FUNDAMENTA
MATHEMATICAE
152 (1997)

The homotopy groups of the L_2-localization of a certain type one finite complex at the prime 3

by

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Abstract. For the Brown–Peterson spectrum BP at the prime 3, v_2 denotes Hazewinkel's second polynomial generator of BP_*. Let L_2 denote the Bousfield localization functor with respect to $v_2^{-1}BP$. A typical example of type one finite spectra is the mod 3 Moore spectrum M. In this paper, we determine the homotopy groups $\pi_*(L_2M \wedge X)$ for the 8 skeleton X of BP.

1. Introduction. Let BP denote the Brown–Peterson spectrum at the prime p and L_2 the Bousfield localization functor from the category of spectra to itself with respect to $v_2^{-1}BP$. Here v_2 is the polynomial generator of $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ with $|v_k| = 2p^k - 2$. A p-local finite spectrum F is said to have *type* n if $K(n)_*(X) \neq 0$ and $K(n-1)_*(X) = 0$ for the Morava K-theories $K(i)_*(-)$ with coefficient ring $K(i)_* = K(i)_*(S^0) = (\mathbb{Z}/p)[v_i, v_i^{-1}]$. The Toda–Smith spectrum $V(n)$ characterized by

$$BP_*(V(n)) = BP_*/(p, v_1, \ldots, v_n)$$

is a typical example of spectrum of type $n+1$ when it exists.

At the prime 3, $V(1)$ exists and the homotopy groups $\pi_*(L_2V(1))$ were recently computed by the second author [9]. Of course, the homotopy groups $\pi_*(L_2V(1))$ at a prime > 3 had already been computed by Ravenel (cf. [5]). In other words, we know the homotopy groups of the L_2-localization of a type two finite complex even at the prime 3. In this paper we compute the homotopy groups of the L_2-localization of a type one finite complex at the prime 3. At a prime > 3, the homotopy groups of the L_2-localization of a type one complex and a type zero complex are computed in [7] and [11], respectively. At the prime $p = 2$, without a finiteness condition, the homotopy groups of the L_2-localization of type two and type one complexes are computed in [3] and [8], respectively.

1991 *Mathematics Subject Classification*: 55Q10, 55Q45, 55P60.

From now on, we fix a prime $p = 3$. Let X denote the 8-skeleton of BP and $M = V(0)$ denote the mod 3 Moore spectrum. Then $M \wedge X$ is a spectrum of type one. Since we have the Adams map $\alpha : \Sigma^4 M \to M$ such that $BP_*(\alpha) = v_1$, we see that $\pi_*(L_2 M \wedge X)$ is a $k(1)_*$-module for $k(1)_* = (\mathbb{Z}/3)[v_1]$, where we identify v_1 and α. In this paper, we determine the homotopy groups $\pi_*(L_2 M \wedge X)$ by using the Adams–Novikov spectral sequence. In order to state our results, consider $k(1)_*$-modules $K^{(i)}$, $L^{(i)}$, $\widehat{K}^{(i)}$, $M^{(i)}$ and $N^{(i)}$ defined as follows:

- $K^{(i)}$ is isomorphic to $K(2)_* - (\mathbb{Z}/3)[v_2^3, v_2^{-3}]$ generated by $g^{(i)}$ of dimensions $-5, 10, 5$ and -12 for $i = 0, 1, 2$ and 3, respectively.
- $L^{(i)}$ is the direct sum of $k(1)_*$-modules isomorphic to

$$k(1)_*/(v_1^{4\cdot 3^k})$$

generated by $g_{k,m}^{(i)}$ for $k \geq 0$ and $m \in \mathbb{Z} - 3\mathbb{Z}$ of dimension $16 \cdot 3^k(3m-1) - 1 + 15i$ for $i = 0, 1$, and $8 \cdot 3^{k+1}(2m-3) - 3 + 15(i-2)$ for $i = 2, 3$.

- $\widehat{K}^{(i)}$ is isomorphic to $(\mathbb{Z}/3)[v_2^3, v_2^{-3}]$ generated by $k^{(i)}$ of dimension $15 - 5i$.
- $M^{(i)}$ is the direct sum of $k(1)_*$-modules isomorphic to

$$k(1)_*/(v_1^{6\cdot 3^k}) \oplus k(1)_*/(v_1^{10\cdot 3^k})$$

generated by ${g'}_{k,m}^{(i)}$ and ${g''}_{k,m}^{(i)}$ for $k \geq 0$, $m \in \mathbb{Z}$ of dimensions $8 \cdot 3^k(18m+3) - 2 + 15(i-1)$ and $8 \cdot 3^k(18m+11) - 2 + 15(i-1)$.

- $N^{(i)}$ is the direct sum of $k(1)_*$-modules isomorphic to

$$k(1)_*/(v_1^3)$$

generated by $g_m^{(i)}$ for $m \in \mathbb{Z}$ of dimension $16 \cdot (3m+2) - 2 + 11(i-1)$.

Let G^s denote a $k(1)_*$-module isomorphic to the finite v_1 torsion part M^s of $H^s M_1^1$ shown in (11.1):

$$\begin{aligned}
G^0 &= K^{(0)} \oplus L^{(0)}, \\
G^1 &= K^{(1)} \oplus L^{(1)} \oplus \widehat{K}^{(1)} \oplus M^{(1)} \oplus N^{(1)}, \\
G^2 &= K^{(2)} \oplus L^{(2)} \oplus \widehat{K}^{(2)} \oplus M^{(2)} \oplus N^{(2)}, \\
G^3 &= K^{(3)} \oplus L^{(3)}, \\
G^s &= 0 \quad \text{for } s > 3.
\end{aligned}$$

Our computation shows that the E_2-term of the Adams–Novikov spectral sequence for $\pi_*(L_2 M \wedge X)$ is zero at the filtration degree greater than 4. Therefore, the spectral sequence collapses to the E_2-term. We compute the E_2-term and obtain:

THEOREM 1.1. *The homotopy group $\pi_*(L_2M \wedge X)$ is isomorphic to the direct sum of a $k(1)_*$-module*

$$k(1)_* \otimes \Lambda(h_{20}),$$

two copies of $K(1)_*/k(1)_*$, and $\bigoplus_{s\ge 0} G^s$. Here $h_{20} \in \pi_{15}(L_2M \wedge X)$ and the direct sum of the two copies of $K(1)_*/k(1)_*$ is generated by the elements $z_j^{(i)} \in \pi_{15i-4j-2}(L_2M \wedge X)$ for $i = 0, 1$ and $0 < j \in \mathbb{Z}$ as a vector space over $\mathbb{Z}/3$ with $v_1 z_j^{(i)} = z_{j-1}^{(i)}$ and $z_0^{(i)} = 0$.

The theorem follows from

THEOREM 1.2. *The homotopy group $\pi_*(L_{K(2)}X \wedge M)$ is isomorphic to the $k_*(1)$-module*

$$k(1)_*^\wedge \otimes \Lambda(h_{20}, \zeta_2) \oplus \bigoplus_{s\ge 0} G^s.$$

Here $k(1)_*^\wedge = \lim_j k(1)_*/(v_1^j)$, and $L_{K(2)}$ denotes the Bousfield localization functor with respect to the Morava K-theory $K(2)_*$.

Note that $L_{K(2)}X \wedge M = \lim_j L_{K(2)}X \wedge V(1)_{(j)} = \lim_j L_2X \wedge V(1)_{(j)}$. This theorem follows from the knowledge of the homotopy groups $\pi_*(L_{K(2)}X \wedge V(1)_{(j)})$ for the cofiber $V(1)_{(j)}$ of $\alpha^j : \Sigma^{4j}M \to M$, whose structures can be read off from the differentials of the Bockstein spectral sequence used to determine $H^*M_1^1$. Note also that we have the duality

$$E_2^{s,*}(X \wedge V(1)_{(j)}) \cong \mathrm{Hom}(E_2^{4-s,*}(X \wedge V(1)_{(j)}), \mathbb{Q}/\mathbb{Z})$$

even at $p = 3$, which was observed at the prime $p > 3$ by Hopkins and Gross [2].

2. The spectrum X. Throughout this paper, we consider everything localized at the prime number 3. Let S^0 denote the sphere spectrum and consider the cofiber $S^0 \cup_{\alpha_1} e^4$ of the map $\alpha_1 : S^3 \to S^0$ which represents the generator $\pi_3(S^0)$. Since the map $\alpha_1 : S^7 \to S^4$ is extended to $\widetilde{\alpha}_1 : S^7 \to S^0 \cup_{\alpha_1} e^4$, the spectrum X is defined to be a cofiber of the map $\widetilde{\alpha}_1$ and denoted by

$$X = S^0 \cup_{\alpha_1} e^4 \cup_{\tilde{\alpha}_1} e^8.$$

Let BP denote the Brown–Peterson spectrum at the prime 3. Then $BP_* = \pi_*(BP) = \mathbb{Z}_{(3)}[v_1, v_2, \ldots]$ with $|v_i| = 2(3^i - 1)$, and $BP_*(BP) = BP_*[t_1, t_2, \ldots]$ with $|t_i| = 2(3^i - 1)$. By the cell structure, we see that X is the 8-skeleton of BP, and so

$$BP_*(X) = BP_*[t_1]/(t_1^3) \subset BP_*(BP),$$

which gives the comodule structure of $BP_*(X)$.

Let M denote the Moore spectrum which is defined as the cofiber of the map $3 : S^0 \to S^0$. Then

$$BP_*(M \wedge X) = (BP_*/(3))[t_1]/(t_1^3)$$

as a comodule.

3. The E_2-term. We note that for the Brown–Peterson spectrum BP, $(BP_*, BP_*(BP))$ is a Hopf algebroid, and the category of $BP_*(BP)$-comodules has enough injectives. Therefore, the Ext group $\mathrm{Ext}^*_{BP_*(BP)}(BP_*, C)$ is defined as a cohomology of the relative injective resolution of a comodule C. Note that $BP_*(W)$ is a comodule for a spectrum W. The E_2-term of the Adams–Novikov spectral sequence for computing the homotopy groups $\pi_*(W)$ is $\mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, BP_*(W))$. Hereafter, spectra are supposed to be 3-local.

Consider a comodule algebra $E(2)_* = v_2^{-1}\mathbb{Z}_{(3)}[v_1, v_2]$ whose structure is read off from that of BP_*. By Landweber's exact functor theorem, $E(2)_*(-) = E(2)_* \otimes_{BP_*} BP_*(-)$ is a homology theory and $E(2)$ denotes the spectrum which represents it by Brown's theorem. Then, $E(2)_*(E(2)) = E(2)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E(2)_*$ is also a Hopf algebroid. We have another Adams–type spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{E(2)_*(E(2))}(E(2)_*, E(2)_*(W)) \Rightarrow \pi_*(L_2W) \tag{3.1}$$

(cf. [6]). Here L_2 denote the Bousfield localization functor with respect to $E(2)$. Using this we will compute first the E_2-term for the spectrum $X \wedge M$ defined in the previous section. Consider the coalgebroid

$$\Sigma = E(2)_*[t_1^3, t_2, t_3, \ldots] \otimes_{BP_*} E(2)_*$$

whose coalgebra structure is inherited from $\Gamma = E(2)_*(E(2))$ by the canonical projection $\Gamma \to \Sigma$. So it is a coalgebra over $E(2)_*$. Let

$$E(2)_*/(3) \to \Sigma \otimes_{E(2)_*} I_0 \to \Sigma \otimes_{E(2)_*} I_1 \to \ldots$$

be a relative Σ-injective resolution, by which we mean a long exact sequence of comodules which is split as $E(2)_*$-modules. Applying the functor $\Gamma \,\square_\Sigma\, -$ to the resolution, we have another Γ-injective resolution

$$(E(2)_*/(3))[t_1]/(t_1^3) \to \Gamma \otimes_{E(2)_*} I_0 \to \Gamma \otimes_{E(2)_*} I_1 \to \ldots,$$

since $\Gamma \,\square_\Sigma\, E(2)_* \cong E(2)_*[t_1]/(t_1^3)$ as a comodule and $\Gamma \,\square_\Sigma\, \Sigma = \Gamma$. These resolutions give the same cohomology and so we have an isomorphism

$$\mathrm{Ext}^{s,t}_\Gamma(E(2)_*, (E(2)_*/(3))[t_1]/(t_1^3)) = \mathrm{Ext}^{s,t}_\Sigma(E(2)_*, E(2)_*/(3)).$$

4. The chromatic spectral sequence. For the sake of simplicity, we use the notation

$$H^{s,t}(C) = \mathrm{Ext}^{s,t}_\Sigma(E(2)_*, C)$$

for a Σ-comodule C. Consider the exact sequence

$$0 \to E(2)_*/(3) \to v_1^{-1}E(2)_*/(3) \to E(2)_*/(3, v_1^\infty) \to 0. \tag{4.1}$$

of comodules, which defines the comodule $E(2)_*/(3, v_1^\infty)$. For short, we write

$$M_1^1 = E(2)_*/(3, v_1^\infty)$$

following [4]. We denote an element of it by

$$x/v_1^j \quad \text{with } x/v_1^j \neq 0 \text{ if and only if } j > 0,$$

for $x \in K(2)_*$. Here $K(2)_* = E(2)_*/(3, v_1) = (\mathbb{Z}/3)[v_2, v_2^{-1}]$. In other words,

$$M_1^1 = \{x/v_1^j : x \in K(2)_*,\ 0 < j \in \mathbb{Z}\}.$$

We also define

$$N_1^0 = E(2)_*/(3) \quad \text{and} \quad M_1^0 = v_1^{-1}E(2)_*/(3).$$

Then the short exact sequence induces a long one:

$$\ldots \to H^{s-1}M_1^1 \xrightarrow{\delta} H^sN_1^0 \to H^sM_1^0 \to H^sM_1^1 \xrightarrow{\delta} \ldots, \tag{4.2}$$

which yields our chromatic spectral sequence. That is, $E_1^{s,t} = H^tM_1^s \Rightarrow H^{s+t}N_1^0$. The structure of the E_1-term $H^*M_1^0$ is given as follows:

$$H^*M_1^0 = \mathrm{Ext}_\Sigma^*(E(2)_*, v_1^{-1}E(2)_*/(3)) = \Lambda_{K(1)_*}(h_{20}). \tag{4.3}$$

This follows from the change of rings theorems:

$$H^*M_1^0 = \mathrm{Ext}_\Sigma^*(E(2)_*, v_1^{-1}E(2)_*/(3)) = \mathrm{Ext}_{\Sigma'}^*(K(1)_*, K(1)_*) \quad \text{and}$$
$$\mathrm{Ext}_{\Sigma'}^*(K(1)_*, K(1)_*) = K(1)_* \otimes \mathrm{Ext}_{S(1,2)}^*(\mathbb{Z}/3, \mathbb{Z}/3)$$

and Ravenel's result (cf. [5, Th. 6.3.7]):

$$\mathrm{Ext}_{S(1,2)}^*(\mathbb{Z}/3, \mathbb{Z}/3) = \Lambda(h_{20}),$$

where $K(1)_* = (\mathbb{Z}/3)[v_1, v_1^{-1}]$,

$$\Sigma' = K(1)_*[t_2, t_3, \ldots]/(v_1t_i^3 - v_1^{3^i}t_i : i > 1) \quad \text{and}$$
$$S(1,2) = \Sigma' \otimes_{K(1)_*} \mathbb{Z}/3 = (\mathbb{Z}/3)[t_2, t_3, \ldots]/(t_i^3 - t_i : i > 1).$$

We further note that the generator h_{20} is represented by the cocycle

$$h_{20} = t_2 - ut_1^3 + uv_1t_1^2 + u^2v_1t_1 \in \Omega_{\Gamma'}^1 K(1)_*\{1, u, u^2\},$$

where Ω denotes the cobar complex and $K(1)_*\{1, u, u^2\} = K(1)_*[t_1]/(t_1^3)$. The exact sequence (4.2) together with (4.3) gives us the desired E_2-term $H^*N_1^0 = H^*(E(2)_*/(3)) = \mathrm{Ext}_\Sigma^*(E(2)_*, E(2)_*/(3))$ of the spectral sequence (3.1), if we know $H^*M_1^1 = H^*(E(2)_*/(3, v_1^\infty))$.

Consider the following short exact sequence:

$$
\begin{tikzcd}
0 \arrow[r] & K(2)_* \arrow[r, "f"] & E(2)_*/(3,v_1^\infty) \arrow[r, "v_1"] \arrow[d, equal] & E(2)_*/(3,v_1^\infty) \arrow[r] \arrow[d, equal] & 0 \\
& & M_1^1 & M_1^1 &
\end{tikzcd}
\tag{4.4}
$$

in which $f(x) = x/v_1$. By the following lemma we will determine $H^*M_1^1$.

LEMMA 4.5 ([4, Remark 3.11]). *Consider the commutative diagram*

$$
\begin{tikzcd}
H^sK(2)_* \arrow[r] & H^sM_1^1 \arrow[r, "v_1"] & H^sM_1^1 \arrow[r, "\delta_s"] & H^{s+1}K(2)_* \\
H^sK(2)_* \arrow[r] \arrow[u, equal] & B \arrow[r, "v_1"] \arrow[u, hook] & B \arrow[r, "\delta_s"] \arrow[u, hook] & H^{s+1}K(2)_* \arrow[u, equal]
\end{tikzcd}
\tag{4.6}
$$

in which the upper sequence is the long exact sequence induced by the short one (4.4). *If the lower sequence is exact, then* $H^*M_1^1 = B$.

5. Definition of some elements. In this section we will work in the cobar complex $\Omega_\Sigma^* M_1^1$, whose homology groups are $H^*M_1^1$. Here the cobar complex consists of the modules

$$\Omega_\Sigma^t M_1^1 = M_1^1 \otimes_{E(2)_*} \overbrace{\Sigma \otimes_{E(2)_*} \ldots \otimes_{E(2)_*} \Sigma}^{t}$$

with differentials $d_t : \Omega_\Sigma^t M_1^1 \to \Omega_\Sigma^{t+1} M_1^1$ given by

$$
\begin{aligned}
d_t(m \otimes \gamma_1 \otimes \ldots \otimes \gamma_t) = \eta_R(m) \otimes \gamma_1 \otimes \ldots \otimes \gamma_t& \\
+ \sum_{i=1}^{t} (-1)^i m \otimes \gamma_1 \otimes \ldots \otimes \Delta(\gamma_i) \otimes \ldots \otimes \gamma_t& \\
- (-1)^t m \otimes \gamma_1 \otimes \ldots \otimes \gamma_t \otimes 1,&
\end{aligned}
$$

where $\eta_R : M_1^1 \to M_1^1 \otimes_{E(2)_*} \Sigma$ and $\Delta : \Sigma \to \Sigma \otimes_{E(2)_*} \Sigma$ are the maps induced from the right unit $\eta_R : E(2)_* \to E(2)_*(E(2))$ and the coproduct $\Delta : E(2)_*(E(2)) \to E(2)_*(E(2)) \otimes_{E(2)_*} E(2)_*(E(2))$, respectively.

Since $\eta_R(v_3) = 0$ and $\eta_R(v_4) = 0$ in $E(2)_*(E(2))$, we have the relations (cf. [10]):

$$
\begin{aligned}
&t_1^9 \equiv v_2^2 t_1 - v_1 v_2^{-1} t_2^3 + v_1^2 v_2 t_1^3 \bmod (3, v_1^3) \quad \text{and} \\
&t_2^9 \equiv v_2^8 t_2 - v_1 v_2^{-1} t_3^3 \bmod (3, v_1^3)
\end{aligned}
\tag{5.1}
$$

in $E(2)_*(E(2))$. Note that in [10] the lower congruence is said to hold mod $(3, v_1^2)$, but a careful computation shows it holds also mod $(3, v_1^3)$.

First, we recall [4] the elements $x_k \in E(2)_*$ for $k \geq 0$ defined inductively by

$$x_i = \begin{cases} v_2^{3^i} & \text{for } i = 0, 1, \\ x_{i-1}^3 + e_i v_1^{4(3^{i-1}-1)} v_2^{2\cdot 3^{i-1}+1} & \text{for } i > 1, \end{cases}$$

where $e_2 = -1$ and $e_i = 1$ for $i > 2$. Then by [4, Prop. 5.4] we have, mod $(3, v_1^{4\cdot 3^{i-1}+1})$,

$$\text{(5.2)} \qquad \begin{aligned} d_0(x_1) &\equiv v_1^3 t_1^9 \equiv v_1^3 v_2^2 (t_1 + v_1(v_2^{-1}(t_2 - t_1^4) - \zeta_2)), \\ d_0(x_i) &\equiv -v_1^{4\cdot 3^{i-1}-1} v_2^{2\cdot 3^{i-1}} (t_1 + v_1 \zeta_2^{3^{i-1}}) \qquad \text{for } i > 1, \end{aligned}$$

where

$$\text{(5.3)} \qquad \zeta_2 = v_2^{-1} t_2 + v_2^{-3}(t_2^3 - t_1^{12}).$$

Here using (5.1), we have

$$\text{(5.4)} \qquad d_1(\zeta_2) \equiv 0 \bmod (3, v_1).$$

Put

$$\widetilde{v_2^u h_{20}} = v_2^u t_2 - u v_1 v_2^{u-9} t_3^3 + u v_1 v_2^{u-3} t_1^3 t_2^3$$

for $u \in \mathbb{Z}$. Recall [10] the element $V = \frac{1}{3v_1}(v_1^3 t_1^9 - v_1^9 t_1^3 - d_0(v_2^3))$. Then

$$\text{(5.5)} \qquad V \equiv -v_2^2 t_1^3 \bmod (3, v_1) \quad \text{and} \quad d_1(V) \equiv v_1^2 b_1 \bmod (3, v_1^8)$$

for

$$\text{(5.6)} \qquad b_k = -t_1^{3^k} \otimes t_1^{2\cdot 3^k} - t_1^{2\cdot 3^k} \otimes t_1^{3^k} \qquad (k \geq 0).$$

We also define the cocycle $\xi = v_2^{-10} t_1^3 \otimes t_3^3 + \ldots$ by the following:

LEMMA 5.7. *In $\Sigma \otimes_{E(2)_*} \Sigma$,*

$$d_0(\widetilde{v_2^2 h_{20}}) \equiv -v_1 V \otimes \zeta_2^3 + v_1^2 v_2^2 \xi \bmod (3, v_1^3).$$

We also have

$$d_0(\widetilde{v_2 h_{20}}) \equiv -v_1 v_2 t_1^3 \otimes \zeta_2^3 \bmod (3, v_1^2).$$

Proof. This follows from a direct computation: mod $(3, v_1^3)$, we have

$$\begin{aligned} d_0(v_2^2 t_2) &\equiv -v_1 v_2 t_1^3 \otimes t_2 + v_1^2 t_1^6 \otimes t_2, \\ d_0(v_1 v_2^{-7} t_3^3) &\equiv -v_1^2 v_2^{-8} t_1^3 \otimes t_3^3 - v_1 v_2^{-7}(t_1^3 \otimes t_2^9 + t_2^3 \otimes t_1^{27} + v_2^3 b_1^3), \\ d_0(-v_1 v_2^{-1} t_1^3 t_2^3) &\equiv v_1^2 v_2^{-2} t_1^3 \otimes t_1^3 t_2^3 \\ &\quad + v_1 v_2^{-1}(t_1^6 \otimes t_1^9 + t_1^3 \otimes t_1^{12} + t_1^3 \otimes t_2^3 + t_2^3 \otimes t_1^3). \end{aligned}$$

Now use the relation (5.1) to obtain the lemma. Note that, as shown in [1], $b_1^3 \equiv b_2 \equiv v_2^6 b_0 \bmod (3, v_1^2)$, which is 0 in our case.

The other case is similar. ∎

Lemma 5.8. *Consider the element* $\widetilde{v_2^{-1}h_{30}} = v_2^{-27}t_3^9 - v_1v_2^2\zeta_2^3 + v_1^2v_2^{-8}t_3^3 - v_1^3v_2t_1^6\zeta_2^3$. *Then* $\widetilde{v_2^{-1}h_{30}} \equiv v_2^{-1}t_3 \bmod (3, v_1)$ *and*

$$d_1(\widetilde{v_2^{-1}h_{30}}) \equiv -b_1 - \zeta_2^3 \otimes t_1^9 + v_1^3v_2\xi \bmod (3, v_1^4)$$

in $\Sigma \otimes_{E(2)_*} \Sigma$.

Proof. The first statement follows from (5.1).

For the second statement, note first that

$$b_1^9 = v_2^{18}b_1 - d_1(v_1^3v_2^9t_1^6t_2^9).$$

Furthermore, notice that

$$v_1t_2^3 \equiv -v_2t_1^9 + v_1^2v_2^2t_1^3 - v_1^3v_2t_1^6$$

and $\zeta_2^3 \equiv v_2^{-3}t_2^3 + v_2^{-9}t_2^9 + v_1v_2^{-4}t_1^3t_2^3 - v_1^2v_2^{-2}t_1^6 \bmod (3, v_1^4)$. Then, mod $(3, v_1^4)$,

$$\begin{aligned}
t_1^9 \otimes t_2^{27} + t_2^9 \otimes t_1^{81} &\equiv (-v_1v_2^{-1}t_2^3 + v_1^2v_2t_1^3 - v_1^3t_1^6) \otimes v_2^{24}t_2^3 + t_2^9 \otimes v_2^{18}t_1^9 \\
&\equiv v_2^{23}t_2^3 \otimes v_2t_1^9 - v_1^2v_2^{23}t_2^3 \otimes v_2^2t_1^3 + v_1^3v_2^{23}t_2^3 \otimes v_2t_1^6 \\
&\quad + v_1^2v_2^{25}t_1^3 \otimes t_2^3 - v_1^3v_2^{24}t_1^6 \otimes t_2^3 + v_2^{18}t_2^9 \otimes t_1^9 \\
&\equiv v_2^{27}\zeta_2^3 \otimes t_1^9 + v_1^2v_2^{25}t_1^6 \otimes t_1^9 + v_1^2v_2^{28}t_1^3 \otimes \zeta_2^3 \\
&\quad - v_1^2v_1^{19}t_1^3 \otimes t_2^9 + v_1^2v_2^{25}t_1^3 \otimes t_1^{12} - v_1^3v_2^{24}t_1^6 \otimes t_2^3 \\
&\quad - v_1^2v_2^{25}t_2^3 \otimes t_1^3 + v_1^3v_2^{24}t_1^3t_2^3 \otimes t_1^3 + v_1^3v_2^{24}t_2^3 \otimes t_1^6.
\end{aligned}$$

Now add the following to obtain the results:

$$\begin{aligned}
d_1(v_2^{-27}t_3^9) &\equiv -v_2^{-27}(t_1^9 \otimes t_2^{27} + t_2^9 \otimes t_1^{81} + v_2^9(t_1^{27} \otimes t_1^{54} + t_1^{54} \otimes t_1^{27})) \\
&\equiv -v_2^{-27}(t_1^9 \otimes t_2^{27} + t_2^9 \otimes t_1^{81} + v_2^9b_1^9), \\
d_1(-v_1v_2^2\zeta_2^3) &\equiv v_1^2v_2t_1^3 \otimes \zeta_2^3 - v_1^3t_1^6 \otimes \zeta_2^3, \\
d_1(v_1^2v_2^{-8}t_3^3) &\equiv v_1^3v_2^{-9}t_1^3 \otimes t_3^3 - v_1^2v_2^{-8}(t_1^3 \otimes t_2^9 + t_2^3 \otimes t_1^{27}), \\
d_1(-v_1^3v_2^{-3}t_1^6t_2^3) &\equiv v_1^3v_2^{-3}(-t_1^3t_2^3 \otimes t_1^3 - t_1^3 \otimes t_1^3t_2^3 + t_2^3 \otimes t_1^6 + t_1^6 \otimes t_2^3). \ \blacksquare
\end{aligned}$$

Lemma 5.9. *Consider the element* $\widetilde{v_2^2h_{11}} = V + v_1^2\widetilde{v_2^{-1}h_{30}}$. *Then* $\widetilde{v_2^2h_{11}} \equiv -v_2^2t_1^3 \bmod (3, v_1)$ *and*

$$d_1(\widetilde{v_2^2h_{11}}) \equiv -v_1^2\zeta_2^3 \otimes t_1^9 + v_1^5v_2\xi \bmod (3, v_1^6)$$

in $\Sigma \otimes_{E(2)_*} \Sigma$.

Proof. The first congruence is seen by (5.5) and the other follows from a direct calculation and the relation

$$d_1(V) \equiv v_1^2b_1 \bmod (3, v_1^8)$$

(see (5.5)). Now the lemma follows from the previous one. $\blacksquare$

Next, we recall [1] the elements $x(n) \in E(2)_*(E(2))$ such that $x(n) \equiv v_2^nt_1 + v_1v_2^n\zeta_2 \bmod (3, v_1^2)$ for $n = 3^ks$ with $k \geq 0$ and for $s \in \mathbb{Z}$ with

$s \equiv 1 \bmod 3$ or $s \equiv -1 \bmod 9$:

$$\begin{aligned}
x(1) &= v_2 t_1 + v_1 \tau, \\
x(3) &= Y, \\
v_1^3 x(3^{k+1}) &= v_1 x(3^k)^3 - d_0(v_2^{3^{k+1}+1}) \\
&\quad + (-1)^k v_1^{3a(k)+1} v_2^{3i(k)} \omega, \qquad k > 0, \\
x(3^k(3t+1)) &= x_{k+1}^t x(3^k), \qquad k \geq 0,\ t \in \mathbb{Z}; \\
x(9t-1) &= -x_2^{t-1} X, \\
v_1^3 x(3(9t-1)) &= v_1 x(9t-1)^3 - d_0(v_2^{3(9t-1)+1}) + v_1^{29} v_2^{27t-12} V, \\
v_1^3 x(3^{k+1}(9t-1)) &= v_1 x(3^k(9t-1))^3 - d_0(v_2^{3^{k+1}(9t-1)+1}) \\
&\quad + v_1^{3a'(k)} v_2^{3i'(t;k)+1} \zeta, \qquad k > 0,
\end{aligned}$$

for the integers $a(k)$, $i(k)$, $a'(k)$ and $i'(t;k)$ for $k \geq 0$ and $t \in \mathbb{Z}$ defined by

$$\begin{gathered}
a(0) = 2, \quad a(k) = 2 \cdot 3^k + 1, \quad i(k) = (3^k - 1)/2, \\
a'(0) = 10, \quad a'(k) = 28 \cdot 3^{k-1}, \\
i'(t;0) = 9t - 4, \quad i'(t;k) = 3^{k-1}(9(3t-1) - 1).
\end{gathered}$$

Here $\tau = t_1^4 - t_2$, ω, X and Y are the elements of $\Omega^1_\Sigma E(2)_* = E(2)_*(E(2))$ defined in [1] such that

$$\begin{aligned}
d_1(\omega) &\equiv \xi^3 + v_2 \xi \bmod (3, v_1), \\
X &\equiv -v_2^8 t_1 \bmod (3, v_1), \\
d_1(X) &\equiv v_1^{10} v_2^5 b_0 + v_1^{10} v_2^5 t_1^3 \otimes \zeta_2^3 \bmod (3, v_1^{11}), \\
Y &\equiv v_2^3 t_1 \bmod (3, v_1), \\
d_1(Y) &\equiv v_1^7 v_2 \xi \bmod (3, v_1^8),
\end{aligned}$$

for a cochain $\xi \in \Omega^2_\Sigma E(2)_*$ defined above Lemma 5.7, which represents the generator ξ of $H^2 K(2)_*$ (see (5.13)) ($K(2)_* = E(2)_*/(3, v_1)$).

Then the following holds:

(5.10) *Let s denote an integer such that either $s \equiv 1 \bmod 3$ or $s \equiv -1 \bmod 9$. Then there exist elements $x(m)$ of $E(2)_*(E(2))/(3)$ for m with $m = 3^k s$ such that*

$$x(m) \equiv v_2^m t_1 + v_1 v_2^m \zeta_2 \bmod (3, v_1^2),$$

and for $k \geq 0$ and $t \in \mathbb{Z}$,

$$d_1(x(3t+1)) \equiv v_1^2 v_2^{3t} b_0 \bmod (3, v_1^3),$$

$$d_1(x(3^k(3t+1))) \equiv -(-1)^k v_1^{a(k)} v_2^{3^{k+1}t + (3^k-1)/2} \xi \bmod (3, v_1^{1+a(k)}) \quad (k > 0);$$

$$d_1(x(9t-1)) \equiv -v_1^{10}v_2^{9t-4}b_0 - v_1^{10}v_2^{9t-4}t_1^3 \otimes \zeta \bmod (3, v_1^{11}),$$
$$d_1(x(3^k(9t-1))) \equiv -v_1^{a'(k)}v_2^{i'(t;k)}t_1 \otimes \zeta \bmod (3, v_1^{1+a'(k)}) \qquad (k>0).$$
Here note that ζ denotes a power of ζ_2.

Now we rewrite this in terms of our complex.

LEMMA 5.11. *In Σ,*
$$x(m) \equiv v_1v_2^m\zeta \bmod (3, v_1^2),$$
and for $k \ge 0$ and $t \in \mathbb{Z}$, in $\Sigma \otimes_{E(2)_*} \Sigma$,
$$d_1(x(3^k(3t+1))) \equiv -(-1)^k v_1^{a(k)}v_2^{3^{k+1}t+i(k)}\xi \bmod (3, v_1^{1+a(k)}) \qquad (k>0);$$
$$d_1(x(3^k(9t-1))) \equiv -v_1^{10\cdot 3^k+1}v_2^{3^k(9t-4)+(3^k-1)/2}\xi \bmod (3, v_1^{2+10\cdot 3^k}) \ (k \ge 0)$$
up to homology.

P r o o f. We get the first congruence by projecting the relations of (5.10) to $\Sigma \otimes_{E(2)_*} \Sigma$. For the second one, with a careful computation using Lemma 5.7, we obtain the congruence for $k=0$. Consider the commutative diagram

$$\begin{tikzcd}
\Gamma \otimes_{E(2)_*} \Gamma \arrow[r, "f"] \arrow[d, "p_1"'] & \Gamma \otimes_{E(2)_*} \Gamma \arrow[d, "p_1"] \\
\Sigma \otimes_{E(2)_*} \Sigma \arrow[d, "p_2"'] & \Sigma \otimes_{E(2)_*} \Sigma \arrow[d, "p_2"] \\
\Gamma(1) \otimes_{E(2)_*} \Gamma(1) \arrow[r, "f"] & \Gamma(1) \otimes_{E(2)_*} \Gamma(1)
\end{tikzcd}$$

in which $f(x) = x^3$ and
$$\Gamma = E(2)_*E(2) = E(2)_*[t_1, t_2, \ldots] \otimes_{BP_*} E(2)_*,$$
$$\Sigma = E(2)_*[t_1^3, t_2, \ldots] \otimes_{BP_*} E(2)_* \quad \text{and} \quad \Gamma(1) = E(2)_*[t_2, \ldots] \otimes_{BP_*} E(2)_*.$$
Suppose that $p_2p_1(X) = v_1^av_2^b\xi$. If $p_1(X^3)$ has a term $v_1^lv_2^mh_{11} \otimes \zeta$, then replace X^3 by $X^3 - d_1(y)$ for y such that $d_1(y) = v_1^lv_2^mh_{11} \otimes \zeta$. Its existence follows from (5.2) and Lemma 5.7. Then $p_2p_1(X^3) = v_1^{3a}v_2^{3b+1}\xi$. On the other hand, if $p_1(X^3) = v_1^{a'}v_2^{b'}g$ for some generator g, then $p_2p_1(X^3) = v_1^{a'}v_2^{b'}g$ since p_2 does not kill generators except for $h_{11}\zeta_2$. Therefore we have $p_1(X^3) = v_1^{3a}v_2^{3b+1}\xi$. Now by the definition of $x(m)$ and induction, we have the lemma. ∎

COROLLARY 5.12. *The elements $v_2^{(3^{k+1}-1)/2}\xi$ and $v_2^{5\cdot 3^k+(3^k-1)/2}\xi$ for $k \ge 0$ are all nontrivial cocycles of $H^2E(2)_*/(3)$.*

P r o o f. Since v_1 acts on the cobar complex $\Omega_\Sigma^*E(2)_*/(3)$ monomorphically, $d_r(v_1^jx) = 0$ implies $d_r(x) = 0$. Thus the elements are cocycles. We also have the projection $H^2E(2)_* \to H^2K(2)_*$ which sends them to elements with the same names which are all nonzero. ∎

For the next lemma, recall [9] the following:

$$H^{*,*}K(2)_* = \Lambda(\zeta_2) \otimes K(2)_*\{1, h_{11}, h_{20}, \xi, \varphi, h_{20}\xi\}. \tag{5.13}$$

Here the bidegrees of the generators are:

$$|\zeta_2| = (1,0), \quad |h_{11}| = (1,12), \quad |h_{20}| = (1,16), \quad |\xi| = (2,8), \quad |\varphi| = (2,12).$$

LEMMA 5.14. *The cocycles $h_{11}\varphi$ and $h_{20}\varphi$ are homologous to $h_{20}\xi + v_2\xi\zeta_2$ and $-v_2\zeta_2\varphi$, respectively.*

Proof. Since $\mathrm{Ext}^3_A(K(2)_*, K(2)_*) = H^3K(2)_*$ is generated by $h_{20}\xi$, $\xi\zeta_2$ and $\varphi\zeta_2$, we may write

$$h_{11}\varphi = \lambda_1 h_{20}\xi + \lambda_2 v_2\xi\zeta_2 \quad \text{and} \quad h_{20}\varphi = \lambda_3 v_2\varphi\zeta_2$$

for some $\lambda_i \in \mathbb{Z}$ $(i = 1,2,3)$. Here $A = K(2)_*[t_1^3, t_2, t_3, \ldots]/(t_1^9, t_i^9 - v_2^{3^i-1}t_i : i > 1)$.

Consider the projection

$$A \to B = (\mathbb{Z}/3)[t_2, t_3, \ldots]/(t_i^9 - t_i : i > 1)$$

sending v_2 to 1, t_1 to 0 and t_i to t_i for $i > 1$. By [5, Th. 6.3.7],

$$\mathrm{Ext}_B(\mathbb{Z}/3, \mathbb{Z}/3) = \Lambda(h_{20}, h_{21}, h_{30}, h_{31}).$$

The projection sends the elements $h_{20}\varphi$ and $\zeta_2\varphi$, respectively, to $h_{20}(h_{20} - h_{21})h_{31} = -h_{20}h_{21}h_{31}$ and $(h_{20}+h_{21})(h_{20}-h_{21})h_{31} = h_{20}h_{21}h_{31}$, which are the generators. Therefore we deduce that $\lambda_3 = -1$.

Similarly send the elements $h_{11}\varphi$, $\lambda_1 h_{20}\xi$ and $\lambda_2 v_2\xi\zeta_2$ to the E_2-term

$$\Lambda(h_{11}) \otimes (\mathbb{Z}/3)[b_1] \otimes \Lambda(h_{20}, h_{21}, h_{30}, h_{31})$$

of the Cartan–Eilenberg spectral sequence computing $H^3K(2)_*$, and we see that $\lambda_1 = 1$ and $\lambda_2 = 1$. ∎

LEMMA 5.15. *There exists a cochain y such that*

$$d_2(v_2^{3m}\xi + v_1 y) \equiv -v_1^3 v_2^{3m-1}\varphi \otimes \zeta_2 \bmod (3, v_1^4).$$

Proof. By Corollary 5.12, $\widetilde{v_2\xi}$ and $\widetilde{v_2^2\xi} = v_2^{-3}(\widetilde{v_2^5\xi})$ are cocycles of $\Omega^2_\Sigma E(2)_*/(3)$ and $\Omega^2_\Sigma E(2)_*/(3, v_1)$, respectively. Put

$$\widetilde{v_2\xi} = v_2\xi - v_1 z_1 \quad \text{and} \quad \widetilde{v_2^2\xi} = v_2^2\xi - v_1 z_2$$

and consider the element

$$w = v_2^2\widetilde{v_2\xi} + v_2\widetilde{v_2^2\xi} \equiv -v_2^3\xi \bmod (3, v_1).$$

Then

$$\begin{aligned} d_2(w) &\equiv -v_1v_2t_1^3\otimes \widetilde{v_2\xi} + v_1^2t_1^6\otimes\widetilde{v_2\xi} + v_1t_1^3\otimes\widetilde{v_2^2\xi} \\ &\equiv v_1^2t_1^3\otimes(v_2z_1-z_2) - v_1^2t_1^6\otimes\widetilde{v_2\xi} \\ &\equiv v_1^2\langle h_{11},h_{11},\widetilde{v_2\xi}\rangle \bmod (3,v_1^4)\end{aligned}$$

by definition of the Massey product, since $d_2(v_2z_1-z_2)\equiv t_1^3\otimes\widetilde{v_2\xi}$. Note that $\langle h_{11},h_{11},\widetilde{v_2\xi}\rangle\in H^2E(2)_*/(3)$.

On the other hand, we may write

$$d_2(w)\equiv \lambda v_1^3v_2^2\varphi\otimes\zeta_2 \bmod (3,v_1^4)$$

for some $\lambda\in\mathbb{Z}/3$. Then in H^3K_2 for $K_2=E(2)_*/(3,v_1^2)$,

$$\langle h_{11},h_{11},\widetilde{v_2\xi}\rangle = \lambda v_1v_2^2\varphi\zeta_2.$$

Sending this by h_{11}, we get

$$\langle h_{11},h_{11},h_{11}\rangle\widetilde{v_2\xi} = \lambda v_1v_2^2h_{11}\varphi\zeta_2$$

by properties of the Massey product. Note that $\langle h_{11},h_{11},h_{11}\rangle=-b_1=-v_1v_2^{-1}\zeta_2h_{21}=-v_1v_2\zeta_2h_{20}$ in H^2K_2 by Lemma 5.8, and that $h_{11}\varphi=h_{20}\xi+v_2\xi\zeta_2$ by Lemma 5.14. We now have

$$-v_1v_2^2\zeta_2h_{20}\xi=\lambda v_1v_2^2h_{20}\xi\zeta_2$$

and $\lambda=1$. Therefore, we have an element y such that $w=-v_2^3\xi-v_1y$ and

$$d_2(v_2^3\xi+v_1y)\equiv -v_1^3v_2^2\varphi\otimes\zeta_2 \bmod (3,v_1^4).$$

Now send this by v_2^{3m-3}, and we have the desired conclusion, since $d_0(v_2^3)\equiv 0 \bmod (3,v_1^4)$. ∎

Lemma 5.16. *There is a cocycle φ in $\Omega_\Sigma^*E(2)_*$ that yields φ in $H^2K(2)_*$ such that*

$$d_2(v_2^2\varphi)\equiv v_1v_2^2\xi\otimes\zeta_2 \bmod (3,v_1^2).$$

Proof. By Lemma 5.15 with $m=1$,

$$d_3(v_2^2\varphi\otimes\zeta_2^3+v_1x)=0 \tag{5.17}$$

for a cocycle φ which represents the homology class $\varphi\in H^2K(2)_*$ and a cochain x, since ζ_2 is homologous to $\zeta_2^3 \bmod (3,v_1)$. Consider $K_2=E(2)_*/(3,v_1^2)$, and the short exact sequence $0\to K(2)_*\xrightarrow{v_1}K_2\xrightarrow{\mathrm{pr}}K(2)_*\to 0$ which yields a long one:

$$\ldots\to H^2K_2\xrightarrow{\mathrm{pr}_*}H^2K(2)_*\xrightarrow{\delta_2'}H^3K(2)_*\xrightarrow{v_1}H^3K_2\to\ldots$$

By virtue of (5.13), we may write

$$\delta_2'(v_2^2\varphi) = \lambda v_2^2\xi\zeta_2 + \mu v_2 h_{20}\xi, \tag{5.18}$$

for some λ and $\mu \in \mathbb{Z}/3$. Since $\zeta_2^3 \in H^1K_2$ and $\zeta_2 = \zeta_2^3 \in H^1K(2)_*$,

$$\delta_3'(v_2^2\varphi\zeta_2) = \delta_2'(v_2^2\varphi)\zeta_2 = \mu v_2 h_{20}\xi\zeta_2 \in H^4K(2)_*$$

by (5.18). By (5.17), the left hand side is 0, and we have $\mu = 0$.

The formula (5.17) also shows

$$d_3(v_2^3\varphi \otimes \zeta_2^3 + v_1v_2x) \equiv v_1t_1^3 \otimes v_2^2\varphi \otimes \zeta_2^3 \bmod (3, v_1^3).$$

Thus we compute

$$\delta_3'(v_2^2\varphi h_{20}) = \delta_3'(-v_2^3\varphi\zeta_2) = -v_2^2h_{11}\varphi\zeta_2 = -v_2^2h_{20}\xi\zeta_2$$

by Lemma 5.14. Notice that $\delta_3'(v_2^2\varphi h_{20}) = \delta_2'(v_2^2\varphi)h_{20} = \lambda v_2^2\xi\zeta_2 h_{20} = -\lambda v_2^2 h_{20}\xi\zeta_2$, and we have $\lambda = 1$. Now the lemma follows from the definition of δ_2'. ∎

6. Calculation of $H^0M_1^1$. We will compute $H^*M_1^1$ by Lemma 4.5, which requires the knowledge of the structure of $H^*K(2)_* = \mathrm{Ext}^*_\Sigma(E(2)_*, K(2)_*)$. Recall again (5.13):

$$H^{*,*}K(2)_* = \Lambda(\zeta_2) \otimes K(2)_*\{1, h_{11}, h_{20}, \xi, \varphi, h_{20}\xi\}.$$

These generators are represented by cocycles given in previous sections, and satisfy

$$\zeta_2 = v_2^{-1}t_2 + v_2^{-3}t_2^3, \quad h_{11} = t_1^3, \quad h_{20} = t_2,$$
$$\xi = v_2^{-10}t_1^3 \otimes t_3^3 + \ldots \quad \text{and} \quad \varphi = v_2^{-12}(t_2^3 - v_2^2t_2) \otimes t_3^3 + \ldots,$$

where $\ldots$ denotes elements of $\Sigma(2) \otimes_{K(2)_*} \Sigma(2)$ for

$$\Sigma(2) = \Lambda_{K(2)_*}(t_1^3) \otimes_{K(2)_*} K(2)_*[t_2]/(t_2^9 - v_2^8t_2).$$

Hereafter we use the notation:

$$k(1)_* = (\mathbb{Z}/3)[v_1] \quad \text{and} \quad K(1)_* = v_1^{-1}k(1)_* = (\mathbb{Z}/3)[v_1, v_1^{-1}],$$

and integers

$$a_0 = 1 \quad \text{and} \quad a_i = 4 \times 3^{i-1} \quad \text{for } i > 0.$$

Now we obtain the following

THEOREM 6.1. *$H^0M_1^1$ is the direct sum of $K(1)_*/k(1)_*$ and cyclic $k(1)_*$-modules isomorphic to $k(1)_*/(v_1^{a_i})$ generated by $x_i^s/v_1^{a_i}$ for $i \geq 0$ and $s \in \mathbb{Z} - 3\mathbb{Z}$.*

Proof. Let B denote the direct sum of $K(1)_*/k(1)_*$ and cyclic $k(1)_*$-modules isomorphic to $k(1)_*/(v_1^{a_i})$ generated by $x_i^s/v_1^{a_i}$ for $i \geq 0$ and $s \in$

$\mathbb{Z} - 3\mathbb{Z}$. Since $H^0K(2)_* = K(2)_*$ and $v_1^{a_i-1}(x_i^s/v_1^{a_i}) = v_2^{3^i s}/v_1$, we have the map $f : H^0K(2)_* \to B$ given by $f(x) = x/v_1$. By definition of B, $v_1 : B \to B$ is well defined. Now by Lemma 4.5, it suffices to show that the sequence

$$H^0K(2)_* \xrightarrow{f} B \xrightarrow{v_1} B \xrightarrow{\delta_0} H^1K(2)_*$$

is exact. Consider the diagram (4.6). By easy diagram chasing, we see that $H^0K(2)_* \xrightarrow{f} B \xrightarrow{v_1} B$ is exact, and that the composition $\delta_0 v_1$ is trivial. Now suppose that $\delta_0(b) = 0$ for $b \in B$. By the definition of B, $b = \sum \lambda_{s,i,j} g_{s,i,j}$ for $\lambda_{s,i,j} \in \mathbb{Z}/3$ and $g_{s,i,j} = x_i^s/v_1^j$. Note that

$$\delta_0(x/v_1^j) = [y] \quad \text{if} \quad d_0(x) \equiv v_1^j y \bmod (3, v_1^{j+1}),$$

where $[y]$ denotes the homology class of y. Then (5.2) yields

$$\delta_0(x_i^s/v_1^{a_i}) = \begin{cases} s v_2^{s-1} h_{11}, & i = 0, \\ s v_2^{3s-1}(v_2^{-1}h_{20} - \zeta_2), & i = 1, \\ s v_2^{3^i s - 3^{i-1}} \zeta_2^{3^{i-1}}, & i > 1. \end{cases} \tag{6.2}$$

Since $[v_1 y] = 0$, $g_{s,i,j} = 0$ if and only if $j = a_i$. Consequently, $\delta_0(b) = \sum_{s,i} \lambda_{s,i,a_i} \delta_0(g_{s,i,a_i})$, and $\lambda_{s,i,a_i} = 0$ for any i since the elements $\delta_0(g_{s,i,a_i})$ are linearly independent over $\mathbb{Z}/3$. Therefore, $b = \sum_{j<a_i} \lambda_{s,i,j} g_{s,i,j}$ and $v_1(b') = b$ for $b' = \sum_{j<a_i} \lambda_{s,i,j} g_{s,i,j+1} \in B$ as desired. ∎

7. Calculation of $H^1M_1^1$.

The results of the last section give

LEMMA 7.1. *The cokernel of* $\delta_0 : H^0M_1^1 \to H^1K(2)_*$ *is a* $\mathbb{Z}/3$*-vector space with the following basis*:

(a) $v_2^{3m+2}h_{11}$ *for* $m \in \mathbb{Z}$,
(b) $v_2^m h_{20}$ *for* $m \in \mathbb{Z}$,
(c) $v_2^{3^k s}\zeta_2$ *for* $k \geq 0$ *and* $s \equiv 1, 4, 7, 8 \bmod 9$ $(s \in \mathbb{Z})$.

In fact, by (5.13), $H^1K(2)_* = K(2)_*\{h_{11}, h_{20}, \zeta_2\}$. Therefore, this follows immediately from (6.2).

In the following, we write $v_2^m h_{11}/v_1^j$, $v_2^m h_{20}/v_1^j$ and $v_2^m \zeta_2/v_1^j$ for the homology classes represented by the cocycles $v_2^m t_1^3/v_1^j + \ldots$, $v_2^m t_2/v_1^j + \ldots$ and $v_2^m \zeta_2/v_1^j + \ldots$, respectively.

PROPOSITION 7.2. *The connecting homomorphism* $\delta_1 : H^1M_1^1 \to H^2K(2)_*$ *acts as follows*:

$$\begin{array}{llll}
\text{(i)} & \delta_1(v_2^{3m+2}h_{11}/v_1^3) \equiv v_2^{3m+1}h_{20}\zeta_2, & m\in\mathbb{Z}; \\
\text{(ii)} & \delta_1(h_{20}/v_1^j) = 0, & j>0, \\
& \delta_1(v_2^m h_{20}/v_1) \equiv -mv_2^m h_{11}\zeta_2, & m\in\mathbb{Z}-3\mathbb{Z}, \\
& \delta_1(v_2^{3^k m}h_{20}/v_1^{4\cdot 3^{k-1}}) \equiv mv_2^{3^{k-1}(3m-1)}h_{20}\zeta_2, & k>0,\ m\in\mathbb{Z}-3\mathbb{Z}; \\
\text{(iii)} & \delta_1(\zeta_2^{3^j}/v_1^j) = 0, & j>0, \\
& \delta_1(v_2^{3m+1}\zeta_2/v_1) = v_2^{3m}h_{11}\zeta_2, & m\in\mathbb{Z}, \\
& \delta_1(v_2^{3^k(3m+1)}\zeta_2/v_1^{6\cdot 3^{k-1}}) = \pm v_2^{3^{k+1}m+(3^k-1)/2}\xi, & k>0,\ m\in\mathbb{Z}, \\
& \delta_1(v_2^{3^k(9m+8)}\zeta_2/v_1^{10\cdot 3^k}) = \pm v_2^{3^k(9m+5)+(3^k-1)/2}\xi, & k\ge 0,\ m\in\mathbb{Z}.
\end{array}$$

P r o o f. First, note that

$$\delta_1(x/v_1^j) = [y] \quad \text{if} \quad d_1(x)\equiv v_1^j y \bmod (3, v_1^{j+1}).$$

Then (i) follows from (5.2) and Lemma 5.9 as follows:

$$\begin{aligned}
d_1(v_2^{3m+2}h_{11}) &\equiv v_2^{3m}d_1(v_2^2h_{11}) && \text{by (5.2)}\\
&\equiv -v_1^3v_2^{3m+1}t_2\otimes\zeta_2^3 && \text{by Lemma 5.9.}
\end{aligned}$$

In fact, $[-\zeta_2^3\otimes t_1^9] = [t_1^9\otimes\zeta_2^3] = [v_1v_2t_2\otimes\zeta_2^3] = v_1v_2h_{20}\zeta_2$.

For (ii), the first formula follows from $d_1(h_{20}) = 0$ and the second one from Lemma 5.7. The other follows from (5.2).

The first formula of (iii) follows from (5.4). The second one follows from (5.2) and (5.4), and the others from Lemma 5.11. ∎

THEOREM 7.3. *$H^1M_1^1$ is isomorphic to the direct sum of*

$$K(1)_*/k(1)_*\oplus K(1)_*/k(1)_*$$

generated by h_{20}/v_1^j and $\zeta_2^{3^j}/v_1^j$ $(j>0)$, and the cyclic $k(1)_*$-modules

(a) $k(1)_*/(v_1^3)\langle v_2^{3m+2}h_{11}\rangle$, $\quad m\in\mathbb{Z}$,

(b) $k(1)_*/(v_1^{a_k})\langle v_2^{3^k m}h_{20}\rangle$, $\quad k\ge 0,\ m\in\mathbb{Z}-3\mathbb{Z}$,

(c) $k(1)_*/(v_1^{A_k})\langle v_2^{3^k(3m+1)}\zeta_2\rangle$, $\quad k\ge 0,\ m\in\mathbb{Z}$,

(d) $k(1)_*/(v_1^{A'_k})\langle v_2^{3^k(9m+8)}\zeta_2\rangle$, $\quad k\ge 0,\ m\in\mathbb{Z}$,

where $a_0 = 1$, $a_k = 4\times 3^{k-1}$, $A_0 = 1$, $A_k = 6\times 3^{k-1}$ $(k>0)$ and $A'_k = 10\times 3^k$ $(k\ge 0)$.

In the same way as the proof of Theorem 6.1, we obtain this theorem from Lemmas 4.5 and 7.1, and Proposition 7.2.

8. Calculation of $H^2M_1^1$. Theorem 7.3 and Proposition 7.2 show the following:

LEMMA 8.1. *The cokernel of the connecting homomorphism* $\delta_1 : H^1M_1^1 \to H^2K(2)_*$ *is the* $\mathbb{Z}/3$*-vector space generated by*

$$\begin{array}{rll}
(0) & h_{20}\zeta_2, & \\
(\mathrm{i}) & v_2^{3^k(9m+8)}h_{20}\zeta_2, & k \geq 0,\ m \in \mathbb{Z}, \\
(\mathrm{ii}) & v_2^{3^k(3m+1)}h_{20}\zeta_2, & k > 0,\ m \in \mathbb{Z}, \\
(\mathrm{iii}) & v_2^u\xi, & u \in G \cup 3\mathbb{Z}, \\
(\mathrm{iv}) & v_2^m\varphi, & m \in \mathbb{Z}.
\end{array}$$

Here, $G = \{3^km + (3^k-1)/2 : k \geq 0, m \equiv 2, 8 \bmod 9\}$.

Note that

$$H^2K(2)_* = K(2)_*\{h_{11}\zeta_2, h_{20}\zeta_2, \xi, \varphi\}$$

by (5.13), and every integer n is written uniquely as

$$n = 3^km + (3^k-1)/2 \quad \text{with } k \geq 0 \text{ and } m \not\equiv 1 \bmod 3.$$

Using the facts shown in the previous sections, we obtain

PROPOSITION 8.2. *The connecting homomorphism* $\delta_2 : H^2M_1^1 \to H^3K(2)_*$ *acts as follows*:

(i)
$$de_2(h_{20}\zeta_2/v_1^j) = 0, \qquad j > 0,$$
$$\delta_2(v_2^{3^k(9m+8)}h_{20}\zeta_2/v_1^{A'_k}) = \pm v_2^{3^k(9m+5)+(3^k-1)/2}h_{20}\xi, \quad k \geq 0,\ m \in \mathbb{Z},$$
$$\delta_2(v_2^{3^k(3m+1)}h_{20}\zeta_2/v_1^{A_k}) = \pm v_2^{3^{k+1}m+(3^k-1)/2}h_{20}\xi, \qquad k > 0,\ m \in \mathbb{Z};$$

(ii)
$$\delta_2(v_2^{3m}\xi/v_1^3) = v_2^{3m-1}\varphi\zeta_2, \qquad m \in \mathbb{Z},$$
$$\delta_2(v_2^{3^k(9m+2)+(3^k-1)/2}\xi/v_1^{4\cdot 3^k}) = -v_2^{3^{k+2}m+(3^{k+1}-1)/2}\xi\zeta_2,$$
$$k \geq 0,\ m \in \mathbb{Z},$$
$$\delta_2(v_2^{3^k(9m+8)+(3^k-1)/2}\xi/v_1^{4\cdot 3^k}) = v_2^{3^{k+1}(3m+2)+(3^{k+1}-1)/2}\xi\zeta_2,$$
$$k \geq 0,\ m \in \mathbb{Z};$$

(iii)
$$\delta_2(v_2^{3m}\varphi/v_1) = v_2^{3m}(v_2^{-1}h_{20} - \zeta_2)\xi, \quad m \in \mathbb{Z},$$
$$\delta_2(v_2^{3m+1}\varphi/v_1) = -v_2^{3m}h_{20}\xi, \qquad m \in \mathbb{Z},$$
$$\delta_2(v_2^{3m+2}\varphi/v_1) = v_2^{3m+2}\xi\zeta_2, \qquad k > 0,\ m \in \mathbb{Z} - 3\mathbb{Z}.$$

Proof. Since $\delta_2(v_2^sh_{20}\zeta_2/v_1^j) = \delta_1(v_2^s\zeta_2/v_1^j)h_{20}$, we obtain (i) from Proposition 7.2.

The first formula of (ii) follows from Lemma 5.15. Let ξ_k for $k \ge 0$ denote the cocycles congruent to $v_2^{3^k\cdot 5+(3^k-1)/2}\xi \bmod (3, v_1)$ appearing in Corollary 5.12. Now compute

$$\begin{aligned}\delta_2(v_2^{3^k(9m+2)+(3^k-1)/2}\xi/v_1^{4\cdot 3^k}) &= \delta_2(v_2^{3^k(9m-3)}\xi_k/v_1^{4\cdot 3^k})\\ &= -v_2^{3^{k+2}m+(3^{k+1}-1)/2}\xi\zeta_2.\end{aligned}$$

In the same way we obtain the third formula.

For (iii), we compute

$$\begin{aligned}\delta_2(v_2^{3m}\varphi/v_1) = \delta_2(v_2^{3m-2}v_2^2\varphi/v_1) &= v_2^{3m-1}h_{11}\varphi + v_2^{3m}\xi\zeta_2\\ &= v_2^{3m-1}(h_{20}\xi + v_2\xi\zeta_2) + v_2^{3m}\xi\zeta_2.\end{aligned}$$

The others are shown in the same way. ■

Now we can state the main theorem of this section:

THEOREM 8.3. *$H^2M_1^1$ is isomorphic to the direct sum of $K(1)_*/k(1)_*$ generated by $h_{20}\otimes\zeta_2^{3^j}/v_1^j$ for $j > 0$, and the cyclic $k(1)_*$-modules*:

(a)	$k(1)_*/(v_1^{2\cdot 3^k})\langle v_2^{3^k(3m+1)}h_{20}\zeta_2\rangle$,	$k > 0,\ m \in \mathbb{Z}$,
	$k(1)_*/(v_1^{10\cdot 3^k})\langle v_2^{3^k(9m+8)}h_{20}\zeta_2\rangle$,	$k \ge 0,\ m \in \mathbb{Z}$,
(b)	$k(1)_*/(v_1^3)\langle v_2^{3m}\xi\rangle$,	$m \in \mathbb{Z}$,
	$k(1)_*/(v_1^{4\cdot 3^k})\langle v_2^{3^k m+(3^k-1)/2}\xi\rangle$,	$k \ge 0,\ m \equiv 2, 8 \bmod 9$,
(c)	$k(1)_*/(v_1)\langle v_2^m\varphi\rangle$,	$m \in \mathbb{Z}$.

The proof is the same as those of Theorems 6.1 and 7.3 by virtue of Proposition 8.2.

9. Calculation of $H^3M_1^1$.

Theorem 8.3 and Proposition 8.2 imply the following:

LEMMA 9.1. *The cokernel of the connecting homomorphism $\delta_2 : H^2M_1^1 \to H^3K(2)_*$ is the $\mathbb{Z}/3$-vector space generated by*

(i)	$v_2^u h_{20}\xi$,	$u \in G$,
(ii)	$v_2^s\varphi\zeta_2$,	$s+1 \in \mathbb{Z} - 3\mathbb{Z}$.

PROPOSITION 9.2. *The connecting homomorphism $\delta_3 : H^3M_1^1 \to H^4K(2)_*$ acts as follows*:

$$\begin{aligned}
&\text{(i)}\quad \delta_3(v_2^{3^k(9m+2)+(3^k-1)/2}h_{20}\xi/v_1^{4\cdot 3^k}) = -v_2^{3^{k+2}m+(3^{k+1}-1)/2}h_{20}\xi\zeta_2,\\
&\hspace{22em} k\ge 0,\ m\in\mathbb{Z},\\
&\quad\ \ \delta_3(v_2^{3^k(9m+8)+(3^k-1)/2}h_{20}\xi/v_1^{4\cdot 3^k}) = v_2^{3^{k+1}(3m+2)+(3^{k+1}-1)/2}h_{20}\xi\zeta_2,\\
&\hspace{22em} k\ge 0,\ m\in\mathbb{Z},\\
&\text{(ii)}\qquad\qquad \delta_3(v_2^{3m}\varphi\zeta_2/v_1) = v_2^{3m-1}h_{20}\xi\zeta_2,\qquad m\in\mathbb{Z},\\
&\qquad\qquad\quad \delta_3(v_2^{3m+1}\varphi\zeta_2/v_1) = -v_2^{3m}h_{20}\xi\zeta_2,\qquad m\in\mathbb{Z}.
\end{aligned}$$

Proof. Since $\delta_3(v_2^s h_{20}\xi/v_1^j) = \delta_2(v_2^s\xi/v_1^j)h_{20}$, (i) follows from Proposition 8.2. In the same way, we obtain (ii) from Proposition 8.2 by $\delta_3(v_2^s\varphi\zeta_2/v_1) = \delta_2(v_2^s\varphi/v_1)\zeta_2$. ■

THEOREM 9.3. *$H^3M_1^1$ is isomorphic to the direct sum of the cyclic $k(1)_*$-modules*

(a) $k(1)_*/(v_1^{4\cdot 3^k})\langle v_2^{3^k m+(3^k-1)/2}\xi h_{20}\rangle$, $\quad k\ge 0,\ m\equiv 2,8 \bmod 9$,
(b) $k(1)_*/(v_1)\langle v_2^m\varphi\zeta_2\rangle$, $\quad m+1\in\mathbb{Z}-3\mathbb{Z}$.

10. Calculation of $H^sM_1^1$ for $s>3$. Theorem 9.3 and Proposition 9.2 imply

LEMMA 10.1. *The cokernel of the connecting homomorphism $\delta_4 : H^3M_1^1 \to H^4K(2)_*$ is trivial.*

THEOREM 10.2. *$H^sM_1^1 = 0$ for all $s>3$.*

11. Calculation of $H^*E(2)/(3)$. We now summarize the results of the previous sections:

$$\begin{aligned}
&H^0M_1^1 = K(1)_*/k(1)_*\oplus M^0 &&\text{(Theorem 6.1)},\\
&H^1M_1^1 = K(1)_*/k(1)_*\oplus K(1)_*/k(1)_*\oplus M^1 &&\text{(Theorem 7.3)},\\
&H^2M_1^1 = K(1)_*/k(1)_*\oplus M^2 &&\text{(Theorem 8.3)},\\
&H^3M_1^1 = M^3 &&\text{(Theorem 9.3)},\\
&H^sM_1^1 = M^s\quad (s>3) &&\text{(Theorem 10.2)}.
\end{aligned}$$

Here M^s denotes the part of finite v_1-torsions of $H^sM_1^1$ for each s, which is expressed explicitly as follows:

$$\begin{aligned}
&M^0 = K\oplus L,\\
&M^1 = K\{h_{20}\}\oplus L\{h_{20}\}\oplus \widehat{K}\{v_2\zeta_2\}\oplus M\{\zeta_2\}\oplus N\{v_2^2h_{11}\},\\
&M^2 = K\{\varphi\}\oplus L\{\xi_k\}\oplus \widehat{K}\{\varphi\}\oplus M\{h_{20}\zeta_2\}\oplus N\{\xi\},\\
&M^3 = K\{v_2^{-1}\varphi\zeta_2\}\oplus L\{\xi_k h_{20}\},\\
&M^s = 0\quad \text{for } s>3,
\end{aligned}\tag{11.1}$$

where $\xi_k = v_2^{-(7\cdot 3^k+1)/2}\xi$, and

$$\widehat{K} = (\mathbb{Z}/3)[v_2^3, v_2^{-3}],$$

$$K = K(2)_* - \widehat{K},$$
$$L = \bigoplus_{k\ge 0} k(1)_*/(v_1^{4\cdot 3^k})\{x_{k+1}^m \mid m \in \mathbb{Z} - 3\mathbb{Z}\},$$
$$M = \bigoplus_{k\ge 0} k(1)_*/(v_1^{6\cdot 3^k})\{x_{k+1}^{3m+1} \mid m \in \mathbb{Z}\} \oplus \bigoplus_{k\ge 0} k(1)_*/(v_1^{10\cdot 3^k})\{x_k^{9m+8} \mid m \in \mathbb{Z}\},$$
$$N = k(1)_*/(v_1^3)\{v_2^{3m} \mid m \in \mathbb{Z}\}.$$

From the long exact sequence (4.2), we have the following:

THEOREM 11.2. *The E_2-term $H^*E(2)_*/(3)$ of the Adams–Novikov spectral sequence for computing $\pi_*(L_2M\wedge X)$ is the direct sum of $k(1)_*$-modules*:

(0) $H^0E(2)_*/(3) = k(1)_*$,
(1) $H^1E(2)_*/(3) = k(1)_*\{h_{20}\} \oplus M^0$,
(2) $H^2E(2)_*/(3) = K(1)_*/k(1)_* \oplus M^1$,
(3) $H^3E(2)_*/(3) = K(1)_*/k(1)_* \oplus M^2 = H^2M_1^1$,
(4) $H^4E(2)_*/(3) = M^3 = H^3M_1^1$,
(5) $H^sE(2)_*/(3) = 0$.

By the sparseness of the spectral sequence, the differential d_r is zero for $r < 5$. Theorem 11.2 shows that $d_r = 0$ for $r \ge 5$. Therefore, the spectral sequence collapses to the E_2-term. Furthermore, there is no extension problem, since $L_2M \wedge X$ is an M-module spectrum. Therefore we obtain the following:

THEOREM 11.3. *The homotopy groups $\pi_*(L_2M \wedge X)$ are isomorphic to the direct sum of the $k(1)_*$-module*

$$k(1)_*\{1, h_{20}\} \oplus K(1)_*/k(1)_*\{\zeta_2, h_{20}\zeta_2\}$$

and finite v_1-torsions $\bigoplus_{k=0}^3 M^k$.

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Received 30 August 1995;
in revised form 22 July 1996