Normal subspaces in products of two ordinals

by

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Abstract. Let $\lambda$ be an ordinal number. It is shown that normality, collectionwise normality and shrinking are equivalent for all subspaces of $(\lambda + 1)^2$.

1. Introduction. It is well known that any ordinal with the order topology is shrinking and collectionwise normal hereditarily. But, in general, products of two ordinals are not. In fact, $(\omega_1 + 1) \times \omega_1$ is not normal. In [KOT], it was proved that the normality, collectionwise normality and shrinking property of $A \times B$, where $A$ and $B$ are subspaces of ordinals, are equivalent. It was asked whether these properties are also equivalent for all subspaces of products of two ordinals [KOT, Problem (i)]. The aim of this paper is to give an affirmative answer.

We recall some basic definitions and introduce some specific notation.

In our discussion, we always assume $X \subset (\lambda + 1)^2$ for some suitably large ordinal $\lambda$. Moreover, in general, the letters $\mu$ and $\nu$ stand for limit ordinals with $\mu \leq \lambda$ and $\nu \leq \lambda$. For each $A \subset \lambda + 1$ and $B \subset \lambda + 1$ put

$$X_A = A \times (\lambda + 1) \cap X, \quad X^B = (\lambda + 1) \times B \cap X,$$

and

$$X_A^B = X_A \cap X^B.$$ 

For each $\alpha \leq \lambda$ and $\beta \leq \lambda$, put

$$V_\alpha(X) = \{\beta \leq \lambda : \langle \alpha, \beta \rangle \in X\}, \quad H_\beta(X) = \{\alpha \leq \lambda : \langle \alpha, \beta \rangle \in X\}.$$

cf $\mu$ denotes the cofinality of the ordinal $\mu$. When $\omega_1 \leq \text{cf} \mu$, a subset $S$ of $\mu$ called stationary in $\mu$ if it intersects all cub (closed and unbounded) sets

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in $\mu$. For each $\mu \leq \lambda$ and $\nu \leq \lambda$ with $\omega_1 \leq \text{cf} \, \mu$ and $\omega_1 \leq \text{cf} \, \nu$, put

$$A_\mu^{\nu} = \{ \alpha < \mu : V_{\alpha}(X) \cap \nu \text{ is stationary in } \nu \},$$

$$B_\mu^{\nu} = \{ \beta < \nu : H_{\beta}(X) \cap \mu \text{ is stationary in } \mu \}.$$ 

Moreover, for each $A \subset \mu$, $\text{Lim}_\mu(A)$ is the set $\{ \alpha < \mu : \alpha = \text{sup}(A \cap \alpha) \}$, in other words, the set of all cluster points of $A$ in $\mu$. Therefore $\text{Lim}_\mu(A)$ is cub in $\mu$ whenever $A$ is unbounded in $\mu$. We will simply denote $\text{Lim}_\mu(A)$ by $\text{Lim}(A)$ if the situation is clear in its context.

A strictly increasing function $M : \text{cf} \, \mu \to \mu$ is said to be normal if $M(\gamma) = \sup\{ M(\gamma') : \gamma' < \gamma \}$ for each limit ordinal $\gamma < \text{cf} \, \mu$, and $\mu = \sup\{ M(\gamma) : \gamma < \text{cf} \, \mu \}$. Note that a normal function on $\text{cf} \, \mu$ always exists if $\text{cf} \, \mu \geq \omega$. So we always fix a normal function $M : \text{cf} \, \mu \to \mu$ for each ordinal $\mu$ with $\text{cf} \, \mu \geq \omega$.

For convenience, we define $M(-1) = -1$. Then $M$ carries $\text{cf} \, \mu$ homeomorphically to the range $\text{ran} \, M$ of $M$ and $\text{ran} \, M$ is closed in $\mu$. Note that for all $S \subset \mu$ with $\omega_1 \leq \text{cf} \, \mu$, $S$ is stationary in $\mu$ if and only if $M^{-1}(S)$ is stationary in $\text{cf} \, \mu$.

Let $\mu$ and $\nu$ be two limit ordinals with $\mu \leq \lambda$ and $\nu \leq \lambda$; moreover, let $M : \text{cf} \, \mu \to \mu$ and $N : \text{cf} \, \nu \to \nu$ be the fixed normal functions on $\text{cf} \, \mu$ and $\text{cf} \, \nu$ respectively. For each $\alpha \in \mu$ and $\beta \in \nu$, define

$$m(\alpha) = \min\{ \gamma < \text{cf} \, \mu : \alpha \leq M(\gamma) \},$$

$$n(\beta) = \min\{ \delta < \text{cf} \, \nu : \beta \leq N(\delta) \},$$

where $\min A$ denotes the minimal ordinal number in $A$. Note that, if $\alpha \in \text{ran} \, M$, then $m(\alpha) = M^{-1}(\alpha)$.

Furthermore, assume $(\mu, \nu) \not\in X$ and $\omega_1 \leq \text{cf} \, \mu = \text{cf} \, \nu = \kappa$. We will use the following notation:

$$X(L, M, N) = \{ (\alpha, \beta) \in X \cap \mu \times \nu : m(\alpha) \leq n(\beta) \} \cup X^{(\nu)}_\mu,$$

$$X(R, M, N) = \{ (\alpha, \beta) \in X \cap \mu \times \nu : m(\alpha) \geq n(\beta) \} \cup X^{(\mu)}_\nu,$$

$$X(\triangle, M, N) = \{ (M(\gamma), N(\gamma)) \in X : \gamma \leq \kappa \},$$

$$\Delta_{MN}(X) = \{ \gamma < \kappa : (M(\gamma), N(\gamma)) \in X \}.$$ 

Intuitively, $X(L, M, N)$ is considered as the upper-left half of $X^{\nu+1}_{\mu+1}$, $X(R, M, N)$ as the lower-right half of $X^{\nu+1}_{\mu+1}$ and $X(\triangle, M, N)$ as the diagonal part of $X^{\nu+1}_{\mu+1}$. Since $M$ and $N$ are homeomorphic closed embeddings, observe that $X(\triangle, M, N)$ and $\Delta_{MN}(X)$ are homeomorphic and that $X(L, M, N)$, $X(R, M, N)$ and $X(\triangle, M, N)$ are closed in $X$.

Let $Y$ be a topological space. Subsets $F$ and $G$ of $Y$ are said to be separated if there are disjoint open sets $U$ and $V$ containing $F$ and $G$ respectively; of course, separated sets are disjoint, and $\emptyset$ and $G$ are separated for each $G \subset Y$. More generally, a collection $\mathcal{H}$ of subsets of $Y$ is said to
be *separated* if there is a pairwise disjoint collection \( \mathcal{U} = \{ U(H) : H \in \mathcal{H} \} \) of open sets in \( Y \) such that each \( U(H) \) contains \( H \). A space \( Y \) is said to be *CWN* (CollectionWise Normal) if any discrete collection of closed sets is separated. Let \( \mathcal{U} \) be an open cover of \( Y \). A collection \( \mathcal{F} = \{ F(U) : U \in \mathcal{U} \} \) of subsets of \( Y \) indexed by \( \mathcal{U} \) is a *shrinking* of \( \mathcal{U} \) if \( F(U) \subseteq U \) for each \( U \in \mathcal{U} \).

A closed shrinking is a shrinking by closed sets. Throughout the paper, for convenience, we do not require \( \mathcal{F} \) to cover \( Y \). We call a space \( Y \) *shrinking* if each open cover of \( Y \) has a closed shrinking which covers \( Y \).

2. Theorem and lemmas. Using the notation described in Section 1, we shall show:

**Theorem.** Assume \( X \subseteq (\lambda + 1)^2 \). The following (1)–(4) are equivalent:

1. \( X \) is shrinking.
2. \( X \) is CWN.
3. \( X \) is normal.
4. For every \( \langle \mu, \nu \rangle \in (\lambda + 1)^2 \setminus X \) with \( \omega \leq \text{cf} \mu \) and \( \omega \leq \text{cf} \nu \), the following (4-1)–(4-5) hold:

   (4-1) \( X(\mu) \) and \( X(\nu) \) are separated.
   (4-2) If \( \omega_1 \leq \text{cf} \nu \) and \( V_\mu(X) \cap \nu \) is not stationary in \( \nu \), then there is a cub set \( D \) in \( \text{cf} \nu \) such that \( X(\mu) \) and \( X^{N(D) \cup \{ \nu \}} \) are separated.
   (4-3) If \( \omega_1 \leq \text{cf} \mu \) and \( H_\nu(X) \cap \mu \) is not stationary in \( \mu \), then there is a cub set \( C \) in \( \text{cf} \mu \) such that \( X(\nu) \) and \( X_{\mu \cup \nu} \) are separated.
   (4-4) If \( \omega_1 \leq \text{cf} \mu < \text{cf} \nu \), \( V_\mu(X) \cap \nu \) is not stationary in \( \nu \), and both \( H_\nu(X) \cap \mu \) and \( A_\mu^\nu \) are non-stationary in \( \mu \) or \( \omega_1 \leq \text{cf} \nu < \text{cf} \mu \), \( H_\nu(X) \cap \mu \) is not stationary in \( \mu \), and both \( V_\mu(X) \cap \nu \) and \( B_\mu^\nu \) are non-stationary in \( \nu \), then there are cub sets \( C \) in \( \text{cf} \mu \) and \( D \) in \( \text{cf} \nu \) such that \( X_{\mu \cup \nu} \) and \( X^{N(D) \cup \{ \nu \}} \) are separated.
   (4-5) If \( \omega_1 \leq \text{cf} \mu = \text{cf} \nu = \kappa \), then (4-5-a) and (4-5-b) hold.
   (4-5-a) \( X(\Delta, M, N) \) and \( X(\mu) \cup X(\nu) \) are separated.
   (4-5-b) If \( \Delta_{MN}(X) \) is not stationary in \( \kappa \), then (b1)–(b4) hold:
   (b1) If \( V_\mu(X) \cap \nu \) is stationary in \( \nu \), then \( X(\mu) \) and any closed set disjoint from \( X(\mu) \) are separated.
   (b2) If \( V_\mu(X) \cap \nu \) is not stationary in \( \nu \), then there is a cub set \( D \) in \( \kappa \) such that the sets \( X(R, M, N)_{M(D) \cup \{ \mu \}} \) and \( X(R, M, N)_{N(D) \cup \{ \nu \}} \) are separated.
   (b3) If \( H_\nu(X) \cap \mu \) is stationary in \( \mu \), then \( X(\nu) \) and any closed set disjoint from \( X(\nu) \) are separated.
   (b4) If \( H_\nu(X) \cap \mu \) is not stationary in \( \mu \), then there is a cub set \( C \) in \( \kappa \) such that the sets \( X(L, M, N)_{N(C) \cup \{ \nu \}} \) and \( X(L, M, N)_{M(C) \cup \{ \mu \}} \) are separated.
To prove the theorem, we need several lemmas. First it is straightforward to show:

**Lemma 1.** Let $X$ be the finite union of closed subspaces $X_i$ ($i \in n$).

1. Let $U$ be an open cover of $X$. If $U|X_i = \{U \cap X_i : U \in U\}$ has a closed shrinking covering $X_i$ for each $i \in n$, then $U$ has a closed shrinking which covers $X$.

2. Let $H$ be a discrete collection of closed sets in $X$. If $H|X_i$ is separated in $X_i$ for each $i \in n$, then $H$ is separated in $X$.

This lemma implies:

**Lemma 2.** If $X$ is the union of two normal (shrinking, CWN) open subspaces $Y$ and $Z$ such that $X \setminus Y$ and $X \setminus Z$ are separated, then $X$ is normal (shrinking, CWN).

**Lemma 3.** Assume $\omega_1 \leq \text{cf} \mu < \text{cf} \nu$ and $X \subset (\mu + 1) \times (\nu + 1) \setminus \{(\mu, \nu)\}$. If $A^\nu_\mu$ is not stationary in $\mu$, then there are cub sets $C$ in $\text{cf} \mu$ and $D$ in $\text{cf} \nu$ such that

$$X \cap M(C) \times N(D) = \emptyset.$$ 

**Proof.** Assume $A^\nu_\mu$ is not stationary in $\mu$. Take a cub set $C$ in $\text{cf} \mu$ such that $M(C) \cap A^\mu_\mu = \emptyset$. For each $\gamma \in C$, by the non-stationarity of $V_{M(\gamma)}(X) \cap \nu$, fix a cub set $D_{\gamma}$ in $\text{cf} \mu$ such that $V_{M(\gamma)}(X) \cap N(D_{\gamma}) = \emptyset$. Put $D = \bigcap_{\gamma \in C} D_{\gamma}$. Since $|C| \leq \text{cf} \mu < \text{cf} \nu$, $D$ is cub in $\text{cf} \nu$. Then these cub sets $C$ and $D$ work.

In an analogous way, we can show:

**Lemma 3’.** Assume $\omega_1 \leq \text{cf} \nu < \text{cf} \mu$ and $X \subset (\mu + 1) \times (\nu + 1) \setminus \{(\mu, \nu)\}$. If $B^\nu_\mu$ is not stationary in $\nu$, then there are cub sets $C$ in $\text{cf} \mu$ and $D$ in $\text{cf} \nu$ such that

$$X \cap M(C) \times N(D) = \emptyset.$$ 

Hereafter, we will not write down such analogous lemmas, but refer to them as “the analogues” of Lemmas 5–9.

**Lemma 4.** Assume $\omega_1 \leq \text{cf} \nu = \text{cf} \mu = \kappa$ and $X \subset (\mu + 1) \times (\nu + 1) \setminus \{(\mu, \nu)\}$. If $X$ is normal and $\Delta_{M,N}(X)$ is not stationary in $\kappa$, then there is a cub set $C$ in $\kappa$ such that

$$X \cap M(C) \times N(C) = \emptyset.$$ 

**Proof.** First we show $A^\nu_\mu$ is not stationary in $\mu$. Assume, on the contrary, that $A^\nu_\mu$ is stationary in $\mu$. Then $A = M^{-1}(A^\mu_\mu) \cap \text{Lim}(\kappa)$ is stationary in $\kappa$. For each $\gamma \in A$, pick

$$h(\gamma) \in N^{-1}(V_{M(\gamma)}(X)) \cap \bigcap_{\gamma' \in A \cap \gamma} \text{Lim}(N^{-1}(V_{M(\gamma')}(X))) \cap \text{Lim}(\kappa)$$

...
with $\gamma < h(\gamma) < \kappa$. This can be done, because $N^{-1}(V_{M(\gamma)}(X))$ is stationary in $\kappa$, $\operatorname{Lim}(N^{-1}(V_{M(\gamma)}(X)))$ is cub in $\kappa$ for each $\gamma' \in A \cap \gamma$, $|A \cap \gamma| < \kappa$ and $\operatorname{Lim}(\kappa) = \operatorname{Lim}_x(\kappa)$ is cub in $\kappa$, so the intersection is stationary in $\kappa$. For each $\gamma \in \kappa \setminus A$, put $h(\gamma) = 0$. Take a cub set $C'$ in $\kappa$ disjoint from $\Delta_{M,N}(X)$, and put

$$C = \{ \gamma < \kappa : \forall \gamma' < \gamma \ (h(\gamma') \not< \gamma) \} \cap C'.$$

Since $C$ is cub in $\kappa$ and $A$ is stationary in $\kappa$, $A' = A \cap \gamma$ is stationary in $\kappa$.

For each $\gamma \in A'$, put $x_\gamma = (M(\gamma), N(h(\gamma)))$. Since, by the definition of $h(\gamma)$, $N(h(\gamma)) \in V_{M(\gamma)}(X)$, we have $x_\gamma \in X$ for each $\gamma \in A'$.

**Claim 1.** $F = \{x_\gamma : \gamma \in A'\}$ is closed discrete in $X$.

**Proof.** Note that $F \subset M(C) \times \operatorname{ran} N$. Let $\langle \alpha, \beta \rangle \in X$. We will find an open neighborhood $U$ of $\langle \alpha, \beta \rangle$ which intersects $F$ in at most one point.

**Case 1.** $\alpha \in \mu \setminus M(C)$ or $\beta \in \nu \setminus \operatorname{ran} N$. If $\alpha \in \mu \setminus M(C)$, then, by the closedness of $M(C)$ in $\mu$, there is $\alpha' < \alpha$ such that $\langle \alpha', \alpha \rangle \cap M(C) = \emptyset$. Then $U = (\alpha', \alpha) \times (\nu + 1) \cap X$ is a neighborhood of $\langle \alpha, \beta \rangle$ missing $F$.

If $\beta \in \nu \setminus \operatorname{ran} N$, then there is $\beta' < \beta$ such that $\langle \beta', \beta \rangle \cap \operatorname{ran} N = \emptyset$. Then $U = (\mu + 1) \times (\beta', \beta) \cap X$ is as desired.

**Case 2.** Otherwise, i.e., $\alpha \in M(C) \cup \{\mu\}$ and $\beta \in \operatorname{ran} N \cup \{\nu\}$. There are two subcases.

(2-1): $\alpha \in M(C) \cup \{\mu\}$ and $\beta \in \operatorname{ran} N$. If $\alpha > M(n(\beta))$, then put $U = (M(n(\beta)), \alpha) \times [0, \beta] \cap X$. Assume $U \ni (M(\gamma), N(h(\gamma)))$ for some $\gamma \in A'$. Then we have $n(\beta) < \gamma$ and $N(h(\gamma)) \leq \beta$ (thus $h(\gamma) \leq n(\beta)$). Therefore $h(\gamma) < \gamma$. But this contradicts the definition of $h(\gamma)$. So $U \cap F = \emptyset$.

If $\alpha \leq M(n(\beta))$, then, since $M(n(\beta)) < \mu$, we have $\alpha \in M(C)$ in this case. Therefore, as $\alpha = M(m(\alpha)) \leq M(n(\beta))$, we have $m(\alpha) \leq n(\beta)$. Assume $m(\alpha) = n(\beta)$. Since $(M(m(\alpha)), N(n(\beta))) = (\alpha, \beta) \in X$, it follows that $m(\alpha) = n(\beta) \in \Delta_{M,N}(X)$. On the other hand, since $m(\alpha) \in C \subset C' \subset \kappa \setminus \Delta_{M,N}(X)$, we get a contradiction. Hence we have $m(\alpha) < n(\beta)$.

Put $U = [0, \alpha] \times (N(m(\alpha)), \beta] \cap X$. Assume $U \ni \gamma = (M(\gamma), N(h(\gamma)))$ for some $\gamma \in A'$ with $m(\alpha) \neq \gamma$. As $M(\gamma) \leq \alpha = M(m(\alpha))$ and $m(\alpha) \neq \gamma$, we have $\gamma < m(\alpha)$. Since $\gamma < m(\alpha) \in C$, we get $h(\gamma) < m(\alpha)$. On the other hand, from $N(m(\alpha)) < N(h(\gamma))$ it follows that $m(\alpha) < h(\gamma)$. This is a contradiction. This argument implies $U \cap F \subset \{x_{m(\alpha)}\}$.

(2-2): $\alpha \in M(C) \cup \{\mu\}$ and $\beta = \nu$. Since $\langle \alpha, \beta \rangle \in X$ but $\langle \mu, \nu \rangle \notin X$, we have $\alpha \in M(C)$. Put $U = [0, \alpha] \times (N(m(\alpha)), \beta] \cap X$. Then $|U \cap F| \leq 1$ as above.

This completes the proof of Claim 1.

Decompose $A'$ into disjoint stationary sets $T_0$ and $T_1$ in $\kappa$, and put $F_i = \{x_\gamma : \gamma \in T_i\}$ for $i \in 2 = \{0, 1\}$. Let $U_i$ be an open set containing $F_i$ for each $i \in 2$. 
Claim 2. $\text{Cl}U_0 \cap \text{Cl}U_1 \neq \emptyset$.

Proof. For each $\gamma \in T_i$ with $i \geq 2$, since $x_\gamma = (M(\gamma), N(h(\gamma))) \in U_i$ and $\gamma$ and $h(\gamma)$ are in $\text{Lim}(\kappa)$, there are $f(\gamma) < \gamma$ and $g(\gamma) < h(\gamma)$ such that $\gamma \leq g(\gamma)$ and

$$(M(f(\gamma)), M(\gamma)) \times (N(g(\gamma)), N(h(\gamma))) \cap X \subset U_i.$$  

By the PDL, for each $i \geq 2$, there are $\zeta_i < \kappa$ and a stationary set $T_i' \subset T_i$ such that $f(\gamma) = \zeta_i$ for each $\gamma \in T_i'$. Put $\gamma_0 = \max\{\zeta_0, \zeta_1\}$. Then

$$(M(\gamma_0), M(\gamma)) \times (N(g(\gamma)), N(h(\gamma))) \cap X \subset U_i$$

for each $i \geq 2$ and $\gamma \in T_i'$.

Take $\gamma_1$ and $\gamma_2$ such that $\gamma_0 < \gamma_1 \in A$ and $\gamma_1 < \gamma_2 \in \bigcap_{i \geq 2} \text{Lim}(T_i')$. We shall show $\langle M(\gamma_1), N(\gamma_2) \rangle \in \text{Cl}U_0 \cap \text{Cl}U_1$. To see this, let $V$ be a neighborhood of $\langle M(\gamma_1), N(\gamma_2) \rangle$. As $\gamma_2 \in \text{Lim}(\kappa)$, there is $\gamma_3 < \gamma_2$ with $\gamma_1 \leq \gamma_3$ such that $\{M(\gamma_1)\} \times (N(\gamma_3), N(\gamma_2)] \cap X \subset V$. Then, since $\gamma_2 \in \text{Lim}(T_0')$, there are $\gamma_4$ and $\gamma_5$ in $T_0'$ with $\gamma_3 < \gamma_4 < \gamma_5 < \gamma_2$. Since $\gamma_5 \in T_0' \subset A' \subset C$, the definition of $C$ yields $\gamma_4 < h(\gamma_4) < \gamma_5$. As $\gamma_1 \in A \cap \gamma_4$, the definition of $h(\gamma_4)$ shows that $h(\gamma_4) \in \text{Lim}(N^{-1}(V_{M(\gamma_1)}(X)))$. Then, since $\gamma_4 \leq g(\gamma_4) < h(\gamma_4)$, there is $\gamma_6 \in N^{-1}(V_{M(\gamma_1)}(X))$ such that $g(\gamma_4) < \gamma_6 < h(\gamma_4)$. Finally,

$$\langle M(\gamma_1), N(\gamma_6) \rangle \in \{M(\gamma_1)\} \times (N(\gamma_3), N(\gamma_2)] \cap (M(\gamma_0), M(\gamma_4)) \times (N(g(\gamma_4)), N(h(\gamma_4))) \cap X \subset V \cap U_0.$$ 

This means $\langle M(\gamma_1), N(\gamma_2) \rangle \in \text{Cl}U_0$. Similarly we have $\langle M(\gamma_1), N(\gamma_2) \rangle \in \text{Cl}U_1$. This completes the proof of Claim 2.

Claim 2 contradicts the normality of $X$. Therefore $A'_\mu$ is not stationary in $\mu$. By a similar argument, $B'_\mu$ is not stationary in $\nu$.

Finally, since $\triangle_{MN}(X)$ is not stationary in $\kappa$, take a cub set $D$ in $\kappa$ such that $D \cap [M^{-1}(A'_\mu) \cup N^{-1}(B'_\mu) \cup \triangle_{MN}(X)] = \emptyset$. For each $\gamma \in D$, since $V_{M(\gamma)}(X) \cap \nu$ is not stationary in $\nu$ and $H_{N(\gamma)}(X) \cap \mu$ is not stationary in $\mu$, we can take a cub set $C_\gamma$ in $\kappa$ disjoint from $N^{-1}(V_{M(\gamma)}(X)) \cup M^{-1}(H_{N(\gamma)}(X))$. Then by an argument similar to [Ku, II, Lemma 6.14], the diagonal intersection

$$E = \{\delta \in D : \forall \gamma \in D \cap \delta (\delta \in C_\gamma)\}$$

is cub in $\kappa$. Assume $\langle M(\gamma), N(\delta) \rangle \in X$ for some $\gamma$ and $\delta$ in $E$. Since $D$ is disjoint from $\triangle_{MN}(X)$ and $E \subset D$, we have $\gamma \neq \delta$. So we may assume $\gamma < \delta$. Then since $\gamma \in D \cap \delta$ and $\delta \in E$, we have $\delta \in C_\gamma$, and thus $N(\delta) \notin V_{M(\gamma)}(X)$. This contradicts $\langle M(\gamma), N(\delta) \rangle \in X$. This means $X \cap M(E) \times N(E) = \emptyset$. This completes the proof of Lemma 4. $\blacksquare$

Lemma 5. Assume $\omega_1 \leq \text{cf} \nu < \text{cf} \mu$ and $X \subset (\mu + 1) \times \nu$. If $V_\mu(X) \cap \nu$ is stationary in $\nu$, then the following hold:
(1) For each open cover $\mathcal{U}$ of $X$, there are $\mu' < \mu$, $\nu' < \nu$ and a shrinking $\mathcal{F}$ of $\mathcal{U}$ by clopen sets in $X$ such that $\bigcup \mathcal{F} = (\mu', \mu] \times (\nu', \nu) \cap X$.

(2) For each discrete collection $\mathcal{H}$ of closed sets in $X$, there are $\mu' < \mu$ and $\nu' < \nu$ such that $(\mu', \mu] \times (\nu', \nu) \cap X$ meets at most one member of $\mathcal{H}$.

**Proof.** (1) For each $\delta \in N^{-1}(\nu(X)) \cap \lim(\text{cf } \nu)$, fix $f(\delta) < \text{cf } \mu$, $g(\delta) < \delta$ and $U(\delta) \in \mathcal{U}$ such that $(M(f(\delta)), \mu] \times (N(g(\delta)), N(\delta)] \cap X \subset U(\delta)$. Applying the PDL, we can find $\delta_0 < \text{cf } \nu$ and a stationary set $S' \subset N^{-1}(\nu(X)) \cap \lim(\text{cf } \nu)$ such that $g(\delta) = \delta_0$ for each $\delta \in S'$. If $\mu > \text{cf } \nu$, then put $\gamma_0 = \sup\{f(\delta) : \delta \in S'\}$ and $S = S'$. If $\mu < \text{cf } \nu$, then, again applying the PDL, we find a stationary set $S \subset S'$ and $\gamma_0 < \text{cf } \mu$ such that $f(\delta) = \gamma_0$ for each $\delta \in S$. In either case, putting $\mu = M(\gamma_0)$ and $\nu' = N(\delta_0)$, we have found a stationary set $S \subset N^{-1}(\nu(X)) \cap \lim(\text{cf } \nu)$ such that $(\mu', \mu] \times (\nu', \nu) \cap X \subset U(\delta)$ for each $\delta \in S$.

For each $\delta$ and $\delta'$ in $S$, define $\delta \sim \delta'$ by $U(\delta) = U(\delta')$. Then $\sim$ is an equivalence relation on $S$, so let $S/\sim$ be its quotient space. For each $E \in S/\sim$, put $U_E = U(\delta)$ for some (any) $\delta \in E$. Note that members of $\{U_E : E \in S/\sim\}$ are all distinct. There are two cases to consider.

First assume that there is $E \in S/\sim$ such that $E$ is unbounded in $\text{cf } \nu$. In this case, since $(\mu', \mu] \times (\nu', \nu) \cap X \subset U(\delta) = U_E$ for each $\delta \in E$, we have $(\mu', \mu] \times (\nu', \nu) \cap X \subset U_E$. For each $U \in \mathcal{U}$, put

$$F(U) = \begin{cases} (\mu', \mu] \times (\nu', \nu) \cap X & \text{if } U = U_E, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ is the desired shrinking of $\mathcal{U}$.

Next assume all $E$’s, $E \in S/\sim$, are bounded in $\text{cf } \nu$. By induction, define $\delta(\eta) \in E(\eta) \in S/\sim$ for each $\eta \in \text{cf } \nu$ so that $\eta + \sup(\bigcup_{\zeta < \eta} E(\zeta)) < \delta(\eta)$. Clearly $E(\eta)$’s are all distinct and $\{\delta(\eta) : \eta < \text{cf } \nu'\}$ is strictly increasing and unbounded in $\text{cf } \nu$. For each $U \in \mathcal{U}$, put

$$F(U) = \begin{cases} (\mu', \mu] \times (\nu', N(\delta(\eta))] \cap X & \text{if } U = U_E(\eta) \text{ for some } \eta < \text{cf } \nu, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\mathcal{F} = \{F(U) : U \in \mathcal{U}\}$ is the desired shrinking of $\mathcal{U}$.

(2) For each $\delta \in N^{-1}(\nu(X)) \cap \lim(\text{cf } \nu)$, fix $f(\delta) < \text{cf } \mu$ and $g(\delta) < \delta$ such that $(M(f(\delta)), \mu] \times (N(g(\delta)), N(\delta)] \cap X$ meets at most one member of $\mathcal{H}$. Then as in (1), we can find desired $\nu' < \nu$ and $\mu' < \mu$.

**Lemma 6.** Assume $\omega_1 \leq \text{cf } \nu \neq \text{cf } \mu$, $X \subset (\mu + 1) \times (\nu + 1) \setminus \{\langle \mu, \nu \rangle\}$ and $V_\mu(X) \cap \nu$ is stationary in $\nu$. If $X_{(\mu)}$ and $X^{(\nu)}$ are separated, then there are $\mu' < \mu$ and $\nu' < \nu$ such that $(\mu', \mu] \times (\nu', \nu) \cap X$ is closed (and trivially open) in $X$.

**Proof.** Since $X_{(\mu)}$ and $X^{(\nu)}$ are separated, take an open set $V$ such that $X_{(\mu)} \subset V \subset \text{Cl } V \subset X \setminus X^{(\nu)}$. For each $\delta \in N^{-1}(\nu(X)) \cap \lim(\text{cf } \nu)$, fix
Lemma 7. Let $\mathcal{P}$ be a topological property which is closed under taking closed subspaces and free unions. Assume $X \subset (\mu + 1) \times (\nu + 1)$ and $X_{\mu+1}$ has the property $\mathcal{P}$ for each $\mu' < \mu$.

(1) If $\text{cf} \mu = \omega$, then $X_\mu$ has the property $\mathcal{P}$.

(2) If $\text{cf} \mu \geq \omega_1$ and $C$ is a cub set in $\text{cf} \mu$ and $V$ is an open set in $X$ containing $X_{\mu(C) \cup \{\mu\}}$, then $X \setminus V$ has the property $\mathcal{P}$.

Proof. (1) Since $X_\mu = \bigoplus_{n \in \omega} X_{\mu(M(n-1),M(n))}$ and $X_{\mu(M(n-1),M(n))}$ is a closed subspace of $X_{\mu(M(n)+1)}$, $X_\mu$ has the property $\mathcal{P}$.

(2) For each $\gamma \in C$, put $h(\gamma) = \text{sup}(C \cap \gamma)$. Note that $h(\gamma) < \gamma$ if $\gamma \in C \setminus \text{Lim}(C)$. For each $\gamma \in C \setminus \text{Lim}(C)$, put $Y(\gamma) = X_{\{h(\gamma),\gamma\}} \setminus V$. Since $Y(\gamma)$ is a closed subspace of $X_{\mu(C)+1}$, it has the property $\mathcal{P}$. Therefore $X \setminus V = \bigoplus_{\gamma \in C \setminus \text{Lim}(C)} Y(\gamma)$ has the property $\mathcal{P}$. ■

Lemma 8. Assume $\omega_1 \leq \text{cf} \mu < \text{cf} \nu$, $X \subset (\mu + 1) \times (\nu + 1) \setminus \{(\mu, \nu)\}$ and $A^\nu_{\mu}$ is stationary in $\mu$. If there are cub sets $C$ in $\text{cf} \mu$ and $D$ in $\text{cf} \nu$ such that $X_{\mu(C) \cup \{\mu\}}$ and $X_{\nu(D)}$ are separated, and $X_{\mu(C) \cup \{\mu\}}$ and $X_{\nu(D)}$ are separated, then there are $\mu' < \mu$ and $\nu' < \nu$ such that $(\mu', \mu) \times (\nu', \nu) \cap X$ is closed (and trivially open) in $X$.

Proof. Take open sets $V$ and $W$ in $X$ such that

\[ X_{\mu(C) \cup \{\mu\}} \subset V \subset \text{Cl}V \subset X \setminus X_{\nu(D)}, \]
\[ X_{\nu(D)} \subset W \subset \text{Cl}W \subset X \setminus X_{\{\mu\}}. \]

First fix $\gamma \in C \cap M^{-1}(A^\nu_{\mu}) \cap \text{Lim}(\text{cf} \mu)$. For each $\delta \in D \cap N^{-1}(V_{M(\gamma)}(X)) \cap \text{Lim}(\text{cf} \nu)$, since $(M(\gamma), N(\delta)) \in V \cap W$, fix $f(\gamma, \delta) < \gamma$ and $g(\gamma, \delta) < \delta$ such that

\[ (M(f(\gamma, \delta)), M(\gamma)] \times (N(g(\gamma, \delta)), N(\delta)] \cap X \subset V \cap W. \]

Since $f(\gamma, \delta) < \gamma$ and $g(\gamma, \delta) < \delta$ for each $\delta \in D \cap N^{-1}(V_{M(\gamma)}(X)) \cap \text{Lim}(\text{cf} \nu)$, noting that $\text{cf} \mu < \text{cf} \nu$ and applying the PDL, we have $f(\gamma) < \gamma$, $g(\gamma) < \text{cf} \nu$ and a stationary set $S_\gamma \subset D \cap N^{-1}(V_{M(\gamma)}(X)) \cap \text{Lim}(\text{cf} \nu)$ such that $f(\gamma, \delta) = f(\gamma)$ and $g(\gamma, \delta) = g(\gamma)$ for each $\delta \in S_\gamma$. Put $\delta_0 = \text{sup}\{g(\gamma) : \gamma \in C \cap M^{-1}(A^\nu_{\mu}) \cap \text{Lim}(\text{cf} \mu)\}$.

Next, since $f(\gamma) < \gamma$ for each $\gamma \in C \cap M^{-1}(A^\nu_{\mu}) \cap \text{Lim}(\text{cf} \mu)$, again applying the PDL, we have $\gamma_0 < \text{cf} \mu$ and a stationary set $T \subset C \cap M^{-1}(A^\nu_{\mu}) \cap \text{Lim}(\text{cf} \mu)$ such that $f(\gamma) = \gamma_0$ for each $\gamma \in T$. Then we have

\[ (M(\gamma_0), \mu) \times (N(\delta_0), \nu) \cap X \subset V \cap W. \]
Put $\mu' = M(\gamma_0)$ and $\nu' = N(\delta_0)$. Since $\text{Cl} V \cap \text{Cl} W$ is disjoint from $X_{\{\mu\}} \cup X^{\{\nu\}}$, we conclude that $(\mu', \mu) \times (\nu', \nu) \cap X$ is closed in $X$. 

**Lemma 9.** Assume $\omega_1 \leq \text{cf} \mu < \text{cf} \nu$, $X \subset \mu \times \nu$ and $A'_{\mu}$ is stationary in $\mu$.

1. If $\mathcal{U}$ is an open cover of $X$, then there are $\mu' < \mu$, $\nu' < \nu$ and a shrinking $\mathcal{F}$ of $\mathcal{U}$ by clopen sets in $X$ such that $\bigcup \mathcal{F} = (\mu', \mu) \times (\nu', \nu) \cap X$.

2. If $\mathcal{H}$ is a discrete collection of closed sets in $X$, there are $\mu' < \mu$ and $\nu' < \nu$ such that $(\mu', \mu) \times (\nu', \nu) \cap X$ meets at most one member of $\mathcal{H}$.

**Proof.** (1) First fix $\gamma \in M^{-1}(A'_{\mu}) \cap \text{Lim}(\text{cf} \mu)$. For each $\delta \in N^{-1}(V_{M(\gamma)}(X)) \cap \text{Lim}(\text{cf} \nu)$, using $\langle M(\gamma), N(\delta) \rangle \in X$, fix $f(\gamma, \delta) < \gamma$, $g(\gamma, \delta) < \delta$ and $U(\gamma, \delta) \in \mathcal{U}$ such that

$$\langle M(f(\gamma, \delta)), M(\gamma) \rangle \times (N(g(\gamma, \delta)), N(\delta)) \cap X \subset U(\gamma, \delta).$$

As in the proof of Lemma 8, applying the PDL twice, we find a stationary set $T \subset M^{-1}(A'_{\mu}) \cap \text{Lim}(\text{cf} \mu)$, a stationary set $S_\gamma \subset N^{-1}(V_{M(\gamma)}(X)) \cap \text{Lim}(\text{cf} \nu)$ for each $\gamma \in T$, $\mu' < \mu$ and $\nu' < \nu$ such that $(\mu', M(\gamma)) \times (\nu', N(\delta)) \cap X \subset U(\gamma, \delta)$ for each $\delta \in S_\gamma$ with $\gamma \in T$.

Put $H = \bigcup_{\gamma \in T} \{ \gamma \} \times S_\gamma$. For each $\langle \gamma, \delta \rangle$ and $\langle \gamma', \delta' \rangle$ in $H$, define $\langle \gamma, \delta \rangle \sim \langle \gamma', \delta' \rangle$ by $U(\gamma, \delta) = U(\gamma', \delta')$. For each $E \in H/\sim$, define $U_E = U(\gamma, \delta)$ for some (any) $\langle \gamma, \delta \rangle \in E$. Then note that

(i) $$\bigcup_{\langle \gamma, \delta \rangle \in E} \{ \mu', M(\gamma) \} \times (\nu', N(\delta)) \cap X \subset U_E.$$ 

For each $\gamma \in T$ and $E \in H/\sim$, put

$$j(E, \gamma) = \sup \{ \delta \in S_\gamma : \langle \gamma, \delta \rangle \in E \}.$$ 

Then put $T(E) = \{ \gamma \in T : j(E, \gamma) = \text{cf} \nu \}$ and $k(E) = \sup T(E)$.

**Claim 1.** $(\mu', M(\gamma)) \times (\nu', \nu) \cap X \subset U_E$ for each $\gamma \in T(E)$.

**Proof.** Assume $\langle \alpha, \beta \rangle \in (\mu', M(\gamma)) \times (\nu', \nu) \cap X$ with $\gamma \in T(E)$. Since $\beta < \nu$ and $\gamma \in T(E)$, there is a $\delta \in S_\gamma$ with $\langle \gamma, \delta \rangle \in E$ such that $\beta < N(\delta)$. Then, by (i), $\langle \alpha, \beta \rangle \in U_E$. This completes the proof of Claim 1.

There are some cases to consider.

**Case 1:** There is an $E \in H/\sim$ such that $k(E) = \text{cf} \mu$. In this case, by Claim 1, $(\mu', \mu) \times (\nu', \nu) \cap X \subset U_E$. So for each $U \in \mathcal{U}$, put

$$F(U) = \begin{cases} (\mu', \mu) \times (\nu', \nu) \cap X & \text{if } U = U_E, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\mathcal{F} = \{ F(U) : U \in \mathcal{U} \}$ is the desired shrinking of $\mathcal{U}$.

**Case 2:** $k(E) < \text{cf} \mu$ for each $E \in H/\sim$. There are two subcases.
(2-1): \( \sup \{ k(E) : E \in H/\sim \} = \text{cf} \mu \). By induction, define two sequences \( \{ E(\xi) : \xi < \text{cf} \mu \} \) in \( H/\sim \) and \( \{ \gamma(\xi) : \xi < \text{cf} \mu \} \) in \( T \) so that \( \xi + \sup_{\eta < \xi} k(E(\eta)) < \gamma(\xi) \in T(E(\xi)). \) Observe that \( E(\xi) \)'s are all distinct and \( \{ \gamma(\xi) : \xi < \text{cf} \mu \} \) is strictly increasing and unbounded in \( \text{cf} \mu \). By Claim 1, \( Z(\xi) = (\mu', M(\gamma(\xi))) \times (\nu', \nu) \cap X \subset U_{E(\gamma(\xi))} \). So for each \( U \in \mathcal{U} \), put
\[
F(U) = \begin{cases} 
Z(\xi) & \text{if } U = U_{E(\xi)} \text{ for some } \xi < \text{cf} \mu, \\
\emptyset & \text{otherwise.}
\end{cases}
\]
Then \( \mathcal{F} = \{ F(U) : U \in \mathcal{U} \} \) is the desired shrinking of \( \mathcal{U} \).

(2-2): \( \gamma_0 = \sup \{ k(E) : E \in H/\sim \} < \text{cf} \mu \). Put \( T' = T \setminus [0, \gamma_0] \), \( H' = \bigcup_{\gamma \in T'} \{ \gamma \} \times S_\gamma \) and \( j(E) = \sup \{ j(E, \gamma) : \gamma \in T' \} \) for each \( E \in H/\sim \). Then, since \( j(E, \gamma) < \text{cf} \nu \) for each \( \gamma \in T' \) and \( |T'| \leq \text{cf} \mu < \text{cf} \nu \), we have
\[
\text{(ii) } j(E) < \text{cf} \nu.
\]

Let \( \prec \) be the co-lexicographic order on \( \text{cf} \mu \times \text{cf} \nu \), that is, \( \langle \zeta', \eta' \rangle < \langle \zeta, \eta \rangle \) is defined by \( \eta' < \eta \) or \( (\eta' = \eta \text{ and } \zeta' < \zeta) \). Since \( \text{cf} \mu < \text{cf} \nu \), the \( \prec \)-order type of \( \text{cf} \mu \times \text{cf} \nu \) is \( \text{cf} \nu \). By \( \prec \)-induction, we shall define two sequences \( \{ E(\eta, \gamma) : \langle \zeta, \eta \rangle \in H/\sim \} \) and \( \{ \gamma(\eta, \gamma) : \langle \zeta, \eta \rangle \in \text{cf} \mu \times \text{cf} \nu \} \) in \( H/\sim \) and \( \{ (\gamma(\eta, \gamma), \delta(\eta, \gamma)) : \langle \zeta, \eta \rangle \in \text{cf} \mu \times \text{cf} \nu \} \) in \( H' \) with \( \gamma(\zeta, \eta), \delta(\zeta, \eta) \in E(\zeta, \eta) \) as follows.

Assume \( E(\xi', \eta'), \gamma(\xi', \eta') \) and \( \delta(\xi', \eta') \) are defined with \( \langle \gamma(\xi', \eta'), \delta(\xi', \eta') \rangle \in E(\xi', \eta') \) for all \( \langle \zeta', \eta' \rangle < \langle \zeta, \eta \rangle \). By (ii), take \( \delta < \text{cf} \nu \) with \( \eta + \sup \{ j(E(\xi', \eta')) : \langle \zeta', \eta' \rangle < \langle \zeta, \eta \rangle \} < \delta \). When \( \zeta = 0 \), take \( \langle \gamma(\zeta, \eta), \delta(\zeta, \eta) \rangle \in H' \) with \( \delta < \delta(\zeta, \eta) \), and let \( E(\zeta, \eta) \) be the equivalence class with \( \langle \gamma(\zeta, \eta), \delta(\zeta, \eta) \rangle \in E(\zeta, \eta) \). When \( \zeta > 0 \), noting that \( \gamma(\zeta', \eta') \) has been defined for all \( \zeta' < \zeta \), take \( \gamma < \text{cf} \mu \) such that \( \zeta + \sup \{ \gamma(\zeta', \eta') : \zeta' < \zeta \} < \gamma \), and take \( \langle \gamma(\zeta, \eta), \delta(\zeta, \eta) \rangle \in H' \) with \( \delta < \delta(\zeta, \eta) \) and \( \gamma < \gamma(\zeta, \eta) \). Finally, let \( E(\zeta, \eta) \) be the equivalence class with \( \langle \gamma(\zeta, \eta), \delta(\zeta, \eta) \rangle \in E(\zeta, \eta) \). This completes the construction.

By the construction, \( E(\zeta, \eta) \)'s are all distinct,

(iii) \( \{ \delta(\zeta, \eta) : \langle \zeta, \eta \rangle \in \text{cf} \mu \times \text{cf} \nu \} \) is strictly increasing and unbounded in \( \text{cf} \nu \),

and

(iv) \( \{ \gamma(\zeta, \eta) : \zeta \in \text{cf} \mu \} \) is also strictly increasing and unbounded in \( \text{cf} \mu \) for each \( \eta < \text{cf} \nu \).

As \( \langle \gamma(\zeta, \eta), \delta(\zeta, \eta) \rangle \in E(\zeta, \eta) \), by (i) we have \( Z(\zeta, \eta) = (\mu', M(\gamma(\zeta, \eta))) \times (\nu', N(\delta(\zeta, \eta))) \subset U_{E(\gamma(\zeta, \eta))}. \) Moreover, by (iii) and (iv), \( \{ Z(\zeta, \eta) : \langle \zeta, \eta \rangle \in \text{cf} \mu \times \text{cf} \nu \} \) covers \( (\mu', \mu) \times (\nu', \nu) \cap X \).

For each \( U \in \mathcal{U} \), put
\[
F(U) = \begin{cases} 
Z(\zeta, \eta) & \text{if } U = U_{E(\zeta, \eta)} \text{ for some } \langle \zeta, \eta \rangle \in \text{cf} \mu \times \text{cf} \nu, \\
\emptyset & \text{otherwise.}
\end{cases}
\]
Then $\mathcal{F} = \{ F(U) : U \in \mathcal{U} \}$ is the desired shrinking of $\mathcal{U}$.

The proof of (2) is easier, so we leave it to the reader. ■

**Lemma 10.** Assume $\omega_\kappa \leq \text{cf } \mu = \text{cf } \nu = \kappa$, $X \subset (\mu + 1) \times (\nu + 1) \setminus \{(\mu, \nu)\}$ and $\triangle_{\mathcal{M}}(X)$ is stationary in $\kappa$. If $X(\triangle, M, N)$ and $X_{(\mu)} \cup X_{(\nu)}$ are separated, then there are $\mu' < \mu$ and $\nu' < \nu$ such that $(\mu', \mu) \times (\nu', \nu) \cap X$ is closed (and trivially open) in $X$.

**Proof.** Take an open set $V$ in $X$ such that $X(\triangle, M, N) \subset V \subset \text{Cl } V \subset X \setminus (X_{(\mu)} \cup X_{(\nu)})$. For each $\gamma \in \triangle_{\mathcal{M}}(X) \cap \text{Lim}(\kappa)$, take $f(\gamma) < \gamma$ such that $(M(f(\gamma)), M(\gamma)] \times (N(f(\gamma)), N(\gamma)] \cap X \subset V$. By the PDL, we find $\mu' < \mu$ and $\nu' < \nu$ such that $(\mu', \mu) \times (\nu', \nu) \cap X \subset V$. Since $\text{Cl } V$ is disjoint from $X_{(\mu)} \cup X_{(\nu)}$, we conclude that $(\mu', \mu) \times (\nu', \nu) \cap X$ is closed in $X$. ■

**Lemma 11.** Assume $\omega_\kappa \leq \text{cf } \mu = \text{cf } \nu = \kappa$, $X \subset \mu \times \nu$ and $\triangle_{\mathcal{M}}(X)$ is stationary in $\kappa$.

1. If $\mathcal{U}$ is an open cover of $X$, then there are $\mu' < \mu$, $\nu' < \nu$ and a shrinking $\mathcal{F}$ of $\mathcal{U}$ by clopen sets in $X$ such that $\bigcup \mathcal{F} = (\mu', \mu) \times (\nu', \nu) \cap X$.

2. If $\mathcal{H}$ is a discrete collection of closed sets in $X$, there are $\mu' < \mu$ and $\nu' < \nu$ such that $(\mu', \mu) \times (\nu', \nu) \cap X$ meets at most one member of $\mathcal{H}$.

**Proof.** (1) For each $\delta \in \triangle_{\mathcal{M}}(X) \cap \text{Lim}(\kappa)$, fix $g(\delta) < \delta$ and $U(\delta) \in \mathcal{U}$ such that $(M(g(\delta)), M(\delta)] \times (N(g(\delta)), N(\delta)] \cap X \subset U(\delta)$. By the PDL, we find $\mu' < \mu$, $\nu' < \nu$ and a stationary set $S \subset \triangle_{\mathcal{M}}(X) \cap \text{Lim}(\kappa)$ such that $(\mu', M(\delta)] \times (\nu', N(\delta)] \cap X \subset U(\delta)$ for each $\delta \in S$. Then by an argument similar to the proof of Lemma 5, making use of the equivalence relation, we can find the desired shrinking of $\mathcal{U}$.

(2) is easy. ■

**Lemma 12.** Let $\mathcal{P}$ be a topological property which is closed under taking closed subspaces and free unions. Assume $\omega_\kappa \leq \text{cf } \mu = \text{cf } \nu = \kappa$, $X \subset (\mu + 1) \times (\nu + 1) \setminus \{(\mu, \nu)\}$, $V_\mu(X)$ is stationary in $\kappa$, but $\triangle_{\mathcal{M}}(X)$ is not stationary in $\kappa$; moreover, $X_{\mu^+}$ and $X_{\nu^+}$ have the property $\mathcal{P}$ for each $\mu' < \mu$ and $\nu' < \nu$. If $V$ is an open set in $X$ containing $X_{(\mu)}$, then $X(R, M, N) \setminus V$ has the property $\mathcal{P}$.

**Proof.** Take a cub set $D$ in $\text{Lim}(\kappa)$ disjoint from $\triangle_{\mathcal{M}}(X)$. For each $\delta \in N^{-1}(V_\mu(X)) \cap D$, fix $f(\delta) < \kappa$ and $g(\delta) < \delta$ such that

$$(M(f(\delta)), \mu] \times (N(g(\delta)), N(\delta)] \cap X \subset V.$$

For each $\delta \in \kappa \setminus [N^{-1}(V_\mu(X)) \cap D]$, put $f(\delta) = 0$. By the PDL, take $\delta_0 < \kappa$ and a stationary set $S \subset N^{-1}(V_\mu(X)) \cap D$ such that $g(\delta) = \delta_0$ for each $\delta \in S$. Put $\nu' = N(\delta_0)$, $D' = \{ \delta < \kappa : \forall \delta' < \delta (f(\delta') < \delta) \}$ and $W = \bigcup_{\delta \in S} (M(f(\delta)), \mu] \times (\nu', N(\delta)] \cap X$. Then $D'$ is cub in $\kappa$ and $W \subset V$. Since $X_{\nu^+} \setminus V$ (and therefore $X(R, M, N)_{\nu^+} \setminus V$) has the property $\mathcal{P}$, it suffices
to represent $Y = X(R, M, N)^{[\nu', \nu]} \setminus W$ as the free union of subspaces having the property $\mathcal{P}$. Here note that $Y$ is closed in $X$ and disjoint from $X_{(\mu)} \cup X^{(\nu)}$. To show this, put $C = \text{Lim}(S) \cap D'$. Then $C$ is cub and $C \subset D \cap D'$. For each $\delta \in C$, put $h(\delta) = \text{sup}(C \cap \delta)$. Then by the closedness of $C$, $h(\delta) \in C$ and $h(\delta) \leq \delta$. For each $\delta \in C \setminus \text{Lim}(C)$ (in other words, $h(\delta) < \delta$), put $Y(\delta) = Y_{(M(h(\delta)), M(\delta))}$. Then each $Y(\delta)$ is clopen in $Y$, and therefore closed in $X$. Moreover, as $Y(\delta) \subset X_{M(\delta) + 1}$, $Y(\delta)$ has the property $\mathcal{P}$. Since $Y(\delta)$’s, $\delta \in C \setminus \text{Lim}(C)$, are pairwise disjoint, it suffices to show $Y = \bigcup_{\delta \in C \setminus \text{Lim}(C)} Y(\delta)$. To show this, let $\langle \alpha, \beta \rangle \in Y$. Note $\alpha < \mu$, $\nu' < \beta < \nu$ and $m(\alpha) \geq n(\beta)$. Let $\delta$ be the minimal ordinal number with $m(\alpha) \leq \delta \in C$. Note that $n(\beta) \leq \delta$.

First assume $n(\beta) = \delta$. Since $\delta = n(\beta) \leq m(\alpha) \leq \delta$, we have $\delta \in \Delta_{MN}(X) \cap C$. This contradicts $C \subset D$. Therefore $n(\beta) < \delta$.

Next assume $\delta \in \text{Lim}(C)$. Then by the minimality of $\delta$, we have $m(\alpha) = \delta$. Using $n(\beta) < \delta$ and $\delta \in C \subset \text{Lim}(S) \cap D'$, pick $\delta' \in S$ with $n(\beta) < \delta' < \delta$. Since $\delta \in D'$, we have $f(\delta') < \delta = m(\alpha)$, and therefore $M(f(\delta')) < \alpha$. Moreover, as $n(\beta) < \delta'$, we have

$$\langle \alpha, \beta \rangle \in (M(f(\delta')), \mu] \times (\nu', N(\delta')] \cap X \subset W.$$  

This contradicts $Y \cap W = \emptyset$. Therefore $\delta \in C \setminus \text{Lim}(C)$. By the minimality of $\delta$, this shows that $h(\delta) < m(\alpha) \leq \delta$. This means $\alpha \in (M(h(\delta)), M(\delta)]$, hence

$$\langle \alpha, \beta \rangle \in Y_{(M(h(\delta)), M(\delta))} = Y(\delta).$$

This completes the proof. ■

3. Proof of the Theorem. The implications (1)$\rightarrow$(3) and (2)$\rightarrow$(3) are evident.

(3)$\rightarrow$(4). Let $X$ be normal and $\langle \mu, \nu \rangle \in (\lambda + 1)^2 \setminus X$ with $\omega \leq \text{cf} \, \mu$ and $\omega \leq \text{cf} \, \nu$. Since $\langle \mu, \nu \rangle \notin X$, $X_{(\mu)}$ and $X^{(\nu)}$ are disjoint closed sets in the normal space $X$. Thus (4-1) holds.

To show (4-2), assume $\omega_1 \leq \text{cf} \, \nu$ and $V_\nu(X) \cap \nu$ is not stationary in $\nu$. Then there is a cub set $D$ in $\text{cf} \, \nu$ such that $V_\nu(X) \cap N(D) = \emptyset$. Since $X_{(\mu)}$ and $X^{(\nu)}$ are disjoint closed sets, (4-2) holds.

(4-3) is similar.

To show (4-4), since the remaining case is similar, we may assume $\omega_1 \leq \text{cf} \, \mu < \text{cf} \, \nu$, $V_\nu(X) \cap \nu$ is not stationary in $\nu$, and both $H_\nu(X) \cap \mu$ and $A_\mu'$ are non-stationary in $\mu$. By the non-stationarity of $A_\mu'$ and Lemma 3, there are cub sets $C'$ in $\text{cf} \, \mu$ and $D'$ in $\text{cf} \, \nu$ such that $X \cap M(C') \times N(D') = \emptyset$. Since $V_\nu(X) \cap \nu$ and $H_\nu(X) \cap \mu$ are non-stationary in $\text{cf} \, \nu$ and $\text{cf} \, \mu$ respectively, take cub sets $C \subset C'$ and $D \subset D'$ such that $M(C) \cap H_\nu(X) = \emptyset$ and $N(D) \cap V_\nu(X) = \emptyset$. Then $X \cap (M(C) \cup \{\mu\}) \times (N(D) \cup \{\nu\}) = \emptyset$. Therefore
$X_{M(C)∪[μ]}$ and $X^{N(D)∪[ν]}$ are disjoint closed sets in the normal space $X$. This shows (4-4).

To show (4-5), assume $ω_1 ≤ cf μ = cf ν = κ$. By $⟨μ, ν⟩ ∉ X$, $X(Δ, M, N)$ and $X_{[μ]}∪X^{[ν]}$ are disjoint closed sets in the normal space $X$. This shows (4-5-a).

To show (4-5-b), assume $Δ_{MN}(X)$ is not stationary in $κ$. Since $X$ is normal, (b1) and (b3) are evident. Assume $V_{μ}(X)∩ν$ is not stationary in $ν$. By Lemma 4 and the non-stationarity of $V_{μ}(X)∩ν$, there is a cub set $D ⊆ κ$ such that $X∩M(D)×N(D) = ∅$ and $N(D)∩V_{μ}(X) = ∅$. Then $X∩(M(D)∪{μ})×N(D) = ∅$. Since $X(R, M, N)$ is disjoint from $X^{[ν]}$, we have $X(R, M, N)∩(M(D)∪{μ})×(N(D)∪{ν}) = ∅$. Since $X(R, M, N)$ is closed in $X$, $X(R, M, N)∩M(D)∪{μ}$ and $X(R, M, N)^{N(D)∪[ν]}$ are disjoint closed sets in the normal space $X$. This shows (b2).

Similarly we can show (b4).

(4)→(1). Assume (4) holds but $X$ is not shrinking. Put

$$μ = min{ζ ≤ λ : X_{ζ+1} is not shrinking},$$

$$ν = min{η ≤ λ : X^{η+1}_{μ+1} is not shrinking}.$$

Note that $X^{ν+1}_{μ+1}$ is not shrinking, but $X^{ν+1}_{μ+1}$ and $X^{ν+1}_{μ+1}$ are shrinking for each $μ' < μ$ and $ν' < ν$. Since $X^{ν+1}_{μ+1}$ is a clopen subspace of $X$, we may assume $X = X^{ν+1}_{μ+1}$. Then again note that $X$ is not shrinking, but $X^{μ'+1}_{μ+1}$ and $X^{ν'+1}_{μ+1}$ are shrinking for each $μ' < μ$ and $ν' < ν$. So there is an open cover $U$ of $X$ which does not have a closed shrinking which covers $X$.

**Claim 1.** $⟨μ, ν⟩ ∉ X$.

**Proof.** Assume $⟨μ, ν⟩ ∈ X$. Then there are $μ' < μ$, $ν' < ν$ and $U ∈ U$ such that $Z = ⟨μ', μ⟩×⟨ν', ν⟩∩X ⊆ U$. Since $Z$ is clopen in $X$ and $X^{μ'+1}_{μ+1}∪X^{ν'+1}_{μ+1}∪Z = X$, and $X^{μ'+1}_{μ+1}$ and $X^{ν'+1}_{μ+1}$ are shrinking, by Lemma 1, $U$ has a closed shrinking which covers $X$, a contradiction. This completes the proof of Claim 1.

**Claim 2.** $ω ≤ cf μ$ and $ω ≤ cf ν$.

**Proof.** Assume $μ = μ' + 1$. Since $X$ is the free union $X_{μ}∪X_{[μ]}$ of shrinking subspaces, $U$ can be shrunk, a contradiction. Therefore $ω ≤ cf μ$. Similarly $ω ≤ cf ν$.

First we consider the following case.

**Case 1.** $cf μ ≠ cf ν$. We may assume $cf μ < cf ν$. We consider two subcases:

(1-1): $V_{μ}(X)∩ν$ is stationary in $ν$. Applying Lemma 5 (1) to $U|X^{ν}$, we find $μ' < μ$, $ν' < ν$ and a shrinking $F$ of $U|X^{ν}$ by closed sets in $X^{ν}$ such
that $\bigcup F = (\mu', \mu] \times (\nu', \nu] \cap X$. Since $X_{\{\mu]\}$ and $X_{\{\nu]\}$ are separated by (4-1), applying Lemma 6, we get $\mu''' < \mu$ and $\nu''' < \nu$ with $\mu' < \mu'''$ and $\nu' < \nu'''$ such that $Z = (\mu'''', \mu] \times (\nu'''', \nu] \cap X$ is closed in $X$. Then $F|Z$ is a shrinking of $U$ by closed sets in $X$ which covers $Z$. Since $X_{\mu''''+1}, X_{\nu''''+1}$ and $X_{\{\nu]\}$ are shrinking closed subspaces and $X = X_{\mu''''+1} \cup X_{\nu''''+1} \cup X_{\{\nu]\} \cup Z$, by Lemma 1, $U$ has a closed shrinking which covers $X$. A contradiction.

(1-2): $V_{\mu}(X) \cap \nu$ is not stationary in $\nu$. In this case, by (4-2), there is a cub set $C$ in $\mu$ such that $X_{\{\mu]\}$ and $X_{\{\nu]\}$ are separated. Take disjoint open sets $V$ and $W$ containing $X_{\{\mu]\}$ and $X_{\{\nu]\}$ respectively. Assume $\mu = \omega$. Then by Lemma 7 (1), $X_{\mu}$ is shrinking, thus $X \setminus V$ is shrinking. Moreover, by (2) of the analogue of Lemma 7, $X \setminus W$ is also shrinking. Therefore by Lemma 1, $X$ is shrinking, a contradiction. Therefore we have $\omega_1 \leq \mu$.

Then by an argument similar to (1-1), assuming $H_{\nu}(X) \cap \mu$ is stationary in $\mu$, we get a contradiction (of course we would use the “analogous” lemmas). So $H_{\nu}(X) \cap \mu$ is not stationary in $\mu$.

Now we are in the situation where $\omega_1 \leq \mu < \nu$, and $H_{\nu}(X) \cap \mu$ and $V_{\mu}(X) \cap \nu$ are not stationary in $\mu$ and $\nu$ respectively. By (4-3), we also have a cub set $C$ in $\mu$ such that $X_{\{\nu]\}$ and $X_{\{\mu]\}$ are separated. Again, we consider two subcases:

(1-2-1): $A''_{\nu}$ is stationary in $\mu$. In this case by Lemmas 8 and 9 (1), we find $\nu' < \nu$ and a shrinking $F$ of $U$ by closed sets in $X$ such that $Z = (\mu', \mu] \times (\nu', \nu] \cap X$ is clopen in $X$ and $\bigcup F = Z$. Since $X_{\mu''+1}, X_{\nu''+1}$ and $X_{\{\nu]\}$ are shrinking closed subspaces and $X = X_{\mu''+1} \cup X_{\nu''+1} \cup X_{\{\nu]\} \cup Z$, by Lemma 1, $U$ has a closed shrinking which covers $X$. A contradiction.

(1-2-2): $A''_{\nu}$ is not stationary in $\mu$. In this case by (4-4), there are cub sets $C$ in $\mu$ and $D$ in $\mu$ such that $X_{\{\mu]\}$ and $X_{\{\nu]\}$ are separated. Take disjoint open sets $V$ and $W$ containing $X_{\{\mu]\}$ and $X_{\{\nu]\}$ respectively. Then by Lemma 7 (2), $X \setminus V$ and $X \setminus W$ are shrinking closed subspaces. Therefore by Lemma 1, $X$ is shrinking, a contradiction.

Next we consider the remaining case.

Case 2: $\mu = \nu = \kappa$. Assume $\kappa = \omega$. Then by Lemma 7 (1), $X_{\mu}$ and $X_{\nu}$ are shrinking. By (4-1), $X_{\{\mu]\}$ and $X_{\{\nu]\}$ are separated. Then by Lemma 2, $X = X_{\mu} \cup X_{\nu}$ is shrinking, a contradiction. Therefore $\omega_1 \leq \kappa$.

Two subcases are now considered:

(2-1): $\Delta_{MN}(X)$ is stationary in $\kappa$. In this case by Lemmas 10 and 11, we have a contradiction as previously.
(2-2): $\triangle_{MN}(X)$ is not stationary in $\kappa$. Since $X$ is the union of the closed subspaces $X(R, M, N)$ and $X(L, M, N)$, we may assume that $U$ does not have a closed shrinking which covers $X(R, M, N)$. Two cases are to consider:

(2-2-1): $V_\mu(X) \cap \nu$ is stationary in $\nu$. As in the proof of Lemma 5 (1), for each $\delta \in N^{-1}(V_\mu(X)) \cap \text{Lim}(\kappa)$, fix $f(\delta) < \kappa$, $g(\delta) < \delta$ and $U(\delta) \in U$ such that $(M(f(\delta)), \mu] \times (N(g(\delta)), N(\delta))] \cap X \subset U(\delta)$. Applying the PDL, we can find $\delta_0 < \kappa$ and a stationary set $S \subset N^{-1}(V_\mu(X)) \cap \text{Lim}(\kappa)$ such that $g(\delta) = \delta_0$ for each $\delta \in S$. Put $\nu' = N(\delta_0)$.

**Claim 3.** There is a closed shrinking $F$ of $U$ such that $\{\mu\} \times (\nu', \nu) \cap X \subset \text{Int}(\bigcup F)$ and $\bigcup F$ is closed in $X$.

**Proof.** As previously, for each $\delta$ and $\delta'$ in $S$, define $\delta \sim \delta'$ by $U(\delta) = U(\delta')$, and let $S/\sim$ be its quotient. For each $E \in S/\sim$, put $U_E = U(\delta)$ for some (any) $\delta \in E$. Observe that $(M(f(\delta)), \mu] \times (\nu', N(\delta))] \cap X \subset U_E$ for each $\delta \in E$.

First, assume there is $E \in S/\sim$ such that $E$ is unbounded in $\kappa$. Put $W = \bigcup_{\delta \in E}(M(f(\delta)), \mu] \times (\nu', N(\delta))] \cap X$. Note that $W \subset U_E$. Since by the condition (b1), $X_{(\mu]}$ and $X \setminus (W \cup X^{\nu'+1})$ are separated, we can find an open set $V$ in $X$ such that $\{\mu\} \times (\nu', \nu) \cap X \subset V \subset \text{Cl} V \subset W$. For each $U \in U$, put

$$F(U) = \begin{cases} \text{Cl} V & \text{if } U = U_E, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $F = \{F(U) : U \in U\}$ is the desired shrinking of $U$.

Next assume all $E$'s, $E \in S/\sim$, are bounded in $\kappa$. As in Lemma 5, define $\delta(\eta) \in E(\eta) \in S/\sim$ for each $\eta \in \kappa$ so that $\eta + \sup(\bigcup_{\zeta < \eta} E(\zeta)) < \delta(\eta)$. For each $U \in U$, put

$$W(U) = \begin{cases} (M(f(\delta(\eta))), \mu] \times (\nu', N(\delta(\eta))] \cap X & \text{if } U = U_{E(\eta)} \text{ for some } \eta < \kappa, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $W = \{W(U) : U \in U\}$ is a shrinking of $U$ by clopen sets in $X$ with $\{\mu\} \times (\nu', \nu) \cap X \subset \bigcup W$. By the condition (b1), take an open set $V$ in $X$ such that $\{\mu\} \times (\nu', \nu) \cap X \subset V \subset \text{Cl} V \subset \bigcup W$.

For each $U \in U$, put

$$F(U) = W(U) \cap \text{Cl} V.$$

Then $F = \{F(U) : U \in U\}$ is the desired shrinking of $U$. This completes the proof of the claim.

Take the shrinking $F$ of $U$ in Claim 3. By Lemma 12,

$$Z = X(R, N, M)^{(\nu', \nu)} \setminus \text{Int}\left(\bigcup F\right)$$
is a shrinking closed subspace. Since \( X(R, M, N) \subset X^{\nu'}+1 \cup \mathcal{Z} \cup \bigcup \mathcal{F} \), by Lemma 1, \( U \) has a closed shrinking which covers \( X(R, M, N) \). A contradiction.

(2-2-2): \( V_\mu(X) \cap \nu \) is not stationary in \( \nu \). Using the clause (b2), take a cub set \( D \) in \( k \) such that \( X(R, M, N)_{M[D]\cup\{\mu\}} \) and \( X(R, M, N)^{(D)\cup\{\nu\}} \) are separated. Take disjoint open sets \( V \) and \( W \) containing \( X(R, M, N)_{M[D]\cup\{\mu\}} \) and \( X(R, M, N)^{(D)\cup\{\nu\}} \) respectively. Then applying Lemma 7 (2) to \( X(R, M, N) \), we see that \( X(R, M, N)\setminus V \) and \( X(R, M, N)\setminus W \) are shrinking. Therefore by Lemma 1, \( X(R, M, N) \) is shrinking, a contradiction.

Thus in all cases, we get contradictions. This completes the proof of \( (4) \rightarrow (1) \).

(4)\rightarrow(2). This proof is almost similar to the one of \( (4) \rightarrow (1) \) except for the case (2-2-1). So we only give a proof of case (2-2-1) for the CWN case.

(2-2-1): \( \omega_1 \leq \text{cf}_t \mu = \text{cf}_t \nu = \kappa, \triangle_{MN}(X) \) is not stationary in \( \kappa \), \( V_\mu(X) \cap \nu \) is stationary in \( \nu \) and \( \mathcal{H} \) is a discrete collection of closed sets in \( X \) which cannot be separated. In this case, for each \( \delta \in \mathcal{N}^{-1}(V_\mu(X)) \cap \text{Lim}(\kappa) \), fix \( g(\delta) < \delta \) such that \( \{\mu\} \times (N(g(\delta)), N(\delta)) \cap X \) meets at most one element of \( \mathcal{H} \). By the PDL, we can take \( \eta' < \nu \) such that \( \{\mu\} \times (\eta',\nu) \cap X \) meets at most one element of \( \mathcal{H} \).

Claim 3'. There is an open set \( V \) such that \( \{\mu\} \times (\eta',\nu) \cap X \subset V \) and \( \text{Cl} \ V \) meets at most one element of \( \mathcal{H} \).

Proof. Put \( \mathcal{H}' = \{H \in \mathcal{H} : H \cap (\{\mu\} \times (\eta',\nu) \cap X) = \emptyset\} \), and \( W = X \setminus \bigcup \mathcal{H}' \). Since \( \{\mu\} \times (\eta',\nu) \cap X \subset W \), take an open set \( V \) such that \( \{\mu\} \times (\eta',\nu) \cap X \subset V \subset \text{Cl} \ V \subset W \) using the clause (b1). Then this \( V \) works.

As \( X(R, M, N) \) is covered by closed sets \( X^{\eta'+1}, Z = X(R, M, N)^{(\eta',\nu)} \setminus V \) and \( \text{Cl} \ V \), we get a contradiction as in case (2-2-1) in the proof of \( (4) \rightarrow (1) \). This completes the proof. ■

4. Non-normal examples and related questions. In [KOT], it is proved that, for subspaces \( A \) and \( B \) of \( \omega_1 \), \( A \times B \) is normal (countably paracompact) if and only if \( A \) is not stationary in \( \omega_1 \), \( B \) is not stationary in \( \omega_1 \) or \( A \cap B \) is stationary.

According to this result, if \( A \) is a countable subspace of \( \omega_1 \), then, since \( A \) is non-stationary, \( A \times B \) is normal for each \( B \subset \omega_1 \). In particular, as is well known, \( (\omega + 1) \times \omega_1 \) is normal. But as is shown in the next example, there is a non-normal subspace of \( (\omega + 1) \times \omega_1 \).

Example 1. Put \( X = \omega \times \omega_1 \cup \{\omega\} \times (\omega_1 \setminus \text{Lim}(\omega_1)) \). Put \( F = \omega \times \text{Lim}(\omega_1) \) and \( H = \{\omega\} \times (\omega_1 \setminus \text{Lim}(\omega_1)) \). Then \( F \) and \( H \) are disjoint closed sets in \( X \). Let \( U \) be an open set containing \( H \). For each \( \alpha \in \omega_1 \setminus \text{Lim}(\omega_1) \), pick \( n(\alpha) \in \omega \)
such that \([n(\alpha), \omega] \times \{\alpha\} \subset U\). Since \(\omega_1 \setminus \text{Lim}(\omega_1)\) is uncountable, there is an uncountable subset \(C \subset \omega_1 \setminus \text{Lim}(\omega_1)\) and \(n \in \omega\) such that \(n(\alpha) = n\) for each \(\alpha \in C\). Observe that \([n, \omega] \times C \subset U\). Pick \(\alpha \in \text{Lim}(C)\). Noting that \(\text{Lim}(C) \subset \text{Lim}(\omega_1)\), we have \(\langle n, \alpha \rangle \in [n, \omega] \times \text{Lim}(C) \cap F \subset \text{Cl} U \cap F\). This argument shows \(X\) is not normal.

Next we give a corollary of the Theorem for subspaces of \(\omega_1^2\). For simplicity, we use the following notation: Let \(X \subset \omega_1^2\), \(\alpha < \omega_1\) and \(\beta < \omega_1\). Put \(V_\alpha(X) = \{\beta < \omega_1 : \langle \alpha, \beta \rangle \in X\}\), \(H_\beta(X) = \{\alpha < \omega_1 : \langle \alpha, \beta \rangle \in X\}\) and \(\triangle(X) = \{\alpha < \omega_1 : \langle \alpha, \alpha \rangle \in X\}\). For subsets \(C\) and \(D\) of \(\omega_1\), put \(X_C = X \cap C \times \omega_1\), \(X^D = X \cap \omega_1 \times D\) and \(X_C^D = X \cap C \times D\).

Consider \(M\) and \(N\) as the identity map on \(\omega_1\) if \(\mu = \nu = \omega_1\) in the Theorem. Then, by checking all clauses in (4) of the Theorem, we can see:

**Corollary.** Let \(X \subset \omega_1^2\). Then the following are equivalent.

1. \(X\) is normal.
2. (2-1-a) If \(\alpha\) is a limit ordinal in \(\omega_1\) and \(V_\alpha(X)\) is not stationary in \(\omega_1\), then there is a cub set \(D \subset \omega_1\) such that \(X_\{\alpha\}\) and \(X^D\) are separated.
3. (2-1-b) If \(\beta\) is a limit ordinal in \(\omega_1\) and \(H_\beta(X)\) is not stationary in \(\omega_1\), then there is a cub set \(C \subset \omega_1\) such that \(X_\{\beta\}\) and \(X_C\) are separated.
4. (2-2) If \(\triangle(X)\) is not stationary in \(\omega_1\), then there is a cub set \(C \subset \omega_1\) such that \(X_C\) and \(X^C\) are separated.

Intuitively, we may consider (2-1-a) to be a condition which guarantees the normality of \(X_{\alpha+1}\) for each \(\alpha < \omega_1\), and (2-1-b) the normality of \(X^{\beta+1}\) for each \(\beta < \omega_1\). If we know that \(X_{\alpha+1}\) and \(X^{\beta+1}\) are normal for each \(\alpha, \beta < \omega_1\), then (2-2) is a condition which guarantees the normality of \(X\).

Consider \(X = \omega_1^2\). Since \(V_\alpha(X)\) and \(H_\beta(X)\) are the stationary set \(\omega_1\) for each \(\alpha, \beta < \omega_1\) and \(\triangle(X)\) is also the stationary set \(\omega_1\), the clause (2) of the Corollary is satisfied. So \(X\) is normal.

**Example 2.** Let \(A\) and \(B\) be disjoint stationary sets in \(\omega_1\) and put \(X = A \times B\). Let \(\alpha\) be a limit ordinal in \(\omega_1\). Then we have

\[
V_\alpha(X) = \begin{cases} B & \text{if } \alpha \in A, \\ \emptyset & \text{otherwise.} \end{cases}
\]

Therefore, if \(V_\alpha(X)\) is not stationary, then necessarily \(\alpha \notin A\) and \(V_\alpha(X) = \emptyset\), so \(X_\{\alpha\} = \emptyset\). Therefore \(X_\{\alpha\}\) and \(X^{\omega_1}\) are separated. This argument proves (2-1-a). Similarly we have (2-1-b). Therefore \(X_{\alpha+1}\) and \(X^{\beta+1}\) are normal for each \(\alpha, \beta < \omega_1\).

Note that \(\triangle(X) = \emptyset\). Let \(C\) be a cub set in \(\omega_1\). Then \(X \cap C^2 = (A \cap C) \times (B \cap C) \neq \emptyset\), equivalently \(X_C \cap X^C \neq \emptyset\). Thus \(X_C\) and \(X^C\) cannot
be separated. Therefore $X$ is not normal, because the clause (2-2) is not satisfied.

**Example 3.** Let $X = \{ (\alpha, \beta) \in \omega^2_1 : \alpha \leq \beta \}$, $Y = \{ (\alpha, \beta) \in \omega^2_1 : \alpha < \beta \}$. Checking (2-1-a) and (2-1-b), we can show that $X_{\alpha+1}$, $X^{\beta+1}$, $Y_{\alpha+1}$ and $Y^{\beta+1}$ are normal for each $\alpha, \beta < \omega_1$.

Since $\Delta(X) = \omega_1$ is stationary, (2-2) for $X$ is satisfied. Thus $X$ is normal (but this is obvious, because $X$ is a closed subspace of $\omega^2_1$). On the other hand, note that $\Delta(Y) = \emptyset$. For each cub set $C$ in $\omega_1$, pick $\alpha$ and $\beta$ in $C$ with $\alpha < \beta$. Then $\langle \alpha, \beta \rangle \in Y \cap C^2$. Therefore (2-2) for $Y$ is not satisfied. Thus $Y$ is not normal.

Let $X = \omega_1 \times (\omega_1 + 1)$. Observe that $X \cap \omega^2_1 = \omega^2_1$ is normal, and $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha, \beta < \omega_1$. Since $\{ (\alpha, \alpha) : \alpha \in \omega_1 \}$ and $X^{\omega_1}$ cannot be separated, $X$ is not normal. Note that both $\Delta(X)$ and $H_{\omega_1}(X)$ are the stationary set $\omega_1$. Next we give a similar example $X \subset \omega_1 \times (\omega_1 + 1)$, but with $\Delta(X)$ and $H_{\omega_1}(X)$ not stationary.

**Example 4.** Let $X = [\omega_1 \setminus \text{Lim}(\omega_1)] \times [(\omega_1 + 1) \setminus \text{Lim}(\omega_1)] \cup \{ (\alpha, \alpha + 1) : \alpha \in \text{Lim}(\omega_1) \}$. Observe that $X \cap \omega^2_1$ is normal, $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha, \beta < \omega_1$ and both $\Delta(X)$ and $H_{\omega_1}(X)$ are the non-stationary set $\omega_1 \setminus \text{Lim}(\omega_1)$.

By an argument similar to that for Claim 1 of Lemma 4, we can see that $F = \{ (\alpha, \alpha + 1) : \alpha \in \text{Lim}(\omega_1) \}$ is closed (discrete). We shall show $F$ and $X^{\omega_1}$ cannot be separated. To see this, let $U$ be an open set containing $F$. For each $\alpha \in \text{Lim}(\omega_1)$, since $\langle \alpha, \alpha + 1 \rangle \in F \subset U$, take $f(\alpha) < \alpha$ such that $\{ f(\alpha), \alpha \} \times \{ \alpha + 1 \} \cap X \subset U$. By the PDL, there are $\alpha_0 < \omega_1$ and a stationary set $S \subset \text{Lim}(\omega_1)$ such that $f(\alpha) = \alpha_0$ for each $\alpha \in S$. Take $\beta \in \omega_1 \setminus \text{Lim}(\omega_1)$ with $\alpha_0 < \beta$. Noting that $\langle \beta, \alpha + 1 \rangle \in X$ for each $\alpha \in S$ with $\alpha > \beta$, we have

$$\langle \beta, \omega_1 \rangle \in \text{Cl}\{ (\beta, \alpha + 1) : \alpha \in S, \alpha > \beta \} \cap X^{\omega_1} \subset \text{Cl} U \cap X^{\omega_1}.$$ 

Thus $F$ and $X^{\omega_1}$ cannot be separated.

In this connection, we have the next question which relates to the clause (4-4) of the Theorem.

**Question 1.** Does there exist a non-normal subspace $X$ of $\omega_1 \times \omega_2$ such that $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha < \omega_1$ and $\beta < \omega_2$?

In this connection, we show:

**Proposition.** If $X = A \times B$ is a subspace of $\omega_1 \times \omega_2$ such that $X_{\alpha+1}$ and $X^{\beta+1}$ are normal for each $\alpha < \omega_1$ and $\beta < \omega_2$, then $X$ is normal.

**Proof.** If $A$ is not stationary in $\omega_1$, then take a cub set $C$ in $\omega_1$ disjoint from $A$. Put $h(\alpha) = \sup(C \cap \alpha)$ for each $\alpha \in C$. Observe that
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\[ X = \bigoplus_{\alpha \in C \setminus \text{Lim}(C)} X_{(h(\alpha), \alpha)} \]

Since \( X_{(h(\alpha), \alpha)} \) is a closed subspace of \( X_{\alpha+1} \), by the inductive assumption, \( X \) is normal. Similarly \( X \) is normal if \( B \) is not stationary in \( \omega_2 \). So we may assume \( A \) and \( B \) are stationary in respectively \( \omega_1 \) and \( \omega_2 \). Let \( U = \{ U_i : i \in 2 \} \) be an open cover of \( X \). Fix \( \alpha \in A \). For each \( \beta \in B \), fix \( f(\alpha, \beta) < \alpha, \ g(\alpha, \beta) < \beta \) and \( i(\alpha, \beta) \in 2 \) such that

\[ (f(\alpha, \beta), \alpha) \times (g(\alpha, \beta), \beta) \cap X \subset U_{i(\alpha, \beta)}. \]

Applying the PDL to \( B \), we find \( f(\alpha) < \alpha, \ g(\alpha) < \omega_2, \ i(\alpha) \in 2 \) and a stationary set \( B(\alpha) \subset B \) in \( \omega_2 \) such that \( f(\alpha, \beta) = f(\alpha), \ g(\alpha, \beta) = g(\alpha) \) and \( i(\alpha, \beta) = i(\alpha) \) for each \( \beta \in B(\alpha) \). Then, applying the PDL to \( A \), we find \( \alpha_0 < \omega_1, \ i_0 \in 2 \) and a stationary set \( A' \subset A \) in \( \omega_1 \) such that \( f(\alpha) = \alpha_0 \) and \( i(\alpha) = i_0 \) for each \( \alpha \in A' \). Put \( \beta_0 = \sup \{ g(\alpha) : \alpha \in A' \} \). Then we have \( Z = (\alpha_0, \omega_1) \times (\beta_0, \omega_2) \cap X \subset U_{i_0} \).

Since \( X \) is the union of closed subspaces, \( X_{\alpha_0+1}, X_{\beta_0+1} \) and \( Z, U \) has a closed shrinking which covers \( X \). Therefore \( X = A \times B \) is normal. \( \square \)

By the result in [KOT], normality and countable paracompactness of \( A \times B \subset \omega_2 \) are equivalent. In this connection, it is natural to ask:

**Question 2.** For any \( X \subset \omega_1^2 \), are normality and countable paracompactness of \( X \) equivalent?

Note that, by [KS], normality implies countable paracompactness in the realm of subspaces of product spaces of two ordinals.

Finally, we restate a question from [KOT]:

**Question 3.** For any subspace \( X \) of the product space of two ordinals, are countable paracompactness, expandability, strong D-property and weak D(\( \omega \))-property of \( X \) equivalent?

References

