operatorname{im}(f_{1\#})\backslash c_{\#}^{-1}(\operatorname{im}(f_{2\#})\operatorname{im}(g_{2\#}) \cap \operatorname{im}(c_{\#}))/\operatorname{im}(g_{1\#})$$

$$\cong \operatorname{im}(f_{2\#}a_{\#})\backslash \operatorname{im}(f_{2\#})\operatorname{im}(g_{2\#})/\operatorname{im}(g_{1\#}b_{\#}).$$

Similarly,

$$\begin{aligned}&\operatorname{Int}(\bar{f},\bar{g})/(\varrho_a \times \varrho_b)(\operatorname{Int}(f_{2\#}, g_{2\#}; J_2))\\ &\qquad \cong (\varrho_a \times \varrho_b)^{-1}(\operatorname{Int}(\bar{f},\bar{g}))/\operatorname{Int}(f_{2\#}, g_{2\#}; J_2)(\operatorname{im}(a_{\#}) \times \operatorname{im}(b_{\#})).\end{aligned}$$

Define $\Psi : (\varrho_a \times \varrho_b)^{-1}(\operatorname{Int}(\bar{f},\bar{g})) \to \operatorname{im}(f_{2\#})\operatorname{im}(g_{2\#}) \cap \operatorname{im}(c_{\#})$ by $\Psi(\alpha_2, \beta_2) = f_{2\#}(\alpha_2)g_{2\#}(\beta_2^{-1})$. Clearly, Ψ maps into $\operatorname{im}(f_{2\#})\operatorname{im}(g_{2\#})$. And, if $(\alpha_2, \beta_2) \in (\varrho_a \times \varrho_b)^{-1}(\operatorname{Int}(\bar{f},\bar{g}))$, then $\varrho_c f_{2\#}(\alpha_2) = \varrho_c g_{2\#}(\beta_2)$, so $f_{2\#}(\alpha_2) = c_{\#}(\gamma_1)g_{2\#}(\beta_2)$, and $f_{2\#}(\alpha_2)g_{2\#}(\beta_2^{-1}) = c_{\#}(\gamma_1) \in \operatorname{im}(c_{\#})$. If $(\alpha, \beta) \in \operatorname{Int}(f_{2\#}, g_{2\#}; J_2)(\operatorname{im}(a_{\#}) \times \operatorname{im}(b_{\#}))$, then $(\alpha, \beta) = (\alpha' a_{\#}(\alpha_1), \beta' b_{\#}(\beta_1))$ with $f_{2\#}(\alpha') = g_{2\#}(\beta')$. Then $\Psi(\alpha, \beta) = f_{2\#}a_{\#}(\alpha_1)g_{2\#}b_{\#}(\beta_1)$, so $\Psi(\alpha, \beta)$ becomes trivial in the double coset $\operatorname{im}(f_{2\#}a_{\#})\backslash(\operatorname{im}(f_{2\#})\operatorname{im}(g_{2\#}) \cap \operatorname{im}(c_{\#}))/\operatorname{im}(g_{1\#}b_{\#})$. There is then a well-defined function

$$\begin{array}{c}(\varrho_a \times \varrho_b)^{-1}(\operatorname{Int}(\bar{f},\bar{g}))/\operatorname{Int}(f_{2\#}, g_{2\#}; J_2)(\operatorname{im}(a_{\#}) \times \operatorname{im}(b_{\#}))\\ \Big\downarrow \psi \\ \operatorname{im}(f_{2\#}a_{\#})\backslash \operatorname{im}(f_{2\#})\operatorname{im}(g_{2\#})/\operatorname{im}(g_{1\#}b_{\#}).\end{array}$$

We claim that ψ is a bijection. Ψ is surjective, so ψ is surjective. To see that ψ is injective, take (α_2, β_2) and $(\alpha_2', \beta_2') \in (\varrho_a \times \varrho_b)^{-1}(\operatorname{Int}(\bar{f},\bar{g}))$ such that $\Psi(\alpha_2, \beta_2)$ and $\Psi(\alpha_2', \beta_2')$ lie in the same double coset. That is,

$$f_{2\#}(\alpha_2')g_{2\#}(\beta_2'^{-1}) = f_{2\#}a_{\#}(\alpha_1)f_{2\#}(\alpha_2)g_{2\#}(\beta_2^{-1})g_{2\#}b_{\#}(\beta_1^{-1})$$

for some $\alpha_1 \in \pi_1(X_1)$, $\beta_1 \in \pi_1(Y_1)$. Then $(\alpha_2^{-1}a_{\#}(\alpha_1^{-1})\alpha_2', \beta_2^{-1}b_{\#}(\beta_1^{-1})\beta_2') \in \operatorname{Int}(f_{2\#}, g_{2\#}; J_2)$. Since $\operatorname{im}(a_{\#}) \lhd \pi_1(X_2)$ and $\operatorname{im}(b_{\#}) \lhd \pi_1(Y_2)$, there exist $\alpha_1' \in \pi_1(X_1)$ and $\beta_1' \in \pi_1(Y_1)$ such that

$$a_{\#}(\alpha_1') = \alpha_2' a_{\#}(\alpha_1){\alpha'}_2^{-1}, \quad b_{\#}(\beta_1') = \beta_2' b_{\#}(\beta_1){\beta'}_2^{-1}.$$

Then

$$(\alpha_2', \beta_2') = (\alpha_2, \beta_2)(\alpha_2^{-1}a_{\#}(\alpha_1^{-1})\alpha_2', \beta_2^{-1}b_{\#}(\beta_1^{-1})\beta_2')(a_{\#}(\alpha_1'), b_{\#}(\beta_1'))$$

and $[\alpha_2', \beta_2'] = [\alpha_2, \beta_2]$ in

$$(\varrho_a \times \varrho_b)^{-1}(\operatorname{Int}(\bar{f},\bar{g}))/\operatorname{Int}(f_{2\#}, g_{2\#}; J_2)(\operatorname{im}(a_{\#}) \times \operatorname{im}(b_{\#})). \ \blacksquare$$

In the settings we will be concerned with in the remaining sections, this will give us sufficient information about the behavior of $\mathcal{I}(a \times b)$ and $\operatorname{RI}(c)$. However, this provides no information about the relation between essential and inessential classes in $\mathcal{I}(f_1, g_1)$ and $\mathcal{I}(f_2, g_2)$. Here too, there do not appear to be any strong general results, but there are some partial results that will be of use.

PROPOSITION 6.5. *Suppose that* $c : Z_1 \to Z_2$ *has the homotopy lifting property. If* J_1 *is an essential intersection class in* $\mathcal{I}(f_1, g_1)$, *then* $J_2 = \mathcal{I}(a \times b)(J_1)$ *is an essential intersection class in* $\mathcal{I}(f_2, g_2)$.

Proof. Suppose J_2 is inessential. Then there exist homotopies F_2 and G_2 such that J_2 is (F_2, G_2)-related to the empty set. Lift $F_2 a$ and $G_2 b$ to homotopies $F_1 : X_1 \times [0,1] \to Z_1$ (based at f_1) and $G_1 : Y_1 \times [0,1] \to Z_1$ (based at g_1). Then J_1 is (F_1, G_1)-related to an intersection class which maps into the empty set. That is, J_1 is (F_1, G_1)-related to the empty set, and so is inessential. ∎

7. Finite covers. As our first application of these functoriality results, we consider how the computation of Nielsen intersection numbers can be facilitated by lifting the problem to a finite cover. This analysis will follow that of [9, §5]. Its results will be of particular interest in two settings. First, when Z has finite fundamental group, we can lift the problem to a compact simply connected space, in which there is only one intersection class. Second, when X, Y and Z are manifolds, but not all are orientable, we can lift the problem to orientable manifolds, where we can employ the intersection index. In general, consider the following diagram:

$$\begin{tikzcd}
\overline{X} \arrow[r, "\bar f"] \arrow[d, "p_X"] & \overline{Z} \arrow[d, "p_Z"] & \overline{Y} \arrow[l, "\bar g"'] \arrow[d, "p_Y"] \\
X \arrow[r, "f"] & Z & Y \arrow[l, "g"']
\end{tikzcd}$$

where all of the vertical maps are finite regular covers. Denote the covering groups by $\mathcal{D}(p_X)$, etc., and the induced maps from $\mathcal{D}(p_X)$ and $\mathcal{D}(p_Y)$ to $\mathcal{D}(p_Z)$ by $f_{p\#}$ and $g_{p\#}$. We want to consider $\mathrm{NI}(f, g)$ as the quantity we are trying to compute, and $\mathrm{NI}(\widetilde{f}, \widetilde{g})$ as a quantity that we know how to compute. The problem is to relate the two.

In this setting, the hypotheses of Propositions 6.4 and 6.5 are satisfied, so we know that

- If $J \in \mathcal{I}(f, g)$, then $\mathrm{RI}(c)^{-1}(J)$ is either empty, or has cardinality $P(f, g; J)$.
- If J is inessential, then every $\overline{J} \in \mathcal{I}(a \times b)^{-1}(J)$ is inessential.

The additional structure supplied by the covering maps allows us to refine these results considerably. If $\widetilde{X}$, $\widetilde{Y}$ and $\widetilde{Z}$ are the universal covers of X, Y and Z respectively, then they are also the universal covers of $\overline{X}$, $\overline{Y}$ and $\overline{Z}$. There are lifts $\widetilde{f}$ and $\widetilde{g}$ of $\bar f$ and $\bar g$ forming a commutative diagram

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "p_{\bar{X}}"] & \widetilde{Z} \arrow[d, "p_{\bar{Z}}"] & \widetilde{Y} \arrow[l, "\tilde{g}"'] \arrow[d, "p_{\bar{Y}}"] \\
\overline{X} \arrow[r, "\bar{f}"] \arrow[d, "p_X"] & \overline{Z} \arrow[d, "p_Z"] & \overline{Y} \arrow[l, "\bar{g}"'] \arrow[d, "p_Y"'] \\
X \arrow[r, "f"] & Z & Y \arrow[l, "g"']
\end{tikzcd}$$

The compositions $\widetilde{p}_X = p_X p_{\bar{X}}$, $\widetilde{p}_Y = p_Y p_{\bar{Y}}$, $\widetilde{p}_Z = p_Z p_{\bar{Z}}$ are of course the universal covering maps of X, Y and Z. Moreover, for any $\gamma \in \mathcal{D}(Z)$, there is a $\overline{\gamma} \in \mathcal{D}(p_Z)$ such that $\overline{\gamma} p_{\bar{Z}} = p_{\bar{Z}}\gamma$.

Thus, if $J = (\widetilde{p}_X \times \widetilde{p}_Y)(\mathrm{Int}(\widetilde{f}, \gamma\widetilde{g}))$ is an intersection class in $\mathcal{I}(f, g)$, then

$$J = (p_X \times p_Y)(p_{\bar{X}} \times p_{\bar{Y}})(\mathrm{Int}(\widetilde{f}, \gamma\widetilde{g}))$$

with $\overline{J} = (p_{\bar{X}} \times p_{\bar{Y}})(\mathrm{Int}(\widetilde{f}, \gamma\widetilde{g}))$ an intersection class in $\mathcal{I}(\bar{f}, \overline{\gamma g})$.

Proposition 7.1. *For every $J \in \mathcal{I}(f, g)$, there is a unique $\overline{\gamma} \in \mathcal{D}(p_Z)$ such that J is in the image of $\mathcal{I}(\bar{f}, \overline{\gamma g})$ under $\mathcal{I}(p_X \times p_Y)$.*

Thus, as we range over $\mathcal{D}(p_Z)$, we range (without repetition) over $\mathcal{I}(f, g)$. We can then concentrate on a single choice of lifts $\bar{f}$ and $\overline{g}$, and determine the relationship between $\mathcal{I}(\bar{f}, \overline{g})$ and its image in $\mathcal{I}(f, g)$ under $\mathcal{I}(p_X \times p_Y)$.

Proposition 7.2. *If $J \in \mathcal{I}(f, g)$ is in the image of $\mathcal{I}(\bar{f}, \overline{g})$, then there are $P(f, g; J)$ intersection classes $\overline{J}$ in $\mathcal{I}(\bar{f}, \overline{g})$ such that $\mathcal{I}(p_X \times p_Y)(\overline{J}) = J$. Each of these sets $\overline{J}$ is a covering space over J, with covering group $(\varrho_X \times \varrho_Y)(\mathrm{Int}(f_\#, g_\#; J)) \cong \mathrm{Int}(f_\#, g_\#; J)/(p_{X\#} \times p_{Y\#})\,\mathrm{Int}(\bar{f}_\#, \overline{g}_\#; \overline{J})$.*

Proof. We know that there are $P(f, g; J)$ Reidemeister classes covering J, and that at least one of them is non-empty. We must show that in fact all of them are non-empty. Consider $\overline{J} = (p_{\tilde{X}} \times p_{\tilde{Y}})(\mathrm{Int}(\widetilde{f}, \widetilde{g}))$ and $\overline{J}' = (p_{\tilde{X}} \times p_{\tilde{Y}})(\mathrm{Int}(\widetilde{f}, \gamma\widetilde{g}))$ for some $\gamma \in \mathcal{D}(\overline{Z})$. Suppose that $\overline{J}$ is non-empty, and that both $\overline{J}$ and $\overline{J}'$ map into J.

Now γ can also be viewed as an element of $\mathcal{D}(Z)$, and

$$\begin{aligned}
(p_X \times p_Y)(\overline{J}') &= (p_X \times p_Y)(p_{\tilde{X}} \times p_{\tilde{Y}})(\mathrm{Int}(\widetilde{f}, \gamma\widetilde{g})) \\
&= (\widetilde{p}_X \times \widetilde{p}_Y)(\mathrm{Int}(\widetilde{f}, \gamma\widetilde{g})) = J.
\end{aligned}$$

That is, $\overline{J}'$ maps *onto* J, and so must be non-empty.

We now want to show that every $\overline{J}$ that maps onto J is in fact a covering space, with covering group $\mathrm{Int}(f_\#, g_\#; J)/(p_{X\#} \times p_{Y\#})\,\mathrm{Int}(\bar{f}_\#, \overline{g}_\#; \overline{J})$. Suppose $(\overline{x}, \overline{y}) \in \overline{J}$, and $(\alpha(\overline{x}), \beta(\overline{y})) \in \overline{J}$ for some $\alpha \in \mathcal{D}(p_X)$, $\beta \in \mathcal{D}(p_Y)$. Then there are paths $\overline{\omega}_1$ from $\overline{x}$ to $\alpha(\overline{x})$ and $\overline{\omega}_2$ from $\overline{y}$ to $\beta(\overline{y})$ such that $\bar{f}(\overline{\omega}_1) \simeq \overline{g}(\overline{\omega}_2)$ in $\overline{Z}$. Thus $[f p_X \overline{\omega}_1] = [g p_Y \overline{\omega}_2]$ in $\pi_1(Z)$. That is,

$([p_X\overline{\omega}_1],[p_Y\overline{\omega}_2]) \in \mathrm{Int}(f_\#,g_\#;J)$. On the other hand, suppose $([\omega_1],[\omega_2]) \in \mathrm{Int}(f_\#,g_\#;J)$. Lift ω_1 to a path $\overline{\omega}_1$ based at $\overline{x}$, and ω_2 to a path $\overline{\omega}_2$ based at $\overline{y}$. Then $\bar f\overline{\omega}_1(0)$ and $\overline{g}\overline{\omega}_2(0)$ coincide, and their projections are endpoint-homotopic. Thus $\bar f\overline{\omega}_1$ and $\overline{g}\overline{\omega}_2$ are endpoint-homotopic, and in particular, $\bar f\overline{\omega}_1(1) = \overline{g}\overline{\omega}_2(1)$. Thus there is an element of $\overline{J}$ corresponding to every $([\omega_1],[\omega_2]) \in \mathrm{Int}(f_\#,g_\#;J)$. It is easy to check that the kernel of this representation is $(p_{X\#}\times p_{Y\#})\,\mathrm{Int}(\bar f_\#,\overline{g}_\#;\overline{J})$. ∎

It remains to determine the relation between essential classes in $\mathcal{I}(f,g)$ and essential classes in $\mathcal{I}(\bar f,\overline{g})$. We already know from Proposition 6.5 that inessential classes in $\mathcal{I}(f,g)$ are covered by inessential classes. Unfortunately, the converse is false: we will see below (Example 7.7) that it is possible for an essential class in $\mathcal{I}(f,g)$ to be covered by an inessential class in $\mathcal{I}(\bar f,\overline{g})$. On the other hand, while it is possible that an essential class could be covered by inessential classes, it cannot occur that some of the $P(f,g;J)$ intersection classes are inessential and some are essential.

PROPOSITION 7.3. *If $\overline{J},\overline{J}'$ are intersection classes in $\mathcal{I}(\bar f,\overline{g})$ such that $(p_X\times p_Y)(\overline{J}) = (p_X\times p_Y)(\overline{J}')$, then $\overline{J}$ is essential if and only if $\overline{J}'$ is.*

Proof. Write $\overline{J}$ as $(p_{\bar X}\times p_{\bar Y})(\mathrm{Int}(\widetilde f,\widetilde g))$ and $\overline{J}'$ as $(p_{\bar X}\times p_{\bar Y})(\mathrm{Int}(\widetilde f,\gamma\widetilde g))$ for some $\gamma\in\mathcal{D}(\overline{Z})$. Now, since $(p_{\bar X}\times p_{\bar Y})(\mathrm{Int}(\widetilde f,\widetilde g)) = (p_X\times p_Y)(\mathrm{Int}(\widetilde f,\gamma\widetilde g))$ in $X\times Y$, Proposition 6.4 implies that, when γ is viewed as an element of $\mathcal{D}(Z)$, $\gamma = f_\#(\alpha^{-1})g_\#(\beta)$ for some $\alpha\in\mathcal{D}(X)$, $\beta\in\mathcal{D}(Y)$. Then $\overline{J}' = (p_{\bar X}\times p_{\bar Y})(\mathrm{Int}(\widetilde f\alpha,\widetilde g\beta))$. Let $\overline{\alpha}\in\mathcal{D}(p_X)$ and $\overline{\beta}\in\mathcal{D}(p_Y)$ be the covering transformations generated by α and β. Then, if J can be deformed to the empty set by homotopies $\overline{F}$ and $\overline{G}$, $\overline{J}$ can be deformed to the empty set by $\overline{F}\overline{\alpha}$ and $\overline{G}\overline{\beta}$. ∎

We are now in a position to compare $\mathrm{NI}(f,g)$ with the intersection numbers $\mathrm{NI}(\bar f,\overline{\gamma}\overline{g})$, with $\overline{\gamma}$ ranging over $\mathcal{D}(p_Z)$. Every essential class in $\mathcal{I}(f,g)$ is covered by either $P(f,g;J)$ essential classes, or by none. Further, $|P(f,g;J)|\le d_Xd_Y$, where $d_X=|\mathcal{D}(p_X)|$ and $d_Y=|\mathcal{D}(p_Y)|$ are the orders of the covering spaces. That is, each essential intersection class in $\mathcal{I}(f,g)$ is covered by at most d_Xd_Y essential intersection classes in the covering spaces. In summary, we have

THEOREM 7.4. *Given a diagram*

$$\begin{tikzcd}
\overline{X} \arrow[r, "\bar f"] \arrow[d, "p_X"] & \overline{Z} \arrow[d, "p_Z"] & \overline{Y} \arrow[l, "\bar g"'] \arrow[d, "p_Y"] \\
X \arrow[r, "f"] & Z & Y \arrow[l, "g"']
\end{tikzcd}$$

in which all of the vertical maps are finite regular covers, we have

$$\mathrm{NI}(f,g) \geq \frac{1}{d_X d_Y} \sum_{\bar{\gamma}\in\mathcal{D}(p_Z)} \mathrm{NI}(\bar{f},\gamma\bar{g}).$$

If this is viewed as a way of estimating $\mathrm{NI}(f,g)$, we can ask how different choices of covering spaces will effect the estimate. That is, what is the dependence of the quantity

$$\frac{1}{d_X d_Y} \sum_{\bar{\gamma}\in\mathcal{D}(p_Z)} \mathrm{NI}(\bar{f},\gamma\bar{g})$$

on the covering spaces $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ chosen? In fact, it is independent of the choice of $\overline{Z}$, in the following sense.

Proposition 7.5. *Given $f : X \to Z$ and $g : Y \to Z$, suppose $p : \overline{Z} \to Z$ is a finite regular cover such that f and g lift to $\bar{f} : X \to \overline{Z}$ and $\overline{g} : Y \to \overline{Z}$. Then* $\mathrm{NI}(f,g) = \sum_{\bar{\gamma}\in\mathcal{D}(p)} \mathrm{NI}(\bar{f},\overline{\gamma g})$.

Proof. We can apply the analysis leading up to Theorem 7.4, with $\overline{X} = X$ and $\overline{Y} = Y$. Then $\mathcal{D}(p_X)$ and $\mathcal{D}(p_Y)$ are trivial, so $P(f,g;J) = 1$ for every intersection class J. Thus every intersection class in $\mathcal{I}(f,g)$ is also an intersection class in $\mathcal{I}(\bar{f},\overline{\gamma g})$ for some (unique) covering transformation $\overline{\gamma}$.

We already know from Proposition 6.5 that if J is essential as an intersection class in $\mathcal{I}(\bar{f},\overline{\gamma g})$, then it is essential as an intersection class in $\mathcal{I}(f,g)$. Conversely, suppose that J is inessential in $\mathcal{I}(\bar{f},\overline{\gamma g})$. Then there are homotopies $\overline{F} : X\times[0,1] \to \overline{Z}$ and $\overline{G} : Y \times [0,1] \to \overline{Z}$ such that J is $(\overline{F},\overline{G})$-related to the empty set. If we compose $\overline{F}$ and $\overline{G}$ with p, then J is $(p\overline{F},p\overline{G})$-related to the empty set, so J is inessential as in intersection class in $\mathcal{I}(f,g)$. ∎

Corollary 7.6. *Consider the diagram*

$$\begin{tikzcd}
 & \overline{Z}_1 \arrow[d, "\bar{p}"] & \\
\overline{X} \arrow[ur, "\bar{f}_1"] \arrow[r, "\bar{f}_2"] \arrow[d, "p_X"] & \overline{Z}_2 \arrow[d, "p_{2Z}"] & \overline{Y} \arrow[ul, "\bar{g}_1"'] \arrow[l, "\bar{g}_2"'] \arrow[d, "p_Y"] \\
X \arrow[r, "f"] & Z & Y, \arrow[l, "g"']
\end{tikzcd}$$

with all of the vertical arrows finite regular covers. If $p_{1Z} = p_{2Z}\overline{p}$, then

$$\frac{1}{d_X d_Y} \sum_{\bar{\gamma}_1\in\mathcal{D}(p_{1Z})} \mathrm{NI}(\bar{f}_1,\overline{\gamma}_1\overline{g}) = \frac{1}{d_X d_Y} \sum_{\bar{\gamma}_2\in\mathcal{D}(p_{2Z})} \mathrm{NI}(\bar{f}_2,\overline{\gamma}_2\overline{g}_2).$$

Having established that the sum is independent of $\overline{Z}$, it is now clear that it cannot be independent of $\overline{X}$ and $\overline{Y}$ in general. For, if it were, then the sum would be independent of all of the lifts involved, and in particular, the inequality in Theorem 7.4 would be an equality for all lifting diagrams. This in turn would imply that essential classes are always covered by essential classes. Again, Example 7.7 will show that this is not true in general.

At the beginning of the section, we indicated two natural applications of these results: first, if Z has finite fundamental group; second, if X, Y and Z are compact manifolds, but not all are orientable. In the first case, we can take $\overline{Z}$ to be the universal cover of Z, and $\overline{X}$ and $\overline{Y}$ to be the covering spaces with $\operatorname{im}(p_{X\#}) = \ker(f_{\#})$ and $\operatorname{im}(p_{Y\#}) = \ker(g_{\#})$. It is then a simple calculation to see that $\operatorname{Int}(\overline{f}, \overline{g}) = (\varrho_X \times \varrho_Y) \operatorname{Int}(f_{\#}, g_{\#}; J)$, so $|P(f, g; J)| = 1$ for every J. That is, we have recovered the construction that defines $\mathrm{NI}(f, g)$: every intersection class $J \in \mathcal{I}(f, g)$ is covered by a single intersection class $\overline{J} \in \mathcal{I}(\overline{f}, \gamma\overline{g})$ for a single $\gamma \in \mathcal{D}(Z)$, and J is essential if and only if $\overline{J}$ is. This gives us the (hardly surprising) formula

$$\mathrm{NI}(f, g) = \sum_{\gamma \in \mathcal{D}(Z)} \mathrm{NI}(\bar{f}, \gamma\overline{g}).$$

A more interesting application arises when all of the spaces X, Y and Z are compact manifolds, but not all are orientable. There are a variety of cases to consider (e.g. Z non-orientable, both X and Y non-orientable; Z non-orientable and one of X, Y orientable; etc.). For the sake of definiteness, we will consider a single case: suppose Z is non-orientable, and X and Y are orientable. In this case, Z has an orientable double cover $\overline{Z}$, so we have the beginning of a lifting diagram:

$$\begin{tikzcd} & \overline{Z} \arrow[d, "p_Z"] & \\ X \arrow[r, "f"] & Z & Y \arrow[l, "g"'] \end{tikzcd}$$

Since $\operatorname{im}(p_{Z\#})$ is a normal subgroup of index 2 in $\pi_1(Z)$, $f_{\#}^{-1}(\operatorname{im}(p_{Z\#}))$ is a normal subgroup of index at most 2 in $\pi_1(X)$, and likewise for $g_{\#}^{-1}(\operatorname{im}(p_{Z\#}))$ in $\pi_1(Y)$. There are finite regular covers $\overline{X}$ and $\overline{Y}$ such that $\operatorname{im}(p_{X\#}) = f_{\#}^{-1}(\operatorname{im}(p_{Z\#}))$ and $\operatorname{im}(p_{Y\#}) = g_{\#}^{-1}(\operatorname{im}(p_{Z\#}))$. Thus the lifting condition is satisfied, and fp_X and gp_Y lift to $\overline{f} : \overline{X} \to \overline{Z}$ and $\overline{g} : \overline{Y} \to \overline{Z}$. We then have a lifting diagram

$$\begin{tikzcd} \overline{X} \arrow[r, "\bar{f}"] \arrow[d, "p_X"] & \overline{Z} \arrow[d, "p_Z"] & \overline{Y} \arrow[l, "\bar{g}"'] \arrow[d, "p_Y"] \\ X \arrow[r, "f"] & Z & Y \arrow[l, "g"'] \end{tikzcd}$$

in which all of the covering spaces are orientable manifolds; p_Z is a double cover; and p_X and p_Y are regular covering spaces of order 1 or 2. If $\dim(X), \dim(Y) \geq 3$, then the intersection index can be used to compute Nielsen intersection numbers in $\overline{Z}$, and so provide a lower bound for Nielsen intersection numbers in Z.

Example 7.7. Consider the map $\alpha : S^1 \to \mathbb{R}P^2$ that generates $\pi_1(\mathbb{R}P^2)$. If we represent $\mathbb{R}P^2$ as a quotient space of the 2-disk D^2, with the antipodal identification on the boundary, then α can be represented by the intersection with D^2 of any line through the origin. Any two distinct lines α_1, α_2 intersect transversally at the origin, and at no other points. Thus $\mathrm{Int}(\alpha_1, \alpha_2)$ consists of a single point, and $\mathcal{I}(\alpha, \alpha)$ consists of a single intersection class. We claim that this class is essential. If α_i' is homotopic to α_i, then α_i' is an essential loop in $\mathbb{R}P^2$. It can be represented by a path β_i in D^2 which intersects the boundary circle in at least one pair of antipodal points. Thus β_1 must separate D^2, and in particular, must separate every pair of antipodal boundary points. In particular, it either separates or coincides with the pair of antipodal boundary points of β_2. In either case, β_1 and β_2 must intersect, so α_1' and α_2' must intersect. That is, we cannot remove the intersection of α_1 and α_2 by a homotopy, so $\mathrm{NI}(\alpha, \alpha) = \mathrm{NI}(\alpha_1, \alpha_2) = 1$.

Now, if we take the double covering $p : S^1 \to S^1$ and the covering $q : S^2 \to \mathbb{R}P^2$, then α_i lifts to $\widetilde{\alpha}_i : S^1 \to S^2$, and there is a commutative diagram

$$\begin{tikzcd}
S^1 \arrow[r, "\tilde{\alpha}_1"] \arrow[d, "p"] & S^2 \arrow[d, "g"] & S^1 \arrow[l, "\tilde{\alpha}_2"'] \arrow[d, "p"] \\
S^1 \arrow[r, "\alpha_1"] & \mathbb{R}P^2 & S^1 \arrow[l, "\alpha_2"']
\end{tikzcd}$$

Since α_1 and α_2 are two distinct lines through the origin in $\mathbb{R}P^2$, their lifts $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ are two distinct great circles in S^2. As such, they intersect twice, with opposite orientation. Since S^2 is simply connected, these two intersection points must lie in the same intersection class. That class must then be algebraically inessential. Moreover, it is topologically inessential: any two images of S^1 in S^2 can be deformed to be disjoint. Thus $\mathrm{NI}(\widetilde{\alpha}_1, \widetilde{\alpha}_2) = 0$, and the same is true if we compose $\widetilde{\alpha}_2$ with the covering transformation $\gamma(x) = -x$: $\mathrm{NI}(\widetilde{\alpha}_1, -\widetilde{\alpha}_2) = 0$.

The formula in Theorem 7.4 in this case yields the result

$$1 = \mathrm{NI}(\alpha, \alpha) \geq \sum_{\gamma \in \mathcal{D}(Z)} \mathrm{NI}(\overline{\alpha}, \gamma\overline{\alpha}) = 0.$$

That is, the formula does not always produce equality. And, in this example, the failure of equality to obtain can be traced back to the failure of essential classes to lift to essential classes: the single transverse intersection of α_1 and

α_2 in $\mathbb{R}P^2$ is lifted to two transverse intersections in S^2. These lay in the same intersection class, but had opposite orientation, so the essential class in $\mathbb{R}P^2$ was covered by an inessential class in S^2.

8. Fibrations. Finally, we want to consider intersection numbers for fibrations. Suppose we have the following diagram:

$$\begin{tikzcd}
D \arrow[r, "f"] \arrow[d] & F \arrow[d] & E \arrow[l, "g"'] \arrow[d] \\
X \arrow[r, "f"] \arrow[d, "p"] & Z \arrow[d, "r"] & Y \arrow[l, "g"'] \arrow[d, "q"] \\
A \arrow[r, "\bar{f}"] & C & B \arrow[l, "\bar{g}"']
\end{tikzcd}$$

The columns are fibrations, and the maps f and g are fiber-preserving. The natural questions are:

• For $(a,b) \in \operatorname{Int}(\bar{f},\bar{g})$, let $f_a : D_a \to F_c$ and $g_b : E_b \to F_c$ denote the corresponding fiber maps. Under what conditions is $\operatorname{NI}(f_a, g_b)$ constant across $\operatorname{Int}(\bar{f},\bar{g})$?

• If $\operatorname{NI}(f_a, g_b)$ is constant across $\operatorname{Int}(\bar{f},\bar{g})$, when does the equality $\operatorname{NI}(f,g) = \operatorname{NI}(\bar{f},\bar{g})\operatorname{NI}(f_a,g_b)$ hold?

• If this naive product formula does not hold, do modifications analogous to those of [12] hold?

We do not by any means attempt here an exhaustive investigation of these questions. Instead, we will content ourselves with some partial results on the first two points. For this first investigation, we will assume that all of the fibrations are locally trivial fiber bundles, and the spaces involved are compact orientable manifolds, with $\dim(A) + \dim(B) = \dim(C)$ and $\dim(D) + \dim(E) = \dim(F)$.

PROPOSITION 8.1. *If (a,b) and (a',b') are in the same intersection class in $\mathcal{I}(\bar{f},\bar{g})$, then $\operatorname{NI}(f_a,g_b) = \operatorname{NI}(f_{a'},g_{b'})$. If $Z \xrightarrow{r} C$ is an orientable fibration, then $\operatorname{NI}(f_a,g_b) = \operatorname{NI}(f_{a'},g_{b'})$ for all (a,b) and (a',b') in $\operatorname{Int}(\bar{f},\bar{g})$.*

Proof. For any two $(a,b),(a',b') \in \operatorname{Int}(\bar{f},\bar{g})$, choose paths α from a to a' and β from b to b'. There is then a homotopy-commutative diagram

$$\begin{tikzcd}
D_a \arrow[r, "f_a"] \arrow[d, "\tau_\alpha"] & F_c \arrow[d, "\tau_{\bar{g}\beta}"] & E_b \arrow[l, "g_b"'] \arrow[d, "\tau_\beta"] \\
D_{a'} \arrow[r, "f_{a'}"'] & F_{c'} & E_{b'} \arrow[l, "g_{b'}"']
\end{tikzcd}$$

with the fiber translation maps τ all homeomorphisms. Now, under either hypothesis, $\tau_{\bar{f}\alpha} \simeq \tau_{\bar{g}\beta}$, so

$$\mathrm{NI}(f_{a'}, g_{b'}) = \mathrm{NI}(\tau_{\bar{f}\alpha} f_a \tau_{\alpha^{-1}}, \tau_{\bar{g}\beta} g_b \tau_{\beta^{-1}}) = \mathrm{NI}(\tau_{\bar{f}\alpha} f_a \tau_{\alpha^{-1}}, \tau_{\bar{f}\alpha} g_b \tau_{\beta^{-1}}).$$

But pre- and post-composition by homeomorphisms does not change the Nielsen number (see Proposition 3.7), so

$$\mathrm{NI}(\tau_{\bar{f}\alpha} f_a \tau_{\alpha^{-1}}, \tau_{\bar{f}\alpha} g_{b'} \tau_{\beta^{-1}}) = \mathrm{NI}(f_a, g_b). \ \blacksquare$$

Corollary 8.2. *If* $\mathrm{NI}(\bar{f}, \bar{g}) = 1$, *or if* $r : Z \to C$ *is orientable*, *then* $\mathrm{RI}(f_a, g_b)$ *and* $\mathrm{NI}(f_a, g_b)$ *are constant across* $(a, b) \in \mathrm{Int}(\bar{f}, \bar{g})$.

We turn now to the "naive product formula"

$$\mathrm{NI}(f, g) = \mathrm{NI}(\bar{f}, \bar{g})\,\mathrm{NI}(f_a, g_b).$$

Certainly, experience with the product formula for Nielsen fixed point numbers suggests that this product formula will not hold in general. We limit ourselves here to some simple sufficient conditions for the naive product formula.

Actually, we will start with the "naive addition formula", analogous to the formula in [6] for fixed points. For this, we do not require $\mathrm{NI}(f_a, g_b)$ to be independent of the intersection class in $\mathcal{I}(\bar{f}, \bar{g})$. Instead, if $\bar{J}_1, \ldots, \bar{J}_N$ are the essential intersection classes of $\mathrm{Int}(\bar{f}, \bar{g})$, choose representatives $(a_i, b_i) \in \bar{J}_i$. The naive addition formula is satisfied if $\mathrm{NI}(f, g) = \sum_i \mathrm{NI}(f_{a_i}, g_{b_i})$. Of course, the naive product formula follows if $\mathrm{NI}(f_a, g_b)$ is independent of i. To obtain conditions that imply the naive addition formula, we need the following to be true:

1. Every intersection class in $\mathrm{Int}(f, g)$ projects to a single intersection class in $\mathrm{Int}(\bar{f}, \bar{g})$, and intersects each fiber in a single intersection class.
2. An intersection class in $\mathrm{Int}(f, g)$ is essential if and only if its projection is essential in $\mathrm{Int}(\bar{f}, \bar{g})$, and its intersection with a fiber is essential in $\mathrm{Int}(f_a, g_b)$.

Of course, if the map $\pi_1(F) \to \pi_1(Z)$ is injective, Proposition 6.4 shows that the first condition is satisfied if

$$\mathrm{Int}(\bar{f}_{\#}, \bar{g}_{\#}; (p \times q)(J)) = (p_{\#} \times q_{\#})(\mathrm{Int}(f_{\#}, g_{\#}; J)).$$

For the second, we know from Proposition 6.5 that if J is an essential class in $\mathcal{I}(f, g)$, then its projection is an essential class in $\mathcal{I}(\bar{f}, \bar{g})$. However, to say more, we need the additional structure of the intersection index.

Proposition 8.3. *Suppose the bases* A, B, C *and the fibers* D, E, F *are all compact orientable manifolds*, *with* $\dim(A) + \dim(B) = \dim(C)$ *and* $\dim(D) + \dim(E) = \dim(F)$. *Let* J *be an intersection class in* $\mathcal{I}(f, g)$ *such that, for every* $(a, b) \in (p \times q)(J)$, $J \cap (D_a \times E_b)$ *is a single intersection class in* $\mathcal{I}(f_a, g_b)$. *Then* J *is an algebraically essential intersection class if and* *only if* $(p\times q)(J)$ *and* $J\cap(D_a\times E_b)$ *are algebraically essential intersection classes.*

Proof. We may assume without loss that $\bar{f}$ and $\bar{g}$ intersect transversely, and that f_a and g_b intersect transversely. Then $(p\times q)(J)=\{(a_1,b_1),\dots,(a_n,b_n)\}$, and $J_i=J\cap(D_{a_i}\times E_{b_i})=\{(d_{i1},e_{i1}),\dots,(d_{im},e_{im})\}$. In a neighborhood of (d_{ij},e_{ij}), f and g can be identified with $\bar{f}\times f_i$ and $\bar{g}\times g_i$, so

$$\operatorname{Ind}(f,g;(d_{ij},e_{ij}))=\operatorname{Ind}(\bar{f},\bar{g};(a_i,b_i))\operatorname{Ind}(f_i,g_i;(d_{ij},e_{ij})).$$

Thus

$$\operatorname{Ind}(f,g;J_i)=\operatorname{Ind}(\bar{f},\bar{g};(a_i,b_i))\operatorname{Ind}(f_i,g_i;J_i)$$

and

$$\operatorname{Ind}(f,g;J)=\sum_{i=1}^{n}\operatorname{Ind}(\bar{f},\bar{g};(a_i,b_i))\operatorname{Ind}(f_i,g_i;J_i).$$

Now $\operatorname{Ind}(f_i,g_i;J_i)$ is constant over $(p\times q)(J)$ (since all of the maps f_i are homotopic, as are all of the maps g_i, and the classes J_i are (F,G)-related), so

$$\begin{aligned}\operatorname{Ind}(f,g;J)&=\sum_{i=1}^{n}\operatorname{Ind}(\bar{f},\bar{g};(a_i,b_i))\operatorname{Ind}(f_i,g_i;J_i)\\&=\operatorname{Ind}(f_i,g_i;J_i)\sum_{i=1}^{n}\operatorname{Ind}(\bar{f},\bar{g};(a_i,b_i))\\&=\operatorname{Ind}(f_i,g_i;J_i)\operatorname{Ind}(\bar{f},\bar{g};(p\times q)(J)).\end{aligned}$$

Clearly, then, J is algebraically essential in $X\times Y$ if and only if $(p\times q)(J)$ is algebraically essential in $A\times B$ and J_i is algebraically essential in $D\times E$. ■

THEOREM 8.4. *Consider the diagram*

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d] & F \arrow[d] & E \arrow[l, "g"'] \arrow[d] \\ X \arrow[r, "f"] \arrow[d, "p"] & Z \arrow[d, "r"] & Y \arrow[l, "g"'] \arrow[d, "q"] \\ A \arrow[r, "\bar{f}"] & C & B \arrow[l, "\bar{g}"'] \end{tikzcd}$$

Suppose that the columns are fibrations, the maps f and g are fiber-preserving, and the bases A, B, C and the fibers D, E, F are all compact orientable manifolds, with $\dim(A)+\dim(B)=\dim(C)$ *and* $\dim(D)+\dim(E)=\dim(F)$. *Let* $\{(a_i,b_i)\}$ *be a set of representatives of the essential intersection* *classes of* $\mathrm{Int}(\bar{f}, \overline{g})$. *Then*

$$\mathrm{NI}(f, g) = \sum_i \mathrm{NI}(f_{a_i}, g_{b_i})$$

provided

1. $\pi_1(F) \to \pi_1(Z)$ *is injective*;
2. $\mathrm{Int}(\bar{f}_\#, \overline{g}_\#; (p \times q)(J)) = (p_\# \times q_\#)(\mathrm{Int}(f_\#, g_\#; J))$ *for every* $J \in \mathcal{I}(f, g)$;
3. $\max\{\dim(A), \dim(B)\}, \max\{\dim(D), \dim(E)\} \geq 3$.

Proof. The first two conditions give the product formula on Reidemeister classes, leaving only the need to show that essential classes in $X \times Y$ correspond to essential classes in $A \times B$ and $D \times E$. The restrictions on the dimensions are of course required to guarantee that topologically essential classes correspond to algebraically essential classes. With that, the previous proposition completes the proof. ∎

COROLLARY 8.5. *If*, *in addition to the hypotheses of Theorem* 8.4, *the fibration* $F \to Z \to C$ *is orientable*, *then*

$$\mathrm{NI}(f, g) = \mathrm{NI}(\bar{f}, \overline{g})\, \mathrm{NI}(f_a, g_b).$$

9. Concluding remarks. These results show that the ideas of Nielsen theory can be applied in a natural way to the study of intersections. However, they by no means establish the entire Nielsen canon in the intersection number setting. A number of further developments and applications suggest themselves at this point.

First, there are several noticeable gaps in the theory presented here. Most obviously, the naive product formula presented can certainly be refined, *à la* You [12], to a more general product formula. Since this will presumably involve mod K intersection classes, the theory of mod K classes, and more generally, the full functoriality of the Nielsen intersection numbers, needs to be established. And finally, as the last hypothesis in Theorem 8.4 reflects, there is an awkward gap in the theory for low-dimensional manifolds. If X, Y and Z are compact orientable manifolds, then we have defined what it means for an intersection class to be algebraically essential, and to be topologically essential. But if $1 \leq \dim(X), \dim(Y) \leq 2$, we do not know if those two concepts agree. Of course, if a class is algebraically essential, then it must be topologically essential. But is the converse true? Similarly, is the Wecken theorem true for intersections on low-dimensional manifolds? Since low-dimensional manifolds are likely to be of particular interest in applications, these are questions that are well worth resolving. At the same time, the well-known difficulties for Nielsen fixed point theory on surfaces suggest that these questions are also likely to prove difficult.

In this paper, an attempt was made to suggest the analogues in the intersection problem for many of the basic results in Nielsen fixed point theory. However, there are some topics in Nielsen fixed point theory that have not been considered here at all: relative Nielsen numbers; equivariant Nielsen numbers; Nielsen numbers for compact maps on non-compact domains; ... All of these are reasonable directions for further work.

Clearly, the Nielsen theory for intersections owes much to the already established theories for coincidences and roots. The definitions, the results presented, and the future directions suggested are all based on analogies with the other domains of Nielsen theory. However, the connection between Nielsen intersection theory and the other subjects is more than just reasoning by analogy. It is possible, in a very explicit way, to transform a problem from any one of these three subjects into a problem in one of the other subjects. That is, given any intersection problem, there is a natural transformation into a coincidence problem so that the intersection set of one corresponds to the coincidence set of the other. How do the Nielsen numbers behave under these transformations? Are Nielsen coincidence numbers, intersection numbers and root numbers all, in some sense, "the same thing"? This will be explored in [10].

To be of real use, Nielsen theory must be able to reach beyond itself, and make contact with other problems. One direction for Nielsen intersection theory is the study of embeddings. Given $f : X \to Z$, when can f be deformed to an embedding? As an indication that Nielsen theory may be able to make a contribution to this problem, we present the following results, which give necessary conditions, in terms of Nielsen intersection numbers, for a map to be deformable to an embedding.

THEOREM 9.1. *If $f : X \to Z$ has $\mathrm{NI}(f,f) \geq 2$, then f is not homotopic to an embedding.*

Proof. If f is an embedding, then $\mathrm{Int}(f,f)$ is the diagonal in $X \times X$. Since $\mathrm{Int}(f,f)$ is thus path-connected, there is only one intersection class, and hence at most one essential class. ∎

THEOREM 9.2. *If X and Z are compact orientable manifolds such that $2\dim(X) = \dim(Z) \geq 6$, and $f : X \to Z$ has $\mathrm{NI}(f,f) \neq 0$, $\mathrm{LI}(f,f) = 0$, then f is not homotopic to an embedding.*

Proof. As in the previous argument, if f is an embedding, then $\mathrm{Int}(f,f)$ is the diagonal in $X \times X$. There is then a single intersection class, and that intersection class has intersection index $\mathrm{LI}(f,f)$. If $\mathrm{LI}(f,f) = 0$, then $\mathcal{I}(f,f)$ has no essential intersection classes, and $\mathrm{NI}(f,f) = 0$. ∎

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