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Approximable dimension and acyclic resolutions

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Abstract. We establish the following characterization of the approximable dimension of the metric space X with respect to the commutative ring R with identity: a-$\dim_R X \le n$ *if and only if there exist a metric space* Z *of dimension at most* n *and a proper* UV^{n-1}-*mapping* $f : Z \to X$ *such that* $\check{H}^n(f^{-1}(x); R) = 0$ *for all* $x \in X$. As an application we obtain some fundamental results about the approximable dimension of metric spaces with respect to a commutative ring with identity, such as the subset theorem and the existence of a universal space. We also show that approximable dimension (with arbitrary coefficient group) is preserved under refinable mappings.

1. Introduction. Approximable dimension was introduced in [10], and is motivated by some results concerning cohomological dimension. The principal motivation for this concept is the following well-known result (and its proof; see [13, Section 6]) about cohomological dimension.

THEOREM 1. *A compactum* X *has cohomological dimension* $\le n$ *if and only if there exist a compactum* Z *of dimension at most* n *and a cell-like mapping* $f : Z \to X$.

In fact, the concepts of cohomological dimension and approximable dimension agree when the coefficient group is $\mathbb{Z}$ or $\mathbb{Z}_p$. Thus approximable dimension is appropriate for extending results about cohomological dimension with coefficients in $\mathbb{Z}$ or $\mathbb{Z}_p$ to other coefficient groups. As an example, consider the following result from [10, Theorem 7.1] about acyclic resolutions.

THEOREM 2. *Let* G *be an abelian group. If a* (*separable*) *metric space* X *has* a-$\dim_G X \le n$, *then there exist a* (*separable*) *metric space* Z *of*

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dimension at most n *and a proper* UV^{n-1}*-mapping* $f: Z \to X$ *such that* $\check{H}^n(f^{-1}(x); G) = 0$ *for all* $x \in X$.

As for the converse, the first author [8, Theorem C] gave the following characterization of the approximable dimension of a compactum with coefficients in a commutative ring with identity.

THEOREM 3. *Let* R *be a commutative ring with identity. A compactum* X *has* $\text{a-dim}_R X \le n$ *if and only if there exist a compactum* Z *of dimension at most* n *and a* UV^{n-1}*-mapping* $f: Z \to X$ *such that* $\check{H}^n(f^{-1}(x); R) = 0$ *for all* $x \in X$.

In Section 2 we will extend Theorem 3 to metric spaces. The corresponding extension of Theorem 1 was obtained by Rubin and Schapiro [12, Theorem 1.3]. Employing a well-known join-method (originally due to Hurewicz [6]), in Section 3 we prove that approximable dimension with coefficients in a commutative ring with identity is actually a dimension-function. In Sections 3 and 4 we obtain several fundamental results about approximable dimension as a dimension theory. For example, we establish the subset theorem and the existence of universal spaces. The main tool for all of this is the Characterization Theorem for approximable dimension (Corollary 1). We conclude by showing, in Section 5, that approximable dimension is preserved under refinable mappings.

The notations and definitions used in this paper may be found in [10]. Nevertheless, for convenience we shall repeat some of them here.

The notation "$\text{c-dim}_G X \le n$" means that every mapping $f: A \to K(G, n)$ of a closed subset A of the space X to an Eilenberg–MacLane space $K(G, n)$ admits an extension over X, and should be read as "*the cohomological dimension of* X *with respect to* G *is at most* n".

DEFINITION 1. Let P and Q be polyhedra. Let G be an abelian group, n be a natural number and $\mathcal{U}$ be an open cover of P. A mapping $\psi: Q \to P$ is $(G, n, \mathcal{U})$-*approximable* if there exists a triangulation T of P such that for any triangulation M of Q there is a mapping $\psi' : |M^{(n)}| \to |T^{(n)}|$ such that

(i) $d(\psi', \psi|_{|M^{(n)}|}) \le \mathcal{U}$, and

(ii) for any mapping $\alpha : |T^{(n)}| \to K(G, n)$, there exists an extension $\beta: Q \to K(G, n)$ of $\alpha \circ \psi'$.

In the above, the notation $d(\psi', \psi|_{|M^{(n)}|}) \le \mathcal{U}$ means that for any point $x \in |M^{(n)}|$, there exists $U \in \mathcal{U}$ containing both $\psi'(x)$ and $\psi(x)$. By a *polyhedron* we mean the space $|K|$ of a simplicial complex K with the Whitehead topology.

DEFINITION 2. A space X has *approximable dimension with respect to a coefficient group* G *of at most* n (abbreviated $\text{a-dim}_G X \le n$) if for every polyhedron P, mapping $f : X \to P$ and open cover $\mathcal{U}$ of P, there exist a polyhedron Q and mappings $\varphi : X \to Q$, $\psi : Q \to P$ such that

(i) $d(f, \psi \circ \varphi) \le \mathcal{U}$, and
(ii) ψ is $(G, n, \mathcal{U})$-approximable.

As regards the relationship between approximable dimension and cohomological dimension, the reader should see [10]. We refer the reader to [2] for the fundamentals of cohomological dimension theory.

Throughout this paper R shall denote a *commutative ring with identity* and G an *abelian group* unless otherwise noted.

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2. A characterization of approximable dimension of metric spaces

THEOREM 4. *If $f : Z \to X$ is a proper UV^{n-1}-mapping from the metric space Z of dimension at most n onto the metric space X such that $\check{H}^n(f^{-1}(x); R) = 0$ for all $x \in X$, then* a-$\dim_R X \le n$.

Proof. Suppose that Z is a closed subset of an ANR Y. Let us take a mapping $g : X \to P$ of X to a polyhedron P and an open cover $\mathcal{U}$ of P. We shall construct a polyhedron Q and mappings $\varphi : X \to Q$, $\psi : Q \to P$ such that $d(g, \psi \circ \varphi) \le \mathcal{U}$ and ψ is $(R, n, \mathrm{st}(\mathcal{U}))$-approximable, thereby showing that a-$\dim_R X \le n$.

Consider a triangulation T of P such that

$$\{\mathrm{st}(v, T) \mid v \in T^{(0)}\} \prec \mathcal{U}', \tag{1}$$

where $\mathcal{U}'$ is an open cover of P and $\mathcal{U}' \prec^* \mathcal{U}$. Here $\prec$ means "is a refinement of" and $\prec^*$ means "is a star refinement of". Since $\dim Z \le n$, there exists a mapping $h : Z \to |T^{(n)}|$ such that $d(h, g \circ f) \le \mathcal{U}'$. Then h can be extended over some neighborhood of Z in Y. Hence we may assume that h is a mapping of a neighborhood O of Z in Y to $|T^{(n)}|$ satisfying

$$d(h|_Z, g \circ f) \le \mathcal{U}'. \tag{2}$$

Let x be a point of X. Note that for each point $z \in O$, there exist an open neighborhood A of z in O and a compact subpolyhedron C of $|T^{(n)}|$ such that $h(A) \subset C$. By this and the compactness of $f^{-1}(x)$, there exist an open neighborhood O_x of $f^{-1}(x)$ in O and a compact subpolyhedron S_x of $|T^{(n)}|$ such that $h(O_x) \subset S_x$. Moreover, by (2) we may assume that

$$h(O_x) \subset U_x \quad \text{for some } U_x \in \mathcal{U}. \tag{3}$$

Because S_x is a compact polyhedron of dimension at most n, $\check{H}^n(S_x; R)$ is a direct sum of k copies of R and the quotient rings $R/m_i R$ for $i = 1, \ldots, l$.

For each $i = 1, \ldots, k+l$, let α_i be the identity of the ith direct summand. Because $\check{H}^n(f^{-1}(x); R) = 0$ we may assume, replacing O_x with a smaller neighborhood if necessary, that $(h|O_x)^*(\alpha_i) = 0$ for all $i = 1, \ldots, k+l$. Consequently,

$$(h|_{O_x})^* = 0. \tag{4}$$

Let us consider the open collection $\{O_x \mid x \in X\}$, which covers Z. Since f is proper and UV^{n-1}, we can find a collection $\mathcal{V}$ of open subsets of O which covers Z and satisfies the following conditions:

(5) $f^{-1}(f(V \cap Z)) = V \cap Z$ for all $V \in \mathcal{V}$, and

(6) for a simplicial pair (K, L) such that $L \supset K^{(0)}$ and a mapping $\zeta : |L| \to O$ such that for each simplex s of K, $\zeta(|s \cap L|) \subset V_s$ for some $V_s \in \mathcal{V}$, there exists an extension $\overline{\zeta} : |K^{(n)} \cup L| \to O$ such that for each simplex s of K, $\overline{\zeta}(|s \cap K^{(n)}|) \subset O_{x(s)}$ for some $x(s) \in X$.

We call the collection $\mathcal{V}$ an *n-refinement* of $\{O_x \mid x \in X\}$ (see [3]).

By (5), there exists an open cover $\mathcal{W}$ of X such that

$$f^{-1}(\mathcal{W}) \prec^* \mathcal{V}, \quad \text{and} \tag{7}$$

$$\mathcal{W} \prec^* g^{-1}(\{\mathrm{st}(v, T) \mid v \in T^{(0)}\}). \tag{8}$$

Then there exist a subcomplex N of the nerve of $\mathcal{W}$ and a surjective mapping $\varphi : X \to |N| = Q$ such that

$$\varphi^{-1}(\mathrm{st}(W, N)) \subset W \quad \text{for each vertex } W \in N. \tag{9}$$

Note that each vertex W of N is a member of the cover $\mathcal{W}$. It then follows from (8) and (9) that for each vertex W of N we can find a vertex $\psi(W)$ of T such that

$$\varphi^{-1}(\mathrm{st}(W, N)) \subset g^{-1}(\mathrm{st}(\psi(W), T)). \tag{10}$$

Clearly, condition (10) implies the existence of a mapping $\psi : Q = |N| \to |T| = P$ such that

$$d(g, \psi \circ \varphi) \leq \{\mathrm{st}(v, T) | v \in T^{(0)}\} \prec \mathcal{U}. \tag{11}$$

To complete the proof we need only show the following:

Claim. *ψ is $(R, n, \mathrm{st}(\mathcal{U}))$-approximable.*

Proof of Claim. It suffices to consider an arbitrary subdivision K of N instead of any triangulation of $|N|$ as in Definition 1. For each vertex k of K, let us take a vertex $W(k)$ of N such that

$$k \in \mathrm{st}(W(k), N). \tag{12}$$

Moreover, choose a point $\mu(k) \in f^{-1}(W(k)) \subset Z$. For any simplex $s = \langle k_0, k_1, \ldots, k_m \rangle$ of K, the collection $\{W(k_0), W(k_1), \ldots, W(k_m)\}$ spans a simplex of N. Hence $\bigcap_{i=0}^m W(k_i) \neq \emptyset$. By (7), we have

$$\mu(k_0), \mu(k_1), \ldots, \mu(k_m) \in \bigcup_{i=0}^{m} f^{-1}(W(k_i)) \subset \mathrm{st}(f^{-1}(W(k_0)), f^{-1}(\mathcal{W})) \subset V_s$$

for some $V_s \in \mathcal{V}$. Then condition (6) implies that there exists an extension $\xi : |K^{(n)}| \to O$ of $\mu : |K^{(0)}| \to O$ such that

(13) for each simplex s of K, $\xi(|s \cap K^{(n)}|) \subset O_{x(s)}$ for some $x(s) \in X$.

Now we shall show that the composition $\psi' = h \circ \xi : |K^{(n)}| \to |T^{(n)}|$ satisfies conditions (i) and (ii) of Definition 1. Let us consider an arbitrary mapping $\alpha : |T^{(n)}| \to K(R, n)$. Then to verify condition (ii) it suffices to show the existence of an extension $\beta : |K^{(n+1)}| \to K(R, n)$ of $\alpha \circ \psi'$. For each $(n+1)$-simplex s of K, it follows from (13) that $\xi(|\partial s|) \subset O_{x(s)}$ for some $x(s) \in X$. Condition (4) implies that $\alpha \circ h | O_{x(s)} \simeq 0$. Hence $\alpha \circ h \circ \xi|_{|\partial s|} \simeq 0$. We therefore have an extension $\xi_s : |s| \to K(R, n)$ of $\alpha \circ h \circ \xi|_{|\partial s|}$. Thus, by simplexwise extension, we obtain an extension $\beta : |K^{(n+1)}| \to K(R, n)$ of $\alpha \circ h \circ \xi = \alpha \circ \psi'$.

As for condition (i), let z be a point of $|K^{(n)}|$ and let $s = \langle k_0, k_1, \ldots, k_t \rangle$, $t \leq n$, be the carrier of z. Then, by (12), (9) and (10),

$$\varphi^{-1}(z) \subset \bigcap_{i=0}^{t} W(k_i) \subset g^{-1}(\mathrm{st}(\psi(W(k_0)), T), \quad \text{and}$$
$$\psi(z) \in \mathrm{st}(\psi(W(k_0)), T).$$

Hence we have

$$\psi(z), g(u) \in \mathrm{st}(\psi(W(k_0)), T) \quad \text{for any } u \in \varphi^{-1}(z).$$

On the other hand, by the construction,

$$\mu(k_0), \mu(k_1), \ldots, \mu(k_t) \in \mathrm{st}(f^{-1}(W(k_0)), f^{-1}(\mathcal{W})) \subset V_s \in \mathcal{V},$$
$$\xi(z) \in \xi(\langle k_0, k_1, \ldots, k_t \rangle) \subset O_{x(s)}, \quad \text{and} \quad V_s \subset O_{x(s)}.$$

Hence, because $f^{-1}(\varphi^{-1}(z)) \subset O_{x(s)} \cap Z$, it follows from (3) that

$$h \circ \xi(z), h(\widetilde{u}) \in h(O_{x(s)}) \subset U_{x(s)} \in \mathcal{U} \quad \text{for any } \widetilde{u} \in f^{-1}(u).$$

Moreover, by (2),

$$h(\widetilde{u}), g(u) = g \circ f(\widetilde{u}) \in U_0 \quad \text{for some } U_0 \in \mathcal{U}.$$

Therefore, by (1), $h \circ \xi(z), \psi(z) \in \mathrm{st}(U_0, \mathcal{U})$. Thus, we have $d(\psi', \psi|_{|K^{(n)}|}) = d(h \circ \xi, \psi|_{|K^{(n)}|}) \leq \mathrm{st}(\mathcal{U})$. This completes the proof of the Claim. ∎

Theorems 2 and 4 imply the following characterization of approximable dimension with coefficients in a commutative ring with identity. This extends Theorem 3 to (separable) metric spaces.

Corollary 1 (Characterization Theorem). *A (separable) metric space X has* a-$\dim_R X \leq n$ *if and only if there exist a (separable) metric space Z of dimension at most n and a proper UV^{n-1}-mapping $f: Z \to X$ such that $\check{H}^n(f^{-1}(x); R) = 0$ for all $x \in X$.*

3. Some fundamental properties of approximable dimension. First we make note of a useful lemma (see [5], Problem 4.3.C and its hint, for an outline of the proof).

Lemma 1. *For every metric space X there exist a 0-dimensional metric space S and a proper surjective mapping $f: S \to X$.*

The following result shows that we may define the *approximable dimension* of the space X with respect to R in the expected way. That is, if n denotes a nonnegative integer and we adopt the convention that a-$\dim_R X = -1$ if and only if $X = \emptyset$, then

(i) a-$\dim_R X = n$ if a-$\dim_R X \leq n$ is true and a-$\dim_R X \leq n-1$ is false, and

(ii) a-$\dim_R X = \infty$ if a-$\dim_R X \leq n$ is false for every n.

Theorem 5. *If X is a metric space and* a-$\dim_R X \leq n$, *then* a-$\dim_R X \leq n+1$.

Proof. By the Characterization Theorem, there exist a metric space Z of dimension at most n and a proper UV^{n-1}-mapping $f: Z \to X$ such that $\check{H}^n(f^{-1}(x); R) = 0$ for all $x \in X$. By Lemma 1, there is a proper mapping $h: S \to X$ from a 0-dimensional metric space S onto X. We then consider the subset

$$\mathrm{Y} = \bigcup \{f^{-1}(x) * h^{-1}(x) \mid x \in X\} \subset Z * S,$$

where $A * B$ denotes the join of the spaces A and B. Since f and h are closed, Y is a closed subset $Z * S$. Clearly, $\dim Y \leq \dim Z * S \leq n+1$.

We define the mapping $\varphi: Y \to X$ by $\varphi(f^{-1}(x) * h^{-1}(x)) = x$ for all $x \in X$. Since $\varphi^{-1}(K) \subset f^{-1}(K) * h^{-1}(K)$ for any compact subset K of X, and since both f and h are proper, φ is also proper. For each point $x \in X$, $\varphi^{-1}(x) = f^{-1}(x) * h^{-1}(x)$. Since $f^{-1}(x)$ has property UV^{n-1} and $h^{-1}(x) \neq \emptyset$, $\varphi^{-1}(x)$ has property UV^n. Moreover, since $\dim f^{-1}(x) \leq n$, $\dim h^{-1}(x) = 0$ and $\check{H}^n(f^{-1}(x); R) = 0$, we have

$$\begin{aligned}\check{H}^{n+1}(f^{-1}(x) * h^{-1}(x); R) &\cong \check{H}^n(f^{-1}(x) \times h^{-1}(x); R) \\ &\cong \bigoplus_{i+j=n} \check{H}^i(f^{-1}(x); R) \otimes \check{H}^j(h^{-1}(x); \mathbb{Z}) = 0.\end{aligned}$$

Thus, φ is a proper UV^n-mapping with R-acyclic point inverses. It follows from the Characterization Theorem that a-$\dim_R X \leq n+1$. ∎

The Characterization Theorem also allows us to immediately obtain the following fundamental property of approximable dimension for metric spaces.

THEOREM 6 (Subset Theorem). *If A is a subspace of the metric space X, then* $\text{a-dim}_R A \le \text{a-dim}_R X$.

As pointed out in [15], if X is a metric space and $\text{a-dim}_R X \le n$, the proof of Theorem 2 provides a proper UV^{n-1}-mapping $\pi : Z \to Y$ from a metric space Z of dimension at most n onto a completely metrizable space Y containing X as a dense subset such that $\check{H}^n(\pi^{-1}(y); R) = 0$ for all $y \in Y$. Then, by Theorem 4, $\text{a-dim}_R Y \le n$. Therefore we have the following.

THEOREM 7 (Completion Theorem). *Every metrizable space X for which* $\text{a-dim}_R X \le n$ *admits a metrizable completion $\widetilde{X}$ such that* $\text{a-dim}_R \widetilde{X} \le n$.

We end this section with the following theorem and an application.

THEOREM 8. *Suppose X is a metric space and $R = \bigoplus_{i=1}^m R_i$. Then* $\text{a-dim}_R X = \max_i\{\text{a-dim}_{R_i} X\}$.

Proof. It suffices to consider the case $R = R_1 \oplus R_2$. Suppose n is a nonnegative integer. By the Characterization Theorem, if $\text{a-dim}_R X \le n$, then $\text{a-dim}_{R_i} X \le n$ for $i = 1, 2$. We shall complete the proof by showing that $\text{a-dim}_R X \le n$ if $\text{a-dim}_{R_i} X \le n$ for $i = 1, 2$.

Consider a mapping $f : X \to P$ of X to a polyhedron P and an open cover $\mathcal{U}$ of P. Since $\text{a-dim}_{R_1} X \le n$, there exist a polyhedron Q_1 and mappings $\varphi_1 : X \to Q_1$, $\psi : Q_1 \to P$ such that

$$d(\psi_1 \circ \varphi_1, f) \le \mathcal{U}, \tag{1}$$

$$\psi_1 \text{ is } (R_1, n, \mathcal{U})\text{-approximable.} \tag{2}$$

Choose an open cover $\mathcal{U}_1$ of Q_1 such that

$$\text{any two } \mathcal{U}_1\text{-close mappings of } Z \text{ to } Q_1 \text{ are homotopic,} \tag{3}$$

$$\mathcal{U}_1 \prec \psi_1^{-1}(\mathcal{U}). \tag{4}$$

Moreover, since $\text{a-dim}_{R_2} X \le n$, there exist a polyhedron Q_2 and mappings $\varphi_2 : X \to Q_1$, $\psi_2 : Q_2 \to Q_1$ such that

$$d(\psi_2 \circ \varphi_2, \varphi_1) \le \mathcal{U}_1, \tag{5}$$

$$\psi_2 \text{ is } (R_2, n, \mathcal{U}_1)\text{-approximable.} \tag{6}$$

By (1), (4) and (5), we have the following (which is not a numerical inequality, but whose meaning should nonetheless be clear):

$$d(\psi_1 \circ \psi_2 \circ \varphi_2, f) \le d(\psi_1 \circ \psi_2 \circ \varphi_2, \psi_1 \circ \varphi_1) + d(\psi_1 \circ \varphi_1, f) \le \mathrm{st}(\mathcal{U}).$$

Now we will show that $\psi_1 \circ \psi_2 : Q_2 \to P$ is $(R_1 \oplus R_2, n, \mathrm{st}(\mathcal{U}))$-approximable. Suppose that T is a triangulation of P demonstrating the $(R_1, n, \mathcal{U})$-approximability of ψ_1, and T_1 is a triangulation of Q_1 demonstrating the $(R_2, n, \mathcal{U}_1)$-approximability of ψ_2. Consider an arbitrary triangulation M of Q_2. Then there exist mappings $\psi_2' : |M^{(n)}| \to |T_1^{(n)}|$ and $\psi_1' : |T_1^{(n)}| \to |T^{(n)}|$ such that

(7) $$d(\psi_1', \psi_1 \mid_{|T_1^{(n)}|}) \le \mathcal{U},$$

(8) $$d(\psi_2', \psi_2 \mid_{|M^{(n)}|}) \le \mathcal{U}_1,$$

(9) for any mapping $\alpha : |T^{(n)}| \to K(R_1, n)$, there exists a mapping $\overline{\alpha} : Q_1 \to K(R_1, n)$ such that $\overline{\alpha}|_{|T_1^{(n)}|} = \alpha \circ \psi_1'$, and

(10) for any mapping $\beta : |T_1^{(n)}| \to K(R_2, n)$, there exists a mapping $\overline{\beta} : Q_2 \to K(R_2, n)$ such that $\overline{\beta}|_{|M^{(n)}|} = \beta \circ \psi_2'$.

By (4), (7) and (8),

$$\begin{aligned} &d(\psi_1' \circ \psi_2', \psi_1 \circ \psi_2|_{|M^{(n)}|}) \\ &\qquad \le d(\psi_1' \circ \psi_2', \psi_1 \circ \psi_2') + d(\psi_1 \circ \psi_2', \psi_1 \circ \psi_2 \mid_{|M^{(n)}|}) \le \mathrm{st}(\mathcal{U}). \end{aligned}$$

Consider now a mapping $\alpha : |T^{(n)}| \to K(R_1, n) \times K(R_2, n)$. By (9), there exists a mapping $\beta_1' : Q_1 \to K(R_1, n)$ such that $\beta_1'|_{|T_1^{(n)}|} = p_1 \circ \alpha \circ \psi_1'$, where $p_i : K(R_1, n) \times K(R_2, n) \to K(R_i, n)$, $i = 1, 2$, is the projection. Then, by (3) and (8), $\beta_1' \circ \psi_2|_{|M^{(n)}|} \simeq p_1 \circ \alpha \circ \psi_1' \circ \psi_2'$. Hence there is a mapping $\beta_1 : Q_2 \to K(R_1, n)$ such that $\beta_1|_{|M^{(n)}|} = p_1 \circ \alpha \circ \psi_1' \circ \psi_2'$. On the other hand, by (10), there exists a mapping $\beta_2 : Q_2 \to K(R_2, n)$ such that $\beta_2|_{|M^{(n)}|} = p_2 \circ \alpha \circ \psi_1' \circ \psi_2'$. Hence we obtain the mapping $\beta = (\beta_1, \beta_2) : Q_2 \to K(R_1, n) \times K(R_2, n)$ such that $\beta|_{|M^{(n)}|} = \alpha \circ (\psi_1' \circ \psi_2')$. Thus, $\psi_1 \circ \psi_2$ is $(R_1 \oplus R_2, n, \mathrm{st}(\mathcal{U}))$-approximable. We conclude that $\text{a-dim}_R X \le n$. ■

In [10] it is shown that if X is a metrizable space, then

$$\text{c-dim}_G X \le \text{a-dim}_G X \le \text{c-dim}_{\mathbb{Z}} X \le \dim X.$$

This result, along with Theorem 8, provides the following (cf. [8, Theorem D]).

COROLLARY 2. *Suppose that X is a metric space and G is a finitely generated abelian group. Then* $\text{a-dim}_G X = \text{c-dim}_G X$.

4. Universal spaces for approximable dimension. We recall the notion of a universal space.

DEFINITION 3. Let $\mathcal{C}$ be a given class of topological spaces. Then a space $X \in \mathcal{C}$ is called a *universal space* for the class $\mathcal{C}$ provided that any $Y \in \mathcal{C}$ can be embedded in X.

For a nonnegative integer n, consider the class

$$\mathcal{C}(G, n) = \{X \mid X \text{ is a metrizable compactum and } \text{c-dim}_G X \le n\}.$$

There is a great deal of interest in the question of the existence of a universal space for the class $\mathcal{C}(G,n)$. This question remains open even in the case $G=\mathbb{Z}$. We note that Dydak and Mogilski [4] have considered the larger class

$$\mathcal{C}_{\mathcal{S}}(G,n)=\{X \mid X \text{ is a separable metrizable space and } \text{c-dim}_G X\le n\}$$

and have shown the existence of a universal space for the class $\mathcal{C}_{\mathcal{S}}(\mathbb{Z},n)$. Yokoi [15] applied Theorem 2 to their method and showed the parallel result for the class $\mathcal{C}_{\mathcal{S}}(\mathbb{Z}_p,n)$. Recently Olszewski [11] made striking progress on the general problem. In particular, he constructed a universal space for the class $\mathcal{C}_{\mathcal{S}}(G,n)$ whenever G is countable.

Here we consider the class

$$\mathcal{A}_{\mathcal{S}}(G,n)=\{X \mid X \text{ is a separable metrizable space and } \text{a-dim}_G X\le n\}.$$

The problem of the existence of a universal space for this class is open for general coefficient groups. However, we may construct a universal space for the class $\mathcal{A}_{\mathcal{S}}(R,n)$. The construction is based on that of Dydak and Mogilski [4], and is thus an application of the Characterization Theorem. Since this construction is quite similar to that of [4], we shall omit the proof.

Theorem 9 (Universal Space Theorem). *For each integer $n\ge 1$, there exists a universal space M_n for the class $\mathcal{A}_{\mathcal{S}}(R,n)$.*

By Theorem 6 we have the following.

Corollary 3. *If X is a separable metric space*, *then* a-dim$_R X\le n$ *if and only if X can be embedded into M_n.*

Corollary 2 and Theorem 9 provide a universal space theorem for cohomological dimension.

Corollary 4 ([4] and [15]). *For a finitely generated abelian group G and an integer $n\ge 1$ there exists a universal space for the class $\mathcal{C}_{\mathcal{S}}(G,n)$.*

5. Refinable mappings and approximable dimension. Let $f:X\to Y$ be a mapping of a metric space X onto a metric space Y. Let $\mathcal{U}$ be an open cover of X. Then the mapping f is called a *$\mathcal{U}$-mapping* if each point $y\in Y$ has a neighborhood O_y such that $f^{-1}(O_y)$ is contained in some element of $\mathcal{U}$. A mapping $r:X\to Y$ of a metric space X onto a metric space Y is *refinable* if for each pair of open covers $\mathcal{U}$ of X and $\mathcal{V}$ of Y there exists a $\mathcal{U}$-mapping $f:X\to Y$ such that $d(f,r)\le\mathcal{V}$. Such a mapping f is called a *$(\mathcal{U},\mathcal{V})$-refinement* of r.

Theorem 10. *Let $r:X\to Y$ be a refinable mapping of a metric space X onto a metric space Y. Then* a-dim$_G X=$ a-dim$_G Y$.

Proof. Assume that a-$\dim_G X \le n$. Consider a mapping $f : Y \to P$ from Y to a polyhedron P, and an open cover $\mathcal{W}$ of P. Since a-$\dim_G X \le n$, there exist a polyhedron Q and mappings $\varphi : X \to Q$, $\psi : Q \to P$ such that

$$d(\psi \circ \varphi, f \circ r) \le \mathcal{W}, \tag{1}$$

$$\psi \text{ is } (G, n, \mathcal{W})\text{-approximable.} \tag{2}$$

Choose a triangulation M of Q such that $\mathcal{V} = \{\mathrm{st}(v, M) \mid v \in M^{(0)}\} \prec \psi^{-1}(\mathcal{W})$. Then there exists a $(\varphi^{-1}(\mathcal{V}), f^{-1}(\mathcal{W}))$-refinement $h : X \to Y$ of r. Since h is a $\varphi^{-1}(\mathcal{V})$-mapping, there exists a locally finite open cover $\mathcal{U}$ of Y such that $h^{-1}(\mathcal{U}) \prec \varphi^{-1}(\mathcal{V})$. Let N be the nerve of the cover $\mathcal{U}$ and $\xi : Y \to N$ be a canonical mapping. Then there is a mapping $\zeta : N \to Q$ induced by the correspondence $h^{-1}(U) \subset \varphi^{-1}(\mathrm{st}(\zeta(U), M))$. It follows that $d(f, \psi \circ \zeta \circ \xi) \le \mathrm{st}(\mathcal{W})$. Combining this with (2), we have a-$\dim_G Y \le n$.

Conversely, assume that a-$\dim_G Y \le n$. Consider a mapping $f : X \to P$ to a polyhedron P and an open cover $\mathcal{W}$ of P. Choose a triangulation L of P such that $\mathcal{U} = \{\mathrm{st}(v, L) \mid v \in L^{(0)}\} \prec \mathcal{W}$. Since r is refinable, there exists an $f^{-1}(\mathcal{U})$-mapping $h : X \to Y$. Then there is a locally finite open cover $\mathcal{V}$ of Y such that $h^{-1}(\mathcal{V}) \prec f^{-1}(\mathcal{U})$. Just as in the first part of the proof, there is a mapping $\zeta : N \to P$, where N is the nerve of $\mathcal{V}$, and a canonical mapping $\xi : Y \to N$ such that $d(\zeta \circ \xi \circ h, f) \le \mathcal{W}$. For the mapping $\zeta \circ \xi : Y \to P$, because a-$\dim_G Y \le n$, there exist a polyhedron Q and mappings $\varphi : Y \to Q$, $\psi : Q \to P$ such that

$$d(\psi \circ \varphi, \zeta \circ \xi) \le \mathcal{W}, \tag{3}$$

$$\psi \text{ is } (G, n, \mathcal{W})\text{-approximable.} \tag{4}$$

Then $d(f, \varphi \circ \psi \circ h) \le \mathrm{st}(\mathcal{W})$. Thus, a-$\dim_G X \le n$. ∎

From Theorem 10 and Corollary 2, we obtain the following.

COROLLARY 5. *Let $r : X \to Y$ be a refinable mapping of the metric space X onto the metric space Y, and let G be a finitely generated abelian group. Then* c-$\dim_G X =$ c-$\dim_G Y$.

We do not know whether the hypothesis that G be finitely generated is necessary in Corollary 5. Related results about refinable mappings and cohomological dimension theory may be found in [7] and [9].

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