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On infinite partitions of lines and space

by

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Abstract. Given a partition $P : L \to \omega$ of the lines in $\mathbb{R}^n$, $n \geq 2$, into countably many pieces, we ask if it is possible to find a partition of the points, $Q : \mathbb{R}^n \to \omega$, so that each line meets at most m points of its color. Assuming Martin's Axiom, we show this is the case for $m \geq 3$. We reduce the problem for $m = 2$ to a purely finitary geometry problem. Although we have established a very similar, but somewhat simpler, version of the geometry conjecture, we leave the general problem open. We consider also various generalizations of these results, including to higher dimension spaces and planes.

1. The m-point property for $m \geq 3$. We consider here several questions concerning infinite partitions of lines, planes, etc. in $\mathbb{R}^n$, in particular, colorings of $\mathbb{R}^n$ with prescribed intersection sizes for the lines and points of a given "color". We are particularly concerned with questions which relate set-theoretic partition properties with the underlying geometry of lines, points, etc., in $\mathbb{R}^n$. The results presented here extend some of those of [2], answer some of the questions raised there, and introduce some new questions as well. In particular, these results lead to some interesting connections between set-theoretic partition questions and purely geometric questions.

Throughout, we use the notions of a partition of a set, $A = A_0 \cup A_1 \cup A_2 \cup \ldots$, and a coloring of the set, $f : A \to \omega$, interchangeably.

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Editorial note: Paul Erdős died of a heart attack on September 20, 1996, at the age of 83, while attending a conference on combinatorics in Warsaw. He published over 1500 papers on set theory, finite and infinite combinatorics, number theory, probability theory, graph theory, topology and other areas of mathematics; the number of his co-authors exceeded 460. Through his brilliant research, stimulating problems and personal influence he made a great contribution to the mathematics of the XXth century.

$[\omega]^{<\omega}$ denotes the finite subsets of ω, the natural numbers. MA denotes Martin's Axiom (cf. [4], [5]), the statement that for any c.c.c. partial order, there is a filter meeting any collection of $< 2^\omega$ many dense sets. We recall that MA is consistent with ZFC and imposes no bound on the size of the continuum.

In [1] it was shown in ZFC that for every infinite partition $L = \bigcup_i L_i$ of L, the set of all lines in $\mathbb{R}^n$, there is a partition, $\mathbb{R}^n = \bigcup_{i\in\omega} S_i$, of the points in $\mathbb{R}^n$ such that $\forall l \in L_i$ ($|l \cap S_i|$ is finite). Furthermore, if $2^\omega \le \omega_m$, then "finite" may be replaced by $m+1$. These results were generalized and extended in [2]. It was also asked in [2] whether the converse must hold. That is, does the partition property with size $m+1$ intersection imply $2^\omega \le \omega_m$, or any bound on 2^ω? We show in Theorem 1.1 that this is not the case.

By the *m-point property* we mean the statement that given any partition $L = L_0 \cup L_1 \cup \ldots$ of the lines in $\mathbb{R}^n$ $(n \ge 2)$, there is a partition $\mathbb{R}^n = S_0 \cup S_1 \cup \ldots$ of the points in $\mathbb{R}^n$ such that $\forall l \in L_i$ $(|l \cap S_i| \le m)$.

THEOREM 1.1. *Assume ZFC + MA. Then for any partition* $L = \bigcup_{i\in\omega} L_i$ *of the lines in* $\mathbb{R}^n$ $(n \ge 2)$, *there is a partition* $\mathbb{R}^n = \bigcup_{i\in\omega} S_i$ *of the points in* $\mathbb{R}^n$ *such that* $\forall l \in L_i$ $(|l \cap S_i| \le 3)$.

A related question is addressed in the next theorem.

THEOREM 1.2. *Assume ZFC + MA. Let* $S \subseteq \mathbb{R}^n$ *be such that any line* l *in* $\mathbb{R}^n$ *meets* S *in a finite set. Then there is a partition* $S = \bigcup_{i\in\omega} S_i$ *such that any line* l *in* $\mathbb{R}^n$ *meets any* S_i *in at most* 3 *points.*

The proofs of Theorems 1.1 and 1.2 are similar. We consider first Theorem 1.1.

LEMMA 1.1. *Assume ZFC + MA. Let* $A = L \cup S$ *be a set of lines* L *and points* S *in* $\mathbb{R}^n$ *with* $|S| < 2^\omega$, *and let* $g : S \to [\omega]^{<\omega}$. *Assume that* $\forall l \in L$ $(|l \cap S|$ *is finite*). *Then there is a partition* $S = S_0 \cup S_1 \cup \ldots$ *such that*:

(1) $\forall x \in S_i$ $(i \notin g(x))$.
(2) $\forall l \in L$ $\forall i$ $(|l \cap S_i| \le 2)$.

Note. g prescribes a finite set of "forbidden colors" which we are to avoid in coloring the points of S.

Proof of Lemma 1.1. Let $A = L \cup S$, $g : S \to [\omega]^{<\omega}$ be as in the statement of the lemma. Let $\mathbb{P} = \{(p, f) : p \in [S]^{<\omega},\ f : p \to \omega,\ \forall x \in p$ $(f(x) \notin g(x)),\ \forall l \in L\ \forall i\ \neg\exists x_1, x_2, x_3 \in p$ $(x_1, x_2, x_3$ are distinct, $f(x_1) = f(x_2) = f(x_3) = i$, and $x_1, x_2, x_3 \in l)\}$. Thus, $\mathbb{P}$ consists of the "finite approximations" to the desired coloring of S. We consider the partial order $<_{\mathbb{P}}$ on $\mathbb{P}$ given by $(p_1, f_1) \prec (p_2, f_2)$ provided $p_1 \supseteq p_2$ and $f_2 = f_1 \restriction p_2$.

If we let, for $x \in S$, $D_x = \{(p,f) \in \mathbb{P} : x \in p\}$, then D_x is clearly dense, since we may extend a condition $(p,f) \in \mathbb{P}$ to $(p \cup \{x\}, f')$ by coloring x any non-forbidden color (i.e. not in $g(x)$) not in range$(f{\restriction}p)$. If G is a filter on $\mathbb{P}$ which meets all of the D_x for $x \in S$, then clearly G defines a coloring $f_G : S \to \omega$ such that $\forall x \in S\ (f_G(x) \notin g(x))$ and $\forall l \in L\ \forall i\ \neg\exists x_1, x_2, x_3$ distinct in S $(f_G(x_1) = f_G(x_2) = f_G(x_3) = i$ and $x_1, x_2, x_3 \in l)$. [Set $f_G(x) = i$ iff $\exists (p,f) \in G\ (x \in p \wedge f(x) = i)$.] This coloring f_G is as required in the lemma.

By MA, such a filter G exists provided $\mathbb{P}$ is c.c.c., which we now show. Suppose, towards a contradiction, that $\mathbb{P}$ is not c.c.c., and let (p_α, f_α), $\alpha < \omega_1$, be an antichain in $\mathbb{P}$. Without loss of generality, we may assume that $|p_\alpha| = k$ for all α, for some fixed $k \in \omega$, and further that the family $\{p_\alpha\}$ forms a Δ-system, that is, there is a "root" $r \in [S]^{<\omega}$ such that $\forall \alpha \neq \beta < \omega_1$ $(p_\alpha \cap p_\beta = r)$. We may also clearly assume that $\forall \alpha, \beta < \omega_1\ (f_\alpha{\restriction}r = f_\beta{\restriction}r)$. Having extracted such a Δ-system, we now consider only the first ω many elements of the antichain: (p_n, f_n).

Let $\ll$ be a fixed well-ordering of $\bigcup_n p_n$ of type ω. If $n < m$, since (p_n, f_n), (p_m, f_m) are incompatible, and since $p_n \cap p_m = r$ and $f_n{\restriction}r = f_m{\restriction}r$, we see that $(p_n \cup p_m, f_n \cup f_m)$ fails to be a condition by virtue of there being, for some line $l \in L$ and $i \in \omega$, distinct x_1, x_2, x_3 in $p_n \cup p_m$ with $f_n \cup f_m(x_1) = f_n \cup f_m(x_2) = f_n \cup f_m(x_3) = i$ and $x_1, x_2, x_3 \in l$. We call such a triple x_1, x_2, x_3 *bad* for l. We clearly cannot have two (or more) of the 3 points in r, since then one of p_n, p_m would contain all three of x_1, x_2, x_3, contradicting $p_n, p_m \in \mathbb{P}$. Thus, whenever $n < m$, at least one of the following holds:

(0) There are two points, say x_1, x_2, in $p_n - r$ and a point $x_3 \in p_m - r$ with x_1, x_2, x_3 bad for some $l \in L$.

(1) There is a point, say x_1, in $p_n - r$ and two points $x_2, x_3 \in p_m - r$ with x_1, x_2, x_3 bad for some $l \in L$.

(2) There is a point, say x_1, in $p_n - r$, a point $x_2 \in p_m - r$, and a point $x_3 \in r$ with x_1, x_2, x_3 bad for some $l \in L$.

For all $n < m$ consider the least case which applies. For this case, we associate with x_1, x_2, x_3 integers $o(x_1), o(x_2), o(x_3)$ which give the ranks of x_1, x_2, x_3 in the ordering $\ll$ restricted to the sets $p_n - r$, $p_m - r$, r (and we assume, for example, that if $x_1, x_2 \in p_n - r$, then $x_1 \ll x_2$). Of course, $o(x_1), o(x_2), o(x_3) \le k$.

We now define a partition $h : (\omega)^2 \to 3 \times k \times k \times k$ by $h(n,m) = (i,a,b,c)$ iff $0 \le i \le 2$ and i is the least case which applies to (p_n, f_n), (p_m, f_m), and $o(x_1) = a, o(x_2) = b, o(x_3) = c$. Since the range of h is finite, by Ramsey's theorem there is an infinite homogeneous set $H \subseteq \omega$ for h. Replacing ω by H, and considering only those (p_n, f_n) for $n \in H$, we may now assume that for all $n < m$, $h(n, m)$ has a constant value. In particular, one of the 3 cases applies for all $n < m$.

Suppose first that case (0) applies for all $n < m$. For each $m \in \omega$, consider (p_0, f_0), (p_m, f_m). Let $x_1(m), x_2(m), x_3(m)$ be the 3 points of case (0) corresponding to the a, b, c of $h(0, m) = (0, a, b, c)$. Thus, $x_1(m), x_2(m) \in p_0 - r$, and $x_3(m) \in p_m - r$. Since $p_0 - r$ is independent of m, we have $x_1(m) = x_1,\ x_2(m) = x_2$ for all m. Also, $\forall m\ \exists i\ \exists l \in L\ (x_1, x_2, x_3(m)$ are bad for $l)$. Since, $x_1, x_2 \in l$, l is determined by x_1, x_2, and is therefore also independent of m. Thus, $x_1, x_2, x_3(m), x_4(m), \ldots$ are all on a single line $l \in L$. This, however, contradicts our assumption that $\forall l \in L$ ($l \cap S$ is finite).

Assume now case (1) applies for all $n < m$. Consider $(p_0, f_0), (p_1, f_1), (p_2, f_2)$. Let x_0, x_1, x_2 be the triple corresponding to (p_0, f_0) and (p_2, f_2), and let x_0', x_1', x_2' be the triple corresponding to $(p_1, f_1), (p_2, f_2)$. Thus, $x_0 \in p_0 - r,\ x_1, x_2 \in p_2 - r,\ x_0' \in p_1 - r,\ x_1', x_2' \in p_2 - r$. Since h is constant, we have $x_1 = x_1',\ x_2 = x_2'$. Thus, both x_0, x_0' are on the line $l \in L$ determined by $x_1, x_2 \in p_2 - r$. (Note $x_0 \neq x_0'$.) For $m \in \omega$, consider the pairs $(p_0, f_0), (p_m, f_m)$ and $(p_1, f_1), (p_m, f_m)$. For the first pair, we get a corresponding triple $x_0(m), x_1(m), x_2(m)$, where $x_0(m) \in p_0 - r,\ x_1(m), x_2(m) \in p_m - r$. We also have $x_0(m) = x_0$ from the constancy of h. Similarly, for the second pair we get $x_0'(m) \in p_1 - r,\ x_1'(m), x_2'(m) \in p_m - r$, and we also obtain $x_0'(m) = x_0'$, and $x_1(m) = x_1'(m), x_2(m) = x_2'(m)$. Thus, the line through $x_1(m), x_2(m)$ also passes through x_0, x_0'. Hence, for all $m \geq 2$, there is a point $x_1(m) \in p_m - r$ on the line $l \in L$ through x_0, x_0', a contradiction.

Finally, the argument for case (2) is essentially identical to that for case (0). In all cases, we contradict the assumption $\mathbb{P}$ is not c.c.c., and this completes the proof of Lemma 1.1. ■

LEMMA 1.2. *Assume ZFC + MA. Let $A = L \cup S$ be a set of lines and points in $\mathbb{R}^n$ of size $< 2^\omega$. Let $L = L_0 \cup L_1 \cup \ldots$ be a partition of the lines in A, and let $g : S \to [\omega]^{<\omega}$. Then there is a partition $S = S_0 \cup S_1 \cup \ldots$ such that*:

(1) $\forall x \in S_i\ (i \notin g(x))$,
(2) $\forall l \in L_i\ (|l \cap S_i| \leq 2)$.

Proof. Let $\omega = B_0 \cup B_1 \cup B_2 \cup \ldots$ be a partition of ω into infinitely many disjoint infinite subsets. For A, g as given in the lemma, consider the new partition of L defined by $L = M_0 \cup M_1 \cup \ldots$, where $l \in M_i$ iff $\exists j\ (l \in L_j \wedge j \in B_i)$.

From Corollary 8 of [2], there is a partition $S = T_0 \cup T_1 \cup \ldots$ such that $\forall l \in M_i$ ($|l \cap T_i|$ is finite). For each $i \in \omega$, consider $A_i = M_i \cup T_i$, so $|A_i| < 2^\omega$. Consider the partition $M_i = L_{i_0} \cup L_{i_1} \cup \ldots$, where $B_i = \{i_0, i_1, \ldots\}$.

By Lemma 1.1 (identifying ω with B_i) there is a partition $T_i = S^i_{i_0} \cup S^i_{i_1} \cup \ldots \cup S^i_{i_k} \cup \ldots$ such that $\forall x \in S^i_{i_k} (i_k \notin g(x))$ and $\forall l \in L_{i_k} (|l \cap S^i_{i_k}| \leq 2)$.

Define the partition of S by $x \in S_k$ iff $\exists i$ $(x \in S_k^i)$. The sets S_k form a partition of S. Also, if $x \in S_k$ then $k \notin g(x)$. Let $l \in L$, say $l \in L_j$. Let i be such that $j \in B_i$, so $l \in M_i$. By construction, l meets at most two points in S_j^i. However, the points in S_j^i are the only points in S which receive color j, since j belongs only to B_i. Thus, l meets at most 2 points from S_j. ∎

Proof of Theorem 1.1. Let $L = \bigcup_i L_i$ be as in the statement of the theorem. We say a set $A = L \cup S$ of lines and points in $\mathbb{R}^n$ is *good* if:

(1) $\forall x \neq y \in S$ (the line $l(x,y)$ determined by x, y is in L).
(2) $\forall l_1 \neq l_2 \in L$ $(l_1 \cap l_2 \in A)$.

Write $L \cup \mathbb{R}^n = \bigcup_{\alpha<2^\omega} A_\alpha$, where each $A_\alpha = L_\alpha \cup S_\alpha$ is good, the A_α are increasing, and $|A_\alpha| < 2^\omega$. We define the coloring $Q : \mathbb{R}^n \to \omega$. We assume that $Q_{<\alpha} = Q \restriction S_{<\alpha}$ has been defined, where $S_{<\alpha} = \bigcup_{\alpha'<\alpha} S_{\alpha'}$. For $x \in S_\alpha - S_{<\alpha}$, let $g_\alpha(x) = \{i \in \omega : \exists l \in L_{<\alpha} \cap L_i\ (x \in l)\}$. Note that $|g(x)| \leq 1$, since if $l_1, l_2 \in L_{<\alpha}$ then $l_1 \cap l_2 \in S_{<\alpha}$.

Consider $B_\alpha = L_\alpha \cup (S_\alpha - S_{<\alpha})$. By Lemma 1.2 applied to L_α, $S_\alpha - S_{<\alpha}$, and g_α, there is a coloring $\widetilde{Q}_\alpha : S_\alpha - S_{<\alpha} \to \omega$ such that $\forall x \in S_\alpha - S_{<\alpha}$ $(\widetilde{Q}_\alpha(x) \notin g(x))$, and $\forall l \in L_\alpha \cap L_i$ (l meets at most 2 points of $S_\alpha - S_{<\alpha}$ of color i). Let $Q_\alpha = Q_{<\alpha} \cup \widetilde{Q}_\alpha$.

Doing this for each $\alpha < 2^\omega$ (using AC) defines the coloring $Q : \mathbb{R}^n \to \omega$. We show Q works. Suppose $l \in (L_\alpha - L_{<\alpha}) \cap L_i$. There is at most one $x \in S_{<\alpha} \cap l$ by goodness. There are at most two $x \in (S_\alpha - S_{<\alpha}) \cap l$ of Q color i. Finally, if $x \in l \cap (S - S_\alpha)$, then $Q(x) \neq i$, since $i \in g_\beta(x)$, where $x \in S_\beta - S_{<\beta}$. ∎

Corollary 1.1. *The "3-point partition property" (i.e. the statement that for any partition $L = \bigcup_i L_i$ of the lines in $\mathbb{R}^n$ there is a partition $\mathbb{R}^n = \bigcup_i S_i$ such that $\forall l \in L_i$ $(|l \cap S_i| \leq 3)$) is consistent with ZFC $+ 2^\omega > \omega_1, \omega_2$, etc.*

We consider now Theorem 1.2; the proof is similar to that of Theorem 1.1, so we will merely outline the differences. Write $S = \bigcup_{\alpha<2^\omega} S_\alpha$, an increasing union, with each S_α closed, that is, if $x, y, z, w \in S_\alpha$ and $l(x,y), l(z,w)$ are distinct, non-parallel lines with $l(x,y) \cap l(z,w) \in S$, then $l(x,y) \cap l(z,w) \in S_\alpha$.

We define by induction on S_α the coloring $Q_\alpha : S_\alpha \to \omega$ (with Q_β extending Q_α if $\alpha < \beta$). At step α, for each $x \in S_\alpha - \bigcup_{\beta<\alpha} S_\beta$, let $g(x) = \{i \in \omega : \exists y, z \in \bigcup_{\beta<\alpha} S_\beta$ ((x,y,z) are collinear and $(\bigcup_{\beta<\alpha} Q_\beta)(y) = (\bigcup_{\beta<\alpha} Q_\beta)(z) = i)\}$. We easily see that $g(x)$ is finite, and we then apply Lemma 1.1 (with L = all lines in $\mathbb{R}^n$) to color the points in $S_\alpha - S_{<\alpha}$. This coloring easily works.

By Corollary 1.1, the three-point partition is consistent with the continuum being "arbitrarily large". It is natural to ask whether this is also true for the two-point property, or indeed whether the two-point property is consistent with $\neg$CH. Consideration of this question leads to a purely geometric question. This analysis is sufficiently detailed to warrant discussion elsewhere ([3]), but we briefly sketch here the main points (though the consistency of the two-point property with $\neg$CH as well as the geometry problem are open).

Assume MA, and let $Q : L \to \omega$ be a given coloring of the lines L in $\mathbb{R}^2$. The basic idea is to first do a preliminary coloring of the points (as in the proof of Lemma 1.2) in $\mathbb{R}^2$, using Theorem 1.1, so that every line in $\mathbb{R}^2$ meets at most 3 points of its color. Given then a set $S \subseteq \mathbb{R}^2$ such that $\forall l \in L\ (|l \cap S| \le 3)$, it suffices to define $P : S \to \omega$ such that $\forall l \in L\ (|\{x \in l \cap S : P(x) = Q(l)\}| \le 2)$. To do this, write $(L, S) = \bigcup_{\alpha < 2^\omega} (L_\alpha, S_\alpha)$, where $|L_\alpha|, |S_\alpha| < 2^\omega$, and each (L_α, S_α) is "sufficiently closed" in (L, S) (e.g., the intersection of (L, S) with an increasing union of models of a large fragment of ZFC). For each $\alpha < 2^\omega$, there is a naturally defined partial order $\mathbb{P}_\alpha$ which attempts to extend the coloring $P_\alpha = P \restriction S_\alpha$ to $P_{\alpha+1}$ maintaining the two-point property. If each $\mathbb{P}_\alpha$ is c.c.c., we can inductively define, using MA, the colorings P_α and complete the proof.

Arguments along the lines of Lemma 1.1 (though more involved) reduce this problem to purely geometric questions. Specifically, we introduce the following geometry conjecture:

Conjecture. *There is an integer $k \in \omega$ such that the following holds. Let $x_1, \ldots, x_k, y_1, \ldots, y_k$ be points in $\mathbb{R}^n$ such that any line $l(x_i, y_j)$ meets no other points of the set. For $1 \le i, j \le k$, let $z_{ij} \in l(x_i, y_j)$. Then there are only finitely many tuples $(x'_1, \ldots, x'_k; y'_1, \ldots, y'_k)$ such that $\forall 1 \le i, j \le k$ $(z_{i,j} \in l(x'_i, y'_j)$, and $l(x'_i, y'_j)$ meets no other point of $(x'_1, \ldots, x'_k; y'_1, \ldots, y'_k))$.*

Thus, this conjecture along with MA implies the two-point partition property. Likewise, consider the second version of the two-point property (corresponding to Theorem 1.2): if $S \subseteq \mathbb{R}^n$ is such that any line l in $\mathbb{R}^n$ meets S in a finite set, then there is a partition $S = \bigcup_{i \in \omega} S_i$ such that any line l in $\mathbb{R}^n$ meets any S_i in at most 2 points. Then MA plus the following somewhat weaker variation of the geometry conjecture suffices:

Conjecture. *There is an integer $k \in \omega$ such that the following holds. Let z_{ij} for each $1 \le i, j \le k$ be points in $\mathbb{R}^n$, no three of which are collinear. Then there are only finitely many tuples $(x_1, \ldots, x_k; y_1, \ldots, y_k)$ of points in $\mathbb{R}^n$ such that $z_{ij} \in l(x_i, y_j)$ for all $1 \le i, j \le k$ and such that every $l(x_i, y_j)$ meets no other point of $(x_1, \ldots, x_k; y_1, \ldots, y_k)$.*

The least integer for which these conjectures are reasonable is $k = 4$, and for this k we refer to them as the "16-point" problem. As a preliminary, one can consider the version of the geometry problem corresponding to the complete graph on k vertices rather than the bipartite graph on $2k$ vertices.

Here it has been shown ([3]) that for $k = 5$ (the smallest reasonable value) the result is true. Specifically:

THEOREM 1.3. *Let $z_{i,j}$ for $1 \le i < j \le 5$ be* 10 *points in* $\mathbb{R}^2$, *no three of which are collinear. Then there are at most finitely many tuples* $(x_1, \ldots, x_5)$ *of distinct points such that* $\forall 1 \le i < j \le 5$ $(z_{ij} \in l(x_i, x_j))$.

This result shows that the bipartite versions of the geometry conjecture are at least plausible, and are of interest in their own right.

2. Higher-dimensional planes. In this section, we extend the previous results concerning lines in $\mathbb{R}^n$ to higher-dimensional hyperplanes in $\mathbb{R}^n$. By a *k-plane* we mean a translate of a k-dimensional subspace of $\mathbb{R}^n$. Let $\mathcal{H}_k$ be the collection of k-planes in $\mathbb{R}^n$ for $1 \le k \le n-1$. Let $h_{x_1,\ldots,x_m}$ or $\mathrm{Span}(x_1, \ldots, x_m)$ denote the smallest plane containing $x_1, \ldots, x_m$.

It was shown in [2] that, in ZF, the "one-point" partition property for lines in $\mathbb{R}^2$ (hence in $\mathbb{R}^n$, $n \ge 2$) is false. That is, there is a coloring $P : L \to \omega$, $L =$ the set of lines in $\mathbb{R}^2$, such that there is no $Q : \mathbb{R}^2 \to \omega$ such that $\forall l \in L$ $(|\{x \in \mathbb{R}^2 : x \in l \wedge Q(x) = P(l)\}| \le 1)$. It was also shown, in ZFC, that there is a set of lines and points in $\mathbb{R}^2$ of size ω_1 for which the one-point partition property fails.

We first extend these negative results to higher dimensions.

THEOREM 2.1. (ZF) *There is a coloring $P : \mathcal{H}_{n-1} \to \omega$ such that for all colorings $Q : \mathbb{R}^n \to \omega$ there is an $h \in \mathcal{H}_{n-1}$ such that* $\mathrm{Span}(\{x \in h : Q(x) = P(h)\}) = h$. *Also*, *any n hyperplanes with distinct P colors meet in at most a point.*

COROLLARY 1.1. (ZF) *There is a coloring $P : \mathcal{H}_{n-1} \to \omega$ such that there is no $Q : \mathbb{R}^n \to \omega$ such that $\forall h \in \mathcal{H}_{n-1}$ $(|\{x \in \mathbb{R}^n : x \in h \wedge Q(x) = P(h)\}| \le n-1)$. Also*, *any n hyperplanes with distinct P colors meet in at most a point.*

Proof. Let $v_1, v_2, v_3, \ldots \in S^{n-1}$ be "directions", and let $v_i \in N_i$ be neighborhoods of S^{n-1} which are pairwise disjoint, and assume that any n distinct vectors from distinct neighborhoods N_j are linearly independent.

Define P by $P(h) = i$ if $v_h \in N_i$, where v_h is the unit normal to h, and P is arbitrary otherwise. Suppose $Q : \mathbb{R}^n \to \omega$ is such that $\forall h \in \mathcal{H}_{n-1}$ $(\mathrm{Span}(\{x \in h : Q(x) = P(h)\}) \subsetneq h)$. We construct a sequence of open balls in $\mathbb{R}^n$, $B_0 \supseteq \overline{B}_1 \supseteq B_1 \supseteq \overline{B}_2 \supseteq B_2 \supseteq \ldots$, such that $B_k \cap \{x : Q(x) = k\} = \emptyset$ for all k. If $x \in \bigcap B_k$, we then have $Q(x) \ne k$ for any $k \in \omega$, a contradiction.

We use the following elementary fact from linear algebra.

LEMMA 2.1. *Let $v \in S^{n-1}$, $N \subseteq S^{n-1}$ an open neighborhood of v, $B \subseteq \mathbb{R}^n$ open*, *and $x_1, \ldots, x_p \in B$, $p \le n-1$, and suppose there is a hyperplane h containing $x_1, \ldots, x_p$ with normal $n_h \in N$. Then there is an open $B' \subseteq B$*

such that every $y \in B'$ lies on a hyperplane also containing $x_1, \ldots, x_p$, and with normal $n_y \in N$.

Set $B_{-1} = \mathbb{R}^n$. Suppose that B_k has been defined, and we define B_{k+1}. Let B'_k be open such that $\overline{B'_k} \subseteq B_k$. If there is no $x \in B'_k$ such that $Q(x) = k+1$, then we let $B_{k+1} = B'_k$. Otherwise let $x^1_{k+1} \in B'_k$, $Q(x^1_{k+1}) = k+1$. Let h_1 be a hyperplane through x^1_{k+1} with normal $n_1 \in N_{k+1}$. By the lemma, there is a ball $C \subseteq B'_k$ such that for all $y \in C$ there is a hyperplane containing x^1_{k+1}, y and with normal in N_{k+1}. If $C \cap \{x : Q(x) = k+1\} = \emptyset$, set $B_{k+1} = C$. Otherwise, let $x^2_{k+1} \in C$, $x^2_{k+1} \neq x^1_{k+1}$, with $Q(x^2_{k+1}) = k+1$, and let h_2 be a hyperplane containing x^1_{k+1}, x^2_{k+2} with normal $n_2 \in N_{k+1}$. Continuing, we define $x^1_{k+1} \neq x^2_{k+1} \neq \ldots \neq x^{n-1}_{k+1}$ (or else B_{k+1} has been defined). We may assume that C is chosen at each step to guarantee $x^{i+1}_{k+1} \notin \mathrm{Span}(x^1_{k+1}, \ldots, x^i_{k+1})$.

By the lemma again, we get $B_{k+1} \subseteq B'_k$ such that for all $y \in B_{k+1}$, there is a hyperplane containing $x^1_{k+1}, \ldots, x^{n-1}_{k+1}, y$ with normal in N_{k+1}. We may assume that for $y \in B_{k+1}$, $y \notin \mathrm{Span}(x^1_{k+1}, \ldots, x^{n-1}_{k+1})$. From the definition of P and the assumed property of Q, it follows that for any $y \in B_{k+1}$, $Q(y) \neq k+1$ (as the points $x^1_{k+1}, \ldots, x^{n-1}_{k+1}$ already span an $(n-2)$-dimensional plane). ∎

As with the case for lines, we can improve this negative result assuming ZFC.

THEOREM 2.2. (ZFC) *There are ω_1 hyperplanes $H = \{h_\alpha : \alpha < \omega_1\}$ in $\mathbb{R}^n$ and ω_1 points $\{x_\alpha : \alpha < \omega_1\}$ in $\mathbb{R}^n$, and a coloring $P : H \to \omega$ such that any n hyperplanes of distinct colors meet in at most a point, and such that for all $Q : \mathbb{R}^n \to \omega$ there is an $h \in \mathcal{H}_{n-1}$ such that* $\mathrm{Span}(\{x \in h : Q(x) = P(h)\}) = h$. *In particular, there is no coloring $Q : \{x_\alpha : \alpha < \omega_1\} \to \omega$ such that $\forall \alpha < \omega_1$ $(|\beta : x_\beta \in h_\alpha \wedge Q(x_\beta) = P(h_\alpha)| \leq n-1)$.*

Proof. We need the following lemma, which is a slight generalization of a theorem of Todorčević [6]. The proof is also a slight generalization of that proof.

LEMMA 2.2. (ZFC) *There is a partial coloring $P : D \to \omega$, $D \subseteq (\omega_1)^n$, such that for any $A \subseteq \omega_1$ of size ω_1, and any $k \in \omega$, $\exists \alpha_1 < \ldots < \alpha_n \in A$ $(P(\alpha_1, \ldots, \alpha_n) = k)$. Furthermore, if $P(\alpha_1, \ldots, \alpha_n) = k$, $P(\beta_1, \ldots, \beta_n) = l$ and $|\{\alpha_1, \ldots, \alpha_n\} \cap \{\beta_1, \ldots, \beta_n\}| \geq 2$, then $k = l$.*

Proof. We proceed by induction on n. For $n = 2$, this is just a result of [6] (and also follows from the argument here, ignoring $\overline{P}, \overline{D}$). Let $\omega_1 = S_0 \cup S_1 \cup S_2 \cup \ldots$, where the S_i are pairwise disjoint and stationary. By induction, let $\overline{P} : \overline{D} \to \omega$, where $\overline{D} \subseteq (\omega_1)^{n-1}$, satisfy the lemma for $n-1$. Following [6], let $r : \omega_1 \to 2^\omega$ be one-to-one, and $e_\alpha : \alpha \to \omega$ a bijection for all $\alpha < \omega_1$. Let $(\alpha, \beta) =$ the least n such that $r(\alpha)(n) \neq r(\beta)(n)$. Let $F_n(\alpha) = \{\beta < \alpha : e_\alpha(\beta) \leq n\}$. We set $P(\alpha_1, \ldots, \alpha_n) = k$ if and only if $\overline{P}(\alpha_1, \ldots, \alpha_{n-1}) = k$, and if $\beta_j = \min\{F_{(\alpha_j, \alpha_n)}(\alpha_n) - \alpha_j\}$, for $1 \leq j \leq n-1$, then $\beta_1 = \beta_2 = \ldots = \beta_{n-1} = \beta \in S_k$.

Let $A \subseteq \omega_1$, $|A| = \omega_1$, and $k \in \omega$. We must show that $\exists \alpha_1, \ldots, \alpha_n \in A$ $(P(\alpha_1, \ldots, \alpha_n) = k)$. Let λ be a sufficiently large regular cardinal. It is enough to show that if $M \prec V_\lambda$ is countable elementary, then M contains $\langle S_i; i \in \omega\rangle$, $\overline{P}$, A, and if $\delta = M \cap \omega_1$, then $\exists \alpha_1, \ldots, \alpha_n \in A$ $(\beta_1 = \ldots = \beta_{n-1} = \delta$ and $\overline{P}(\alpha_1, \ldots, \alpha_{n-1}) = k)$. Fix such δ, M, and let $\alpha_n \in A$, $\alpha_n > \delta$. Let n_0 be large enough such that $\delta \in F_{n_0}(\alpha_n)$. Let $n_1 \geq n_0$ be such that there are ω_1 many $\gamma \in A$ such that $r(\gamma){\restriction}n_1 = r(\alpha_n){\restriction}n_1$ but $a = r(\gamma)(n_1) \neq r(\alpha_n)(n_1)$. Let $\varepsilon < \delta$, $\varepsilon > \sup R_{n_1}(\alpha_n) \cap \delta$. Since $M \models$ "theorem is true for $n-1$ using $\overline{P}$", $\delta = \omega_1 \cap M$, and $M \models$ "$A \cap \{\gamma : r(\gamma){\restriction}n_1 = r(\alpha_n){\restriction}n_1 \wedge r(\gamma)(n_1) = a\}$ has size ω_1", let $\varepsilon < \alpha_1 < \ldots < \alpha_{n-1} < \delta$ be in A such that $\overline{P}(\alpha_1, \ldots, \alpha_{n-1}) = k$ and $r(\alpha_1){\restriction}n_1 = \ldots = r(\alpha_{n-1}){\restriction}n_1 = r(\alpha_n){\restriction}n_1$, $r(\alpha_1)(n_1) = \ldots = r(\alpha_{n-1})(n_1) = a \neq r(\alpha_n)(n_1)$. Then clearly $\beta_1 = \ldots = \beta_{n-1} = \delta$. ∎

If now we choose ω_1 points $\{x_\alpha : \alpha < \omega_1\}$ in $\mathbb{R}^n$ in sufficiently general position, then it is easy to see that for any n tuples $t_1 = (x^1_1, \ldots, x^1_n), \ldots, t_n = (x^n_1, \ldots, x^n_n)$ from the x_α such that $|t_i \cap t_j| \leq 1$ for all $i \neq j$, the n hyperplanes $h_1, \ldots, h_n$ determined by $t_1, \ldots, t_n$ satisfy $|h_1 \cap \ldots \cap h_n| \leq 1$. Also, for distinct $x_{\alpha_1}, \ldots, x_{\alpha_n}$, $\mathrm{Span}(x_{\alpha_1}, \ldots, x_{\alpha_n})$ is $(n-1)$-dimensional.

Fix such points $R = \{x_\alpha : \alpha < \omega_1\}$ in $\mathbb{R}^n$, and fix a function $P : D \to \omega$, $D \subseteq (\omega_1)^n$, as in Lemma 2.2. Consider the set H of hyperplanes $h_{x_{\alpha_1}, \ldots, x_{\alpha_n}}$ determined by $t = (\alpha_1, \ldots, \alpha_n) \in (\omega_1)^n$ such that $P(\alpha_1, \ldots, \alpha_n)$ is defined. Color these hyperplanes by $P(h_{x_{\alpha_1}, \ldots, x_{\alpha_n}}) = P(\alpha_1, \ldots, \alpha_n)$. Given n hyperplanes $h_1, \ldots, h_n \in H$ of distinct P color, by the lemma we deduce that the corresponding tuples of points $t_1, \ldots, t_n$ satisfy $|t_i \cap t_j| \leq 1$ for $i \neq j$. We then have $|h_1 \cap \ldots \cap h_n| \leq 1$ by the property of the x_{α_i}. Thus, any n of the hyperplanes in H of distinct P color meet in at most one point.

Suppose $Q : R \to \omega$ is a coloring of R. Fix $k \in \omega$ such that $\{\gamma : Q(x_\gamma) = k\}$ has size ω_1. By the lemma, there are $\gamma_1 < \ldots < \gamma_n$ such that $Q(\gamma_1) = \ldots = Q(\gamma_n) = k$, and $P(\gamma_1, \ldots, \gamma_n) = k$. Then $P(h_{x_{\gamma_1}, \ldots, x_{\gamma_n}}) = k$, and hence there is a hyperplane in $\mathcal{H}_{n-1}$ meeting n points of its color in R which span it. ∎

R e m a r k 2.1. It follows from Theorem 2.6 below that one cannot strengthen Theorem 2.2 for $n > 2$ by requiring that any n distinct hyperplanes in H meet in at most one point.

R e m a r k 2.2. Theorem 2.1 has an extension to Hilbert space as well: There is a coloring P of the co-dimension 1 planes in ℓ^2 such that for any $Q : \ell^2 \to \omega$ there is a plane h such that $\mathrm{cl}(\mathrm{Span}(\{x \in h : Q(x) = P(h)\})) = h$.

To see this, fix an orthonormal basis $N_0, N_1, \ldots \in \ell^2$ for ℓ^2. For h a hyperplane with unit normal n_h, let $i_h \in \omega$ be least such that $n_h \cdot N_{i_h} \neq 0$. Set $P(h) = i$ iff $n_h \cdot N_{i_h} \in U_i$, where $\{U_i\}$ are fixed, pairwise disjoint, open subsets of $(0,1)$ all having 0 as a limit point. Suppose $Q : \ell^2 \to \omega$ were such that $\forall h\ (h \neq \mathrm{cl}(\mathrm{Span}(\{x \in h : Q(x) = k+1\})))$. We follow the outline of Theorem 2.1. Suppose B_k has been defined, and let B'_k be open of diameter $< 2^{-k}$ such that $\overline{B'_k} \subseteq B_k$. If $\mathrm{cl}(\mathrm{Span}(\{x \in B'_k : Q(x) = k+1\})) \neq \ell^2$, then let $B_{k+1} \subseteq B'_k$, and $B_{k+1} \cap \mathrm{cl}(\mathrm{Span}(\{x \in B'_k : Q(x) = k+1\})) = \emptyset$. Otherwise, let H be a co-dimension 2 plane such that $H = \mathrm{cl}(\mathrm{Span}(\{x \in H \cap B'_k : Q(x) = k+1\}))$. Fixing an origin within H, we may identify H with a co-dimension 2 subspace of ℓ^2. Let x, y extend a basis for H to a basis for ℓ^2. Let j be least so that at least one of $x \cdot N_j$, $y \cdot N_j$ is non-zero. We may then find a unit vector of the form $n = \alpha x + \beta y$ so that $n \cdot N_j \in U_{k+1}$. Let h have normal n (and contain our new origin). Thus, $P(h) = k+1$. Also, there is an open $B_{k+1} \subseteq B'_k - H$ such that all $x \in B_{k+1}$ lie in a co-dimension 1 plane with normal $\alpha' x + \beta' y \in U_{k+1}$. From the assumed property of Q, $Q(x) \neq k+1$ for all $x \in B_{k+1}$. Continuing, we reach a contradiction.

We now consider the positive partition results for higher dimensions. First we extend Corollary 8 of [2] from lines in $\mathbb{R}^n$ to hyperplanes. Clearly, if there are hyperplanes of every color whose intersection contains a subspace of dimension ≥ 1, then there is no coloring of the points of this subspace such that every hyperplane meets only finitely many points of its color. Thus, restriction on the coloring P of the hyperplanes is necessary.

Definition 2.1. If $H \subseteq \bigcup_{k=1}^{n-1} \mathcal{H}_k$ and $P : H \to [\omega]^{<\omega}$, we say P is *acceptable* if $\forall x \neq y \in \mathbb{R}^n$ $(\bigcup\{P(h) : h \in H \wedge x, y \in h\}$ is finite).

Theorem 2.3. (ZFC) *Let $P : \bigcup_{k=1}^{n-1} \mathcal{H}_k \to \omega$ be an acceptable coloring of the k-planes, $1 \leq k \leq n-1$. Then there is a coloring $Q : \mathbb{R}^n \to \omega$ such that any $h \in \bigcup_{k-1}^{n-1} \mathcal{H}_k$ meets only finitely many points of its color.*

The following definition, and variations of it, will be used frequently.

Definition 2.2. If $A = H \cup S$, where $H \subseteq \bigcup_{k=1}^{n-1} \mathcal{H}_k$, $S \subseteq \mathbb{R}^n$, we say A is *good* provided:

(1) If $x_1, \ldots, x_n \in S$, then $h_{x_1,\ldots,x_n} \in H$.
(2) If $h_1, \ldots, h_p \in H$ and $|h_1 \cap \ldots \cap h_p| = 1$, then $h_1 \cap \ldots \cap h_p \in S$.

If $A = H \cup S \subseteq \bigcup_{k=1}^{n-1} \mathcal{H}_k \cup \mathbb{R}^n$ and $P : H \to [\omega]^{<\omega}$ is acceptable, then there is a good $A^1 \supseteq A$ such that $|A| = |A^1|$. Define $P^1 : A^1 \cap \bigcup_{k=1}^{n-1} \mathcal{H}_k \to [\omega]^{<\omega}$ by $P^1(h^1) = \bigcup\{P(h) : h \in H,\ h^1 \subseteq h\}$. Then P^1 is an acceptable coloring of A^1, and if $h \subseteq h'$, then $P(h') \subseteq P(h)$.

To prove Theorem 2.3, it thus suffices to prove the following lemma.

LEMMA 2.3. *Suppose* $A = H\cup S \subseteq \bigcup_{k=1}^{n-1}\mathcal{H}_k \cup \mathbb{R}^n$ *is good*, $P : H \to [\omega]^{<\omega}$ *is acceptable*, *and* $P(h') \subseteq P(h)$ *whenever* $h \subseteq h'$. *Suppose also* $g : S \to [\omega]^{<\omega}$ (*giving "forbidden colors"*) *is given. Then there is a* $Q : S \to \omega$ *such that* $\forall x \in S$ $(Q(x) \notin g(x))$ *and* $\forall h \in H$ $(\{x : x \in h \cap S \wedge Q(x) \in P(h)\}$ *is finite*).

Proof. We proceed by induction on $\kappa = |A|$. If $\kappa \le \omega$, the lemma is obvious (letting Q be 1-1 and avoiding g). If $|A| > \omega$, let $A = \bigcup_{\alpha<\kappa} A_\alpha$ be strictly increasing, where each $A_\alpha = H_\alpha \cup S_\alpha$ is good. Note that each $(H_\alpha, P\restriction H_\alpha)$ is also acceptable. Let $A_{<\alpha}$ denote $\bigcup_{\beta<\alpha} A_\beta$, and similarly for $H_{<\alpha}, S_{<\alpha}$. Suppose, inductively, that $Q\restriction S_{<\alpha}$ has been defined and $A_{<\alpha}$, $P\restriction H_{<\alpha}$, $Q\restriction S_{<\alpha}$ satisfy the conclusion of the lemma. For $x \in S_\alpha$ define

$$g'(x) = \begin{cases} g(x) \cup \bigcup\{P(h) : h \in H_{<\alpha} \text{ and } x \in h\} & \text{if } x \in S_\alpha - S_{<\alpha}, \\ g(x) & \text{if } x \in S_{<\alpha}. \end{cases}$$

By acceptability and goodness, $g'(x)$ is finite for all $x \in S_\alpha$. By induction, let A_α, $P\restriction H_\alpha$, Q'_α satisfy the conclusion of the lemma using g'.

Let

$$Q_\alpha(x) = \begin{cases} Q'_\alpha(x) & \text{if } x \in S_\alpha - S_{<\alpha}, \\ Q_{<\alpha}(x) & \text{if } x \in S_{<\alpha}. \end{cases}$$

Let $Q = \bigcup_{\alpha<\kappa} Q_\alpha$; we show Q satisfies the conclusion of the lemma for A, g. Clearly, if $x \in S$, then $Q(x) \notin g(x)$.

Let $h \in H_\alpha - H_{<\alpha}$, and suppose $x_1, x_2, x_3, \ldots$ are distinct points in S with $x_i \in h$ and $Q(x_i) \in P(h)$. Say, without loss of generality, $Q(x_i) = r$ for all i. If $x_i \notin S_\alpha$, then $Q(x_i) \notin P(h)$, since at the stage where $Q(x_i)$ is defined, we have $P(h) \subseteq g'(x_i)$. Also, by induction, only finitely many of the x_i are in $S_\alpha - S_{<\alpha}$. So assume without loss of generality that all $x_i \in S_{<\alpha}$. Let $\alpha_0 < \alpha$ be least such that at least two of the x_i are in A_{α_0}. Let $h_0 = \mathrm{Span}\{x_i : x_i \in S_{\alpha_0}\}$. Then $h_0 \in H_{\alpha_0}$, and $r \in P(h_0)$. By induction, only finitely many of the x_i lie in S_{α_0}.

However, if $x_i \in S_\alpha - S_{\alpha_0}$, then $x_i \notin h_0$, since otherwise at the stage $\beta > \alpha_0$ where $Q(x_i)$ is defined, $r \in g'(x_i)$. Let $\alpha_1 > \alpha_0$ be least such that some $x_i \in S_{\alpha_1} - S_{<\alpha_1}$. Let $h_1 = \mathrm{Span}\{x_i : x_i \in S_{\alpha_1}\}$. By induction, only finitely many of the x_i lie in S_{α_1}. Continuing, we produce $h_0 \subsetneq h_1 \subsetneq \ldots \subseteq h$, a contradiction. ∎

Theorem 2.3 implies a result concerning simultaneous colorings of the points and lines.

THEOREM 2.4. (ZFC) *Let* $P : \bigcup_{k=m}^{n-1}\mathcal{H}_k \to \omega$ *be an acceptable coloring of the* k-*planes in* $\mathbb{R}^n$, $m \le k \le n-1$. *Then there is a coloring* $Q : \mathbb{R}^n \cup \bigcup_{k=1}^{m-1}\mathcal{H}_k \to \omega$ *such that any* $h \in \bigcup_{k=1}^{n-1}\mathcal{H}_k$ *meets only finitely many points of its color*, *and contains only finitely many* $h' \in \bigcup_{k=1}^{m-1}\mathcal{H}_k$ *of its color.*

P r o o f. Let $P : \bigcup_{k=m}^{n-1} \mathcal{H}_k \to \omega$ be an acceptable coloring. Extend P to $P' : \bigcup_{k=1}^{n-1} \mathcal{H}_k \to [\omega]^{<\omega}$ by $P'(h') = \bigcup\{P(h) + i : \dim(h) \geq m,\ h' \subseteq h,\ 0 \leq i \leq m - \dim(h')\}$. Easily, P' is acceptable, and $h_1 \subseteq h_2$ implies $P'(h_2) \subseteq P'(h_1)$. Define Q on $\bigcup_{k=1}^{m-1} \mathcal{H}_k$ by defining, for $h' \in \bigcup_{k=1}^{m-1} \mathcal{H}_k$, $Q(h') = \sup(P'(h'))$. Lemma 2.3 extends Q to $\mathbb{R}^n$ so that $\forall h \in \bigcup_{k=1}^{n-1} \mathcal{H}_k$ ($\{x \in h : Q(x) \in P'(h)\}$ is finite). Note also that if $h \in \bigcup_{k=1}^{n} \mathcal{H}_k$, then h properly contains no $h' \in \bigcup_{k=1}^{m-1} \mathcal{H}_k$ with $Q(h') \in P'(h)$. ■

The next theorem strengthens the previous one in that we may prescribe the cardinality of the intersections of the planes with points of same color (with "finite" as a lower bound).

THEOREM 2.5. (ZFC) *Let $P : \bigcup_{k=1}^{n-1} \mathcal{H}_k \to \omega$ be a coloring of the planes in $\mathbb{R}^n$ which is acceptable. Let $c : \bigcup_{k=1}^{n-1} \mathcal{H}_k \to \{-1\} \cup \{\alpha \in \mathrm{ON} : \omega_\alpha \leq c\}$ be such that if $h_1 \subseteq h_2$ and $P(h_1) = P(h_2)$, then $c(h_1) \leq c(h_2)$. Assume also that $c(h) \geq \sum_{h'} c(h')$, the sum ranging over $h' \subsetneq h$ such that $c(h') > -1$ and h' is c-minimal, that is, $\neg \exists h'' \subsetneq h'$ $(c(h'') = c(h'))$. Then there is a coloring $Q : \mathbb{R}^n \to \omega$ such that for all $h \in \bigcup_{k=1}^{n-1} H_k$, h meets exactly $\omega_{c(h)}$ many points x such that $Q(x) = P(h)$ (where ω_{-1} means "finite").*

As before, we proceed by showing a stronger, but more technical lemma.

LEMMA 2.4. *There is a function F which assigns to each $h \in \bigcup_{k=1}^{n-1} \mathcal{H}_k$ a set $F(h) \subseteq h$ of size 2^ω such that:*

(1) *If $h_1 \neq h_2$ then $F(h_1) \cap F(h_2) = \emptyset$.*
(2) *For all $h_1 \subsetneq h_2$, $h_1 \cap F(h_2)$ is finite.*

P r o o f. Let $\widetilde{F}(h) \subseteq h$ be a set of size 2^ω such that for all $h' \subsetneq h$, $h' \cap \widetilde{F}(h)$ is finite [may assume $h = \mathbb{R}^k$, in which case let $\widetilde{F}(h) =$ range of the map $t \to (t, t^2, \ldots, t^k)$]. Let h_α, $\alpha < 2^\omega$, be an enumeration of $\bigcup_{k=1}^{n-1} \mathcal{H}_k$. We define $F(h_\alpha) \subseteq \widetilde{F}(h_\alpha)$ by induction on α. Assume $F(h_{\alpha'})$ defined for all $\alpha' < \alpha$. For all $\alpha' < \alpha$, $F(h_{\alpha'}) \cap \widetilde{F}(h_\alpha)$ is finite using the fact that if $h_{\alpha'} \not\supseteq h_\alpha$ then $F(h_{\alpha'}) \cap \widetilde{F}(h_\alpha) \subseteq (h_{\alpha'} \cap h_\alpha) \cap \widetilde{F}(h_\alpha)$, and if $h_{\alpha'} \supsetneq h_\alpha$ then $F(h_{\alpha'}) \cap \widetilde{F}(h_\alpha) \subseteq h_\alpha \cap \widetilde{F}(h_{\alpha'})$. Thus, $\bigcup_{\alpha' < \alpha} F(h_{\alpha'}) \cap \widetilde{F}(h_\alpha)$ has size $< 2^\omega$, and we let $F(h_\alpha) = \widetilde{F}(h_\alpha) - \bigcup_{\alpha' < \alpha} F(h_{\alpha'})$. ■

The function F of Lemma 2.4 is fixed for the remainder of the paper. The next lemma immediately implies Theorem 2.5.

LEMMA 2.5. *Let $A = H \cup S \subseteq \bigcup_{k=1}^{n-1} \mathcal{H}_k \cup \mathbb{R}^n$ be good of size $\kappa \geq \omega$, and $P : H \to [\omega]^{<\omega}$ be acceptable. Assume that $\forall h \in H$ $(|F(h) \cap S| = \kappa)$. Let d be a (partial) function which assigns to $h \in H$ and $l \in P(h)$ a value $d(h, l) \in \{-1\} \cup \{\alpha \in \mathrm{ON} : \omega_\alpha \leq \kappa\}$ satisfying:*

(1) *If $h_1 \subseteq h_2$ and $d(h_1, l)$, $d(h_2, l)$ are defined, then $d(h_1, l) \leq d(h_2, l)$.*

(2) *For all h, l such that $d(h,l)$ is defined, if $d(h,l) > -1$ then*

$$\omega_{d(h,l)} \geq \sum_{\substack{h' \subsetneq h \\ h' \text{ is } l\text{-minimal}}} \omega_{d(h',l)}.$$

Here we say $h \in H$ is l-minimal if $d(h,l)$ is defined and $\neg\exists h' \subsetneq h$ $(d(h,l) = d(h',l))$. Also, we ignore terms in the sum of the form $\omega_{d(h',l)} = -1$.

Then there is a coloring $Q : S \to \omega$ such that $\forall h \in H$ $\forall l \in P(h)$ $(|\{x \in S \cap h : Q(x) = l\}| = \omega_{d(h,l)})$.

Proof. We may assume $h_1 \subseteq h_2 \to P(h_1) \supseteq P(h_2)$ for all $h_1, h_2 \in H$. Let F be as in Lemma 2.4, and we may assume (by considering $F(h) \cap S$) that $F(h) \subseteq h \cap S$, and $|F(h)| = \kappa$ for all $h \in H$. Fix a bijection $\alpha \to (\alpha_0, \alpha_1, k_\alpha)$ between κ and $\kappa^2 \times \omega$.

Write $A = \bigcup_{\alpha<\kappa} A_\alpha$, where:

(1) Each $A_\alpha = H_\alpha \cup S_\alpha$ is good and has size $\kappa_\alpha < \kappa$.

(2) For all $\alpha < \kappa$, if the α_0th plane h_{α_0} (in some fixed enumeration of H) is in $H_{<\alpha}$, then $\exists z \in S_\alpha - S_{<\alpha}$ $(z \in F(h_{\alpha_0}) - \bigcup\{h' : h' \in H_{<\alpha}, h' \not\supseteq h_{\alpha_0}\})$.

For each α as in (2), we pick a point $z_\alpha \in S_\alpha - S_{<\alpha}$ which is as in (2).

We define now $Q_\alpha = Q\restriction S_\alpha$ by induction on $\alpha < \kappa$. Assume $Q_{<\alpha}$ has been defined. Define $g_\alpha : S_\alpha \to [\omega]^{<\omega}$ by $g_\alpha(x) = \emptyset$ if $x \in S_{<\alpha}$, and for $x \in S_\alpha - S_{<\alpha}$, $g_\alpha(x) = \bigcup\{P(h') : h' \in H_{<\alpha},\ x \in h'\}$. By acceptability and goodness, $g_\alpha(x)$ is a finite set. From Lemma 2.3, let $\widetilde{Q}_\alpha$ be a coloring extending $Q_{<\alpha}$ of S_α such that $\forall x \in S_\alpha - S_{<\alpha}$ $(\widetilde{Q}_\alpha(x) \notin g_\alpha(x))$ and any $h \in H_\alpha$ meets only finitely many $x \in S_\alpha$ with $\widetilde{Q}_\alpha(x) \in P(h)$.

If z_α is not defined, we set $Q_\alpha = \widetilde{Q}_\alpha$. If z_α is defined, we also set $Q_\alpha = \widetilde{Q}_\alpha$ for all points except z_α. If $k_\alpha \notin P(h_{\alpha_0})$ or $d(h_{\alpha_0}, k_\alpha)$ is not defined, or if h_{α_0} is not k_α-minimal, we set $Q_\alpha(z_\alpha) = \widetilde{Q}_\alpha(z_\alpha)$. If h_{α_0} is k_α-minimal, and $|\{x \in h_{\alpha_0} \cap S_{<\alpha} : Q_{<\alpha}(x) = k_\alpha\}| = \omega_{d(h_{\alpha_0},k_\alpha)}$ then we set $Q_\alpha(z_\alpha) = \widetilde{Q}_\alpha(z_\alpha)$, and if $|\{x \in h_{\alpha_0} \cap S_{<\alpha} : Q_{<\alpha}(x) = k_\alpha\}| < \omega_{d(h_{\alpha_0},k_\alpha)}$ then we set $Q_\alpha(z_\alpha) = k_\alpha$.

To see this works, fix $\alpha < \kappa$, and $h \in H_\alpha - H_{<\alpha}$, and $l \in P(h)$ with $d(h,l)$ defined. We must show that $|\{x \in h \cap S : Q(x) = l\}| = \omega_{d(h,l)}$.

As in Lemma 2.3, there are only finitely many points $x \in h \cap S$ not of the form z_β with $Q(x) = l$. Thus, we need only consider points of the form z_β for some $\beta \neq \alpha$. Clearly, $|\{z_\beta : z_\beta \in h \wedge Q(z_\beta) = l\}| \geq \omega_{d(h,l)}$ as there are $\kappa \geq \omega_{d(h,l)}$ many β for which z_β is on $\widetilde{h}$ and $k_\beta = l$, where $\widetilde{h} \subseteq h$ is l-minimal.

Suppose $|\{z_\beta : z_\beta \in h \wedge Q(z_\beta) = l\}| > \omega_{d(h,l)}$. We assume h is chosen with $\dim(h)$ minimal. Thus, for all $h' \subsetneq h$ which are l-minimal, $|\{z_\beta : z_\beta \in h'$

$\wedge\, Q(z_\beta) = l\}| = \omega_{d(h',l)}$ and hence

$$\Big|\Big\{z_\beta : z_\beta \in \bigcup_{\substack{h' \subsetneq h \\ h' \text{ is } l\text{-minimal}}} h' \wedge Q(z_\beta) = l\Big\}\Big| \le \sum \omega_{d(h',l)} \le \omega_{d(h,l)}.$$

Thus, we need only consider z_β which do not lie in an l-minimal subspace h' of h. Then $z_\beta \in F(h')$ for some l-minimal h', and this h' is not a proper subspace of h. We may also assume $h' \neq h$ as easily $\le \omega_{d(h,l)}$ points in $S(h)$ have color l. Thus we may assume $h' \cap h$ is a proper subspace of h' for each z_β.

If $\beta > \alpha$, it then follows from the definition of z_β that $z_\beta \notin h$. So assume $\beta < \alpha$. Let $\beta_0 < \alpha$ be least such that two of the z_β, say z_1, z_2, are in S_{β_0}. Thus $h_{z_1,z_2} \in H_{\beta_0}$ by goodness. Easily, at most $\omega_{d(h,l)}$ many of the z_β of color l are in h_{z_1,z_2}. Let $\beta_1 < \alpha$ be least such that some such z_β, say z_3, lies in $S_{\beta_1} - h_{z_1,z_2}$. Thus, $h_{z_1,z_2,z_3} \in H_{\beta_1}$. Again, at most $\omega_{d(h,l)}$ many of the z_β of color l lie in h_{z_1,z_2,z_3}. Continuing, we produce $h_{z_1,z_2} \subsetneq h_{z_1,z_2,z_3} \subsetneq \ldots \subsetneq h$, a contradiction. ∎

As an immediate corollary we have:

Corollary 2.2. *Suppose $P : \mathcal{H}_k \to [\omega]^{<\omega}$ is an acceptable coloring of the k-planes in $\mathbb{R}^n$, and d assigns to each k-plane h and each $l \in P(h)$ a value $d(h,l) \in \{-1\} \cup \{\alpha : \omega_\alpha \le 2^\omega\}$. Then there is a coloring $Q : \mathbb{R}^n \to \omega$ such that $\forall h \in \mathcal{H}_k\ \forall l \in P(h)\ (|\{x : x \in h \wedge Q(x) = l\}| = \omega_{d(h,l)})$.*

Theorem 2.3 shows that the hypothesis of acceptability on the coloring of planes in $\mathbb{R}^n$ is enough to get a coloring of the points of $\mathbb{R}^n$ with the "finite intersection property". We turn now to the problem of getting a uniform bound for the finite size of their intersections, as discussed for lines in §1.

Before discussing the ZFC problem, however, we consider the corresponding results assuming bounds on 2^ω. The first theorem below uses a stronger hypothesis on the planes than acceptability, but gets a stronger bound. The hypothesis applies, for example, to a partition of planes perpendicular to a coordinate axis. The second theorem requires just acceptability.

We introduce some notation for the theorems. Suppose $H \subseteq \bigcup_{k=1}^{n-1} \mathcal{H}_k$ is a family of planes in $\mathbb{R}^n$, $P : H \to [\omega]^{<\omega}$, and d is a partial function from $\{(h,l) : h \in H,\ l \in P(h)\}$ to the cardinals. We say $h \in H$ is *l-minimal* if $d(h,l)$ is defined and $\neg\exists h' \subsetneq h\ (d(h',l) = d(h,l))$. If

$$\sum_{\substack{h' \subsetneq h \\ h' \text{ is } l\text{-minimal}}} d(h',l) \quad \text{is infinite,}$$

we define

$$\sideset{}{^*}\sum_{\substack{h' \subsetneq h \\ h' \text{ is } l\text{-minimal}}} d(h',l) = \sum_{\substack{h' \subsetneq h \\ h' \text{ is } l\text{-minimal}}} d(h',l).$$

Otherwise, we define

$$\sideset{}{^*}\sum_{\substack{h'\subsetneq h\\ h' \text{ is } l\text{-minimal}}} d(h',l)$$

to be the maximum size of

$$Z \subseteq \bigcup_{\substack{h'\subsetneq h\\ h' \text{ is } l\text{-minimal}}} h'$$

such that $|Z\cap h'| = d(h',l)$ for all l-minimal $h' \subseteq h$. For example, if $h = \mathbb{R}^2$, l_1, l_2, l_3 are three lines in $\mathbb{R}^2$ forming a triangle, and $d(l_1,l) = 3$, $d(l_2,l) = 3$, $d(l_3,l) = 3$, $d(l_1 \cap l_2, l) = 1$, then

$$\sideset{}{^*}\sum_{\substack{h'\subsetneq h\\ h' \text{ is } l\text{-minimal}}} d(h',l) = 8.$$

For all h, l,

$$\sideset{}{^*}\sum_{\substack{h'\subsetneq h\\ h' \text{ is } l\text{-minimal}}} d(h',l) \le \sum_{\substack{h'\subsetneq h\\ h' \text{ is } l\text{-minimal}}} d(h',l).$$

THEOREM 2.6. *Assume* $2^\omega \le \omega_m$.

(A) *Let* $H \subseteq \bigcup_{k=1}^{n-1} \mathcal{H}_k$ *be a family of planes in* $\mathbb{R}^n$ *such that the intersection of any infinite subset of* H *contains at most one point. Let* $P : H \to [\omega]^{<\omega}$. *Then there is a coloring* $Q : \mathbb{R}^n \to \omega$ *such that* $\forall h \in H\ \forall l \in P(h)$ (h *meets at most* $(m+1)$ *points in* $\mathbb{R}^n$ *of* Q *color* l).

(B) *Let* H *and* P *be as above. Let* d *be a partial function from* $\{(h,l) : h \in H,\ l \in P(h)\}$ *to the set of cardinals* $\ge m+1$ *and* $\le 2^\omega$. *Assume that if* $d(h,l)$ *is defined*, *then*

$$d(h,l) \ge (m+1) + \sideset{}{^*}\sum_{\substack{h'\subsetneq h\\ h' \text{ is } l\text{-minimal}}} d(h',l).$$

Then there is a coloring $Q : \mathbb{R}^n \to \omega$ *such that* $\forall h \in H\ \forall l \in P(h)$ (*if* $d(h,l)$ *is defined then* $|\{x \in h : Q(x) = l\}| = d(h,l)$).

Remark 2.3. The m-term in (B) may seem peculiar, but (B) is false assuming only

$$d(h,l) \ge \sideset{}{^*}\sum_{\substack{h'\subsetneq h\\ h' \text{ is } l\text{-minimal}}} d(h',l).$$

Theorem 2.6(A) follows from the following lemma.

Lemma 2.6. *Let* $A = H \cup S \subseteq \bigcup_{k=1}^{n-1} \mathcal{H}_k \cup \mathbb{R}^n$, $|A| = \omega_m$, *be such that the intersection of any infinite subset of* H *contains at most one point. Let* $P : H \to [\omega]^{<\omega}$, *and* $g : S \to [\omega]^{<\omega}$. *Then there is a coloring* $Q : S \to \omega$ *such that* $\forall x \in S$ $(Q(x) \notin g(x))$ *and* $\forall h \in H$ $\forall l \in P(h)$ $(h$ *meets at most* $m+1$ *points in* S *of* Q *color* $l)$. *Furthermore*, *if* $x_0 \in S$, $l_0 \in \omega$ *are fixed*, *and* $l_0 \notin g(x_0)$, *then there is a* Q *as above also satisfying* $Q(x_0) = l_0$.

The proof of Lemma 2.6 is exactly like that for lines (cf. Corollary 9 of [2]) so we omit it (the "furthermore" clause is trivial when $m = 0$; for $m > 0$, when writing $A = \bigcup_{\alpha<\omega_m} A_\alpha$, require that $x_0 \in A_0$ and proceed inductively).

Theorem 2.6(B) follows immediately from the following lemma.

Lemma 2.7. *Let*

$$A = H \cup S \subseteq \bigcup_{k=1}^{n-1} \mathcal{H}_k \cup \mathbb{R}^n, \qquad |A| = \omega_m.$$

Assume the intersection of infinitely many distinct planes in H *contains at most one point*, P, d *are as in* (B), *and* $\forall h \in H$ $(|F(h) \cap S| = \omega_m)$. *Then there is a* $Q : S \to \omega$ *as in the conclusion of* (B).

Proof. The lemma is true, but not needed, for $m = 0$ by a similar argument, which we therefore leave to the reader. So assume $m \geq 1$. Fix a bijection $\alpha \to (\alpha_0, \alpha_1, k_\alpha)$ between ω_m and $(\omega_m)^2 \times \omega$. Write $A = \bigcup_{\alpha<\omega_m} A_\alpha$ as an increasing union of sets $A_\alpha = H_\alpha \cup S_\alpha$ of size $< \omega_m$, where:

(1) Each A_α is good, which means here that if $x, y \in S_\alpha$ then the finitely many planes in H which contain x, y are also in H_α, and if $H_1, \ldots, H_p \in H_\alpha$ intersect in a point z, then $z \in S_\alpha$.

(2) If the α_0th plane h_{α_0} lies in $H_{<\alpha}$ then $\exists z_\alpha \in (S_\alpha - S_{<\alpha})$ $(z_\alpha \in F(h_{\alpha_0}) - \bigcup\{h' \in H_{<\alpha} : h' \not\supseteq h_{\alpha_0}\})$.

Assume $Q_{<\alpha}$ is defined, and we define Q_α.

Case I: z_α is not defined, $k_\alpha \notin P(h_{\alpha_0})$ or $d(h_{\alpha_0}, k_\alpha)$ is not defined. Let $g_\alpha(x) = \bigcup\{P(h) : h \in H_{<\alpha}, x \in h\}$ for $x \in S_\alpha - S_{<\alpha}$, and $g_\alpha(x) = \emptyset$ otherwise. Let $\widetilde{Q}_\alpha$ be the restriction to $S_\alpha - S_{<\alpha}$ of the coloring given by Lemma 2.6 applied to $H_\alpha, S_\alpha, g_\alpha$.

In the remaining cases, assume z_α, $d(h_{\alpha_0}, k_\alpha)$ are defined.

Case II: $d(h_{\alpha_0}, k_\alpha)$ is finite. For $h \in H_{<\alpha}$ let $r(h) = |\{x \in S_{<\alpha} : x \in h \wedge Q_{<\alpha}(x) = k_\alpha\}|$. If for all l-minimal $\widetilde{h}$ such that $\widetilde{h} \subsetneq h_{\alpha_0}$ we have $\widetilde{h} \in H_{<\alpha}$ and $r(\widetilde{h}) = d(\widetilde{h}, k_\alpha)$, and if $r(h_{\alpha_0}) < d(h_{\alpha_0}, k_\alpha)$, we let g_α be as in Case I, except we set $g_\alpha(z_\alpha) = \emptyset$. We then let $\widetilde{Q}_\alpha$ be given by Lemma 2.6

applied to H_α, S_α, g_α, requiring $\widetilde{Q}_\alpha(z_\alpha) = k_\alpha$. Otherwise, we define $\widetilde{Q}_\alpha$ as in Case I.

Case III: $d(h_{\alpha_0}, k_\alpha)$ is infinite. If $r(h_{\alpha_0}) < d(h_{\alpha_0}, k_\alpha)$ we let g_α be as in Case I, except we set $g_\alpha(z_\alpha) = \emptyset$. We let $\widetilde{Q}_\alpha$ be given by Lemma 2.6 applied to H_α, S_α, g_α, requiring $\widetilde{Q}_\alpha(z_\alpha) = k_\alpha$. If $r(h_{\alpha_0}) = d(h_{\alpha_0}, k_\alpha)$, we define $\widetilde{Q}_\alpha$ as in Case I.

To see this works, suppose $h \in H_\alpha - H_{<\alpha}$, $l \in P(h)$, and $d(h,l)$ is defined. We consider the case

$$\sum_{\substack{h' \subsetneq h \\ h' \text{ is } l\text{-minimal}}}^{*} d(h', l) \quad \text{is finite,}$$

the other case being similar but easier. Note that in all of the above cases, h meets at most m points in $S_\alpha - S_{<\alpha}$ of Q_α color l. Also, h contains at most one point $x \in S_{<\alpha}$ by goodness. Thus, h meets at most $m+1$ points in S_α of Q_α color l. Any $x \in h \cap (S - S_\alpha)$ of Q color l must be of the form z_β for some $\beta > \alpha$. An initial segment of these z_β, say $z_{\beta_1}, \ldots, z_{\beta_p}$, are such that $h_{(\beta_i)_0}$ is a proper l-minimal subspace of h. By induction on $\dim(h)$, we therefore have

$$|\{x \in h \cap S_{\beta_p} : Q(x) = l\}| \le (m+1) + \sum_{\substack{h' \subsetneq h \\ h' \text{ is } l\text{-minimal}}}^{*} d(h', l) \le d(h, l).$$

The only z_β for $\beta > \beta_p$ of Q color l which are added to h are such that $h_{\beta_0} = h$. It follows that $|\{x \in h \cap S : Q(x) = l\}| \le d(h,l)$.

We also easily have $|\{x \in h \cap S : Q(x) = l\}| \ge d(h,l)$, as there are κ many β such that $\beta_0 = \alpha$ and $k_\beta = l$. ∎

We now consider the second version of this theorem.

Theorem 2.7. *Assume* $2^\omega \le \omega_m$.

(A) *Let* $H \subseteq \bigcup_{k=1}^{n-1} \mathcal{H}_k$ *and* $P : H \to [\omega]^{<\omega}$ *be acceptable. Then there is a coloring* $Q : \mathbb{R}^n \to \omega$ *such that* $\forall h \in H$ $\forall l \in P(h)$ (h *meets at most* $\varrho(\dim(h), m)$ *many points of* Q *color* l), *where* $\varrho : \omega^+ \times \omega \to \omega^+$ *is defined by* $\varrho(a,0) = 1$, $\varrho(a,b) = \sum_{a' \le a} \varrho(a', b-1) + 1$.

(B) *Let* $H \subset \bigcup_{k=1}^{n-1} \mathcal{H}_k$ *and* $P : H \to [\omega]^{<\omega}$ *be acceptable. Suppose* d *is a partial function from* $\{(h,l) : h \in H,\ l \in P(h)\}$ *to the set of cardinals with*

$$2^\omega \ge d(h,l) \ge \varrho(\dim(h), m) + \sum_{\substack{h' \subsetneq h \\ h' \text{ is } l\text{-minimal}}}^{*} d(h', l).$$

Then there is a coloring $Q : \mathbb{R}^n \to \omega$ *such that* $\forall h \in H$ $\forall l \in P(h)$ (*if* $d(h,l)$ *is defined then* $|\{x \in h : Q(x) = l\}| = d(h,l)$).

The following table gives some values for the ϱ function.

	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$
$\dim(h)=1$	1	2	3	4	5
$\dim(h)=2$	1	3	6	10	15
$\dim(h)=3$	1	4	10	20	35
$\dim(h)=4$	1	5	15	35	70

Consider first Theorem 2.7(A). We may assume without loss of generality that $H=\bigcup_{k=1}^{n-1}\mathcal{H}_k$, and that if $h_1\subseteq h_2$ then $P(h_1)\supseteq P(h_2)$. If $A\subseteq \bigcup_{k=1}^{n-1}\mathcal{H}_k\cup\mathbb{R}^n$ we define A being good as in Definition 2.2. It now suffices to prove the following lemma.

Lemma 2.8. *Let $A=H\cup S\subseteq\bigcup_{k=1}^{n-1}\mathcal{H}_k\cup\mathbb{R}^n$ be good, $|A|\le\omega_k$, and $P:H\to[\omega]^{<\omega}$ be acceptable. Let $g:S\to[\omega]^{<\omega}$. Then there is a coloring $Q:S\to\omega$ such that $\forall x\in S\ (Q(x)\notin g(x))$ and $\forall h\in H\ \forall l\in P(h)\ (|\{x\in h\cap S:Q(x)=l\}|\le\varrho(\dim(h),k))$. Furthermore, if $x_0\in S$ and $l_0\notin g(x_0)$, then there is a Q as above with $Q(x_0)=l_0$.*

Proof. Write $A=\bigcup_{\alpha<\omega_k}A_\alpha$ as an increasing union of good sets $A_\alpha=H_\alpha\cup S_\alpha$, each of cardinality $<\omega_k$. Assume $Q_{<\alpha}$ is defined. Define g_α on $S_\alpha-S_{<\alpha}$ by $g_\alpha(x)=g(x)\cup\bigcup\{P(h):h\in H_{<\alpha},\ x\in h\}$, and set $g_\alpha=g$ on $S_{<\alpha}$. By induction, there is coloring $\widetilde{Q}_\alpha$ of $S_\alpha-S_{<\alpha}$ such that $\widetilde{Q}_\alpha(x)\notin g_\alpha(x)$ and $\forall h\in H_\alpha\ \forall l\in P(h)\ (|\{x\in S_\alpha-S_{<\alpha}:x\in h\wedge\widetilde{Q}_\alpha(x)=l\}|\le\varrho(\dim(h),k-1))$. Let $Q_\alpha=Q_{<\alpha}\cup\widetilde{Q}_\alpha$.

To see this works, fix $h\in H_\alpha-H_{<\alpha}$, $l\in P(h)$. There are at most $\varrho(\dim(h),k-1)$ points $x\in S_\alpha-S_{<\alpha}$ on h of color l. If $x\in h\cap(S-S_\alpha)$, then $Q_\alpha(x)\ne l$, since l was "forbidden" at the step where x was colored.

We consider $x\in S_{<\alpha}$. Let $e_0=\dim(h)$. Let $B=\{x\in S_{<\alpha}:x\in h\wedge Q(x)=l\}$. Let e_1 be the dimension of $\mathrm{Span}(B)$. Note that $e_1<e_0$ by goodness. Let $\alpha_1<\alpha$ be least such that $\mathrm{Span}(B\cap S_{\alpha_1})=\mathrm{Span}(B)$. Note that $\mathrm{Span}(B)\in H_{\alpha_1}$ and $l\in P(h)\subseteq P(\mathrm{Span}(B))$. By induction on α, there are at most $\varrho(e_1,k)$ many points $x\in\mathrm{Span}(B)\cap S$ of Q color l. Also, if $\alpha_1<\beta<\alpha$ and $x\in h\cap(S_\beta-S_{<\beta})$, then $x\in\mathrm{Span}(B)$ and so $Q(x)\ne l$. Thus, at most $\varrho(e_1,k)+\varrho(e_0,k-1)\le\varrho(e_0-1,k)+\varrho(e_0,k-1)=\varrho(e_0,k)$ many points $x\in S$ of Q color l lie on h (a minor variation is required when $e_0=1$).

If $x_0\in S$ and $l_0\notin g(x_0)$ are fixed, we again proceed as above, except we require $x_0\in S_0$, and use induction (when $k=0$ the result is easy). ■

Consider now Theorem 2.7(B). Let F be as in Lemma 2.4, and define being good as in Definition 2.2. It suffices to show the following lemma.

Lemma 2.9. *Suppose $A=H\cup S\subseteq\bigcup_{k=1}^{n-1}\mathcal{H}_k\cup\mathbb{R}^n$ is good of size $\le\omega_k$, $P:H\to[\omega]^{<\omega}$ is acceptable, d is a partial function from $\{(h,l):h\in H,\ l\in$*

$P(h)\}$ to the cardinals $\le \omega_k$,

$$d(h,l) \ge \varrho(\dim(h),k) + \sideset{}{^*}\sum_{\substack{h' \subsetneq h \\ h' \text{ is } l\text{-minimal}}} d(h',l),$$

and $\forall h \in H$ $(|F(h) \cap S| = \omega_k)$. Then there is a coloring $Q : S \to \omega$ such that $\forall h \in H$ $\forall l \in P(h)$ (if $d(h,l)$ is defined then $|\{x \in h \cap S : Q(x) = l\}| = d(h,l)$).

Proof. Let $\alpha \to (\alpha_0, \alpha_1, k_\alpha)$ be a bijection between ω_k and $\omega_k^2 \times \omega$. Write $A = \bigcup_{\alpha<\omega_k} A_\alpha$ as an increasing union of good sets $A_\alpha = H_\alpha \cup S_\alpha$, each of size $< \omega_k$, such that for all $\alpha < \omega_k$, if the α_0th plane h_{α_0} in H lies in $H_{<\alpha}$, then $\exists z_\alpha \in S_\alpha - S_{<\alpha}$ $(z_\alpha \in F(h_{\alpha_0}) - \bigcup\{h' : h' \not\supseteq h_{\alpha_0}, h' \in H_{<\alpha}\})$.

Assuming $Q_{<\alpha}$ is defined, we define Q_α exactly as in Lemma 2.7.

To see this works, fix $h \in H_\alpha - H_{<\alpha}$ and $l \in P(h)$ with $d(h,l)$ defined. We again consider the case $d(h,l)$ finite, as the other case is similar but easier. Let $B_1 = \{x \in h \cap S_{<\alpha} : Q_{<\alpha}(x) = l\}$. Let $\alpha_1 < \alpha$ be least such that $\mathrm{Span}(B_1 \cap S_{\alpha_1}) = \mathrm{Span}(B_1)$. Note that $\mathrm{Span}(B_1) \in H_{\alpha_1}$, $l \in P(\mathrm{Span}(B_1))$, and $e_1 = \dim(\mathrm{Span}(B_1)) < \dim(h) = e_0$. If $\alpha_1 < \beta < \alpha$, and $x \in h \cap (S_\beta - S_{<\beta})$ has Q color l, then $x = z_\beta$ and h_{β_0} is an l-minimal subspace of $\mathrm{Span}(B_1) \subseteq h$. Also, $|\{x \in h \cap (S_{\alpha_1} - S_{<\alpha_1}) : Q(x) = l\}| \le \varrho(e_1, k-1)$. Let $B_2 = \{x \in h \cap S_{<\alpha_1} : Q_{<\alpha}(x) = l\}$. Let $\alpha_2 < \alpha_1$ be least such that $\mathrm{Span}(B_2 \cap S_{\alpha_2}) = \mathrm{Span}(B_2)$, and let $e_2 = \dim(\mathrm{Span}(B_2))$. Thus, $e_2 < e_1$. If $\alpha_2 < \beta < \alpha_1$, and $x \in h \cap (S_\beta - S_{<\beta})$ has Q color l, then $x = z_\beta$ and h_{β_0} is an l-minimal subspace of $\mathrm{Span}(B_2) \subseteq h$. Also, $|\{x \in h \cap (S_{\alpha_2} - S_{<\alpha_2}) : Q(x) = l\}| \le \varrho(e_2, k-1)$. Continuing, let $C = \{x \in h \cap S : Q(x) = l\}$ $\cap((S_{\alpha_1} - S_{<\alpha_1}) \cup (S_{\alpha_2} - S_{<\alpha_2}) \cup \ldots)$. If $x \in h \cap (S - S_\alpha)$ has Q color l, then $x = z_\beta$ for some $\beta > \alpha$ such that h_{β_0} is an l-minimal subspace of h. An initial segment of these, say $z_{\beta_1}, \ldots, z_{\beta_p}$ are such that $h_{(\beta_i)_0}$ is a proper subspace of h. Thus we can write $\{x \in h \cap S_{\beta_p} : Q(x) = l\} = C \cup D$, where $|C| \le 1 + \varrho(1, k-1) + \ldots + \varrho(e_0 - 1, k-1) = \varrho(e_0, k)$, and every $x \in D$ lies in an l-minimal proper subspace of h. By induction on $\dim(h)$, it follows that

$$|D| \le \sideset{}{^*}\sum_{\substack{h' \subsetneq h \\ h' \text{ is } l\text{-minimal}}} d(h',l).$$

Thus,

$$|\{x \in h \cap S_{\beta_p} : Q(x) = l\}| \le \varrho(e_0,k) + \sideset{}{^*}\sum_{\substack{h' \subsetneq h \\ h' \text{ is } l\text{-minimal}}} d(h',l) \le d(h,l).$$

It follows easily that $|\{x \in h \cap S : Q(x) = l\}| = d(h,l)$. ∎

We now turn to consistency results for planes in $\mathbb{R}^n$.

THEOREM 2.8. *Assume ZFC + MA.*

(A) *Let* $H \subseteq \bigcup_{k=1}^{n-1} \mathcal{H}_k$ *and* $P : H \to [\omega]^{<\omega}$, *and assume that the intersection of any infinite subset of* H *contains at most one point. Then there is a* $Q : \mathbb{R}^n \to \omega$ *such that* $\forall h \in H\ \forall l \in P(h)\ (|\{x \in h : Q(x) = l\}| \le 3)$.

(B) *Let* $H \subseteq \bigcup_{k=1}^{n-1} \mathcal{H}_k$ *and* $P : H \to [\omega]^{<\omega}$ *be acceptable. Then there is a* $Q : \mathbb{R}^n \to \omega$ *such that* $\forall h \in H\ \forall l \in P(h)\ (|\{x \in h : Q(x) = l\}| \le 2^{\dim(h)+1} - 1)$.

The proof of Theorem 2.8(A) is entirely similar to that of Theorem 1.1, so we omit it.

LEMMA 2.10. *Assume ZFC + MA. Let* $H \subseteq \bigcup_{k=1}^{n-1} \mathcal{H}_k$, *and let* $P : H \to [\omega]^{<\omega}$ *be acceptable. Let* $S \subseteq \mathbb{R}^n$ *have size* $< 2^\omega$, *and let* $g : S \to [\omega]^{<\omega}$. *Then there is a* $Q : S \to \omega$ *such that* $\forall x \in S\ (Q(x) \notin g(x))$ *and* $\forall h \in H\ \forall l \in P(h)\ (|\{x \in h \cap S : Q(x) = l\}| \le 2^{\dim(h)})$.

Proof. From Theorem 2.3 and the argument of Lemma 1.2, we may assume that $\forall h \in H$ ($h \cap S$ is finite). We may further assume that $\forall x_1, \ldots, x_p \in S$ (if $h_{x_1,\ldots,x_p} \subseteq h \in H$, then $h_{x_1,\ldots,x_p} \in H$) and $\forall h_1 \subseteq h_2$ in H $(P(h_1) \supseteq P(h_2))$. Let $\mathbb{P} = \{(p, f) : p \in [S]^{<\omega},\ f : p \to \omega,\ \forall x \in p\ (f(x) \notin g(x)), \forall h \in H\ \forall l \in P(h)\ (|h \cap \{x \in p : f(x) = l\}| \le 2^{\dim(h)})\}$. As usual, set $(p_1, f_1) <_{\mathbb{P}} (p_2, f_2)$ iff $p_1 \supseteq p_2$ and $f_2 = f_1 \restriction p_2$. It suffices to show that $\mathbb{P}$ is c.c.c. Assume not, and let $(p_\alpha, f_\alpha), \alpha < \omega_1$, be an antichain. We may assume $|p_\alpha| = n_0$ for all $\alpha < \omega_1$, the p_α form a $\boldsymbol{\Delta}$-system with root $r \in [S]^{<\omega}$, and $\forall \alpha, \beta\ (p_\alpha \restriction r = p_\beta \restriction r)$. Consider then the first ω elements (p_n, f_n) of the antichain. By Ramsey's theorem, we may assume that for some $1 \le d_0 \le n-1$, $\forall i < j\ \exists h_{i,j} \in H\ \exists l_{i,j}\ (\dim(h_{i,j}) = d_0 \wedge |h_{i,j} \cap \{x \in p_i : f_i(x) = l_{i,j}\}| = l_1$ $\wedge |h_{i,j} \cap \{x \in p_j : f_j(x) = l_{i,j}\}| = l_2$, and $l_1 + l_2 > 2^{d_0}$), but for all $d < d_0$, $\forall i < j\ \forall h \in H\ \forall l \in P(h)\ (|h \cap \{x \in p_i \cup p_j : (f_i \cup f_j)(x) = l\} \le 2^d)$. We may further assume that $\forall i < j$ (the l_1 points in p_i have fixed ranks in $\ll \restriction p_i$) and similarly for the l_2 points in p_j, where $\ll$ denotes a fixed well-order of $\mathbb{R}^n$. We assume $l_1 \le l_2$, the other case being easier. Since $l_1 + l_2 > 2^{d_0}$, $l_2 > 2^{d_0 - 1}$. Fix a $j \in \omega$, and consider the planes $h_{1,j}, h_{2,j}, \ldots, h_{j-1,j}$. Let $h(j)$ be the span of the corresponding l_2 points in p_j. Since $l_{j-1,j} \in P(h_{j-1,j}) \subseteq P(h(j))$, and $l_2 > 2^{d_0 - 1}$, we must have $\dim(h(j)) = d_0$, and hence $h(j) = h_{1,j} = h_{2,j} = \ldots = h_{j-1,j}$. Let B_j be the span of the union of the l_1 points from $p_1, \ldots, p_j$. Let j be large enough so that $B_j = B_{j'}$ for all $j' > j$. However, $h(j)$ then contains infinitely many points of S, a contradiction. ∎

Proof of Theorem 2.8(B). Let H, P be as in (B), and let $A = H \cup \mathbb{R}^n$. We may assume $H = \bigcup_{k=1}^{n-1} \mathcal{H}_k$. Write $A = \bigcup_{\alpha < 2^\omega} A_\alpha$, where each $A_\alpha = H_\alpha \cup S_\alpha$ is good, and $|A_\alpha| < 2^\omega$. Assuming $Q_{<\alpha}$ defined, let $g(x) = \bigcup\{P(h) : h \in H_{<\alpha}, x \in h\}$ for $x \in S_\alpha - S_{<\alpha}$. Apply Lemma 2.10 to get a coloring $\widetilde{Q} : (S_\alpha - S_{<\alpha}) \to \omega$ such that $\forall x \in S_\alpha - S_{<\alpha}\ (\widetilde{Q}(x) \notin g_\alpha(x))$

and for any $h \in H_\alpha$ and $l \in P(h)$, h meets at most $2^{\dim(h)}$ points of $S_\alpha - S_{<\alpha}$ of color l. Let $Q_\alpha = Q_{<\alpha} \cup \widetilde{Q}_\alpha$. Easily, if $h \in H_\alpha - H_{<\alpha}$ and $l \in P(h)$, then h meets at most $1 + 2 + 2^2 + \ldots + 2^{\dim(h)} = 2^{\dim(h)+1} - 1$ many points in S of color l. ■

As for the case with lines, we conjecture that the CH result is consistent with ¬CH. That is:

Conjecture. *The following is consistent with ZFC + ¬CH. For any* $P : H \subseteq \bigcup_{k=1}^{n-1} \mathcal{H}_k \to \omega$ *which is acceptable, there is a* $Q : \mathbb{R}^n \to \omega$ *such that* $\forall h \in H\ \forall l \in P(h)\ (|\{x \in h : Q(x) = l\}| \leq \dim(h) + 1)$.

Notice that the gap between the CH results and those of Theorem 2.8 widens as $\dim(h)$ increases. Thus, for lines only the consistency of the 2-point property with ¬CH is open, but for 2-planes (and acceptable colorings), it is open for intersections of sizes 3, 4.

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