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Normal subspaces in products of two ordinals

by

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Abstract. Let λ be an ordinal number. It is shown that normality, collectionwise normality and shrinking are equivalent for all subspaces of $(\lambda+1)^2$.

1. Introduction. It is well known that any ordinal with the order topology is shrinking and collectionwise normal hereditarily. But, in general, products of two ordinals are not. In fact, $(\omega_1+1)\times\omega_1$ is not normal. In [KOT], it was proved that the normality, collectionwise normality and shrinking property of $A\times B$, where A and B are subspaces of ordinals, are equivalent. It was asked whether these properties are also equivalent for *all* subspaces of products of two ordinals [KOT, Problem (i)]. The aim of this paper is to give an affirmative answer.

We recall some basic definitions and introduce some specific notation.

In our discussion, we always assume $X\subset(\lambda+1)^2$ for some suitably large ordinal λ. Moreover, in general, the letters μ and ν stand for limit ordinals with $\mu\le\lambda$ and $\nu\le\lambda$. For each $A\subset\lambda+1$ and $B\subset\lambda+1$ put

$$X_A=A\times(\lambda+1)\cap X,\quad X^B=(\lambda+1)\times B\cap X,$$

and

$$X_A^B=X_A\cap X^B.$$

For each $\alpha\le\lambda$ and $\beta\le\lambda$, put

$$V_\alpha(X)=\{\beta\le\lambda:\langle\alpha,\beta\rangle\in X\},\quad H_\beta(X)=\{\alpha\le\lambda:\langle\alpha,\beta\rangle\in X\}.$$

cf μ denotes the cofinality of the ordinal μ. When $\omega_1\le\operatorname{cf}\mu$, a subset S of μ called *stationary in* μ if it intersects all cub (closed and unbounded) sets

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in μ. For each $\mu \le \lambda$ and $\nu \le \lambda$ with $\omega_1 \le \mathrm{cf}\,\mu$ and $\omega_1 \le \mathrm{cf}\,\nu$, put

$$A_\mu^\nu = \{\alpha < \mu : V_\alpha(X) \cap \nu \text{ is stationary in } \nu\},$$
$$B_\mu^\nu = \{\beta < \nu : H_\beta(X) \cap \mu \text{ is stationary in } \mu\}.$$

Moreover, for each $A \subset \mu$, $\mathrm{Lim}_\mu(A)$ is the set $\{\alpha < \mu : \alpha = \sup(A \cap \alpha)\}$, in other words, the set of all cluster points of A in μ. Therefore $\mathrm{Lim}_\mu(A)$ is cub in μ whenever A is unbounded in μ. We will simply denote $\mathrm{Lim}_\mu(A)$ by $\mathrm{Lim}(A)$ if the situation is clear in its context.

A strictly increasing function $M : \mathrm{cf}\,\mu \to \mu$ is said to be *normal* if $M(\gamma) = \sup\{M(\gamma') : \gamma' < \gamma\}$ for each limit ordinal $\gamma < \mathrm{cf}\,\mu$, and $\mu = \sup\{M(\gamma) : \gamma < \mathrm{cf}\,\mu\}$. Note that a normal function on $\mathrm{cf}\,\mu$ always exists if $\mathrm{cf}\,\mu \ge \omega$. So we always fix a normal function $M : \mathrm{cf}\,\mu \to \mu$ for each ordinal μ with $\mathrm{cf}\,\mu \ge \omega$.

For convenience, we define $M(-1) = -1$. Then M carries $\mathrm{cf}\,\mu$ homeomorphically to the range $\mathrm{ran}\,M$ of M and $\mathrm{ran}\,M$ is closed in μ. Note that for all $S \subset \mu$ with $\omega_1 \le \mathrm{cf}\,\mu$, S is stationary in μ if and only if $M^{-1}(S)$ is stationary in $\mathrm{cf}\,\mu$.

Let μ and ν be two limit ordinals with $\mu \le \lambda$ and $\nu \le \lambda$; moreover, let $M : \mathrm{cf}\,\mu \to \mu$ and $N : \mathrm{cf}\,\nu \to \nu$ be the fixed normal functions on $\mathrm{cf}\,\mu$ and $\mathrm{cf}\,\nu$ respectively. For each $\alpha \in \mu$ and $\beta \in \nu$, define

$$m(\alpha) = \min\{\gamma < \mathrm{cf}\,\mu : \alpha \le M(\gamma)\},$$
$$n(\beta) = \min\{\delta < \mathrm{cf}\,\nu : \beta \le N(\delta)\},$$

where $\min A$ denotes the minimal ordinal number in A. Note that, if $\alpha \in \mathrm{ran}\,M$, then $m(\alpha) = M^{-1}(\alpha)$.

Furthermore, assume $\langle \mu, \nu\rangle \notin X$ and $\omega_1 \le \mathrm{cf}\,\mu = \mathrm{cf}\,\nu = \kappa$. We will use the following notation:

$$X(L, M, N) = \{\langle \alpha, \beta\rangle \in X \cap \mu \times \nu : m(\alpha) \le n(\beta)\} \cup X_\mu^{\{\nu\}},$$
$$X(R, M, N) = \{\langle \alpha, \beta\rangle \in X \cap \mu \times \nu : m(\alpha) \ge n(\beta)\} \cup X_{\{\mu\}}^\nu,$$
$$X(\triangle, M, N) = \{\langle M(\gamma), N(\gamma)\rangle \in X : \gamma < \kappa\},$$
$$\triangle_{MN}(X) = \{\gamma < \kappa : \langle M(\gamma), N(\gamma)\rangle \in X\}.$$

Intuitively, $X(L, M, N)$ is considered as the upper-left half of $X_{\mu+1}^{\nu+1}$, $X(R, M, N)$ as the lower-right half of $X_{\mu+1}^{\nu+1}$ and $X(\triangle, M, N)$ as the diagonal part of $X_{\mu+1}^{\nu+1}$. Since M and N are homeomorphic closed embeddings, observe that $X(\triangle, M, N)$ and $\triangle_{MN}(X)$ are homeomorphic and that $X(L, M, N)$, $X(R, M, N)$ and $X(\triangle, M, N)$ are closed in X.

Let Y be a topological space. Subsets F and G of Y are said to be *separated* if there are disjoint open sets U and V containing F and G respectively; of course, separated sets are disjoint, and $\emptyset$ and G are separated for each $G \subset Y$. More generally, a collection $\mathcal{H}$ of subsets of Y is said to be *separated* if there is a pairwise disjoint collection $\mathcal{U} = \{U(H) : H \in \mathcal{H}\}$ of open sets in Y such that each $U(H)$ contains H. A space Y is said to be *CWN* (CollectionWise Normal) if any discrete collection of closed sets is separated. Let $\mathcal{U}$ be an open cover of Y. A collection $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ of subsets of Y indexed by $\mathcal{U}$ is a *shrinking* of $\mathcal{U}$ if $F(U) \subset U$ for each $U \in \mathcal{U}$. A closed shrinking is a shrinking by closed sets. Throughout the paper, for convenience, we do not require $\mathcal{F}$ to cover Y. We call a space Y *shrinking* if each open cover of Y has a closed shrinking which covers Y.

2. Theorem and lemmas. Using the notation described in Section 1, we shall show:

THEOREM. *Assume $X \subset (\lambda+1)^2$. The following* (1)–(4) *are equivalent*:

(1) *X is shrinking.*
(2) *X is CWN.*
(3) *X is normal.*
(4) *For every $\langle \mu,\nu\rangle \in (\lambda+1)^2 \setminus X$ with $\omega \le \operatorname{cf}\mu$ and $\omega \le \operatorname{cf}\nu$, the following* (4-1)–(4-5) *hold*:

(4-1) *$X_{\{\mu\}}$ and $X^{\{\nu\}}$ are separated.*

(4-2) *If $\omega_1 \le \operatorname{cf}\nu$ and $V_\mu(X) \cap \nu$ is not stationary in ν, then there is a cub set D in $\operatorname{cf}\nu$ such that $X_{\{\mu\}}$ and $X^{N(D)\cup\{\nu\}}$ are separated.*

(4-3) *If $\omega_1 \le \operatorname{cf}\mu$ and $H_\nu(X) \cap \mu$ is not stationary in μ, then there is a cub set C in $\operatorname{cf}\mu$ such that $X^{\{\nu\}}$ and $X_{M(C)\cup\{\mu\}}$ are separated.*

(4-4) *If ($\omega_1 \le \operatorname{cf}\mu < \operatorname{cf}\nu$, $V_\mu(X)\cap\nu$ is not stationary in ν, and both $H_\nu(X)\cap\mu$ and A^ν_μ are non-stationary in μ) or ($\omega_1 \le \operatorname{cf}\nu < \operatorname{cf}\mu$, $H_\nu(X)\cap\mu$ is not stationary in μ, and both $V_\mu(X)\cap\nu$ and B^ν_μ are non-stationary in ν), then there are cub sets C in $\operatorname{cf}\mu$ and D in $\operatorname{cf}\nu$ such that $X_{M(C)\cup\{\mu\}}$ and $X^{N(D)\cup\{\nu\}}$ are separated.*

(4-5) *If $\omega_1 \le \operatorname{cf}\mu = \operatorname{cf}\nu = \kappa$, then* (4-5-a) *and* (4-5-b) *hold.*

(4-5-a) *$X(\triangle, M, N)$ and $X_{\{\mu\}} \cup X^{\{\nu\}}$ are separated.*

(4-5-b) *If $\triangle_{MN}(X)$ is not stationary in κ, then* (b1)–(b4) *hold*:

(b1) *If $V_\mu(X)\cap\nu$ is stationary in ν, then $X_{\{\mu\}}$ and any closed set disjoint from $X_{\{\mu\}}$ are separated.*

(b2) *If $V_\mu(X)\cap\nu$ is not stationary in ν, then there is a cub set D in κ such that the sets $X(R,M,N)_{M(D)\cup\{\mu\}}$ and $X(R,M,N)^{N(D)\cup\{\nu\}}$ are separated.*

(b3) *If $H_\nu(X)\cap\mu$ is stationary in μ, then $X^{\{\nu\}}$ and any closed set disjoint from $X^{\{\nu\}}$ are separated.*

(b4) *If $H_\nu(X)\cap\mu$ is not stationary in μ, then there is a cub set C in κ such that the sets $X(L,M,N)^{N(C)\cup\{\nu\}}$ and $X(L,M,N)_{M(C)\cup\{\mu\}}$ are separated.*

To prove the theorem, we need several lemmas. First it is straightforward to show:

LEMMA 1. *Let X be the finite union of closed subspaces X_i $(i \in n)$.*

(1) *Let $\mathcal{U}$ be an open cover of X. If $\mathcal{U}|X_i = \{U \cap X_i : U \in \mathcal{U}\}$ has a closed shrinking covering X_i for each $i \in n$, then $\mathcal{U}$ has a closed shrinking which covers X.*

(2) *Let $\mathcal{H}$ be a discrete collection of closed sets in X. If $\mathcal{H}|X_i$ is separated in X_i for each $i \in n$, then $\mathcal{H}$ is separated in X.*

This lemma implies:

LEMMA 2. *If X is the union of two normal* (*shrinking*, *CWN*) *open subspaces Y and Z such that $X \setminus Y$ and $X \setminus Z$ are separated*, *then X is normal* (*shrinking*, *CWN*).

LEMMA 3. *Assume $\omega_1 \le \operatorname{cf}\mu < \operatorname{cf}\nu$ and $X \subset (\mu+1)\times(\nu+1)\setminus\{\langle\mu,\nu\rangle\}$. If A^ν_μ is not stationary in μ, then there are cub sets C in $\operatorname{cf}\mu$ and D in $\operatorname{cf}\nu$ such that*

$$X \cap M(C) \times N(D) = \emptyset.$$

Proof. Assume A^ν_μ is not stationary in μ. Take a cub set C in $\operatorname{cf}\mu$ such that $M(C) \cap A^\nu_\mu = \emptyset$. For each $\gamma \in C$, by the non-stationarity of $V_{M(\gamma)}(X) \cap \nu$, fix a cub set D_γ in $\operatorname{cf}\mu$ such that $V_{M(\gamma)}(X) \cap N(D_\gamma) = \emptyset$. Put $D = \bigcap_{\gamma\in C} D_\gamma$. Since $|C| \le \operatorname{cf}\mu < \operatorname{cf}\nu$, D is cub in $\operatorname{cf}\nu$. Then these cub sets C and D work. ∎

In an analogous way, we can show:

LEMMA 3′. *Assume $\omega_1 \le \operatorname{cf}\nu < \operatorname{cf}\mu$ and $X \subset (\mu+1)\times(\nu+1)\setminus\{\langle\mu,\nu\rangle\}$. If B^ν_μ is not stationary in ν, then there are cub sets C in $\operatorname{cf}\mu$ and D in $\operatorname{cf}\nu$ such that*

$$X \cap M(C) \times N(D) = \emptyset.$$

Hereafter, we will not write down such analogous lemmas, but refer to them as "the analogues" of Lemmas 5–9.

LEMMA 4. *Assume $\omega_1 \le \operatorname{cf}\nu = \operatorname{cf}\mu = \kappa$ and $X \subset (\mu+1)\times(\nu+1)\setminus\{\langle\mu,\nu\rangle\}$. If X is normal and $\triangle_{MN}(X)$ is not stationary in κ, then there is a cub set C in κ such that*

$$X \cap M(C) \times N(C) = \emptyset.$$

Proof. First we show A^ν_μ is not stationary in μ. Assume, on the contrary, that A^ν_μ is stationary in μ. Then $A = M^{-1}(A^\nu_\mu) \cap \operatorname{Lim}(\kappa)$ is stationary in κ. For each $\gamma \in A$, pick

$$h(\gamma) \in N^{-1}(V_{M(\gamma)}(X)) \cap \bigcap_{\gamma'\in A\cap\gamma} \operatorname{Lim}(N^{-1}(V_{M(\gamma')}(X))) \cap \operatorname{Lim}(\kappa)$$

with $\gamma < h(\gamma) < \kappa$. This can be done, because $N^{-1}(V_{M(\gamma)}(X))$ is stationary in κ, $\mathrm{Lim}(N^{-1}(V_{M(\gamma')}(X)))$ is cub in κ for each $\gamma' \in A \cap \gamma$, $|A \cap \gamma| < \kappa$ and $\mathrm{Lim}(\kappa) = \mathrm{Lim}_\kappa(\kappa)$ is cub in κ, so the intersection is stationary in κ. For each $\gamma \in \kappa \setminus A$, put $h(\gamma) = 0$. Take a cub set C' in κ disjoint from $\triangle_{MN}(X)$, and put

$$C = \{\gamma < \kappa : \forall \gamma' < \gamma\, (h(\gamma') < \gamma)\} \cap C'.$$

Since C is cub in κ and A is stationary in κ, $A' = A \cap C$ is stationary in κ. For each $\gamma \in A'$, put $x_\gamma = \langle M(\gamma), N(h(\gamma))\rangle$. Since, by the definition of $h(\gamma)$, $N(h(\gamma)) \in V_{M(\gamma)}(X)$, we have $x_\gamma \in X$ for each $\gamma \in A'$.

Claim 1. *$F = \{x_\gamma : \gamma \in A'\}$ is closed discrete in X.*

Proof. Note that $F \subset M(C) \times \mathrm{ran}\, N$. Let $\langle \alpha, \beta\rangle \in X$. We will find an open neighborhood U of $\langle \alpha, \beta\rangle$ which intersects F in at most one point.

Case 1. *$\alpha \in \mu \setminus M(C)$ or $\beta \in \nu \setminus \mathrm{ran}\, N$.* If $\alpha \in \mu \setminus M(C)$, then, by the closedness of $M(C)$ in μ, there is $\alpha' < \alpha$ such that $(\alpha', \alpha] \cap M(C) = \emptyset$. Then $U = (\alpha', \alpha] \times (\nu + 1) \cap X$ is a neighborhood of $\langle \alpha, \beta\rangle$ missing F.

If $\beta \in \nu \setminus \mathrm{ran}\, N$, then there is $\beta' < \beta$ such that $(\beta', \beta] \cap \mathrm{ran}\, N = \emptyset$. Then $U = (\mu + 1) \times (\beta', \beta] \cap X$ is as desired.

Case 2. *Otherwise*, *i.e.*, *$\alpha \in M(C) \cup \{\mu\}$ and $\beta \in \mathrm{ran}\, N \cup \{\nu\}$.* There are two subcases.

(2-1): *$\alpha \in M(C) \cup \{\mu\}$ and $\beta \in \mathrm{ran}\, N$.* If $\alpha > M(n(\beta))$, then put $U = (M(n(\beta)), \alpha] \times [0, \beta] \cap X$. Assume $U \ni \langle M(\gamma), N(h(\gamma))\rangle$ for some $\gamma \in A'$. Then we have $n(\beta) < \gamma$ and $N(h(\gamma)) \le \beta$ (thus $h(\gamma) \le n(\beta)$). Therefore $h(\gamma) < \gamma$. But this contradicts the definition of $h(\gamma)$. So $U \cap F = \emptyset$.

If $\alpha \le M(n(\beta))$, then, since $M(n(\beta)) < \mu$, we have $\alpha \in M(C)$ in this case. Therefore, as $\alpha = M(m(\alpha)) \le M(n(\beta))$, we have $m(\alpha) \le n(\beta)$. Assume $m(\alpha) = n(\beta)$. Since $\langle M(m(\alpha)), N(n(\beta))\rangle = \langle \alpha, \beta\rangle \in X$, it follows that $m(\alpha) = n(\beta) \in \triangle_{MN}(X)$. On the other hand, since $m(\alpha) \in C \subset C' \subset \kappa \setminus \triangle_{MN}(X)$, we get a contradiction. Hence we have $m(\alpha) < n(\beta)$. Put $U = [0, \alpha] \times (N(m(\alpha)), \beta] \cap X$. Assume $U \ni x_\gamma = \langle M(\gamma), N(h(\gamma))\rangle$ for some $\gamma \in A'$ with $m(\alpha) \ne \gamma$. As $M(\gamma) \le \alpha = M(m(\alpha))$ and $m(\alpha) \ne \gamma$, we have $\gamma < m(\alpha)$. Since $\gamma < m(\alpha) \in C$, we get $h(\gamma) < m(\alpha)$. On the other hand, from $N(m(\alpha)) < N(h(\gamma))$ it follows that $m(\alpha) < h(\gamma)$. This is a contradiction. This argument implies $U \cap F \subset \{x_{m(\alpha)}\}$.

(2-2): *$\alpha \in M(C) \cup \{\mu\}$ and $\beta = \nu$.* Since $\langle \alpha, \beta\rangle \in X$ but $\langle \mu, \nu\rangle \notin X$, we have $\alpha \in M(C)$. Put $U = [0, \alpha] \times (N(m(\alpha)), \beta] \cap X$. Then $|U \cap F| \le 1$ as above.

This completes the proof of Claim 1.

Decompose A' into disjoint stationary sets T_0 and T_1 in κ, and put $F_i = \{x_\gamma : \gamma \in T_i\}$ for $i \in 2 = \{0, 1\}$. Let U_i be an open set containing F_i for each $i \in 2$.

Claim 2. $\mathrm{Cl}\, U_0 \cap \mathrm{Cl}\, U_1 \neq \emptyset$.

Proof. For each $\gamma \in T_i$ with $i \in 2$, since $x_\gamma = \langle M(\gamma), N(h(\gamma))\rangle \in U_i$ and γ and $h(\gamma)$ are in $\mathrm{Lim}(\kappa)$, there are $f(\gamma) < \gamma$ and $g(\gamma) < h(\gamma)$ such that $\gamma \leq g(\gamma)$ and

$$(M(f(\gamma)), M(\gamma)] \times (N(g(\gamma)), N(h(\gamma))] \cap X \subset U_i.$$

By the PDL, for each $i \in 2$, there are $\zeta_i < \kappa$ and a stationary set $T_i' \subset T_i$ such that $f(\gamma) = \zeta_i$ for each $\gamma \in T_i'$. Put $\gamma_0 = \max\{\zeta_0, \zeta_1\}$. Then

$$(M(\gamma_0), M(\gamma)] \times (N(g(\gamma)), N(h(\gamma))] \cap X \subset U_i$$

for each $i \in 2$ and $\gamma \in T_i'$.

Take γ_1 and γ_2 such that $\gamma_0 < \gamma_1 \in A$ and $\gamma_1 < \gamma_2 \in \bigcap_{i\in 2} \mathrm{Lim}(T_i')$. We shall show $\langle M(\gamma_1), N(\gamma_2)\rangle \in \mathrm{Cl}\, U_0 \cap \mathrm{Cl}\, U_1$. To see this, let V be a neighborhood of $\langle M(\gamma_1), N(\gamma_2)\rangle$. As $\gamma_2 \in \mathrm{Lim}(\kappa)$, there is $\gamma_3 < \gamma_2$ with $\gamma_1 \leq \gamma_3$ such that $\{M(\gamma_1)\} \times (N(\gamma_3), N(\gamma_2)] \cap X \subset V$. Then, since $\gamma_2 \in \mathrm{Lim}(T_0')$, there are γ_4 and γ_5 in T_0' with $\gamma_3 < \gamma_4 < \gamma_5 < \gamma_2$. Since $\gamma_5 \in T_0' \subset A' \subset C$, the definition of C yields $\gamma_4 < h(\gamma_4) < \gamma_5$. As $\gamma_1 \in A \cap \gamma_4$, the definition of $h(\gamma_4)$ shows that $h(\gamma_4) \in \mathrm{Lim}(N^{-1}(V_{M(\gamma_1)}(X)))$. Then, since $\gamma_4 \leq g(\gamma_4) < h(\gamma_4)$, there is $\gamma_6 \in N^{-1}(V_{M(\gamma_1)}(X))$ such that $g(\gamma_4) < \gamma_6 < h(\gamma_4)$. Finally,

$$\langle M(\gamma_1), N(\gamma_6)\rangle \in \{M(\gamma_1)\} \times (N(\gamma_3), N(\gamma_2)] \cap (M(\gamma_0), M(\gamma_4)] \times (N(g(\gamma_4)), N(h(\gamma_4))] \cap X \subset V \cap U_0.$$

This means $\langle M(\gamma_1), N(\gamma_2)\rangle \in \mathrm{Cl}\, U_0$. Similarly we have $\langle M(\gamma_1), N(\gamma_2)\rangle \in \mathrm{Cl}\, U_1$. This completes the proof of Claim 2.

Claim 2 contradicts the normality of X. Therefore A^ν_μ is not stationary in μ. By a similar argument, B^ν_μ is not stationary in ν.

Finally, since $\triangle_{MN}(X)$ is not stationary in κ, take a cub set D in κ such that $D \cap [M^{-1}(A^\nu_\mu) \cup N^{-1}(B^\nu_\mu) \cup \triangle_{MN}(X)] = \emptyset$. For each $\gamma \in D$, since $V_{M(\gamma)}(X) \cap \nu$ is not stationary in ν and $H_{N(\gamma)}(X) \cap \mu$ is not stationary in μ, we can take a cub set C_γ in κ disjoint from $N^{-1}(V_{M(\gamma)}(X)) \cup M^{-1}(H_{N(\gamma)}(X))$. Then by an argument similar to [Ku, II, Lemma 6.14], the diagonal intersection

$$E = \{\delta \in D : \forall \gamma \in D \cap \delta\, (\delta \in C_\gamma)\}$$

is cub in κ. Assume $\langle M(\gamma), N(\delta)\rangle \in X$ for some γ and δ in E. Since D is disjoint from $\triangle_{MN}(X)$ and $E \subset D$, we have $\gamma \neq \delta$. So we may assume $\gamma < \delta$. Then since $\gamma \in D \cap \delta$ and $\delta \in E$, we have $\delta \in C_\gamma$, and thus $N(\delta) \notin V_{M(\gamma)}(X)$. This contradicts $\langle M(\gamma), N(\delta)\rangle \in X$. This means $X \cap M(E) \times N(E) = \emptyset$. This completes the proof of Lemma 4. ■

Lemma 5. *Assume $\omega_1 \leq \mathrm{cf}\, \nu \neq \mathrm{cf}\, \mu$ and $X \subset (\mu + 1) \times \nu$. If $V_\mu(X) \cap \nu$ is stationary in ν, then the following hold*:

(1) *For each open cover* $\mathcal{U}$ *of* X, *there are* $\mu' < \mu$, $\nu' < \nu$ *and a shrinking* $\mathcal{F}$ *of* $\mathcal{U}$ *by clopen sets in* X *such that* $\bigcup \mathcal{F} = (\mu', \mu] \times (\nu', \nu) \cap X$.

(2) *For each discrete collection* $\mathcal{H}$ *of closed sets in* X, *there are* $\mu' < \mu$ *and* $\nu' < \nu$ *such that* $(\mu', \mu] \times (\nu', \nu) \cap X$ *meets at most one member of* $\mathcal{H}$.

Proof. (1) For each $\delta \in N^{-1}(V_\mu(X)) \cap \mathrm{Lim}(\mathrm{cf}\,\nu)$, fix $f(\delta) < \mathrm{cf}\,\mu$, $g(\delta) < \delta$ and $U(\delta) \in \mathcal{U}$ such that $(M(f(\delta)), \mu] \times (N(g(\delta)), N(\delta)] \cap X \subset U(\delta)$. Applying the PDL, we can find $\delta_0 < \mathrm{cf}\,\nu$ and a stationary set $S' \subset N^{-1}(V_\mu(X)) \cap \mathrm{Lim}(\mathrm{cf}\,\nu)$ such that $g(\delta) = \delta_0$ for each $\delta \in S'$. If $\mathrm{cf}\,\mu > \mathrm{cf}\,\nu$, then put $\gamma_0 = \sup\{f(\delta) : \delta \in S'\}$ and $S = S'$. If $\mathrm{cf}\,\mu < \mathrm{cf}\,\nu$, then, again applying the PDL, we find a stationary set $S \subset S'$ and $\gamma_0 < \mathrm{cf}\,\mu$ such that $f(\delta) = \gamma_0$ for each $\delta \in S$. In either case, putting $\mu' = M(\gamma_0)$ and $\nu' = N(\delta_0)$, we have found a stationary set $S \subset N^{-1}(V_\mu(X)) \cap \mathrm{Lim}(\mathrm{cf}\,\nu)$ such that $(\mu', \mu] \times (\nu', N(\delta)] \cap X \subset U(\delta)$ for each $\delta \in S$.

For each δ and δ' in S, define $\delta \sim \delta'$ by $U(\delta) = U(\delta')$. Then $\sim$ is an equivalence relation on S, so let $S/{\sim}$ be its quotient space. For each $E \in S/{\sim}$, put $U_E = U(\delta)$ for some (any) $\delta \in E$. Note that members of $\{U_E : E \in S/{\sim}\}$ are all distinct. There are two cases to consider.

First assume that there is $E \in S/{\sim}$ such that E is unbounded in $\mathrm{cf}\,\nu$. In this case, since $(\mu', \mu] \times (\nu', N(\delta)] \cap X \subset U(\delta) = U_E$ for each $\delta \in E$, we have $(\mu', \mu] \times (\nu', \nu) \cap X \subset U_E$. For each $U \in \mathcal{U}$, put

$$F(U) = \begin{cases} (\mu', \mu] \times (\nu', \nu) \cap X & \text{if } U = U_E, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ is the desired shrinking of $\mathcal{U}$.

Next assume all E's, $E \in S/{\sim}$, are bounded in $\mathrm{cf}\,\nu$. By induction, define $\delta(\eta) \in E(\eta) \in S/{\sim}$ for each $\eta \in \mathrm{cf}\,\nu$ so that $\eta + \sup(\bigcup_{\zeta<\eta} E(\zeta)) < \delta(\eta)$. Clearly $E(\eta)$'s are all distinct and $\{\delta(\eta) : \eta < \mathrm{cf}\,\nu\}$ is strictly increasing and unbounded in $\mathrm{cf}\,\nu$. For each $U \in \mathcal{U}$, put

$$F(U) = \begin{cases} (\mu', \mu] \times (\nu', N(\delta(\eta))] \cap X & \text{if } U = U_{E(\eta)} \text{ for some } \eta < \mathrm{cf}\,\nu, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ is the desired shrinking of $\mathcal{U}$.

(2) For each $\delta \in N^{-1}(V_\mu(X)) \cap \mathrm{Lim}(\mathrm{cf}\,\nu)$, fix $f(\delta) < \mathrm{cf}\,\mu$ and $g(\delta) < \delta$ such that $(M(f(\delta)), \mu] \times (N(g(\delta)), N(\delta)] \cap X$ meets at most one member of $\mathcal{H}$. Then as in (1), we can find desired $\nu' < \nu$ and $\mu' < \mu$. ∎

LEMMA 6. *Assume* $\omega_1 \le \mathrm{cf}\,\nu \ne \mathrm{cf}\,\mu$, $X \subset (\mu+1) \times (\nu+1) \setminus \{\langle \mu, \nu \rangle\}$ *and* $V_\mu(X) \cap \nu$ *is stationary in* ν. *If* $X_{\{\mu\}}$ *and* $X^{\{\nu\}}$ *are separated*, *then there are* $\mu' < \mu$ *and* $\nu' < \nu$ *such that* $(\mu', \mu] \times (\nu', \nu) \cap X$ *is closed* (*and trivially open*) *in* X.

Proof. Since $X_{\{\mu\}}$ and $X^{\{\nu\}}$ are separated, take an open set V such that $X_{\{\mu\}} \subset V \subset \mathrm{Cl}\,V \subset X \setminus X^{\{\nu\}}$. For each $\delta \in N^{-1}(V_\mu(X)) \cap \mathrm{Lim}(\mathrm{cf}\,\nu)$, fix

$f(\delta) < \operatorname{cf}\mu$ and $g(\delta) < \delta$ such that $(M(f(\delta)), \mu] \times (N(g(\delta)), N(\delta)] \cap X \subset V$. Then as in Lemma 5, we can find $\mu' < \mu$ and $\nu' < \nu$ such that $(\mu', \mu] \times (\nu', \nu) \cap X \subset V$. Since $\operatorname{Cl} V \cap X^{\{\nu\}} = \emptyset$, we conclude that $(\mu', \mu] \times (\nu', \nu) \cap X$ is closed in X. ■

LEMMA 7. *Let $\mathcal{P}$ be a topological property which is closed under taking closed subspaces and free unions. Assume $X \subset (\mu + 1) \times (\nu + 1)$ and $X_{\mu'+1}$ has the property $\mathcal{P}$ for each $\mu' < \mu$.*

(1) *If $\operatorname{cf}\mu = \omega$, then X_μ has the property $\mathcal{P}$.*

(2) *If $\operatorname{cf}\mu \geq \omega_1$ and C is a cub set in $\operatorname{cf}\mu$ and V is an open set in X containing $X_{M(C)\cup\{\mu\}}$, then $X \setminus V$ has the property $\mathcal{P}$.*

Proof. (1) Since $X_\mu = \bigoplus_{n\in\omega} X_{(M(n-1),M(n)]}$ and $X_{(M(n-1),M(n)]}$ is a closed subspace of $X_{M(n)+1}$, X_μ has the property $\mathcal{P}$.

(2) For each $\gamma \in C$, put $h(\gamma) = \sup(C \cap \gamma)$. Note that $h(\gamma) < \gamma$ if $\gamma \in C \setminus \operatorname{Lim}(C)$. For each $\gamma \in C \setminus \operatorname{Lim}(C)$, put $Y(\gamma) = X_{(M(h(\gamma)),M(\gamma)]} \setminus V$. Since $Y(\gamma)$ is a closed subspace of $X_{M(\gamma)+1}$, it has the property $\mathcal{P}$. Therefore $X \setminus V = \bigoplus_{\gamma\in C\setminus\operatorname{Lim}(C)} Y(\gamma)$ has the property $\mathcal{P}$. ■

LEMMA 8. *Assume $\omega_1 \leq \operatorname{cf}\mu < \operatorname{cf}\nu$, $X \subset (\mu + 1) \times (\nu + 1) \setminus \{\langle\mu, \nu\rangle\}$ and A^ν_μ is stationary in μ. If there are cub sets C in $\operatorname{cf}\mu$ and D in $\operatorname{cf}\nu$ such that $X_{M(C)\cup\{\mu\}}$ and $X^{\{\nu\}}$ are separated, and $X^{N(D)\cup\{\nu\}}$ and $X_{\{\mu\}}$ are separated, then there are $\mu' < \mu$ and $\nu' < \nu$ such that $(\mu', \mu) \times (\nu', \nu) \cap X$ is closed* (*and trivially open*) *in X.*

Proof. Take open sets V and W in X such that

$$X_{M(C)\cup\{\mu\}} \subset V \subset \operatorname{Cl} V \subset X \setminus X^{\{\nu\}},$$
$$X^{N(D)\cup\{\nu\}} \subset W \subset \operatorname{Cl} W \subset X \setminus X_{\{\mu\}}.$$

First fix $\gamma \in C \cap M^{-1}(A^\nu_\mu) \cap \operatorname{Lim}(\operatorname{cf}\mu)$. For each $\delta \in D \cap N^{-1}(V_{M(\gamma)}(X)) \cap \operatorname{Lim}(\operatorname{cf}\nu)$, since $\langle M(\gamma), N(\delta)\rangle \in V \cap W$, fix $f(\gamma, \delta) < \gamma$ and $g(\gamma, \delta) < \delta$ such that

$$(M(f(\gamma, \delta)), M(\gamma)] \times (N(g(\gamma, \delta)), N(\delta)] \cap X \subset V \cap W.$$

Since $f(\gamma, \delta) < \gamma$ and $g(\gamma, \delta) < \delta$ for each $\delta \in D \cap N^{-1}(V_{M(\gamma)}(X)) \cap \operatorname{Lim}(\operatorname{cf}\nu)$, noting that $\operatorname{cf}\mu < \operatorname{cf}\nu$ and applying the PDL, we have $f(\gamma) < \gamma$, $g(\gamma) < \operatorname{cf}\nu$ and a stationary set $S_\gamma \subset D \cap N^{-1}(V_{M(\gamma)}(X)) \cap \operatorname{Lim}(\operatorname{cf}\nu)$ such that $f(\gamma, \delta) = f(\gamma)$ and $g(\gamma, \delta) = g(\gamma)$ for each $\delta \in S_\gamma$. Put $\delta_0 = \sup\{g(\gamma) : \gamma \in C \cap M^{-1}(A^\nu_\mu) \cap \operatorname{Lim}(\operatorname{cf}\mu)\}$.

Next, since $f(\gamma) < \gamma$ for each $\gamma \in C \cap M^{-1}(A^\nu_\mu) \cap \operatorname{Lim}(\operatorname{cf}\mu)$, again applying the PDL, we have $\gamma_0 < \operatorname{cf}\mu$ and a stationary set $T \subset C \cap M^{-1}(A^\nu_\mu) \cap \operatorname{Lim}(\operatorname{cf}\mu)$ such that $f(\gamma) = \gamma_0$ for each $\gamma \in T$. Then we have

$$(M(\gamma_0), \mu) \times (N(\delta_0), \nu) \cap X \subset V \cap W.$$

Put $\mu' = M(\gamma_0)$ and $\nu' = N(\delta_0)$. Since $\mathrm{Cl}\, V \cap \mathrm{Cl}\, W$ is disjoint from $X_{\{\mu\}} \cup X^{\{\nu\}}$, we conclude that $(\mu', \mu) \times (\nu', \nu) \cap X$ is closed in X. ∎

LEMMA 9. *Assume $\omega_1 \le \mathrm{cf}\, \mu < \mathrm{cf}\, \nu$, $X \subset \mu \times \nu$ and A^ν_μ is stationary in μ.*

(1) *If $\mathcal{U}$ is an open cover of X, then there are $\mu' < \mu$, $\nu' < \nu$ and a shrinking $\mathcal{F}$ of $\mathcal{U}$ by clopen sets in X such that $\bigcup \mathcal{F} = (\mu', \mu) \times (\nu', \nu) \cap X$.*

(2) *If $\mathcal{H}$ is a discrete collection of closed sets in X, there are $\mu' < \mu$ and $\nu' < \nu$ such that $(\mu', \mu) \times (\nu', \nu) \cap X$ meets at most one member of $\mathcal{H}$.*

Proof. (1) First fix $\gamma \in M^{-1}(A^\nu_\mu) \cap \mathrm{Lim}(\mathrm{cf}\, \mu)$. For each $\delta \in N^{-1}(V_{M(\gamma)}(X)) \cap \mathrm{Lim}(\mathrm{cf}\, \nu)$, using $\langle M(\gamma), N(\delta)\rangle \in X$, fix $f(\gamma, \delta) < \gamma$, $g(\gamma, \delta) < \delta$ and $U(\gamma, \delta) \in \mathcal{U}$ such that

$$(M(f(\gamma, \delta)), M(\gamma)] \times (N(g(\gamma, \delta)), N(\delta)] \cap X \subset U(\gamma, \delta).$$

As in the proof of Lemma 8, applying the PDL twice, we find a stationary set $T \subset M^{-1}(A^\nu_\mu) \cap \mathrm{Lim}(\mathrm{cf}\, \mu)$, a stationary set $S_\gamma \subset N^{-1}(V_{M(\gamma)}(X)) \cap \mathrm{Lim}(\mathrm{cf}\, \nu)$ for each $\gamma \in T$, $\mu' < \mu$ and $\nu' < \nu$ such that $(\mu', M(\gamma)] \times (\nu', N(\delta)] \cap X \subset U(\gamma, \delta)$ for each $\delta \in S_\gamma$ with $\gamma \in T$.

Put $H = \bigcup_{\gamma \in T} \{\gamma\} \times S_\gamma$. For each $\langle \gamma, \delta\rangle$ and $\langle \gamma', \delta'\rangle$ in H, define $\langle \gamma, \delta\rangle \sim \langle \gamma', \delta'\rangle$ by $U(\gamma, \delta) = U(\gamma', \delta')$. For each $E \in H/{\sim}$, define $U_E = U(\gamma, \delta)$ for some (any) $\langle \gamma, \delta\rangle \in E$. Then note that

$$\text{(i)} \qquad \bigcup_{\langle \gamma, \delta\rangle \in E} (\mu', M(\gamma)] \times (\nu', N(\delta)] \cap X \subset U_E.$$

For each $\gamma \in T$ and $E \in H/{\sim}$, put

$$j(E, \gamma) = \sup\{\delta \in S_\gamma : \langle \gamma, \delta\rangle \in E\}.$$

Then put $T(E) = \{\gamma \in T : j(E, \gamma) = \mathrm{cf}\, \nu\}$ and $k(E) = \sup T(E)$.

CLAIM 1. $(\mu', M(\gamma)] \times (\nu', \nu) \cap X \subset U_E$ *for each* $\gamma \in T(E)$.

Proof. Assume $\langle \alpha, \beta\rangle \in (\mu', M(\gamma)] \times (\nu', \nu) \cap X$ with $\gamma \in T(E)$. Since $\beta < \nu$ and $\gamma \in T(E)$, there is a $\delta \in S_\gamma$ with $\langle \gamma, \delta\rangle \in E$ such that $\beta < N(\delta)$. Then, by (i), $\langle \alpha, \beta\rangle \in U_E$. This completes the proof of Claim 1.

There are some cases to consider.

Case 1: *There is an $E \in H/{\sim}$ such that $k(E) = \mathrm{cf}\, \mu$.* In this case, by Claim 1, $(\mu', \mu) \times (\nu', \nu) \cap X \subset U_E$. So for each $U \in \mathcal{U}$, put

$$F(U) = \begin{cases} (\mu', \mu) \times (\nu', \nu) \cap X & \text{if } U = U_E, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ is the desired shrinking of $\mathcal{U}$.

Case 2: $k(E) < \mathrm{cf}\, \mu$ *for each* $E \in H/{\sim}$. There are two subcases.

(2-1): $\sup\{k(E) : E \in H/{\sim}\} = \operatorname{cf}\mu$. By induction, define two sequences $\{E(\zeta) : \zeta < \operatorname{cf}\mu\}$ in $H/{\sim}$ and $\{\gamma(\zeta) : \zeta < \operatorname{cf}\mu\}$ in T so that $\zeta + \sup_{\eta<\zeta} k(E(\eta)) < \gamma(\zeta) \in T(E(\zeta))$. Observe that $E(\zeta)$'s are all distinct and $\{\gamma(\zeta) : \zeta < \operatorname{cf}\mu\}$ is strictly increasing and unbounded in $\operatorname{cf}\mu$. By Claim 1, $Z(\zeta) = (\mu', M(\gamma(\zeta))] \times (\nu', \nu) \cap X \subset U_{E(\gamma(\zeta))}$. So for each $U \in \mathcal{U}$, put

$$F(U) = \begin{cases} Z(\zeta) & \text{if } U = U_{E(\zeta)} \text{ for some } \zeta < \operatorname{cf}\mu, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ is the desired shrinking of $\mathcal{U}$.

(2-2): $\gamma_0 = \sup\{k(E) : E \in H/{\sim}\} < \operatorname{cf}\mu$. Put $T' = T \setminus [0, \gamma_0]$, $H' = \bigcup_{\gamma\in T'}\{\gamma\} \times S_\gamma$ and $j(E) = \sup\{j(E,\gamma) : \gamma \in T'\}$ for each $E \in H/{\sim}$. Then, since $j(E,\gamma) < \operatorname{cf}\nu$ for each $\gamma \in T'$ and $|T'| \le \operatorname{cf}\mu < \operatorname{cf}\nu$, we have

$$j(E) < \operatorname{cf}\nu. \tag{ii}$$

Let $\prec$ be the co-lexicographic order on $\operatorname{cf}\mu \times \operatorname{cf}\nu$, that is, $\langle\zeta',\eta'\rangle \prec \langle\zeta,\eta\rangle$ is defined by $\eta' < \eta$ or ($\eta' = \eta$ and $\zeta' < \zeta$). Since $\operatorname{cf}\mu < \operatorname{cf}\nu$, the $\prec$-order type of $\operatorname{cf}\mu \times \operatorname{cf}\nu$ is $\operatorname{cf}\nu$. By $\prec$-induction, we shall define two sequences $\{E(\zeta,\eta) : \langle\zeta,\eta\rangle \in \operatorname{cf}\mu \times \operatorname{cf}\nu\}$ in $H/{\sim}$ and $\{\langle\gamma(\zeta,\eta), \delta(\zeta,\eta)\rangle : \langle\zeta,\eta\rangle \in \operatorname{cf}\mu \times \operatorname{cf}\nu\}$ in H' with $\langle\gamma(\zeta,\eta), \delta(\zeta,\eta)\rangle \in E(\zeta,\eta)$ as follows.

Assume $E(\zeta',\eta')$, $\gamma(\zeta',\eta')$ and $\delta(\zeta',\eta')$ are defined with $\langle\gamma(\zeta',\eta'), \delta(\zeta',\eta')\rangle \in E(\zeta',\eta')$ for all $\langle\zeta',\eta'\rangle \prec \langle\zeta,\eta\rangle$. By (ii), take $\delta < \operatorname{cf}\nu$ with $\eta + \sup\{j(E(\zeta',\eta')) : \langle\zeta',\eta'\rangle \prec \langle\zeta,\eta\rangle\} < \delta$. When $\zeta = 0$, take $\langle\gamma(\zeta,\eta), \delta(\zeta,\eta)\rangle \in H'$ with $\delta < \delta(\zeta,\eta)$, and let $E(\zeta,\eta)$ be the equivalence class with $\langle\gamma(\zeta,\eta), \delta(\zeta,\eta)\rangle \in E(\zeta,\eta)$. When $\zeta > 0$, noting that $\gamma(\zeta',\eta)$ has been defined for all $\zeta' < \zeta$, take $\gamma < \operatorname{cf}\mu$ such that $\zeta + \sup\{\gamma(\zeta',\eta) : \zeta' < \zeta\} < \gamma$, and take $\langle\gamma(\zeta,\eta), \delta(\zeta,\eta)\rangle \in H'$ with $\delta < \delta(\zeta,\eta)$ and $\gamma < \gamma(\zeta,\eta)$. Finally, let $E(\zeta,\eta)$ be the equivalence class with $\langle\gamma(\zeta,\eta), \delta(\zeta,\eta)\rangle \in E(\zeta,\eta)$. This completes the construction.

By the construction, $E(\zeta,\eta)$'s are all distinct,

(iii) $\{\delta(\zeta,\eta) : \langle\zeta,\eta\rangle \in \operatorname{cf}\mu \times \operatorname{cf}\nu\}$ is strictly increasing and unbounded in $\operatorname{cf}\nu$,

and

(iv) $\{\gamma(\zeta,\eta) : \zeta \in \operatorname{cf}\mu\}$ is also strictly increasing and unbounded in $\operatorname{cf}\mu$ for each $\eta < \operatorname{cf}\nu$.

As $\langle\gamma(\zeta,\eta), \delta(\zeta,\eta)\rangle \in E(\zeta,\eta)$, by (i) we have $Z(\zeta,\eta) = (\mu', M(\gamma(\zeta,\eta))] \times (\nu', N(\delta(\zeta,\eta))] \subset U_{E(\zeta,\eta)}$. Moreover, by (iii) and (iv), $\{Z(\zeta,\eta) : \langle\zeta,\eta\rangle \in \operatorname{cf}\mu \times \operatorname{cf}\nu\}$ covers $(\mu',\mu) \times (\nu',\nu) \cap X$.

For each $U \in \mathcal{U}$, put

$$F(U) = \begin{cases} Z(\zeta,\eta) & \text{if } U = U_{E(\zeta,\eta)} \text{ for some } \langle\zeta,\eta\rangle \in \operatorname{cf}\mu \times \operatorname{cf}\nu, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ is the desired shrinking of $\mathcal{U}$.

The proof of (2) is easier, so we leave it to the reader. ∎

LEMMA 10. *Assume* $\omega_1 \le \operatorname{cf} \mu = \operatorname{cf} \nu = \kappa$, $X \subset (\mu+1) \times (\nu+1) \setminus \{\langle \mu, \nu \rangle\}$ *and* $\triangle_{MN}(X)$ *is stationary in* κ. *If* $X(\triangle, M, N)$ *and* $X_{\{\mu\}} \cup X^{\{\nu\}}$ *are separated*, *then there are* $\mu' < \mu$ *and* $\nu' < \nu$ *such that* $(\mu', \mu) \times (\nu', \nu) \cap X$ *is closed* (*and trivially open*) *in* X.

Proof. Take an open set V in X such that $X(\triangle, M, N) \subset V \subset \operatorname{Cl} V \subset X \setminus (X_{\{\mu\}} \cup X^{\{\nu\}})$. For each $\gamma \in \triangle_{MN}(X) \cap \operatorname{Lim}(\kappa)$, take $f(\gamma) < \gamma$ such that $(M(f(\gamma)), M(\gamma)] \times (N(f(\gamma)), N(\gamma)] \cap X \subset V$. By the PDL, we find $\mu' < \mu$ and $\nu' < \nu$ such that $(\mu', \mu) \times (\nu', \nu) \cap X \subset V$. Since $\operatorname{Cl} V$ is disjoint from $X_{\{\mu\}} \cup X^{\{\nu\}}$, we conclude that $(\mu', \mu) \times (\nu', \nu) \cap X$ is closed in X. ∎

LEMMA 11. *Assume* $\omega_1 \le \operatorname{cf} \mu = \operatorname{cf} \nu = \kappa$, $X \subset \mu \times \nu$ *and* $\triangle_{MN}(X)$ *is stationary in* κ.

(1) *If* $\mathcal{U}$ *is an open cover of* X, *then there are* $\mu' < \mu$, $\nu' < \nu$ *and a shrinking* $\mathcal{F}$ *of* $\mathcal{U}$ *by clopen sets in* X *such that* $\bigcup \mathcal{F} = (\mu', \mu) \times (\nu', \nu) \cap X$.

(2) *If* $\mathcal{H}$ *is a discrete collection of closed sets in* X, *there are* $\mu' < \mu$ *and* $\nu' < \nu$ *such that* $(\mu', \mu) \times (\nu', \nu) \cap X$ *meets at most one member of* $\mathcal{H}$.

Proof. (1) For each $\delta \in \triangle_{MN}(X) \cap \operatorname{Lim}(\kappa)$, fix $g(\delta) < \delta$ and $U(\delta) \in \mathcal{U}$ such that $(M(g(\delta)), M(\delta)] \times (N(g(\delta)), N(\delta)] \cap X \subset U(\delta)$. By the PDL, we find $\mu' < \mu$, $\nu' < \nu$ and a stationary set $S \subset \triangle_{MN}(X) \cap \operatorname{Lim}(\kappa)$ such that $(\mu', M(\delta)] \times (\nu', N(\delta)] \cap X \subset U(\delta)$ for each $\delta \in S$. Then by an argument similar to the proof of Lemma 5, making use of the equivalence relation, we can find the desired shrinking of $\mathcal{U}$.

(2) is easy. ∎

LEMMA 12. *Let* $\mathcal{P}$ *be a topological property which is closed under taking closed subspaces and free unions. Assume* $\omega_1 \le \operatorname{cf} \mu = \operatorname{cf} \nu = \kappa$, $X \subset (\mu+1) \times (\nu+1) \setminus \{\langle \mu, \nu \rangle\}$, $V_\mu(X)$ *is stationary in* κ, *but* $\triangle_{MN}(X)$ *is not stationary in* κ; *moreover*, $X_{\mu'+1}$ *and* $X^{\nu'+1}$ *have the property* $\mathcal{P}$ *for each* $\mu' < \mu$ *and* $\nu' < \nu$. *If* V *is an open set in* X *containing* $X_{\{\mu\}}$, *then* $X(R, M, N) \setminus V$ *has the property* $\mathcal{P}$.

Proof. Take a cub set D in $\operatorname{Lim}(\kappa)$ disjoint from $\triangle_{MN}(X)$. For each $\delta \in N^{-1}(V_\mu(X)) \cap D$, fix $f(\delta) < \kappa$ and $g(\delta) < \delta$ such that

$$(M(f(\delta)), \mu] \times (N(g(\delta)), N(\delta)] \cap X \subset V.$$

For each $\delta \in \kappa \setminus [N^{-1}(V_\mu(X)) \cap D]$, put $f(\delta) = 0$. By the PDL, take $\delta_0 < \kappa$ and a stationary set $S \subset N^{-1}(V_\mu(X)) \cap D$ such that $g(\delta) = \delta_0$ for each $\delta \in S$. Put $\nu' = N(\delta_0)$, $D' = \{\delta < \kappa : \forall \delta' < \delta\, (f(\delta') < \delta)\}$ and $W = \bigcup_{\delta \in S} (M(f(\delta)), \mu] \times (\nu', N(\delta)] \cap X$. Then D' is cub in κ and $W \subset V$. Since $X^{\nu'+1} \setminus V$ (and therefore $X(R, M, N)^{\nu'+1} \setminus V$) has the property $\mathcal{P}$, it suffices to represent $Y = X(R,M,N)^{(\nu',\nu]} \setminus W$ as the free union of subspaces having the property $\mathcal{P}$. Here note that Y is closed in X and disjoint from $X_{\{\mu\}} \cup X^{\{\nu\}}$. To show this, put $C = \mathrm{Lim}(S) \cap D'$. Then C is cub and $C \subset D \cap D'$. For each $\delta \in C$, put $h(\delta) = \sup(C \cap \delta)$. Then by the closedness of C, $h(\delta) \in C$ and $h(\delta) \le \delta$. For each $\delta \in C \setminus \mathrm{Lim}(C)$ (in other words, $h(\delta) < \delta$), put $Y(\delta) = Y_{(M(h(\delta)),M(\delta)]}$. Then each $Y(\delta)$ is clopen in Y, and therefore closed in X. Moreover, as $Y(\delta) \subset X_{M(\delta)+1}$, $Y(\delta)$ has the property $\mathcal{P}$. Since $Y(\delta)$'s, $\delta \in C \setminus \mathrm{Lim}(C)$, are pairwise disjoint, it suffices to show $Y = \bigcup_{\delta \in C\setminus \mathrm{Lim}(C)} Y(\delta)$. To show this, let $\langle \alpha, \beta\rangle \in Y$. Note $\alpha < \mu$, $\nu' < \beta < \nu$ and $m(\alpha) \ge n(\beta)$. Let δ be the minimal ordinal number with $m(\alpha) \le \delta \in C$. Note that $n(\beta) \le \delta$.

First assume $n(\beta) = \delta$. Since $\delta = n(\beta) \le m(\alpha) \le \delta$, we have $\delta \in \triangle_{MN}(X) \cap C$. This contradicts $C \subset D$. Therefore $n(\beta) < \delta$.

Next assume $\delta \in \mathrm{Lim}(C)$. Then by the minimality of δ, we have $m(\alpha) = \delta$. Using $n(\beta) < \delta$ and $\delta \in C \subset \mathrm{Lim}(S) \cap D'$, pick $\delta' \in S$ with $n(\beta) < \delta' < \delta$. Since $\delta \in D'$, we have $f(\delta') < \delta = m(\alpha)$, and therefore $M(f(\delta')) < \alpha$. Moreover, as $n(\beta) < \delta'$, we have

$$\langle \alpha, \beta\rangle \in (M(f(\delta')), \mu] \times (\nu', N(\delta')] \cap X \subset W.$$

This contradicts $Y \cap W = \emptyset$. Therefore $\delta \in C \setminus \mathrm{Lim}(C)$. By the minimality of δ, this shows that $h(\delta) < m(\alpha) \le \delta$. This means $\alpha \in (M(h(\delta)), M(\delta)]$, hence

$$\langle \alpha, \beta\rangle \in Y_{(M(h(\delta)),M(\delta)]} = Y(\delta).$$

This completes the proof. ∎

3. Proof of the Theorem. The implications (1)→(3) and (2)→(3) are evident.

(3)→(4). Let X be normal and $\langle \mu, \nu\rangle \in (\lambda+1)^2 \setminus X$ with $\omega \le \mathrm{cf}\,\mu$ and $\omega \le \mathrm{cf}\,\nu$. Since $\langle \mu,\nu\rangle \notin X$, $X_{\{\mu\}}$ and $X^{\{\nu\}}$ are disjoint closed sets in the normal space X. Thus (4-1) holds.

To show (4-2), assume $\omega_1 \le \mathrm{cf}\,\nu$ and $V_\mu(X) \cap \nu$ is not stationary in ν. Then there is a cub set D in $\mathrm{cf}\,\nu$ such that $V_\mu(X) \cap N(D) = \emptyset$. Since $X_{\{\mu\}}$ and $X^{N(D)\cup\{\nu\}}$ are disjoint closed sets, (4-2) holds.

(4-3) is similar.

To show (4-4), since the remaining case is similar, we may assume $\omega_1 \le \mathrm{cf}\,\mu < \mathrm{cf}\,\nu$, $V_\mu(X) \cap \nu$ is not stationary in ν, and both $H_\nu(X) \cap \mu$ and A^ν_μ are non-stationary in μ. By the non-stationarity of A^ν_μ and Lemma 3, there are cub sets C' in $\mathrm{cf}\,\mu$ and D' in $\mathrm{cf}\,\nu$ such that $X \cap M(C') \times N(D') = \emptyset$. Since $V_\mu(X) \cap \nu$ and $H_\nu(X) \cap \mu$ are non-stationary in $\mathrm{cf}\,\nu$ and $\mathrm{cf}\,\mu$ respectively, take cub sets $C \subset C'$ and $D \subset D'$ such that $M(C) \cap H_\nu(X) = \emptyset$ and $N(D) \cap V_\mu(X) = \emptyset$. Then $X \cap (M(C) \cup \{\mu\}) \times (N(D) \cup \{\nu\}) = \emptyset$. Therefore

$X_{M(C)\cup\{\mu\}}$ and $X^{N(D)\cup\{\nu\}}$ are disjoint closed sets in the normal space X. This shows (4-4).

To show (4-5), assume $\omega_1 \le \mathrm{cf}\,\mu = \mathrm{cf}\,\nu = \kappa$. By $\langle\mu,\nu\rangle \notin X$, $X(\triangle, M, N)$ and $X_{\{\mu\}} \cup X^{\{\nu\}}$ are disjoint closed sets in the normal space X. This shows (4-5-a).

To show (4-5-b), assume $\triangle_{MN}(X)$ is not stationary in κ. Since X is normal, (b1) and (b3) are evident. Assume $V_\mu(X) \cap \nu$ is not stationary in ν. By Lemma 4 and the non-stationarity of $V_\mu(X) \cap \nu$, there is a cub set $D \subset \kappa$ such that $X \cap M(D) \times N(D) = \emptyset$ and $N(D) \cap V_\mu(X) = \emptyset$. Then $X \cap (M(D) \cup \{\mu\}) \times N(D) = \emptyset$. Since $X(R, M, N)$ is disjoint from $X^{\{\nu\}}$, we have $X(R, M, N) \cap (M(D) \cup \{\mu\}) \times (N(D) \cup \{\nu\}) = \emptyset$. Since $X(R, M, N)$ is closed in X, $X(R, M, N)_{M(D)\cup\{\mu\}}$ and $X(R, M, N)^{N(D)\cup\{\nu\}}$ are disjoint closed sets in the normal space X. This shows (b2).

Similarly we can show (b4).

(4)→(1). Assume (4) holds but X is not shrinking. Put

$$\mu = \min\{\zeta \le \lambda : X_{\zeta+1} \text{ is not shrinking}\},$$
$$\nu = \min\{\eta \le \lambda : X^{\eta+1}_{\mu+1} \text{ is not shrinking}\}.$$

Note that $X^{\nu+1}_{\mu+1}$ is not shrinking, but $X^{\nu+1}_{\mu'+1}$ and $X^{\nu'+1}_{\mu+1}$ are shrinking for each $\mu' < \mu$ and $\nu' < \nu$. Since $X^{\nu+1}_{\mu+1}$ is a clopen subspace of X, we may assume $X = X^{\nu+1}_{\mu+1}$. Then again note that X is not shrinking, but $X_{\mu'+1}$ and $X^{\nu'+1}$ are shrinking for each $\mu' < \mu$ and $\nu' < \nu$. So there is an open cover $\mathcal{U}$ of X which does not have a closed shrinking which covers X.

CLAIM 1. $\langle\mu,\nu\rangle \notin X$.

Proof. Assume $\langle\mu,\nu\rangle \in X$. Then there are $\mu' < \mu$, $\nu' < \nu$ and $U \in \mathcal{U}$ such that $Z = (\mu',\mu] \times (\nu',\nu] \cap X \subset U$. Since Z is clopen in X and $X_{\mu'+1} \cup X^{\nu'+1} \cup Z = X$, and $X_{\mu'+1}$ and $X^{\nu'+1}$ are shrinking, by Lemma 1, $\mathcal{U}$ has a closed shrinking which covers X, a contradiction. This completes the proof of Claim 1.

CLAIM 2. $\omega \le \mathrm{cf}\,\mu$ *and* $\omega \le \mathrm{cf}\,\nu$.

Proof. Assume $\mu = \mu' + 1$. Since X is the free union $X_\mu \oplus X_{\{\mu\}}$ of shrinking subspaces, $\mathcal{U}$ can be shrunk, a contradiction. Therefore $\omega \le \mathrm{cf}\,\mu$. Similarly $\omega \le \mathrm{cf}\,\nu$.

First we consider the following case.

Case 1: $\mathrm{cf}\,\mu \ne \mathrm{cf}\,\nu$. We may assume $\mathrm{cf}\,\mu < \mathrm{cf}\,\nu$. We consider two subcases:

(1-1): *$V_\mu(X) \cap \nu$ is stationary in ν.* Applying Lemma 5 (1) to $\mathcal{U}|X^\nu$, we find $\mu' < \mu$, $\nu' < \nu$ and a shrinking $\mathcal{F}$ of $\mathcal{U}|X^\nu$ by closed sets in X^ν such that $\bigcup \mathcal{F} = (\mu', \mu] \times (\nu', \nu) \cap X$. Since $X_{\{\mu\}}$ and $X^{\{\nu\}}$ are separated by (4-1), applying Lemma 6, we get $\mu'' < \mu$ and $\nu'' < \nu$ with $\mu' < \mu''$ and $\nu' < \nu''$ such that $Z = (\mu'', \mu] \times (\nu'', \nu) \cap X$ is closed in X. Then $\mathcal{F}|Z$ is a shrinking of $\mathcal{U}$ by closed sets in X which covers Z. Since $X_{\mu''+1}$, $X^{\nu''+1}$ and $X^{\{\nu\}}$ are shrinking closed subspaces and $X = X_{\mu''+1} \cup X^{\nu''+1} \cup X^{\{\nu\}} \cup Z$, by Lemma 1, $\mathcal{U}$ has a closed shrinking which covers X. A contradiction.

(1-2): *$V_\mu(X) \cap \nu$ is not stationary in ν.* In this case, by (4-2), there is a cub set D in $\mathrm{cf}\,\nu$ such that $X_{\{\mu\}}$ and $X^{N(D)\cup\{\nu\}}$ are separated. Take disjoint open sets V and W containing $X_{\{\mu\}}$ and $X^{N(D)\cup\{\nu\}}$ respectively. Assume $\mathrm{cf}\,\mu = \omega$. Then by Lemma 7 (1), X_μ is shrinking, thus $X \setminus V$ is shrinking. Moreover, by (2) of the analogue of Lemma 7, $X \setminus W$ is also shrinking. Therefore by Lemma 1, X is shrinking, a contradiction. Therefore we have $\omega_1 \le \mathrm{cf}\,\mu$.

Then by an argument similar to (1-1), assuming $H_\nu(X) \cap \mu$ is stationary in μ, we get a contradiction (of course we would use the "analogous" lemmas). So $H_\nu(X) \cap \mu$ is not stationary in μ.

Now we are in the situation where $\omega_1 \le \mathrm{cf}\,\mu < \mathrm{cf}\,\nu$, and $H_\nu(X) \cap \mu$ and $V_\mu(X) \cap \nu$ are not stationary in μ and ν respectively. By (4-3), we also have a cub set C in $\mathrm{cf}\,\mu$ such that $X^{\{\nu\}}$ and $X_{M(C)\cup\{\mu\}}$ are separated. Again, we consider two subcases:

(1-2-1): *A^ν_μ is stationary in μ.* In this case by Lemmas 8 and 9 (1), we find $\mu' < \mu$, $\nu' < \nu$ and a shrinking $\mathcal{F}$ of $\mathcal{U}$ by closed sets in X such that $Z = (\mu', \mu) \times (\nu', \nu) \cap X$ is clopen in X and $\bigcup \mathcal{F} = Z$. Since $X_{\mu'+1}$, $X^{\nu'+1}$, $X_{\{\mu\}}$ and $X^{\{\nu\}}$ are shrinking closed subspaces and $X = X_{\mu'+1} \cup X^{\nu'+1} \cup X_{\{\mu\}} \cup X^{\{\nu\}} \cup Z$, by Lemma 1, $\mathcal{U}$ has a closed shrinking which covers X. A contradiction.

(1-2-2) : *A^ν_μ is not stationary in μ.* In this case by (4-4), there are cub sets C in $\mathrm{cf}\,\mu$ and D in $\mathrm{cf}\,\nu$ such that $X_{M(C)\cup\{\mu\}}$ and $X^{N(D)\cup\{\nu\}}$ are separated. Take disjoint open sets V and W containing $X_{M(C)\cup\{\mu\}}$ and $X^{N(D)\cup\{\nu\}}$ respectively. Then by Lemma 7 (2), $X \setminus V$ and $X \setminus W$ are shrinking closed subspaces. Therefore by Lemma 1, X is shrinking, a contradiction.

Next we consider the remaining case.

Case 2: $\mathrm{cf}\,\mu = \mathrm{cf}\,\nu = \kappa$. Assume $\kappa = \omega$. Then by Lemma 7 (1), X_μ and X^ν are shrinking. By (4-1), $X_{\{\mu\}}$ and $X^{\{\nu\}}$ are separated. Then by Lemma 2, $X = X_\mu \cup X^\nu$ is shrinking, a contradiction. Therefore $\omega_1 \le \kappa$. Two subcases are now considered:

(2-1): *$\triangle_{MN}(X)$ is stationary in κ.* In this case by Lemmas 10 and 11, we have a contradiction as previously.

(2-2): $\triangle_{MN}(X)$ *is not stationary in* κ. Since X is the union of the closed subspaces $X(R,M,N)$ and $X(L,M,N)$, we may assume that $\mathcal{U}$ does not have a closed shrinking which covers $X(R,M,N)$. Two cases are to consider:

(2-2-1): $V_\mu(X)\cap\nu$ *is stationary in* ν. As in the proof of Lemma 5 (1), for each $\delta\in N^{-1}(V_\mu(X))\cap\operatorname{Lim}(\kappa)$, fix $f(\delta)<\kappa$, $g(\delta)<\delta$ and $U(\delta)\in\mathcal{U}$ such that $(M(f(\delta)),\mu]\times(N(g(\delta)),N(\delta)]\cap X\subset U(\delta)$. Applying the PDL, we can find $\delta_0<\kappa$ and a stationary set $S\subset N^{-1}(V_\mu(X))\cap\operatorname{Lim}(\kappa)$ such that $g(\delta)=\delta_0$ for each $\delta\in S$. Put $\nu'=N(\delta_0)$.

Claim 3. *There is a closed shrinking* $\mathcal{F}$ *of* $\mathcal{U}$ *such that* $\{\mu\}\times(\nu',\nu)\cap X\subset \operatorname{Int}(\bigcup\mathcal{F})$ *and* $\bigcup\mathcal{F}$ *is closed in X.*

Proof. As previously, for each δ and δ' in S, define $\delta\sim\delta'$ by $U(\delta)=U(\delta')$, and let $S/{\sim}$ be its quotient. For each $E\in S/{\sim}$, put $U_E=U(\delta)$ for some (any) $\delta\in E$. Observe that $(M(f(\delta)),\mu]\times(\nu',N(\delta)]\cap X\subset U_E$ for each $\delta\in E$.

First, assume there is $E\in S/{\sim}$ such that E is unbounded in κ. Put $W=\bigcup_{\delta\in E}(M(f(\delta)),\mu]\times(\nu',N(\delta)]\cap X$. Note that $W\subset U_E$. Since by the condition (b1), $X_{\{\mu\}}$ and $X\setminus(W\cup X^{\nu'+1})$ are separated, we can find an open set V in X such that $\{\mu\}\times(\nu',\nu)\cap X\subset V\subset \operatorname{Cl}V\subset W$. For each $U\in\mathcal{U}$, put

$$F(U)=\begin{cases}\operatorname{Cl}V & \text{if } U=U_E,\\ \emptyset & \text{otherwise.}\end{cases}$$

Then $\mathcal{F}=\{F(U):U\in\mathcal{U}\}$ is the desired shrinking of $\mathcal{U}$.

Next assume all E's, $E\in S/{\sim}$, are bounded in κ. As in Lemma 5, define $\delta(\eta)\in E(\eta)\in S/{\sim}$ for each $\eta\in\kappa$ so that $\eta+\sup(\bigcup_{\zeta<\eta}E(\zeta))<\delta(\eta)$. For each $U\in\mathcal{U}$, put

$$W(U)=\begin{cases}(M(f(\delta(\eta))),\mu]\times(\nu',N(\delta(\eta))]\cap X\\ \qquad \text{if } U=U_{E(\eta)} \text{ for some } \eta<\kappa,\\ \emptyset \qquad \text{otherwise.}\end{cases}$$

Then $\mathcal{W}=\{W(U):U\in\mathcal{U}\}$ is a shrinking of $\mathcal{U}$ by clopen sets in X with $\{\mu\}\times(\nu',\nu)\cap X\subset\bigcup\mathcal{W}$. By the condition (b1), take an open set V in X such that $\{\mu\}\times(\nu',\nu)\cap X\subset V\subset\operatorname{Cl}V\subset\bigcup\mathcal{W}$.

For each $U\in\mathcal{U}$, put

$$F(U)=W(U)\cap\operatorname{Cl}V.$$

Then $\mathcal{F}=\{F(U):U\in\mathcal{U}\}$ is the desired shrinking of $\mathcal{U}$. This completes the proof of the claim.

Take the shrinking $\mathcal{F}$ of $\mathcal{U}$ in Claim 3. By Lemma 12,

$$Z=X(R,N,M)^{(\nu',\nu]}\setminus\operatorname{Int}\Big(\bigcup\mathcal{F}\Big)$$

is a shrinking closed subspace. Since $X(R,M,N) \subset X^{\nu'+1} \cup Z \cup \bigcup \mathcal{F}$, by Lemma 1, $\mathcal{U}$ has a closed shrinking which covers $X(R,M,N)$. A contradiction.

(2-2-2): $V_\mu(X) \cap \nu$ *is not stationary in* ν. Using the clause (b2), take a cub set D in κ such that $X(R,M,N)_{M(D)\cup\{\mu\}}$ and $X(R,M,N)^{N(D)\cup\{\nu\}}$ are separated. Take disjoint open sets V and W containing $X(R,M,N)_{M(D)\cup\{\mu\}}$ and $X(R,M,N)^{N(D)\cup\{\nu\}}$ respectively. Then applying Lemma 7 (2) to $X(R,M,N)$, we see that $X(R,M,N)\setminus V$ and $X(R,M,N)\setminus W$ are shrinking. Therefore by Lemma 1, $X(R,M,N)$ is shrinking, a contradiction.

Thus in all cases, we get contradictions. This completes the proof of (4)→(1).

(4)→(2). This proof is almost similar to the one of (4) → (1) except for the case (2-2-1). So we only give a proof of case (2-2-1) for the CWN case.

(2-2-1): $\omega_1 \le \operatorname{cf}\mu = \operatorname{cf}\nu = \kappa$, $\triangle_{MN}(X)$ *is not stationary in* κ, $V_\mu(X)\cap\nu$ *is stationary in* ν *and* $\mathcal{H}$ *is a discrete collection of closed sets in* X *which cannot be separated.* In this case, for each $\delta \in N^{-1}(V_\mu(X)) \cap \operatorname{Lim}(\kappa)$, fix $g(\delta) < \delta$ such that $\{\mu\} \times (N(g(\delta)), N(\delta)] \cap X$ meets at most one element of $\mathcal{H}$. By the PDL, we can take $\nu' < \nu$ such that $\{\mu\} \times (\nu',\nu) \cap X$ meets at most one element of $\mathcal{H}$.

Claim 3′. *There is an open set* V *such that* $\{\mu\} \times (\nu',\nu) \cap X \subset V$ *and* $\operatorname{Cl} V$ *meets at most one element of* $\mathcal{H}$.

Proof. Put $\mathcal{H}' = \{H \in \mathcal{H} : H \cap (\{\mu\} \times (\nu',\nu) \cap X) = \emptyset\}$, and $W = X \setminus \bigcup \mathcal{H}'$. Since $\{\mu\} \times (\nu',\nu) \cap X \subset W$, take an open set V such that $\{\mu\} \times (\nu',\nu) \cap X \subset V \subset \operatorname{Cl} V \subset W$ using the clause (b1). Then this V works.

As $X(R,M,N)$ is covered by closed sets $X^{\nu'+1}$, $Z = X(R,M,N)^{(\nu',\nu]} \setminus V$ and $\operatorname{Cl} V$, we get a contradiction as in case (2-2-1) in the proof of (4)→(1). This completes the proof. ∎

4. Non-normal examples and related questions. In [KOT], it is proved that, for subspaces A and B of ω_1, $A \times B$ is normal (countably paracompact) if and only if A is not stationary in ω_1, B is not stationary in ω_1 or $A \cap B$ is stationary.

According to this result, if A is a countable subspace of ω_1, then, since A is non-stationary, $A \times B$ is normal for each $B \subset \omega_1$. In particular, as is well known, $(\omega+1) \times \omega_1$ is normal. But as is shown in the next example, there is a non-normal subspace of $(\omega+1) \times \omega_1$.

Example 1. Put $X = \omega \times \omega_1 \cup \{\omega\} \times (\omega_1 \setminus \operatorname{Lim}(\omega_1))$. Put $F = \omega \times \operatorname{Lim}(\omega_1)$ and $H = \{\omega\} \times (\omega_1 \setminus \operatorname{Lim}(\omega_1))$. Then F and H are disjoint closed sets in X. Let U be an open set containing H. For each $\alpha \in \omega_1 \setminus \operatorname{Lim}(\omega_1)$, pick $n(\alpha) \in \omega$

such that $[n(\alpha), \omega] \times \{\alpha\} \subset U$. Since $\omega_1 \setminus \mathrm{Lim}(\omega_1)$ is uncountable, there is an uncountable subset $C \subset \omega_1 \setminus \mathrm{Lim}(\omega_1)$ and $n \in \omega$ such that $n(\alpha) = n$ for each $\alpha \in C$. Observe that $[n, \omega] \times C \subset U$. Pick $\alpha \in \mathrm{Lim}(C)$. Noting that $\mathrm{Lim}(C) \subset \mathrm{Lim}(\omega_1)$, we have $\langle n, \alpha\rangle \in [n, \omega] \times \mathrm{Lim}(C) \cap F \subset \mathrm{Cl}\, U \cap F$. This argument shows X is not normal.

Next we give a corollary of the Theorem for subspaces of ω_1^2. For simplicity, we use the following notation: Let $X \subset \omega_1^2$, $\alpha < \omega_1$ and $\beta < \omega_1$. Put $V_\alpha(X) = \{\beta < \omega_1 : \langle \alpha, \beta\rangle \in X\}$, $H_\beta(X) = \{\alpha < \omega_1 : \langle \alpha, \beta\rangle \in X\}$ and $\triangle(X) = \{\alpha < \omega_1 : \langle \alpha, \alpha\rangle \in X\}$. For subsets C and D of ω_1, put $X_C = X \cap C \times \omega_1$, $X^D = X \cap \omega_1 \times D$ and $X_C^D = X \cap C \times D$.

Consider M and N as the identity map on ω_1 if $\mu = \nu = \omega_1$ in the Theorem. Then, by checking all clauses in (4) of the Theorem, we can see:

Corollary. *Let $X \subset \omega_1^2$. Then the following are equivalent.*

(1) *X is normal.*
(2) (2-1-a) *If α is a limit ordinal in ω_1 and $V_\alpha(X)$ is not stationary in ω_1, then there is a cub set $D \subset \omega_1$ such that $X_{\{\alpha\}}$ and X^D are separated.*
(2-1-b) *If β is a limit ordinal in ω_1 and $H_\beta(X)$ is not stationary in ω_1, then there is a cub set $C \subset \omega_1$ such that $X^{\{\beta\}}$ and X_C are separated.*
(2-2) *If $\triangle(X)$ is not stationary in ω_1, then there is a cub set $C \subset \omega_1$ such that X_C and X^C are separated.*

Intuitively, we may consider (2-1-a) to be a condition which guarantees the normality of $X_{\alpha+1}$ for each $\alpha < \omega_1$, and (2-1-b) the normality of $X^{\beta+1}$ for each $\beta < \omega_1$. If we know that $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha, \beta < \omega_1$, then (2-2) is a condition which guarantees the normality of X.

Consider $X = \omega_1^2$. Since $V_\alpha(X)$ and $H_\beta(X)$ are the stationary set ω_1 for each $\alpha, \beta < \omega_1$ and $\triangle(X)$ is also the stationary set ω_1, the clause (2) of the Corollary is satisfied. So X is normal.

Example 2. Let A and B be disjoint stationary sets in ω_1 and put $X = A \times B$. Let α be a limit ordinal in ω_1. Then we have

$$V_\alpha(X) = \begin{cases} B & \text{if } \alpha \in A, \\ \emptyset & \text{otherwise.} \end{cases}$$

Therefore, if $V_\alpha(X)$ is not stationary, then necessarily $\alpha \notin A$ and $V_\alpha(X) = \emptyset$, so $X_{\{\alpha\}} = \emptyset$. Therefore $X_{\{\alpha\}}$ and X^{ω_1} are separated. This argument proves (2-1-a). Similarly we have (2-1-b). Therefore $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha, \beta < \omega_1$.

Note that $\triangle(X) = \emptyset$. Let C be a cub set in ω_1. Then $X \cap C^2 = (A \cap C) \times (B \cap C) \neq \emptyset$, equivalently $X_C \cap X^C \neq \emptyset$. Thus X_C and X^C cannot be separated. Therefore X is not normal, because the clause (2-2) is not satisfied.

Example 3. Let $X = \{\langle\alpha,\beta\rangle \in \omega_1^2 : \alpha \le \beta\}$, $Y = \{\langle\alpha,\beta\rangle \in \omega_1^2 : \alpha < \beta\}$. Checking (2-1-a) and (2-1-b), we can show that $X_{\alpha+1}$, $X^{\beta+1}$, $Y_{\alpha+1}$ and $Y^{\beta+1}$ are normal for each $\alpha,\beta < \omega_1$.

Since $\triangle(X) = \omega_1$ is stationary, (2-2) for X is satisfied. Thus X is normal (but this is obvious, because X is a closed subspace of ω_1^2). On the other hand, note that $\triangle(Y) = \emptyset$. For each cub set C in ω_1, pick α and β in C with $\alpha < \beta$. Then $\langle\alpha,\beta\rangle \in Y \cap C^2$. Therefore (2-2) for Y is not satisfied. Thus Y is not normal.

Let $X = \omega_1 \times (\omega_1 + 1)$. Observe that $X \cap \omega_1^2 = \omega_1^2$ is normal, and $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha,\beta < \omega_1$. Since $\{\langle\alpha,\alpha\rangle : \alpha \in \omega_1\}$ and $X^{\{\omega_1\}}$ cannot be separated, X is not normal. Note that both $\triangle(X)$ and $H_{\omega_1}(X)$ are the stationary set ω_1. Next we give a similar example $X \subset \omega_1 \times (\omega_1+1)$, but with $\triangle(X)$ and $H_{\omega_1}(X)$ not stationary.

Example 4. Let

$$X = [\omega_1 \setminus \mathrm{Lim}(\omega_1)] \times [(\omega_1 + 1) \setminus \mathrm{Lim}(\omega_1)] \cup \{\langle\alpha,\alpha+1\rangle : \alpha \in \mathrm{Lim}(\omega_1)\}.$$

Observe that $X \cap \omega_1^2$ is normal, $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha,\beta < \omega_1$ and both $\triangle(X)$ and $H_{\omega_1}(X)$ are the non-stationary set $\omega_1 \setminus \mathrm{Lim}(\omega_1)$. By an argument similar to that for Claim 1 of Lemma 4, we can see that $F = \{\langle\alpha,\alpha+1\rangle : \alpha \in \mathrm{Lim}(\omega_1)\}$ is closed (discrete). We shall show F and $X^{\{\omega_1\}}$ cannot be separated. To see this, let U be an open set containing F. For each $\alpha \in \mathrm{Lim}(\omega_1)$, since $\langle\alpha,\alpha+1\rangle \in F \subset U$, take $f(\alpha) < \alpha$ such that $(f(\alpha),\alpha] \times \{\alpha+1\} \cap X \subset U$. By the PDL, there are $\alpha_0 < \omega_1$ and a stationary set $S \subset \mathrm{Lim}(\omega_1)$ such that $f(\alpha) = \alpha_0$ for each $\alpha \in S$. Take $\beta \in \omega_1 \setminus \mathrm{Lim}(\omega_1)$ with $\alpha_0 < \beta$. Noting that $\langle\beta,\alpha+1\rangle \in X$ for each $\alpha \in S$ with $\alpha > \beta$, we have

$$\langle\beta,\omega_1\rangle \in \mathrm{Cl}\{\langle\beta,\alpha+1\rangle : \alpha \in S,\ \alpha > \beta\} \cap X^{\{\omega_1\}} \subset \mathrm{Cl}\, U \cap X^{\{\omega_1\}}.$$

Thus F and $X^{\{\omega_1\}}$ cannot be separated.

In this connection, we have the next question which relates to the clause (4-4) of the Theorem.

Question 1. Does there exist a non-normal subspace X of $\omega_1 \times \omega_2$ such that $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha < \omega_1$ and $\beta < \omega_2$?

In this connection, we show:

Proposition. *If $X = A \times B$ is a subspace of $\omega_1 \times \omega_2$ such that $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha < \omega_1$ and $\beta < \omega_2$, then X is normal.*

Proof. If A is not stationary in ω_1, then take a cub set C in ω_1 disjoint from A. Put $h(\alpha) = \sup(C \cap \alpha)$ for each $\alpha \in C$. Observe that

$X = \bigoplus_{\alpha \in C \setminus \mathrm{Lim}(C)} X_{(h(\alpha),\alpha]}$. Since $X_{(h(\alpha),\alpha]}$ is a closed subspace of $X_{\alpha+1}$, by the inductive assumption, X is normal. Similarly X is normal if B is not stationary in ω_2. So we may assume A and B are stationary in respectively ω_1 and ω_2. Let $\mathcal{U} = \{U_i : i \in 2\}$ be an open cover of X. Fix $\alpha \in A$. For each $\beta \in B$, fix $f(\alpha,\beta) < \alpha$, $g(\alpha,\beta) < \beta$ and $i(\alpha,\beta) \in 2$ such that $(f(\alpha,\beta),\alpha] \times (g(\alpha,\beta),\beta] \cap X \subset U_{i(\alpha,\beta)}$. Applying the PDL to B, we find $f(\alpha) < \alpha$, $g(\alpha) < \omega_2$, $i(\alpha) \in 2$ and a stationary set $B(\alpha) \subset B$ in ω_2 such that $f(\alpha,\beta) = f(\alpha)$, $g(\alpha,\beta) = g(\alpha)$ and $i(\alpha,\beta) = i(\alpha)$ for each $\beta \in B(\alpha)$. Then, applying the PDL to A, we find $\alpha_0 < \omega_1$, $i_0 \in 2$ and a stationary set $A' \subset A$ in ω_1 such that $f(\alpha) = \alpha_0$ and $i(\alpha) = i_0$ for each $\alpha \in A'$. Put $\beta_0 = \sup\{g(\alpha) : \alpha \in A'\}$. Then we have $Z = (\alpha_0, \omega_1) \times (\beta_0, \omega_2) \cap X \subset U_{i_0}$. Since X is the union of closed subspaces, X_{α_0+1}, X^{β_0+1} and Z, $\mathcal{U}$ has a closed shrinking which covers X. Therefore $X = A \times B$ is normal. ∎

By the result in [KOT], normality and countable paracompactness of $A \times B \subset \omega_1^2$ are equivalent. In this connection, it is natural to ask:

QUESTION 2. For any $X \subset \omega_1^2$, are normality and countable paracompactness of X equivalent?

Note that, by [KS], normality implies countable paracompactness in the realm of subspaces of product spaces of two ordinals.

Finally, we restate a question from [KOT]:

QUESTION 3. For any subspace X of the product space of two ordinals, are countable paracompactness, expandability, strong D-property and weak D(ω)-property of X equivalent?

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