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An ordinal version of some applications of the classical interpolation theorem

by

Benoît Bossard (Paris)

Abstract. Let E be a Banach space with a separable dual. Zippin's theorem asserts that E embeds in a Banach space E_1 with a shrinking basis, and W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński have shown that E is a quotient of a Banach space E_2 with a shrinking basis. These two results use the interpolation theorem established by W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński. Here, we prove that the Szlenk indices of E_1 and E_2 can be controlled by the Szlenk index of E, where the Szlenk index is an ordinal index associated with a separable Banach space which provides a transfinite measure of the separability of the dual space.

Introduction. Let E be a Banach space with a separable dual. Zippin's theorem ([Z]) shows that E embeds in a Banach space E_1 with a shrinking basis, and in [D-F-J-P] it is shown that E is a quotient of a Banach space E_2 with a shrinking basis. These two results use the interpolation scheme of [D-F-J-P]. Close to the index introduced by W. Szlenk in [S], the *Szlenk index* of E, denoted by $\mathrm{Sz}(E)$, is defined by slicing the dual unit ball of E with w^*-open sets. Here, we show that we can control the Szlenk indices of E_1 and E_2 by the Szlenk index of E. More precisely, there exist universal maps $\varphi_1 : \omega_1 \to \omega_1$ and $\varphi_2 : \omega_1 \to \omega_1$ such that if $\mathrm{Sz}(E) \le \alpha < \omega_1$ then we can choose E_1 and E_2 with $\mathrm{Sz}(E_1) \le \varphi_1(\alpha)$ and $\mathrm{Sz}(E_2) \le \varphi_2(\alpha)$ (Theorems 3.1 and 4.2). We do not know φ_1 and φ_2 more precisely, in particular we do not know if φ_1 or φ_2 can be the identity map.

We use tools from descriptive set theory (see [K-L]) and some results from [B1] (see also [B2]). This study is closely related to the Borel regularity of the interpolation scheme of [D-F-J-P].

The first section is devoted to notations and recalls, and the second one to preliminary lemmas. In the third section, we prove that φ_1 exists, following [G-M-S] in the proof of Zippin's theorem. As a corollary, we obtain

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the control of Sz when embedding a reflexive separable space in a reflexive space with a basis.

In the fourth section, we prove the existence of φ_2.

This work answers some questions that were formulated by G. Godefroy, and the author would like to thank him for his invaluable suggestions and encouragement.

I. Notations and preliminaires. We will denote by $\omega = \{0, 1, 2, \ldots\}$ the first infinite ordinal, by ω^* the set $\omega \setminus \{0\}$, by ω_1 the first uncountable ordinal. Let A be a set. We will denote by A^ω (resp. $A^{<\omega}$) the set of all infinite (resp. finite) sequences in A, and by $\mathcal{P}_{\mathrm{f}}(A)$ the set of all finite subsets of A. If a is an element of A^ω or $A^{<\omega}$, we will write $a = (a_i)_i$, and when A is a topological space, $\overline{a} = \overline{\{a_i : i\}}$. Concatenation is denoted by $\frown$.

Let $C(I)$ be the Banach space of all continuous functions on the Cantor set $I = \{0, 1\}^\omega$. It is classical that every separable Banach space is isometric to a subspace of $C(I)$. Let X be a Banach space. Then B_X is its closed unit ball. If $A \subseteq X$, then $\mathrm{conv}(A)$ denotes its convex hull, $\mathrm{sp}(A)$ (resp. $\mathrm{sp}_{\mathbb{Q}}(A)$) the vector (resp. $\mathbb{Q}$-vector) space spanned by A, $\overline{\mathrm{conv}}(A)$ and $\overline{\mathrm{sp}}(A)$ their closures, $A^\perp$ the orthogonal of A and $\mathrm{diam}(A) = \sup\{\|x - y\| : x \in A, y \in A\}$. If $A \subseteq X^*$, then $\overline{A}^*$ denotes its w^*-closure.

If λ and $\mathbf{x}$ are finite or infinite sequences respectively in $\mathbb{R}$ and X, we will write $\lambda\mathbf{x} = \sum_i \lambda_i x_i$. If $\mathbf{x} \in X^\omega$ and $\mathbf{y} \in Y^\omega$ where Y is a Banach space, and $k \in [1, \infty)$, then $\mathbf{x} \overset{k}{\sim} \mathbf{y}$ will mean

$$\forall \lambda \in \mathbb{R}^{<\omega}, \qquad \frac{1}{k}\|\lambda\mathbf{x}\| \le \|\lambda\mathbf{y}\| \le k\|\lambda\mathbf{x}\|$$

and we will write $\mathbf{x} \sim \mathbf{y}$ if there exists some $k \in [1, \infty)$ such that $\mathbf{x} \overset{k}{\sim} \mathbf{y}$. If X and Y are isomorphic (resp. isometric), we will write $X \simeq Y$ (resp. $X \equiv Y$).

We recall the definition of the Szlenk index $\mathrm{Sz}(X)$ when X is a separable Banach space. Let F be a w^*-closed subset of B_{X^*}. For $\varepsilon > 0$, we set

$$F'_\varepsilon = \{x^* \in F : \text{for any } w^*\text{-neighborhood } V \text{ of } x^*,\ \mathrm{diam}(V \cap F) > \varepsilon\},$$
$$F_\varepsilon^{(0)} = F,$$

and we define by transfinite induction

$$F_\varepsilon^{(\alpha+1)} = (F_\varepsilon^{(\alpha)})'_\varepsilon \qquad \text{if } \alpha \text{ is a countable ordinal,}$$
$$F_\varepsilon^{(\alpha)} = \bigcap_{\beta<\alpha} F_\varepsilon^{(\beta)} \qquad \text{if } \alpha \text{ is a limit countable ordinal.}$$

Then we set

$$\zeta_\varepsilon(F) = \begin{cases} \inf\{\alpha : F_\varepsilon^{(\alpha)} = \emptyset\} & \text{if it exists,} \\ \omega_1 & \text{if not,} \end{cases}$$
$$\zeta(F) = \sup_{\varepsilon>0} \zeta_\varepsilon(F), \qquad \mathrm{Sz}(X) = \zeta(B_{X^*}).$$

If $X \simeq Y$, we have $\mathrm{Sz}(X) = \mathrm{Sz}(Y)$. It is classical (see [D-G-Z], Theorem I-5-2, for instance) that X is a Banach space with a separable dual iff $\mathrm{Sz}(X) < \omega_1$. It is not difficult to see that if Y is a Banach subspace of X with a finite codimension, then $\mathrm{Sz}(Y) = \mathrm{Sz}(X)$.

Let P be a Polish space, and $\mathcal{O}$ a basis of open subsets of P. We denote by $\mathcal{F}(P)$ the set of all closed subsets of P equipped with the Effros–Borel structure (i.e. the canonical Borel structure generated by the family $\{\{F \in \mathcal{F}(P) : F \cap O \neq \emptyset\} : O \in \mathcal{O}\}$ (see [C]). We have the following easy result (see [B-1], Lemma 2.6, for instance) where $\mathcal{SE} \subseteq \mathcal{F}(C(I))$ is the subset consisting of the Banach subspaces.

FACT 1.1. *The following subsets are Borel sets*:

(*i*) $\{(F,G) \in \mathcal{F}(P)^2 : F \subseteq G\}$,
(*ii*) $\{(x,F) \in P \times \mathcal{F}(P) : x \in F\}$,
(*iii*) $\{(\mathbf{x},F) \in P^\omega \times \mathcal{F}(P) : \overline{\mathbf{x}} = \mathbf{F}\}$,
(*iv*) $\{(\mathbf{x},X) \in C(I)^\omega \times \mathcal{SE} : \overline{\mathrm{sp}}(\mathbf{x}) = X\}$.

If in addition P is compact, the Effros–Borel structure of $\mathcal{F}(P)$ is generated by the Hausdorff topology, thus by the family

$$\{\{F \in \mathcal{F}(P) : F \subseteq O\} : O \in \mathcal{O}\}.$$

We use the notation $\boldsymbol{\Sigma}_1^1$ (resp. $\boldsymbol{\Pi}_1^1$) for analytic (resp. coanalytic) subsets and we refer to [K-L] for definitions and results in descriptive set theory.

Let $\mathcal{SE}$ (resp. $\mathcal{SE}(\ell_1)$) be the set of all closed vector subspaces of $C(I)$ (resp. ℓ_1). We will denote by $\mathbf{e} = (e_i)_{i\in\omega}$ the canonical basis of ℓ_1. If $H \in \mathcal{SE}(\ell_1)$ and $e \in \ell_1$, then $\overset{\bullet}{e}{}^H$ will be the class of e in ℓ_1/H, and $\overset{\bullet}{\mathbf{e}}{}^H = (\overset{\bullet}{e}{}_i^H)_{i\in\omega}$. It is a classical result that the spaces $(\ell_1/H)^*$ and $H^\perp$ are isometric and w^*-isomorphic via the map I_{s_H} defined by $I_{s_H}(y^*)(e) = y^*(\overset{\bullet}{e}{}^H)$ for $y^* \in (\ell_1/H)^*$ and $e \in \ell_1$.

We recall some results without proof (see [B1] or [B2]). The subset $\mathcal{SE}$ (resp. $\mathcal{SE}(\ell_1)$) is a Borel subset of $\mathcal{F}(C(I))$ (resp. $\mathcal{F}(\ell_1)$), thus a standard Borel space (i.e. its Borel structure is generated by a Polish topology).

We write $B_\infty = (B_{\ell_\infty}, w^*)$ and we fix a countable basis $\mathcal{O}_\infty = (O_n)_{n\in\omega}$ of open subsets of B_∞. We equip the set $\mathcal{K} = \mathcal{F}(B_\infty)$ with the Hausdorff topology and if α is a countable ordinal, we define

$$\mathcal{K}_\alpha = \{K \in \mathcal{K} : \zeta(K) \le \alpha\}, \qquad \mathcal{H}_\alpha = \{H \in \mathcal{SE}(\ell_1) : \mathrm{Sz}(\ell_1/H) \le \alpha\}.$$

If $H \in \mathcal{SE}(\ell_1)$, we define $K(H) = B_{H^\perp}$, and we have $\zeta(K(H)) = \mathrm{Sz}(\ell_1/H)$, thus $H \in \mathcal{H}_\alpha$ implies $K(H) \in \mathcal{K}_\alpha$. The index ζ is a $\boldsymbol{\Pi}_1^1$-rank on $\{K \in \mathcal{K} : \zeta(K) < \omega_1\} \in \boldsymbol{\Pi}_1^1$, the index Sz is a $\boldsymbol{\Pi}_1^1$-rank on $\{X \in \mathcal{SE} : \mathrm{Sz}(X) < \omega_1\} \in \boldsymbol{\Pi}_1^1$ and the index defined by $H \mapsto \mathrm{Sz}(\ell_1/H)$ is a $\boldsymbol{\Pi}_1^1$-rank on $\{H \in \mathcal{SE}(\ell_1) : \mathrm{Sz}(\ell_1/H) < \omega_1\} \in \boldsymbol{\Pi}_1^1$ (see [B1], Ch. 4, or [B2]). Here, we will use the following direct consequences.

Proposition 1.2. *Let α be a countable ordinal.*

(i) *The sets $\mathcal{K}_\alpha$ and $\mathcal{H}_\alpha$ are Borel sets, thus standard Borel spaces.*

(ii) *If $A \subseteq \{X \in \mathcal{SE} : \mathrm{Sz}(X) < \omega_1\}$ is $\boldsymbol{\Sigma}_1^1$, then there exists a countable ordinal β such that $\mathrm{Sz}(X) \le \beta$ for any $X \in A$.*

We recall the interpolation scheme of Davis–Figiel–Johnson–Pełczyński (see [D-F-J-P]). Let Y be a Banach space, and W a closed convex symmetric and bounded subset of Y. For every $n \in \omega$, $U_n(W)$ is $\overline{2^n W + 2^{-n} B_Y}$, and j_n is the gauge of $U_n(W)$. We denote by $Z(W)$ the vector subspace of Y consisting of those y's for which $\|y\|^2_{Z(W)} = \sum_{n\in\omega} j_n^2(y)$ is finite. Then $Z(W)$ equipped with the norm $\|\cdot\|_{Z(W)}$ is a Banach space containing W, and its unit ball is

$$C(W) = \{y \in Y : \|y\|_{Z(W)} \le 1\}.$$

Fact 1.3. (i) *If Y is a subspace of a Banach space X, then the results of the interpolation scheme in Y and in X starting from W is the same.*

(ii) *If $k \in [1,\infty)$, then the identity is an isomorphism between $Z(W)$ and $Z(kW)$.*

Proof. (i) For any $n \in \omega$, let ℓ_n be the gauge of $\overline{2^n W + 2^{-n} B_X}$ and $C' = \{y \in X : \sum_{n\in\omega} \ell_n^2(y) \le 1\}$. We have

$$C' \subseteq \bigcap_{n\in\omega} \overline{2^n W + 2^{-n} B_X} \subseteq \bigcap_{n\in\omega} \overline{\overline{\mathrm{sp}}(W) + 2^{-n} B_X} \subseteq \overline{\mathrm{sp}}(W) \subseteq Y.$$

Consequently, $C' = C(W)$, and (i) follows.

(ii) We have

$$2^n W + 2^{-n} B_Y \subseteq 2^n kW + 2^{-n} B_Y \subseteq k[2^n W + 2^{-n} B_Y].$$

Thus $C(W) \subseteq C(kW) \subseteq kC(W)$ and (ii) follows. ∎

Finally, let $\mathbf{x}$ be a basic sequence in a Banach space X. Then $\mathbf{x}$ is *shrinking* if $(\overline{\mathrm{sp}}(\mathbf{x}))^* = \overline{\mathrm{sp}}(\mathbf{x}^*)$, where $\mathbf{x}^*$ is the sequence of biorthogonal functionals of $\mathbf{x}$. And $\mathbf{x}$ is *boundedly complete* if $(\overline{\mathrm{sp}}(\mathbf{x}^*))^* = \overline{\mathrm{sp}}(\mathbf{x})$.

II. Some preliminary lemmas. We state some definitions and lemmas which will be useful in the following sections.

Lemma 2.1. *The map $H \mapsto K(H) = B_{H^\perp}$ from $\mathcal{SE}(\ell_1)$ into $\mathcal{K}$ is Borel.*

P r o o f. First we have

CLAIM. *The map $k : \ell_1^\omega \to \mathcal{K}$ defined by $k(\mathbf{w}) = B_{\mathbf{w}^\perp}$ is Borel.*

Indeed, let $O \in \mathcal{O}_\infty$ and $A(O) = \{\mathbf{w} : k(\mathbf{w}) \subseteq O\}$. We have

$$\begin{aligned} A(O) &= \{\mathbf{w} : \forall y \notin O,\ \exists n \in \omega, \exists \varepsilon \in Q^{*+},\ |y(w_n)| \geq 2\varepsilon\} \\ &= \{\mathbf{w} : \forall y \notin O,\ \exists m \in \omega,\ y \in O_m, \\ &\qquad \exists \varepsilon \in \mathbb{Q}^{*+},\ \exists n \in \omega,\ \forall y' \in O_m,\ |y'(w_n)| \geq \varepsilon\}. \end{aligned}$$

As ${}^cO$ is compact, we see that $\mathbf{w} \in A(O)$ iff there exists $I \in \mathcal{P}_{\mathrm{f}}(\omega)$ such that

(i) $\forall m \in I$, $\exists \varepsilon \in \mathbb{Q}^{*+}$, $\exists n \in \omega$, $\forall y \in O_m$, $|y(w_n)| \geq \varepsilon$,
(ii) ${}^cO \subseteq \bigcup_{m \in I} O_m$.

If m, ε and n are fixed, the set $\bigcap_{y \in O_m} \{\mathbf{w} : |y(w_n)| \geq \varepsilon\}$ is closed, thus $A(O)$ is Borel, and the claim follows.

With this claim and Fact 1.1(iii), the subset $\{(H, \mathbf{w}, K) : \overline{\mathbf{w}} = H, k(\mathbf{w}) = K\}$) of $\mathcal{SE}(\ell_1) \times \ell_1^\omega \times \mathcal{K}$ is Borel, therefore its projection $\{(H, K) \in \mathcal{SE}(\ell_1) \times \mathcal{K} : K = K(H)\}$ is $\boldsymbol{\Sigma}_1^1$, and thus Borel by the separation theorem. Lemma 2.1 follows. ∎

For every $K \in \mathcal{K}$ we define by transfinite induction $\{K_m^{\beta,n} : m, n \in \omega, \beta$ countable ordinal$\}$ as follows: for every $m, n \in \omega$, $K_m^{0,0} = K$, for $\beta < \omega_1$ and $n \in \omega$,

$$K_m^{\beta,n+1} = \begin{cases} K_m^{\beta,n} & \text{if } \operatorname{diam}(O_n \cap K_m^{\beta,n}) > 2^{-m}, \\ K_m^{\beta,n} \setminus O_n & \text{if not}, \end{cases}$$
$$K_m^{\beta+1,0} = \bigcap_{n \in \omega} K_m^{\beta,n},$$

and if β is a limit ordinal, $K_m^{\beta,0} = \bigcap_{\gamma < \beta} K_m^{\gamma,0}$.

Let α be a countable ordinal. If $K \in \mathcal{K}_\alpha$, then $\zeta(K) \leq \alpha$, and clearly for every $m \in \omega$ there exist $\beta < \alpha$ and $n \in \omega$ such that $K_m^{\beta,n} = \emptyset$. We have

LEMMA 2.2. *Let $m, n \in \omega$, and $\beta < \alpha$ fixed. The map $K \mapsto K_m^{\beta,n}$ from $\mathcal{K}_\alpha$ into $\mathcal{K}_\alpha$ is Borel.*

First we will use two lemmas.

LEMMA 2.3. *Let $m \in \omega$ and $O \in \mathcal{O}_\infty$. The map from $\mathcal{K}$ into $\mathcal{K}$ defined by*

$$K \mapsto K' = \begin{cases} K & \textit{if } \operatorname{diam}(K \cap O) > 2^{-m}, \\ K \setminus O & \textit{if not}, \end{cases}$$

is Borel.

LEMMA 2.4. *Let $\beta < \omega_1$. The map from $\mathcal{K}^\beta$ into $\mathcal{K}$ defined by $(K_\gamma)_{\gamma < \beta} \mapsto \bigcap_{\gamma < \beta} K_\gamma$ is Borel. In particular, the map $\mathcal{K}^2 \to \mathcal{K}$ defined by $(F, G) \mapsto F \cap G$ is Borel.*

Proof of Lemma 2.2. It follows from Lemmas 2.3 and 2.4 by transfinite induction. ∎

Proof of Lemma 2.3. Let $\Omega \in \mathcal{O}_\infty$. We have

$$\{K : K' \subseteq \Omega\} = \{K : K \subseteq \Omega\} \cup [\{K : K \setminus O \subseteq \Omega\} \cap \{K : \mathrm{diam}(K \cap O) \subseteq 2^{-m}\}].$$

Clearly, $\{K : K \subseteq \Omega\}$ is Borel, and so is $\{K : K \setminus O \subseteq \Omega\} = \{K : K \subseteq O \cup \Omega\}$.

Let

$$\mathcal{V} = \{(V_1, V_2) \in \mathcal{O}_\infty^2 : \forall (x_1^*, x_2^*) \in V_1 \times V_2,\ \|x_1^* - x_2^*\| > 2^{-m}\}.$$

Claim. *We have*

$$\{K : \mathrm{diam}(K \cap O) > 2^{-m}\} = \bigcup_{(V_1,V_2)\in\mathcal{V}} \{K : K \cap V_1 \cap O \neq \emptyset,\ K \cap V_2 \cap O \neq \emptyset\}.$$

By this claim, $\{K : \mathrm{diam}(K \cap O) \le 2^{-m}\}$ is a Borel set, thus so is $\{K : K' \subseteq \Omega\}$, and the lemma follows.

We prove the claim. Suppose $\mathrm{diam}(K \cap O) > 2^{-m}$. There exist $x^*, y^* \in K \cap O$ and $x \in B_{\ell_1}$ such that $(x^* - y^*)(x) > 2^{-m}$. Let $\lambda = x^*(x)$, $\mu = y^*(x)$ and $\varepsilon_1 > 0$ be such that $\lambda - \mu > \varepsilon + \varepsilon_1$. Then the two subsets of B_{ℓ_∞},

$$L_1 = \{z^* : z^*(x) > \lambda - \varepsilon_1/2\}, \quad L_2 = \{z^* : z^*(x) < \mu + \varepsilon_1/2\},$$

are w^*-open, and $x^* \in L_1$, $y^* \in L_2$, thus $K \cap L_1 \cap O \neq \emptyset$ and $K \cap L_2 \cap O \neq \emptyset$. If $x_1^* \in L_1$ and $x_2^* \in L_2$, we have

$$x_1^*(x) - x_2^*(x) > \lambda - \varepsilon_1/2 - \mu - \varepsilon_1/2 > \varepsilon + \varepsilon_1 - \varepsilon_1 = \varepsilon,$$

thus $\|x_1^* - x_2^*\| > \varepsilon$ and there exists $(V_1, V_2) \in \mathcal{V}$ such that $V_1 \subseteq L_1$ and $V_2 \subseteq L_2$. Consequently,

$$\{K : \mathrm{diam}(K \cap O) > 2^{-m}\} \subseteq \bigcup_{(V_1,V_2)\in\mathcal{V}} \{K : K \cap V_i \cap O \neq \emptyset,\ i \in \{1,2\}\}.$$

The other inclusion is clear and the claim is proved. ∎

Proof of Lemma 2.4. Let $\Omega \in \mathcal{O}_\infty$ and

$$h(\Omega) = \Big\{(K_\gamma)_{\gamma<\beta} : \bigcap_{\gamma<\beta} K_\gamma \subseteq \Omega\Big\}.$$

We have

$$\begin{aligned} h(\Omega) &= \{(K_\gamma)_{\gamma<\beta} : \forall \delta < \beta,\ \forall x \in K_\delta,\ x \in \Omega,\ \text{or } \exists \gamma' < \beta,\ x \notin K_{\gamma'}\} \\ &= \bigcap_{\delta<\beta} \{(K_\gamma)_{\gamma<\beta} : \forall x \in K_\delta,\ \exists n \in \omega,\ x \in O_n \text{ and} \\ &\qquad (O_n \subseteq \Omega \text{ or } \exists \gamma' < \beta,\ O_n \cap K_{\gamma'} = \emptyset)\}. \end{aligned}$$

As K_δ is compact, we obtain $(K_\gamma)_{\gamma<\beta} \in h(\Omega)$ if and only if for any $\delta < \beta$, there exists $J \in \mathcal{P}_{\mathrm{f}}(\omega)$ such that

(i) $K_\delta \subseteq \bigcup_{n\in J} O_n$,
(ii) $\forall n \in J$ such that $O_n \not\subseteq O$, $\exists \gamma' < \beta$, $O_n \cap K_{\gamma'} = \emptyset$.

It follows easily that $h(\Omega)$ is Borel and that proves the lemma. ∎

Let $\alpha < \omega_1$. The set

$$\mathcal{L}_\alpha = \{(K,F) \in \mathcal{K}^2 : \zeta(K) \le \alpha,\ F \subseteq K,\ F \neq \emptyset\}$$

is Borel (use Fact 1.1 and Proposition 1.2).

We now use the so-called "dessert selection" ([G-M-S]). With $(K,F) \in \mathcal{L}_\alpha$ we associate $s_K(F) \in F$ in the following way. For any $m \in \omega$, there exist $\beta < \alpha$ and $n \in \omega$ such that $K_m^{\beta,n} = \emptyset$, thus there exist $\alpha_0 < \alpha$ and $n_0 \in \omega$ such that $F \cap K_m^{\alpha_0,n_0} \neq \emptyset$ and $F \cap K_m^{\alpha_0,n_0+1} = \emptyset$. We write $\Lambda_m(K,F) = F \cap K_m^{\alpha_0,n_0}$. Then we have $\mathrm{diam}(\Lambda_m(K,F)) \le 2^{-m}$. By induction, we define $(\Sigma_m(K,F))_{m\in\omega}$ by

$$\Sigma_0(K,F) = F, \qquad \Sigma_{m+1}(K,F) = \Lambda_{m+1}(K, \Sigma_m(K,F)).$$

For any $m \in \omega$, $\Sigma_m(K,F) \neq \emptyset$ and $\mathrm{diam}(\Sigma_m(K,F)) \le 2^{-m}$, thus $\bigcap_{m\in\omega} \sum_m(K,F)$ has a single element that we denote by $s_K(F)$. We have

LEMMA 2.5. *The map from $\mathcal{L}_\alpha$ into B_∞ defined by $(K,F) \mapsto s_K(F)$ is Borel.*

Proof.

CLAIM. *Let $m \in \omega$. The map $\Lambda_m : \mathcal{L}_\alpha \to \mathcal{K}$ defined by $(K,F) \mapsto \Lambda_m(K,F)$ is Borel.*

Indeed, let $\Omega \in \mathcal{O}_\infty$. We have

$$\{(K,F) : \Lambda_m(K,F) \subseteq \Omega\} = \{(K,F) : \exists \beta < \alpha,\ \exists n \in \omega,\ F \cap K_m^{\beta,n} \neq \emptyset \text{ and } F \cap K_m^{\beta,n} \subseteq \Omega\}$$

and this last subset is Borel by Lemmas 2.2 and 2.4. The claim follows.

Then an induction proves that the map $\mathcal{L}_\alpha \ni (K,F) \mapsto \Sigma_m(K,F)$ is Borel, and by Lemma 2.4, so is the map defined by $\mathcal{L}_\alpha \ni (K,F) \mapsto \bigcap_{m\in\omega} \Sigma_m(K,F)$.

Consequently, if $O \in \mathcal{O}_\infty$, we have

$$\{(K,F) : s_K(F) \in O\} = \Big\{(K,F) : \bigcap_{m\in\omega} \Sigma_m(K,F) \subseteq O\Big\}$$

and this last subset is Borel. The lemma follows. ∎

If $\mathbf{x}$ is a basic sequence in a Banach space X, we denote by $\mathbf{x}^* = (x_i^*)_{i\in\omega}$ the sequence of its biorthogonal functionals.

LEMMA 2.6. *Let X be a separable Banach space, $S(X)$ the subset of X^ω consisting of basic sequences, and $\mathcal{B}(X)$ the subset of normalized bases, when X has a basis.*

(i) *The set $\mathcal{A}(X) = \{(\mathbf{x}, y) \in S(X) \times X : y \in \overline{\mathrm{sp}}(\mathbf{x})\}$ is Borel, and the map from this set into X, with $m \in \omega$ fixed, defined by $(\mathbf{x}, y) \mapsto \sum_{i\le m} x_i^*(y)x_i$ is Borel.*

(ii) *The set $\mathcal{B}(X)$ is Borel, thus a standard Borel space, and the map from $\mathcal{B}(X)$ into $(B_{X^*}, w^*)^\omega$ defined by $\mathbf{x} \mapsto \mathbf{x}^*$ is Borel.*

Proof. (i) First, $S(X)$ is Borel because

$$\mathbf{x} \in S(X) \Leftrightarrow \exists M \in \omega,\ \forall n, p \in \omega,\ \forall \lambda \in \mathbb{Q}^{<\omega},\ \Big\|\sum_{i=0}^{n} \lambda_i x_i\Big\| \le M\Big\|\sum_{i=0}^{n+p} \lambda_i x_i\Big\|.$$

Thus, by Fact 1.1, $\mathcal{A}(X)$ is Borel. In $\mathcal{A}(X) \times X \times \mathbb{R}^\omega$, the subset $\{((\mathbf{x}, y), z, (y(i))_{i\in\omega}) : z = \sum_{i\le m} y(i)x_i$, and $\forall \varepsilon \in \mathbb{Q}^{*+},\ \exists N \in \omega,\ \forall n \ge N, \|\sum_{i\le n} y(i)x_i - y\| \le \varepsilon\}$ is clearly Borel. Consequently, its projection

$$\Big\{((\mathbf{x}, y), z) : z = \sum_{i\le m} x_i^*(y)x_i\Big\}$$

is $\boldsymbol{\Sigma}_1^1$, thus Borel by the separation theorem, and (i) is proved.

(ii) Let ξ be a dense sequence in X. Then $\mathbf{x} \in \mathcal{B}(X)$ iff $\mathbf{x} \in S(X)$, $\|x_i\| = 1$ for all $i \in \omega$ and

$$\forall \varepsilon \in \mathbb{Q}^{*+},\ \forall i \in \omega,\ \exists \lambda \in \mathbb{Q}^{<\omega},\quad \|\lambda\mathbf{x} - y_i\| \le \varepsilon.$$

It follows that $\mathcal{B}(X)$ is Borel.

Now, let $(\mathbf{x}^l)_{l\in\omega}$ be a sequence of elements of $\mathcal{B}(X)$, and $\mathbf{x} \in \mathcal{B}(X)$ such that $\mathbf{x}^l \to \mathbf{x}$ in X^ω. We are going to show that w^*-$\lim_l x_i^{l*} = x_i^*$ for every $i \in \omega$. As $\mathbf{x}$ is a basis, it is enough to show that

$$\lim_l |x_i^{l*}(\lambda\mathbf{x}) - x_i^*(\lambda\mathbf{x})| = 0$$

for any $\lambda \in \mathbb{Q}^{<\omega}$. We have

$$\begin{aligned}|x_i^{l*}(\lambda\mathbf{x}) - x_i^*(\lambda\mathbf{x})| &\le |x_i^{l*}(\lambda\mathbf{x}) - x_i^{l*}(\lambda\mathbf{x}^l)| + |x_i^{l*}(\lambda\mathbf{x}^l) - x_i^*(\lambda\mathbf{x})| \\ &\le \|\lambda\mathbf{x} - \lambda\mathbf{x}^l\| + |\lambda_i - \lambda_i|.\end{aligned}$$

As λ is a finite sequence, $\lim_l \|\lambda\mathbf{x} - \lambda\mathbf{x}^l\| = 0$, thus $\lim_l |x_i^{l*}(\lambda\mathbf{x}) - x_i^*(\lambda\mathbf{x})| = 0$ and w^*-$\lim_l x_i^{l*} = x_i^*$. The lemma follows. ■

III. On Zippin's theorem. In [Z], M. Zippin shows the following theorem:

THEOREM. *Every Banach space with a separable dual embeds in a Banach space with a shrinking basis.*

The aim of this section is to give a "quantitative" refinement of this theorem.

THEOREM 3.1. *There exists a universal map* $\varphi_1 : \omega_1 \to \omega_1$ *such that for every Banach space* E *with a separable dual and every countable ordinal* α, *if* $\mathrm{Sz}(E) \le \alpha$, *then* E *embeds in a Banach space* Z *with a shrinking basis which satisfies* $\mathrm{Sz}(Z) \le \varphi_1(\alpha)$.

We will follow the proof of Zippin's theorem given in [G-M-S] to which we refer for some results.

Let $f_0 \in C(I)$ be a fixed function that separates points in I, and $\mathbf{1}$ be the constant function which is equal to 1 everywhere. First we define a standard Borel space.

LEMMA 3.2. *Let* α *be a countable ordinal. In* $\mathcal{SE}(\ell_1) \times \ell_1^\omega \times C(I)^\omega \times \mathcal{SE}$ *the subset*

$$\mathcal{S}_\alpha = \{(H, \mathbf{h}, \mathbf{x}, X) : \mathrm{Sz}(X) \le \alpha,\ \overline{\mathrm{sp}}(\mathbf{x}) = X,\\ \overline{\mathrm{sp}}(\mathbf{h}) = H,\ \mathbf{x} \overset{1}{\sim} \overset{\bullet}{\mathbf{e}}{}^H,\ \mathbf{1} \in X,\ f_0 \in X\}$$

is Borel, *thus a standard Borel space.*

Proof. This is clearly a consequence of Fact 1.1, Proposition 1.2 and the following.

CLAIM. *In* $C(I)^\omega \times \ell_1^\omega$, *the subset* $A_1 = \{(\mathbf{x}, \mathbf{h}) : \mathbf{x} \overset{1}{\sim} \overset{\bullet}{\mathbf{e}}{}^H$ *with* $H = \overline{\mathrm{sp}}(\mathbf{h})\}$ *is Borel.*

Indeed, for $(\mathbf{x}, \mathbf{h}) \in C(I)^\omega \times \ell_1^\omega$, we have the equivalence: $(\mathbf{x}, \mathbf{h}) \in A_1$ if and only if for any $\lambda \in \mathbb{Q}^{<\omega}$, $\|\lambda \mathbf{x}\| = \|\lambda \overset{\bullet}{\mathbf{e}}{}^H\|$. Thus $(\mathbf{x}, \mathbf{h}) \in A_1$ if and only if for any $\lambda \in \mathbb{Q}^{<\omega}$,

(i) $\forall \mu \in \mathbb{Q}^{<\omega}$, $\|\lambda \mathbf{x}\| \le \|\lambda \mathbf{e} + \mu \mathbf{h}\|$,
(ii) $\forall \varepsilon \in \mathbb{Q}^{*+}$, $\exists \nu \in \mathbf{Q}^{<\omega}$, $\|\lambda \mathbf{e} + \nu \mathbf{h}\| \le \|\lambda \mathbf{x}\| + \varepsilon$.

Then it is not difficult to prove the claim, and the lemma follows. ∎

For $a \in \mathcal{S}_\alpha$, we write $a = (H(a), \mathbf{h}(a), \mathbf{x}(a), X(a))$ with $\mathbf{h}(a) = (h_i(a))_{i\in\omega}$ and $\mathbf{x}(a) = (x_i(a))_{i\in\omega}$. The proof of Theorem 3.1 is a straightforward consequence of the following central lemma which will be proved afterwards.

LEMMA 3.3. *Let* $\alpha < \omega_1$. *In the set* $\{Y \in \mathcal{SE} : Y$ *has a shrinking basis*$\}$, *there exists a* $\boldsymbol{\Sigma}_1^1$ *subset* $\mathcal{T}_\alpha$ *such that for any* $a \in \mathcal{S}_\alpha$, *there is some* $V \in \mathcal{T}_\alpha$ *in which* $X(a)$ *embeds.*

Proof of Theorem 3.1. For any $\alpha < \omega_1$, as $\mathcal{T}_\alpha \subseteq \{X \in \mathcal{SE} : \mathrm{Sz}(X) < \omega_1\}$, by Proposition 1.2 we can choose $\beta < \omega_1$ such that for any $V \in \mathcal{T}_\alpha$ we have $\mathrm{Sz}(V) \le \beta$ and we define φ_1 by $\varphi_1(\alpha) = \beta$. It remains to check that φ_1 satisfies the required conditions.

Let E be a separable Banach space such that $\mathrm{Sz}(E) \le \alpha$. We may suppose that $E \in \mathcal{SE}$, and we define $X(E) \in \mathcal{SE}$ by

$$X(E) = \{x + \lambda f_0 + \mu \mathbf{1} : x \in E,\ (\lambda, \mu) \in \mathbb{R}^2\}.$$

As in $X(E)$, $\mathrm{codim}(E) \le 2$, we have $\mathrm{Sz}(X(E)) = \mathrm{Sz}(E) \le \alpha$. There exists $H \in \mathcal{SE}(\ell_1)$ such that $X(E)$ is isometric to ℓ_1/H, thus there exists $a \in \mathcal{S}_\alpha$ such that $X(a) = X(E)$. Then by Lemma 3.3 there exists a Banach space $V \in \mathcal{T}_\alpha$ with a shrinking basis such that $\mathrm{Sz}(V) \le \varphi_1(\alpha)$, and in which $X(E)$ embeds, thus E too, and Theorem 3.1 is proved. ∎

The proof of Lemma 3.3 follows the proof of Zippin's theorem in [G-M-S].

Let α be a fixed countable ordinal, and $a = (H, \mathbf{h}, \mathbf{x}, X) \in \mathcal{S}_\alpha$. We denote by T_a, or T, the map from ℓ_1/H into $C(I)$ defined by $T(\overset{\bullet}{e}{}_i^H) = x_i$. Without proof we will use some results of the proof of Zippin's theorem given in [G-M-S] to obtain a Banach space $Z(a)$ with a shrinking basis in which X embeds isomorphically.

The set of Radon measures on I is denoted by $\mathcal{M}(I)$. As $\mathrm{Sz}(\ell_1/H) = \mathrm{Sz}(X) \le \alpha$ and $T^*(\mathcal{M}(I)) \subseteq (\ell_1/H)^*$, $T^*(\mathcal{M}(I))$ is separable. Since $f_0 \in X$ separates points in I, T^* is one-to-one on the set $\{\delta_t : t \in I\}$ of Dirac measures. Moreover, f_0 and $\mathbf{1}$ belong to $T(\ell_1/H) = X$. We consider on I the following metric:

$$\Delta(s,t) = \sup\{|\varphi(s) - \varphi(t)| : \varphi \in T(B_{\ell_1/H}) = B_X\} = \|T^*(\delta_s) - T^*(\delta_t)\|.$$

The w^*-topology of $(\ell_1/H)^*$ induces the usual topology of I via the map $I \ni t \mapsto T^*(\delta_t)$. Thus I is separable for the metric Δ, and for any $\varepsilon > 0$ every closed subset of I contains a non-empty relatively open subset with Δ-diameter less than ε (see [G-M-S]).

We denote by ψ_a or ψ the map from I into the unit ball of $(\ell_1/H)^*$ defined by $\psi(t) = T^*(\delta_t)$, and $\widetilde{\psi}_a = \widetilde{\psi} = I_{s_H} \circ \psi$. Then we have

$$\widetilde{\psi}^{-1}(K(H)) = \widetilde{\psi}^{-1}(B_{H^\perp}) = I.$$

Claim. *For any $m, n \in \omega$ and $\beta < \omega_1$, the subset*

$$D_m^{\beta,n} = \widetilde{\psi}^{-1}[(K(H))_m^{\beta,n}] \setminus \widetilde{\psi}^{-1}[(K(H))_m^{\beta,n+1}]$$

of I has a Δ-diameter less than 2^{-m}.

Indeed, let $s, t \in D_m^{\beta,n}$. Then

$$\Delta(s,t) = \|\psi(s) - \psi(t)\| = \|\widetilde{\psi}(s) - \widetilde{\psi}(t)\|.$$

As $\widetilde{\psi}(s)$ and $\widetilde{\psi}(t)$ belong to $(K(H))_m^{\beta,n} \setminus (K(H))_m^{\beta,n+1}$, we have $\Delta(s,t) \le 2^{-m}$.

With the definitions used in [G-M-S], it is easy to build a "Δ-fragmentation" $(f_m)_{m\in\omega}$, of I where, for any $m \in \omega$, f_m is a "well ordered slicing" defined from the set of difference sets $\{D_m^{\beta,n} : (\beta, n) \in A_m\}$ with

$A_m = \{(\beta, n) : D_m^{\beta,n} \neq \emptyset\}$ equipped with the lexicographical order. Then we consider the "dessert selection" ([G-M-S]) which associates with a closed subset $A \subseteq I$ an element $s_a(A) = s(A) \in A$ in the following way. For any $m \in \omega$, there exist $\beta < \alpha$ and $n \in \omega$ such that $A \cap \widetilde{\psi}^{-1}[(K(H))_m^{\beta,n}] \neq \emptyset$ and $A \cap \widetilde{\psi}^{-1}[(K(H))_m^{\beta,n+1}] = \emptyset$. We set $L_m(A) = A \cap \widetilde{\psi}^{-1}[(K(H))_m^{\beta,n}]$. We define $S_0(A) = A$ and by induction for any $m \in \omega$, $S_{m+1}(A) = L_{m+1}(S_m(A))$. Then $s(A)$ is the single element of $\bigcap_{m\in\omega} S_m(A)$. We will prove the next lemma later.

LEMMA 3.4. *Let A be a closed subset of I. The map from $\mathcal{S}_\alpha$ into I defined by $a \mapsto s_a(A)$ is Borel.*

For every $\sigma \in 2^{<\omega}$, we set $A_\sigma = \{t \in I : \sigma \prec t\}$. By a property of the "dessert selection" ([G-M-S], Theorem (A)), for $i \in \{0,1\}$, if $s(A_\sigma) \in A_{\sigma^\frown(i)}$, then $s(A_{\sigma^\frown(i)}) = s(A_\sigma)$. We define

$$\sigma^+ = \begin{cases} \sigma_a^+ = \sigma^\frown(1) & \text{if } s(A_{\sigma^\frown(0)}) = s(A_\sigma), \\ \sigma^\frown(0) & \text{if not.} \end{cases}$$

Let $(B_n(a))_{n\in\omega}$ be the sequence of elements of $\{A_\emptyset\} \cup \{A_{\sigma^+} : \sigma \in 2^{<\omega}\}$ equipped with the following order: for any $\sigma, \tau \in 2^{<\omega}$, A_σ is before A_τ if the length of σ is less than the length of τ, or if they have the same length and σ is before τ in the lexicographical order. In [G-M-S] it is shown that the sequence $\mathbf{b}(a) = (b_n(a))_{n\in\omega} = (\mathbf{1}_{B_n(a)})_{n\in\omega}$ is a monotone basis for $C(I)$.

LEMMA 3.5. *The map from $\mathcal{S}_\alpha$ into $C(I)^\omega$ defined by $a \mapsto b(a)$ is Borel.*

Proof. First by Lemma 3.4 the map from $\mathcal{S}_\alpha$ into the product space $\prod(A_\sigma : \sigma \in 2^{<\omega})$ defined by $a \mapsto (s_a(A_\sigma))_{\sigma\in 2^{<\omega}}$ is Borel.

CLAIM. *The map ξ from $\prod(A_\sigma : \sigma \in 2^{<\omega})$ into $(2^{<\omega})^{2^{<\omega}}$ defined by $(s_\sigma)_{\sigma\in 2^{<\omega}} \mapsto (s'_\sigma)_{\sigma\in 2^{<\omega}}$ with*

$$s'_\sigma = \begin{cases} \sigma^\frown(1) & \textit{if } s_{\sigma^\frown(0)} = s_\sigma, \\ \sigma^\frown(0) & \textit{if not,} \end{cases}$$

is Borel.

Indeed, fix $\tau, \tau' \in 2^{<\omega}$ and let $M = \{(s_\sigma)_{\sigma\in 2^{<w}} : s'_\tau = \tau'\}$. Then M is Borel because

- if $\tau' = \tau^\frown(0)$, then $M = \{(s_\sigma)_\sigma : s_{\tau^\frown(0)} \neq s_\tau\}$,
- if $\tau' = \tau^\frown(1)$, then $M = \{(s_\sigma)_\sigma : s_{\tau^\frown(0)} = s_\tau\}$,

and $M = \emptyset$ in the other situations. The claim follows.

The image of $(s_a(A_\sigma))_{\sigma\in 2^{<\omega}}$ by the map ξ is $(\sigma_a^+)_{\sigma\in 2^{<\omega}}$. The map $2^{<\omega} \ni \sigma \mapsto \mathbf{1}_{A_\sigma}$ is clearly Borel, thus so is the map $\mathcal{S}_\alpha \ni a \mapsto (b_n(a))_{n\in\omega}$. ■

Let $a = (H, \mathbf{h}, \mathbf{x}, X) \in \mathcal{S}_\alpha$. For any $i \in \omega$, we denote by $P_i = P_i(a)$ the projection from $C(I)$ onto $\mathrm{sp}(\{b_n(a) : n \leq i\})$ corresponding to the basis

$b(a)$ of $C(I)$, and we set

$$W(a) = \overline{\bigcup_{i\in\omega} P_i T(B_{\ell_1/H})} = \overline{\bigcup_{i\in\omega} P_i(B_X)}.$$

LEMMA 3.6. *The map from $\mathcal{S}_\alpha$ into $\mathcal{F}(C(I))$ defined by $a \mapsto W(a)$ is Borel.*

Proof. It is enough to show that the set $w(F) = \{a \in \mathcal{S}_\alpha : W(a) \subseteq F\}$ is a Borel subset when F is a closed subset of $C(I)$. We write $(b_n)_{n\in\omega} = (b_n(a))_{n\in\omega}$ and $(b_n^*)_{n\in\omega}$ is the sequence of the biorthogonal functionals. We have $a \in w(F)$ iff

$$\forall i \in \omega, \quad P_i(B_X) \subseteq F,$$

iff

$$\forall i \in \omega,\ \forall \lambda \in \mathbb{Q}^{<\omega}, \quad P_i(\lambda \mathbf{x}) \in F \text{ or } \|\lambda \mathbf{x}\| > 1,$$

that is to say, iff

$$\forall i \in \omega,\ \forall \lambda \in \mathbb{Q}^{<\omega}, \quad \sum_{n=0}^{i} b_n^*(\lambda \mathbf{x}) b_n \in F \text{ or } \|\lambda \mathbf{x}\| > 1.$$

By Lemmas 3.5 and 2.6, $w(F)$ is Borel, and Lemma 3.6 follows. ■

Now, we apply the interpolation scheme to $W(a)$. In [G-M-S], it is shown that $\mathbf{b}(a)$ defines a shrinking basis of the Banach space $Z(W(a)) = Z(a)$, and ℓ_1/H embeds in $Z(a)$, thus so does X. We denote by $(\widetilde{\mathbf{b}}_n(a))_{n\in\omega} = \widetilde{\mathbf{b}}(a)$ the sequence $\mathbf{b}(a)$ regarded as a basis of $Z(a)$. We define

$$\mathcal{T}_\alpha = \{V \in \mathcal{SE} : \exists a \in \mathcal{S}_\alpha,\ V \equiv Z(a)\}.$$

LEMMA 3.7. *In $\mathcal{SE}$, $\mathcal{T}_a$ is $\boldsymbol{\Sigma}_1^1$.*

End of proof of Lemma 3.3. For any $a \in \mathcal{S}_\alpha$, there exists $V \in \mathcal{SE}$ such that $V \equiv Z(a)$, thus $V \in \mathcal{T}_\alpha$ and $X(a)$ embeds in V. By Lemma 3.7, Lemma 3.3 is proved. ■

Proof of Lemma 3.7. The set $\mathcal{T}_\alpha$ is a projection of the following subset of $\mathcal{S}_\alpha \times C(I)^\omega \times \mathcal{SE}$:

$$R = \{(a, \mathbf{v}, V) : \overline{\mathrm{sp}}(\mathbf{v}) = V,\ \mathbf{v} \overset{1}{\sim} \widetilde{\mathbf{b}}(a)\}.$$

The following assertions (i), (ii) and (iii) are equivalent, where $j_n(a)$ is the gauge of $U_n(W(a)) = \overline{2^n W(a) + 2^{-n} B_{C(I)}}$:

(i) $\mathbf{v} \overset{1}{\sim} \widetilde{\mathbf{b}}(a)$,
(ii) $\forall \lambda \in \mathbb{Q}^{<\omega}$, $\|\lambda \mathbf{v}\| = \|\lambda \widetilde{\mathbf{b}}(a)\|$,
(iii) $\forall \lambda \in \mathbb{Q}^{<\omega}$,

$$\forall N \in \omega, \quad \sum_{n \le N} j_n^2(a)(\lambda \mathbf{b}(a)) \le \|\lambda \mathbf{v}\|,$$
$$\forall \varepsilon \in \mathbb{Q}^{*+},\ \exists M \in \omega, \quad \sum_{n \le M} j_n^2(a)(\lambda \mathbf{b}(a)) \le \|\lambda \mathbf{v}\| - \varepsilon.$$

CLAIM. *Let $N \in \omega$. The map $(a, y) \mapsto \sum_{n \le N} j_n^2(a)(y)$ from $\mathcal{S}_\alpha \times C(I)$ into $\mathbb{R}$ is Borel.*

By this claim, Lemma 3.5 and Fact 1.1, R is clearly Borel, thus $\mathcal{T}_\alpha$ is $\boldsymbol{\Sigma}_1^1$. We prove the claim. Let $r \in \mathbb{R}$. We have

$$\begin{aligned} A(r) &= \Big\{(a, y) : \sum_{n \le N} j_n^2(a)(y) < r\Big\} \\ &= \Big\{a \in \mathcal{S}_\alpha : \exists \mu \in \mathbb{Q}^{<\omega},\ \sum_n \mu_n^2 < r \text{ and } \forall n \le N,\ y \in \mu_n U_n(W(a))\Big\}. \end{aligned}$$

The map from $\mathcal{S}_\alpha$ into $\mathcal{F}(C(I))$ defined by $a \mapsto U_n(W(a))$ is Borel. Indeed, let O be an open subset of $C(I)$. We have

$$\begin{aligned} \{a : U_n(W(a)) \cap O = \emptyset\} &= \{a : (2^n W(a) + 2^{-n} B_{C(I)}) \cap O = \emptyset\} \\ &= \{a : W(a) \cap 2^{-n}(O + 2^{-n} B_{C(I)}) = \emptyset\}. \end{aligned}$$

Since $a \mapsto W(a)$ is Borel (Lemma 3.6), this last set is Borel, and $a \mapsto U_n(W(a))$ is Borel.

Consequently, by Fact 1.1, $A(r)$ is Borel and the claim follows. ■

Proof of Lemma 3.4. We fix a closed subset A of I, and let $a = (H, \mathbf{h}, \mathbf{x}, X) \in \mathcal{S}_\alpha$. For any $m \in \omega$, we have easily

$$L_m(A) = \widetilde{\psi}^{-1}(\Lambda_m(\widetilde{\psi}(A))), \quad S_m(A) = \widetilde{\psi}^{-1}(\Sigma_m(\widetilde{\psi}(A))).$$

Thus we obtain

$$s(A) = \widetilde{\psi}^{-1}[s_{K(H)}(\widetilde{\psi}(A))].$$

CLAIM. *The map from $\mathcal{S}_\alpha$ into $\mathcal{K}$ defined by $a \mapsto \widetilde{\psi}_a(A)$ is Borel.*

Indeed, let $(t_i)_{i\in\omega}$ be a dense sequence in A, and F be a w^*-closed subset of B_∞. Then

$$\{a : \widetilde{\psi}_a(A) \subseteq F\} = \bigcap_{i \in \omega} \{a : \widetilde{\psi}_\alpha(t_i) \in F\}.$$

The claim will be shown if we prove that for any $t \in I$ the map $a \mapsto \widetilde{\psi}_a(t)$ is Borel. Let $t \in I$. The Borel structure of B_∞ is generated by the subsets $\{f : f(e_j) \le \mu\}$ where $\mu \in \mathbb{R}$ and e_j is a vector of the canonical basis of ℓ_1. Then it suffices to show, for $\mu \in \mathbb{R}$ and e_j fixed, that the subset $\{a : \widetilde{\psi}_a(t)(e_j) \le \mu\}$ is Borel. For $a = (H, \mathbf{h}, \mathbf{x}, X)$, we have

$$\widetilde{\psi}(t)(e_j) = \psi(t)(\overset{\bullet}{e}{}_j^H) = T^*(\delta_t(\overset{\bullet}{e}{}_j^H)) = \delta_t(T(\overset{\bullet}{e}{}_j^H)) = \delta_t(x_j) = x_j(t).$$

Thus

$$\{a : \widetilde{\psi}_a(t)(e_j) \le \mu\} = \{a : x_j(t) \le \mu\}$$

and this last set is clearly a Borel set. The claim is proved.

Now, let F be a w^*-closed subset of B_∞ and $\ell(F) = \{a \in \mathcal{S}_a : s_a(A) \notin F\}$. Then

$$\ell(F) = \{a \in \mathcal{S}_\alpha : s_{K(H)}(\widetilde{\psi}_a(A)) \notin \widetilde{\psi}_a(F)\}.$$

Using the claim, Lemma 2.1, Lemma 2.5 and Fact 1.1, it is not difficult to see that $\ell(F)$ is a Borel set, and the lemma follows. ■

Using [D-F-J-P], Corollary 6, M. Zippin proves, as a corollary of his theorem, that a separable reflexive space embeds in a reflexive space with a basis ([Z]). Here we have

Corollary 3.8. *Let α be a countable ordinal. For every separable reflexive space E such that* $\mathrm{Sz}(E) \le \alpha$, *there exists a reflexive space Z with a basis such that* $\mathrm{Sz}(Z) \le \varphi_1(\alpha)$ *and E embeds in Z.*

Remark. With the notations of this corollary, $\mathrm{Sz}(Z)$ is controlled by $\mathrm{Sz}(E)$, whereas Sz is not a $\boldsymbol{\Pi}^1_1$-rank on the subset of reflexive separable subspaces of $C(I)$ with a basis. A $\boldsymbol{\Pi}^1_1$-rank on this set is $\sup(\mathrm{Sz}(Z), \mathrm{Sz}(Z^*))$ ([B1]), whereas the set of separable reflexive Banach subspaces of $C(I)$ with Szlenk index less than ω is not $\boldsymbol{\Sigma}^1_1$ (this follows from [L1], Proposition 4.3).

Proof of Corollary 3.8. Let Y be a separable reflexive Banach space such that $\mathrm{Sz}(Y) \le \alpha < \omega_1$. There exists $a = (H, \mathbf{h}, \mathbf{x}, X) \in \mathcal{S}_\alpha$ such that $X(E) \simeq \ell_1/H$, with the notation of the proof of Theorem 3.1. Clearly $X(E)$ is reflexive. As $\widetilde{\mathbf{b}}(a)$ is a shrinking basis of $Z(a), W(a)$ is $\sigma(Z(a), Z(a)^*)$-compact, thus $\sigma(C(I), C(I)^*)$-compact, and consequently $Z(a)$ is reflexive (use [D-F-J-P], Lemma 2 and Lemma 1(iv), (vii)). The corollary follows from Theorem 3.1. ■

As mentioned in the introduction, we do not know if φ_1 can be the identity map. It is shown in [L1], Proposition 3.1, (or see [L2]) that a Banach space is superreflexive iff its dentability index (an ordinal index close to the Szlenk index) is less than ω. We do not know if a separable superreflexive space embeds in a superreflexive space with a basis.

Some slight modifications of the proof of Theorem 3.1 allow one to show the following refinement of Theorem III.1 of [G-M-S].

Theorem 3.9. *There exists a universal map $\varphi'_1 : \omega_1 \to \omega_1$ which satisfies the following. If a bounded linear operator T from a separable Banach space X into $C(I)$ has an adjoint such that $\zeta(T^*(B_{X^*})) \le \alpha < \omega_1$, then T factors through a Banach space Z with a shrinking basis such that* $\mathrm{Sz}(Z) \le \varphi'_1(\alpha)$.

IV. On a result of W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński. In [D-F-J-P] (Corollary 8), the following result is shown:

THEOREM 4.1. *If E is a Banach space with a separable dual, then E is a quotient of a Banach space with a shrinking basis.*

Following a similar approach as in the third section we will give a "quantitative" refinement of this theorem.

THEOREM 4.2. *There exists a universal map $\varphi_2 : \omega_1 \to \omega_1$ such that for any Banach space E with a separable dual and for any countable ordinal α, if $\mathrm{Sz}(E) \le \alpha$, then E is a quotient of a Banach space X with a shrinking basis which satisfies $\mathrm{Sz}(X) \le \varphi_2(\alpha)$.*

Let $H \in \mathcal{SE}(\ell_1)$ and $E = \ell_1/H$ such that E^* is separable. Without proof, we give the scheme used in [D-F-J-P] to build a space X with a shrinking basis with E a quotient of X. We denote by Q_H the quotient map $\ell_1 \to E$. By Remark 4.10 of [J-R] there exists an element $\mathbf{x}$ of the set $\mathcal{B}(\ell_1)$ of normalized bases of ℓ_1 such that

$$Q_H^*(E) = H^\perp \subseteq L = \overline{\mathrm{sp}}^{\|\cdot\|}(\mathbf{x}^*).$$

Using the notations of Lemmas 2, 3, 1 of [D-F-J-P], we set $V = B_{H^\perp}$ and $V_S = V \cup (\bigcup_{m\in\omega} \pi_m^*(\mathbf{x})(V))$ where $\pi_m(\mathbf{x})$ is the natural projection from ℓ_1 onto the space $\mathrm{sp}\{x_i : i \le m\}$ and $\pi_m^*(\mathbf{x})$ is its dual. Then W is $\overline{\mathrm{conv}}^{\sigma(L,\ell_1)}(V_S)$ and $C = C(W)$. The subsets V, V_S, W and C of ℓ_∞ are w^*-compact. The subsequence $\mathbf{x}'$ of $\mathbf{x}^*$ formed by the elements of $\mathbf{x}^*$ which are in $Z(W)$ is a boundedly complete basis of $Z(W)$, and the sequence of the biorthogonal functionals of $\mathbf{x}'$ is a shrinking basis of a space X with E a quotient of X.

Let $\mathbf{x} \in \mathcal{B}(\ell_1)$. We denote by $c_{\mathbf{x}}$ the best basis constant of $\mathbf{x}$ and set

$$K(H) = B_{H^\perp},$$
$$K_S(H,\mathbf{x}) = \frac{1}{c_{\mathbf{x}}}\Big[K(H) \cup \Big(\bigcup_{m\in\omega} \pi_m^*(\mathbf{x})K(H)\Big)\Big],$$
$$W(K,\mathbf{x}) = \overline{\mathrm{conv}}^*(K_S(H,\mathbf{x})).$$

The constants used ensure that these three sets are subsets of B_∞, thus elements of $\mathcal{K}$. By Fact 1.3(ii), $C(W(H,\mathbf{x}))$ is the unit ball of a Banach space we denote by $Y(H,\mathbf{x})$, and this space is isomorphic to $Z(W)$. We denote by $\xi(H,\mathbf{x}) = (\xi_i(H,\mathbf{x}))_i$ the subsequence consisting of the elements of $\mathbf{x}^*$ which are in $Y(H,\mathbf{x})$.

As above, if $\mathbf{x}$ is such that $H^\perp \subseteq \overline{\mathrm{sp}}^{\|\cdot\|}(\mathbf{x}^*)$, then $\xi(H,\mathbf{x})$ is a boundedly complete basis of $Y(H,\mathbf{x})$ and the sequence $\xi^*(H,\mathbf{x}) = (\xi_i^*(H,\mathbf{x}))_i$ of its biorthogonal functionals is a shrinking basis of a space $X(H,\mathbf{x})$ with E as a quotient. Connected with this construction, we give three lemmas that will be proved later. Let α be a fixed countable ordinal. Here $\mathbf{y}$ is an element of $B_\infty^{\omega\times\alpha\times\omega}$, and we write

$$\mathbf{y} = \{y(n,\beta,m) : m,n \in \omega,\ \beta < \alpha\}.$$

We use some notations of the second section. In $\mathcal{K}_\alpha \times B_\infty^{\omega\times\alpha\times\omega}$, D_α is the subset consisting of the elements $(K,\mathbf{y})$ such that for any $m,n \in \omega$ and $\beta < \alpha$,

$$\text{if } K_m^{\beta,n} \neq K_m^{\beta,n+1},\ \text{then } y(n,\beta,m) \in K_m^{\beta,n} \setminus K_m^{\beta,n+1},$$

otherwise, $y(n,\beta,m) = 0$.

LEMMA 4.3. *The set $\mathcal{D}_\alpha$ is Borel.*

We set $\mathcal{D}_\alpha(K) = \{\mathbf{y} : K(\mathbf{y}) \in \mathcal{D}_\alpha\}$.

Remark 1. For any $K \in \mathcal{K}_\alpha$, $\mathcal{D}_\alpha(K) \neq \emptyset$.

Remark 2. As $\zeta(K) \le \alpha$, we easily show that if $\mathbf{y} \in \mathcal{D}_\alpha(K)$, then

$$K = \overline{\mathbf{y}}^{\|\cdot\|} = \overline{\{\mathbf{y}(\mathbf{n},\beta,\mathbf{m}) : \mathbf{n},\mathbf{m} \in \omega, \beta < \alpha\}}^{\|\cdot\|}.$$

We define in $\mathcal{H}_\alpha \times \mathcal{B}(\ell_1)$,

$$\mathcal{B}_\alpha = \{(H,\mathbf{x}) : \exists \mathbf{y} \in \mathcal{D}_\alpha(K(H)),\ \overline{\mathbf{y}}^{\|\cdot\|} \subseteq \overline{\mathrm{sp}}^{\|\cdot\|}(\mathbf{x}^*)\}.$$

LEMMA 4.4. *The set $\mathcal{B}_\alpha$ is a $\boldsymbol{\Sigma}_1^1$ subset.*

Remark 3. For any $H \in \mathcal{H}_\alpha$, the subset $\{\mathbf{x} \in \mathcal{B}(\ell_1) : (H,\mathbf{x}) \in \mathcal{B}_\alpha\}$ is non-empty. This is clear by Remark 4.10 of [J-R] and Remarks 1 and 2 above.

LEMMA 4.5. *The following subset of $\mathcal{SE}$ is $\boldsymbol{\Sigma}_1^1$:*

$$Q_\alpha = \{G : \exists (H,\mathbf{x}) \in \mathcal{B}_\alpha,\ G \equiv Z(H,\mathbf{x})\}.$$

Proof of Theorem 4.2. For any $\alpha < \omega_1$, we have $Q_\alpha \subseteq \{G \in \mathcal{SE} : G^* \text{ separable}\}$, and Q_α is $\boldsymbol{\Sigma}_1^1$. By Proposition 1.2, we can choose $\beta < \omega_1$ such that for any $G \in Q_\alpha$ we have $\mathrm{Sz}(G) \le \beta$. We define φ_2 by $\varphi_2(\alpha) = \beta$. It remains to check that φ_2 satisfies the required conditions. Let E be a separable Banach space such that $\mathrm{Sz}(E) \le \alpha$. For some $H \in \mathcal{H}_\alpha$, we have $E \simeq \ell_1/H$. By Remark 3 there exists $\mathbf{x} \in \mathcal{B}(\ell_1)$ such that $(H,\mathbf{x}) \in \mathcal{B}_\alpha$, and there exists $G \in \mathcal{SE}$ such that $G \equiv Z(H,\mathbf{x})$. Thus $G \in Q_\alpha$, $\mathrm{Sz}(G) \le \varphi_2(\alpha)$ and G is a Banach space with a shrinking basis with E a quotient of X. ∎

Proof of Lemma 4.3. We fix $n,m \in \omega$ and $\beta < \alpha$. We define the subset $\mathcal{D}_\alpha(n,\beta,m)$ of $\mathcal{K}_\alpha \times B_\infty^{\omega\times\alpha\times\omega}$ by: $(K,\mathbf{y}) \in \mathcal{D}_\alpha(n,\beta,m)$ iff

$$K_m^{\beta,n} = K_m^{\beta,n+1} \quad \text{and} \quad y(n,\beta,m) = 0, \quad \text{or}$$
$$K_m^{\beta,n} \neq K_m^{\beta,n+1} \quad \text{and} \quad y(n,\beta,m) \in K_m^{\beta,n} \setminus K_m^{\beta,n+1}.$$

Using Lemma 2.2 and Fact 1.1, it is not difficult to see that $\mathcal{D}_\alpha(n,\beta,m)$ is Borel, thus so is $\mathcal{D}_\alpha$. ∎

P r o o f   o f   L e m m a 4.4. The set $\mathcal{B}_\alpha$ is a projection of the following subset of $\mathcal{H}_\alpha \times \mathcal{B}(\ell_1) \times B_\infty^{\omega\times\alpha\times\omega} \times \mathcal{K}_\alpha$:

$$J_\alpha = \{(H,\mathbf{x},\mathbf{y},K) : K = K(H),\ (K,\mathbf{y}) \in \mathcal{D}_\alpha,\ \overline{\mathbf{y}}^{\|\cdot\|} \subseteq \overline{\mathrm{sp}}^{\|\cdot\|}(\mathbf{x}^*)\}.$$

Claim. *The subset* $\{(\mathbf{x}^*,\mathbf{y}) : \overline{\mathbf{y}}^{\|\cdot\|} \subseteq \overline{\mathrm{sp}}^{\|\cdot\|}(\mathbf{x}^*)\}$ *of* $B_\infty^\omega \times B_\infty^{\omega\times\alpha\times\omega}$ *is Borel.*

Indeed, this subset is equal to

$$\{(\mathbf{x}^*,\mathbf{y}) : \forall(n,\beta,m) \in \omega\times\alpha\times\omega,\ \forall\varepsilon \in \mathbb{Q}^{*+},\ \exists\lambda \in \mathbb{Q}^{<\omega},\ \|\lambda\mathbf{x}^* - y(n,\beta,m)\|_\infty \le \varepsilon\}.$$

As $\|\cdot\|_\infty$ is w^*-lower semicontinuous, the claim follows.

Now, using Lemmas 2.1, 2.6, 4.3 and this claim, we conclude that J_α is Borel, thus $\mathcal{B}_\alpha$ is $\boldsymbol{\Sigma}^1_1$. ∎

Before proving Lemma 4.5, we study the Borel regularity of some maps.

Lemma 4.6. *The map* $K \mapsto \overline{\mathrm{conv}}^*(K)$ *from* $\mathcal{K}$ *into* $\mathcal{K}$ *is Borel.*

P r o o f. We define

$$\Lambda = \Big\{\lambda \in \mathbb{Q}^{<\omega} : \sum_i \lambda_i = 1,\ \lambda_i \ge 0 \text{ for any } i \in \omega\Big\}.$$

We have

$$\begin{aligned}\{(K,\mathbf{k},F) \in \mathcal{K}\times B_\infty^\omega \times \mathcal{K} : \overline{\mathbf{k}}^* = K, \overline{\mathrm{conv}}^*(\mathbf{k}) = F\} \\ = \{(K,\mathbf{k},F) : \overline{\mathbf{k}}^* = K,\ \forall\lambda \in \Lambda,\ \lambda\mathbf{k} \in F, \\ \text{and } \forall O \in \mathcal{O}_\infty,\ O \cap F = \emptyset \text{ or } \exists\lambda \in \Lambda,\ \lambda\mathbf{k} \in O\}.\end{aligned}$$

This set is Borel by Fact 1.1, and its projection $\{(K,F) : \overline{\mathrm{conv}}^*(K) = F\}$ is $\boldsymbol{\Sigma}^1_1$, thus Borel by the separation theorem. Lemma 4.6 follows. ∎

Lemma 4.7. *The map* $(H,\mathbf{x}) \mapsto K_S(H,\mathbf{x})$ *from* $\mathcal{SE}(\ell_1)\times\mathcal{B}(\ell_1)$ *into* $\mathcal{K}$ *is Borel, and so is the map* $(H,\mathbf{x}) \mapsto W(H,\mathbf{x})$ *from* $\mathcal{SE}(\ell_1)\times\mathcal{B}(\ell_1)$ *into* $\mathcal{K}$.

P r o o f. First we prove that the map from $B_\infty^\omega \times \mathcal{B}(\ell_1)$ into $\mathcal{K}$ defined by

$$(\mathbf{k},\mathbf{x}) \mapsto L(\mathbf{k},\mathbf{x}) = \frac{1}{c_\mathbf{x}}\Big[\overline{\mathbf{k}}^* \cup \bigcup_{m\in\omega} \pi_m^*(\mathbf{x})(\overline{\mathbf{k}}^*)\Big]$$

is Borel. Let F be a w^*-closed subset of B_∞. We have

$$\{(\mathbf{k},\mathbf{x}) : L(\mathbf{k},\mathbf{x}) \subseteq F\} = \Big\{(\mathbf{k},\mathbf{x}) : \forall j \in \omega,\ \forall m \in \omega,\ \frac{1}{c_\mathbf{x}}\sum_{i\le m} k_j(x_i)x_i^* \in F\Big\}.$$

Claim. *Let* $n \in \omega$. *The map from* $B_\infty \times \mathcal{B}(\ell_1)$ *into* B_∞ *defined by*

$$(k,\mathbf{x}) \mapsto \frac{1}{c_\mathbf{x}}\sum_{i\le m} k(x_i)x_i^*$$

is Borel.

Indeed, the map $(k,\mathbf{x}) \mapsto \sum_{i\le m} k(x_i)x_i^*$ from $(\ell_\infty, w^*) \times \mathcal{B}(\ell_1)$ into (ℓ_∞, w^*) is Borel by Lemma 2.6(ii). For any $b\in\mathbb{R}$, we have

$$\{\mathbf{x} : c_{\mathbf{x}} \le b\} = \Big\{\mathbf{x} : \forall n,p\in\omega,\ \forall \mu\in\mathbb{Q}^{<\omega},\ \Big\|\sum_{i\le n}\mu_i x_i\Big\| \le b\Big\|\sum_{i\le n+p}\mu_i x_i\Big\|\Big\}$$

and this last subset is closed. Thus the map $\mathcal{B}(\ell_1)\ni \mathbf{x}\mapsto c_{\mathbf{x}}$ is Borel. The claim follows.

Now $\{(\mathbf{k},\mathbf{x}) : L(\mathbf{k},\mathbf{x}) \subseteq F\}$ is clearly Borel, and the map $(\mathbf{k},\mathbf{x}) \mapsto L(\mathbf{k},\mathbf{x})$ is Borel. Consequently, by Lemma 2.1 and Fact 1.1, the subset

$$\{(H,\mathbf{x},K,\mathbf{k},L) : L = L(\mathbf{k},\mathbf{x}),\ \overline{\mathbf{k}}^* = K = K(H))\}$$

of $\mathcal{SE}(\ell_1)\times\mathcal{B}(\ell_1)\times\mathcal{K}\times B_\infty^\omega\times\mathcal{K}$ is Borel, its projection $\{(H,\mathbf{x},K) : K = K_S(H,\mathbf{x})\}$ is $\boldsymbol{\Sigma}_1^1$, thus Borel by the separation theorem. Therefore the map $(H,\mathbf{x})\mapsto K_S(H,\mathbf{x})$ is Borel.

The last assertion of Lemma 4.7 follows from Lemma 4.6. ∎

We denote by $\mathcal{K}_{\mathrm{c}}$ the subset of $\mathcal{K}$ consisting of w^*-closed symmetric convex subsets of B_∞, and for all $n\in\omega$ we set

$$U'_n(W) = \frac{1}{2^{n+1}}U_n(W) = \frac{1}{2}(W + 2^{-2n}B_\infty).$$

Lemma 4.8. *The map from $\mathcal{K}_{\mathrm{c}}$ into $\mathcal{K}_{\mathrm{c}}^\omega$ defined by $W\mapsto (U'_n(W))_{n\in\omega}$ is Borel.*

Proof. Let F be a w^*-closed subset of B_∞ and $n\in\omega$. We have

$$\begin{aligned}\{W : U'_n(W)\cap F = \emptyset\} &= \{W : W\cap(2F + 2^{-2n}B_\infty) = \emptyset\}\\ &= \{W : W\cap[(2F+2^{-2n}B_\infty)\cap B_\infty] = \emptyset\}\end{aligned}$$

and this last subset is clearly Borel. The lemma follows. ∎

Lemma 4.9. *Let $m\in\omega$. The map f from $\mathcal{SE}(\ell_1)\times\mathcal{B}(\ell_1)$ into B_∞ defined by $f(H,\mathbf{x}) = \xi_m(H,\mathbf{x})$ if it exists, and 0 if not, is Borel.*

Proof. Let $O\in\mathcal{O}_\infty$. Then $(H,\mathbf{x})\in f^{-1}(O)$ iff (i) or (ii), where

(i) $\exists l_0 < l_1 < \ldots < l_{m-1} < l_m$, $x^*_{l_m}\in O$, $\sum_{n\in\omega} j_n^2(x^*_{l_i}) < \infty$ if $0\le i \le m$, $\sum_{n\in\omega} j_n^2(x^*_q) = \infty$ if $q < l_m$ and $q\notin\{l_i : 0\le i\le m-1\}$.

(ii) $0\in O$ and $\forall l_0 < l_1 < \ldots < l_{m-1} < l_m$, $\exists i$, $0\le i\le m$, $\sum_{n\in\omega} j_n^2(x^*_{l_i}) = \infty$, where j_n is the gauge of $U_n(W(H,\mathbf{x}))$.

Claim 4.10. *We fix $M\in\mathbb{R}, N\in\omega$. The following subset is Borel*:

$$\Big\{(H,\mathbf{x},y)\in\mathcal{SE}(\ell_1)\times\mathcal{B}(\ell_1)\times B_\infty : \sum_{n\le N} j_n^2(y) < M\Big\}.$$

Indeed, $\sum_{n\le N} j_n^2(y) < M$ iff $\exists\gamma\in\mathbb{Q}^{<\omega}$, $\sum_{n\le N}\gamma_n^2 < M$ and $y\in\gamma_n U_n(H,\mathbf{x})$ for all $n\le N$. As $U_n(H,\mathbf{x}) = 2^{n+1}U'_n(W(H,\mathbf{x}))$, by Lemma 4.7, Lemma 4.8 and Fact 1.1 the claim follows.

Now with Lemma 2.6 it is not difficult to see that $f^{-1}(O)$ is Borel, and Lemma 4.9 follows. ∎

Proof of Lemma 4.5. The subset Q_α is a projection of the subset

$$\mathcal{R}_\alpha = \{(G,\mathbf{g},H,\mathbf{x}) : \overline{\mathrm{sp}}(\mathbf{g}) = G,\ (H,\mathbf{x}) \in \mathcal{B}_\alpha,\ \mathbf{g} \overset{1}{\sim} \xi^*(H,\mathbf{x})\}$$

of $\mathcal{SE} \times C(I)^\omega \times \mathcal{H}_\alpha \times \mathcal{B}(\ell_1)$. The following assertions (i), (ii) and (iii) are equivalent:

(i) $\mathbf{g} \overset{1}{\sim} \xi^*(H,\mathbf{x})$.

(ii) $\forall\lambda \in \mathbb{Q}^{<\omega}$, $\|\lambda\mathbf{g}\| = \|\lambda\xi^*(H,\mathbf{x})\| = \sup\{\sum_i \lambda_i\mu_i : \mu \in \mathbb{Q}^{<\omega}, \sum_{n\in\omega} j_n^2(\mu\xi(H,\mathbf{x})) \le 1\}$ where j_n is the gauge of $U_n(W(H,\mathbf{x}))$.

(iii) $\forall\lambda \in \mathbb{Q}^{<\omega}$,

$$\forall\mu \in \mathbb{Q}^{<\omega}, \quad \sum_{n\in\omega} j_n^2(\mu\xi(H,\mathbf{x})) \le 1 \Rightarrow \sum_i \lambda_i\mu_i \le \|\lambda\mathbf{g}\|,$$

$$\forall\varepsilon \in \mathbb{Q}^{*+},\ \exists\nu \in \mathbb{Q}^{<\omega}, \quad \sum_{n\in\omega} j_n^2(\nu\xi(H,\mathbf{x})) \le 1 \text{ and } \|\lambda\mathbf{g}\| - \varepsilon \le \sum \lambda_i\mu_i.$$

By Lemma 4.8, Lemma 4.9 and Claim 4.10, (i) defines a Borel relation, and by Lemma 4.4 and Fact 1.1, $\mathcal{R}_\alpha$ is $\boldsymbol{\Sigma}_1^1$, and thus so is Q_α. ∎

Remark on the Borel regularity of the interpolation scheme. Let X be a separable Banach space, and $\mathcal{F}_{\mathrm{c}}(X)$ the subset of $\mathcal{F}(X)$ consisting of the bounded convex symmetric subsets, equipped with the Borel structure inherited from the Effros Borel structure. It is not clear whether the map $\mathcal{F}_{\mathrm{c}}(X) \ni W \mapsto C(W) \in \mathcal{F}_{\mathrm{c}}(X)$ is Borel, but we can prove that the map $\mathcal{F}_{\mathrm{c}}(X) \ni W \mapsto C'(W) = \overline{\mathrm{sp}(W) \cap C(W)}$ is Borel. The unit ball of the Banach subspace $Z'(W)$ of $Z(W)$ spanned by W is $C'(W)$, and $Z(W)$ has the same properties as $Z(W)$. Then we can prove that, if $A \subseteq \mathcal{F}_{\mathrm{c}}(X)$ is $\boldsymbol{\Sigma}_1^1$, then so is the subset $\{Z : \exists W \in A,\ Z \simeq Z'(W)\}$ of $\mathcal{SE}$.

Now let $\mathcal{K}(X)$ be the set of w^*-closed subsets of B_{X^*} equipped with the Hausdorff topology and $\mathcal{K}_{\mathrm{c}}(X) \subseteq \mathcal{K}(X)$ be the subset of w^*-closed convex symmetric subsets. Then we can prove that the map $\mathcal{K}_{\mathrm{c}}(X) \ni W \mapsto \frac{1}{2}C(W) \in \mathcal{K}_{\mathrm{c}}(X)$ is Borel.

References

[B1] B. Bossard, Mémoire de Thèse, Université Paris VI, 1994.

[B2] —, *Codage des espaces de Banach séparables. Familles analytiques ou coanalytiques d'espaces de Banach*, C. R. Acad. Sci. Paris 316 (1993), 1005–1010.

[C] J. P. R. Christensen, *Topology and Borel Structure*, North-Holland Math. Stud. 10, North-Holland, 1974.

[D-G-Z] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach spaces*, Pitman Monographs and Surveys 64, Longman, 1993.

[D-F-J-P] W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński, *Factoring weakly compact operators*, J. Funct. Anal. 17 (1974), 311–327.

[G-M-S] N. Ghoussoub, B. Maurey and W. Schachermayer, *Slicings, selections, and their applications*, Canad. J. Math. 44 (1992), 483–504.

[J-R] W. B. Johnson and H. P. Rosenthal, *On ω^*-basic sequences and their applications to the study of Banach spaces*, Studia Math. 43 (1972), 77–92.

[K-L] A. S. Kechris and A. Louveau, *Descriptive Set Theory and the Structure of Sets of Uniqueness*, London Math. Soc. Lecture Note Ser. 128, Cambridge Univ. Press, 1987.

[L1] G. Lancien, *Théorie de l'indice et problèmes de renormage en géométrie des espaces de Banach*, Thèse de doctorat de l'Université Paris VI, 1992.

[L2] —, *Dentability indices and locally uniformly convex renormings*, Rocky Mountain J. Math. 23 (1993), 635–647.

[S] W. Szlenk, *The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces*, Studia Math. 30 (1968), 53–61.

[Z] M. Zippin, *Banach spaces with separable duals*, Trans. Amer. Math. Soc. 310 (1988), 371–379.

Equipe d'Analyse
Université Paris VI
Boîte 186
4, place Jussieu
75252 Paris Cedex 05, France
E-mail: stelia@ccr.jussieu.fr

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