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Parabolic Cantor sets

by

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Abstract. The notion of a parabolic Cantor set is introduced allowing in the definition of hyperbolic Cantor sets some fixed points to have derivatives of modulus one. Such difference in the assumptions is reflected in geometric properties of these Cantor sets. It turns out that if the Hausdorff dimension of this set is denoted by h, then its h-dimensional Hausdorff measure vanishes but the h-dimensional packing measure is positive and finite. This latter measure can also be dynamically characterized as the only h-conformal measure. It is relatively easy to see that any two parabolic Cantor sets formed with the help of the same alphabet are canonically topologically conjugate and we then discuss the rigidity problem of what are the possibly weakest sufficient conditions for this topological conjugacy to be "smoother". It turns out that if the conjugating homeomorphism preserves the moduli of the derivatives at periodic points, then the dimensions of both sets are equal and the homeomorphism is shown to be absolutely continuous with respect to the corresponding h-dimensional packing measures. This property in turn implies the conjugating homeomorphism to be Lipschitz continuous. Additionally the existence of the scaling function is shown and a version of the rigidity theorem, expressed in terms of scaling functions, is proven. We also study the real-analytic Cantor sets for which the stronger rigidity can be shown, namely that the absolute continuity of the conjugating homeomorphism alone implies its real analyticity.

1. Introduction; preliminaries. The goal of this paper is to classify parabolic Cantor sets up to bi-Lipschitz and real-analytic conjugacy. This is done in the last three sections of the paper. The first 5 sections have mostly survey character and collect basic dynamical and geometric properties of a single parabolic Cantor set. The theory of parabolic Cantor sets has its roots in the theory of parabolic rational maps and expanding cookie-cutter Cantor sets. The former one is a model and prototype for exploring properties of a single map. The proofs in both settings are very similar and most of them are skipped as they can be found in one of the papers [ADU], [DU1]–[DU4], [U1], and [U2].

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From the theory of expanding Cantor sets we mostly adopted to our setting the concept of scaling function and the rigidity problem. In contrast to what is going on in the case of expanding Cantor sets, the geometry of parabolic Cantor sets fails to be bounded. Nevertheless it continues to be determined, up to the level of bi-Lipschitz conjugacy, by the scaling function. The geometry is also determined (again up to bi-Lipschitz conjugacy) by the packing measure class and the Hausdorff dimension of the Cantor set. This is much less evident than in the case of expanding sets. The point is that for expanding sets there is an extremely simple relation between the conformal (equivalently packing) measure of a ball and the power of its radius, with exponent being the Hausdorff dimension of the Cantor set under consideration. Namely, these two quantities are almost proportional—their ratio stays bounded away from zero and infinity. For parabolic Cantor sets the relation between radii of balls and their conformal measures is more complex. Proving Lipschitz conjugacy becomes technically more involved. Of special importance is Section 9, where dealing with real-analytic systems, employing the methods of complex-analytic functions and indirectly the concept of nonlinearity (see [Su1] and [Pr3]), we prove a stronger version of rigidity that the absolute continuity (with respect to packing measures) of the conjugating homeomorphism alone implies its real analyticity.

Concluding this introduction we would like to mention that Mathematica Gottingensis preprint "Parabolic Cantor sets" 21 (1995) contains a more complete version of this paper, especially in the part dealing with a single parabolic Cantor set.

To introduce notation, let S^1 denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ and let l be the normalized Lebesgue measure on S^1, $l(S^1) = 1$. Let I be a finite set consisting of at least two elements and let $\{\Delta_j : j \in I\}$ be a finite collection of closed nondegenerate and not overlapping subarcs (their intersections contain at most one point) of S^1. Finally, let $f : \bigcup_{j\in I} \Delta_j \to S^1$ be a C^1 map, open onto its image, with the following properties:

(1.1) If $i, j \in I$ and $\Delta_i \cap \Delta_j \neq \emptyset$, then $f|_{\Delta_i \cup \Delta_j}$ is injective.

(1.2) For every $j \in I$ the restriction $f|_{\Delta_j}$ is $C^{1+\theta}$ differentiable, that is, the derivative function $f'|_{\Delta_j}$ is Hölder continuous with an exponent $\theta > 0$, which means that
$$|f'(y) - f'(x)| \le Q|y - x|^\theta$$
for some constant $Q > 0$ and all $x, y \in \Delta_j$.

(1.3) $|f'(x)| \ge 1$ for all $x \in \bigcup_{j\in I} \Delta_j$ but $|f'(x)| = 1$ may hold only if $f(x) = x$.

(1.4) If $f(\omega) = \omega$ and $|f'(\omega)| = 1$, then the derivative f' is monotone on each sufficiently small one-sided neighborhood of ω.

(1.5) There exists $L \geq 2$ such that if $f(\omega) = \omega$ and $|f'(\omega)| = 1$, then there exists $0 < \beta = \beta(\omega) < \theta/(1-\theta)(= \infty$ if $\theta = 1)$ such that

$$\frac{2}{L} \leq \liminf_{x\to\omega} \frac{|f'(x)| - 1}{|x-\omega|^\beta} \leq \limsup_{x\to\omega} \frac{|f'(x)| - 1}{|x-\omega|^\beta} \leq \frac{L}{2}.$$

(1.6) For every $i \in I$ there exists $I(i) \subset I$ such that $f(\Delta_i) \cap \bigcup_{j\in I} \Delta_j = \bigcup_{k\in I(i)} \Delta_k$.

The reader should notice that in the case when the intervals I_j are mutually disjoint, then without loosing generality the circle S^1 can be replaced by a compact subinterval of $\mathbb{R}$. In this case also the openness of $f : \bigcup_{j\in I} \Delta_j \to S^1$ and (1.1) follow automatically from other assumptions. In the general case, property (1.3) describes a kind of hyperbolicity and requirement (1.6) establishes the Markov property which always gives rise to a nice symbolic representation of f.

In the sequel we will need f to satisfy one more condition. In order to express it let $A : I \times I \to \{0,1\}$ be the matrix (called *incidence matrix*) defined by the requirement that $A_{ij} = 1$ if and only if $f(\Delta_i) \supset \Delta_j$. The last condition we need is that the matrix A is *primitive*, which means that

(1.7) there exists $q \geq 1$ such that all entries of A^q are positive.

Let next $\Sigma_A^\infty \subset I^\infty$ be the space of all one-sided infinite sequences $\tau = \tau_0\tau_1\tau_2\ldots$ acceptable by A, that is, such that $A_{\tau_j\tau_{j+1}} = 1$ for all $j = 0,1,2,\ldots$ and let Σ_A^* be the set of all finite sequences acceptable by A. We put $\Sigma_A = \Sigma_A^* \cup \Sigma_A^\infty$ and for every integer $n \geq 0$ we let Σ_A^n be the subset of Σ_A^* consisting of all words of length $n+1$. Given $\tau \in \Sigma_A$ and $n \geq 0$ we define $\tau|_n = \tau_0\tau_1\ldots\tau_n$ to consist of the first $n+1$ initial letters of τ; if $n+1$ exceeds the length of τ, then $\tau|_n$ is just τ. Notice that Σ_A^∞ is compact and by primitiveness of A it is nonempty. Notice also that Σ_A^∞ is forward invariant under the left-sided shift map (cutting out the first coordinate) which will be denoted by σ. For all words $\tau \in \Sigma_A^n$, $n \geq 0$, define

$$\Delta(\tau) = \Delta_{\tau_0} \cap f^{-1}(\Delta_{\tau_1}) \cap \ldots \cap f^{-n}(\Delta_{\tau_n}).$$

Observe that $\Delta(\tau)$ is a nonempty closed subinterval of S^1. Fix $\tau \in \Sigma_A^\infty$ and consider the descending sequence $\{\Delta(\tau|_n) : n \geq 0\}$ of compact nonempty subintervals of S^1. Then the intersection $\bigcap_{n\geq 0} \Delta(\tau|_n)$ is a closed nonempty subinterval of S^1. We shall prove the following.

LEMMA 1.1. *For every $\tau \in \Sigma_A^\infty$ the set $\Delta(\tau) = \bigcap_{n\geq 0} \Delta(\tau|_n)$ is a singleton. Even more, the diameters of $\Delta(\tau|_n)$ tend to zero uniformly with respect to τ.*

Proof. Let $\Sigma_A^+ = \{\tau \in \Sigma_A^\infty : l(\Delta(\tau)) > 0\}$ and suppose that $\Sigma_A^+ \neq \emptyset$. Since for any two distinct elements $\tau, \tau' \in \Sigma_A^\infty$ the intersection $\Delta(\tau) \cap \Delta(\tau')$ is either an empty set or a point, the family Σ_A^+ contains an element of largest length. So, the remark that if $\tau \in \Sigma_A^\infty$, then also $\sigma(\tau) \in \Sigma_A^\infty$ and $l(\Delta(\sigma(\tau))) = l(f(\Delta(\tau))) > l(\Delta(\tau))$, gives a contradiction and finishes the proof of the first part of the lemma.

In order to prove the second part suppose to the contrary that $\exists \varepsilon > 0$ $\forall n \ge 0$ $\exists \tau^{(n)} \in \Sigma_A^\infty$ $\exists k_n \ge n$ such that $l(\Delta\tau^{(n)}|_{k_n}) \ge \varepsilon$. By compactness of Σ_A^∞ we can find an accumulation point $\tau \in \Sigma_A^\infty$ of the sequence $\{\tau^{(n)} : n \ge 1\}$. But since the sequence of lengths $\{l(\Delta(\tau|_n)) : n \ge 1\}$ is decreasing this yields $l(\Delta(\tau|_n)) \ge \varepsilon$ for all $n \ge 1$ and consequently $l(\Delta(\tau)) \ge \varepsilon$. This, however, contradicts the first part of the lemma and completes the proof. ∎

In view of Lemma 1.1 we can define a continuous map $\pi : \Sigma_A^\infty \to S^1$ putting $\pi(\tau) = \Delta(\tau)$. The range of this map, $J = J(f) = \pi(\Sigma_A^\infty)$, is called the *dynamical Cantor set* (DCS) associated with the dynamical system $(f, I; \Delta_j, j \in I)$. Although J may happen to be an interval, nevertheless we still choose the name Cantor set since we consider an interval as a degenerate Cantor set, and since, what is perhaps a more important reason, in some sense J is an interval in exceptional cases only (Lemma 2.4). Let us now formulate the following obvious lemma.

LEMMA 1.2. (a) $J = \bigcap_{n\ge 0} \bigcup_{\tau \in \Sigma_A^n} \Delta(\tau)$.

(b) *J can be characterized as the set of those points of S^1 whose positive iterates under f are all defined* (*and therefore contained in $\bigcup_{j\in I} \Delta_j$*).

(c) $f^{-1}(J) = J = f(J)$.

(d) $f \circ \pi = \pi \circ \sigma$.

(e) *π is at most* 2-*to*-1.

Proof. Properties (a) and (b) are obvious. The relations $f(J) \subset J = f^{-1}(J)$ follow immediately from (b), and the inclusion $f(J) \supset J$ follows from (b) and primitiveness of the matrix A. The properties (d) and (e) follow from the definition of J. ∎

Let

$$\Omega = \Omega(f) = \{\omega \in J : f(\omega) = \omega \text{ and } |f'(\omega)| = 1\}$$

Each $\omega \in \Omega$ is called a fixed parabolic point or briefly a *parabolic point*. For every $q \ge 1$ consider now the system $(f^q, I^q; \Delta(\tau), \tau \in I^q)$. One can prove the following.

LEMMA 1.3. *The set I^q consists of at least two elements*, $\{\Delta(\tau) : \tau \in I^q\}$ *is a finite collection of nonoverlapping closed intervals*, *and $f^q : \bigcup_{\tau\in I^q} \Delta(\tau) \to S^1$ is continuous. Moreover*,

(a) *The system $(f^q, I^q; \Delta(\tau), \tau \in I^q)$ satisfies the conditions* (1.1)–(1.7).

(b) $J(f^q) = J(f)$.

(c) $\Omega(f^q) = \Omega(f)$.

(d) *If $\tau \in I^2$ and $\omega \in \Omega(f) \cap \Delta(\tau)$, then $f^2|_{\Delta(\tau)}$ is orientation preserving.*

Besides the formal value of Lemma 1.3 its practical advantage is that passing to the second iterate of f one keeps the same Cantor set, the same set of parabolic points, and f^2 "preserves" one-sided neighborhoods of parabolic points. Therefore from now on we will assume that already f itself satisfies condition (d) of Lemma 1.3. The next lemma is an immediate consequence of Lemma 1.2(d), (e).

LEMMA 1.4. *For every* $n \geq 1$ *the set* $\mathrm{Per}_n(f) = \{x \in J : f^n(x) = x\}$ *is finite.*

Using our assumptions (1.1)–(1.7) and Lemma 1.4 we conclude that the number

$$\delta_1 = \frac{1}{2} \min \begin{cases} \min\{l(\Delta_i) : i \in I\} \\ \min\{\mathrm{dist}(\Delta_i, \Delta_j) : i, j \in I,\ \Delta_i \cap \Delta_j = \emptyset\} \\ \min\{|x-y| : x, y \in \mathrm{Per}_2(f),\ x \neq y\} \end{cases}$$

is positive. One can easily check that

LEMMA 1.5. *If* $0 < \delta \leq \delta_1$ *and* $x \in B(\mathrm{Per}_1(f), \delta) \setminus \mathrm{Per}_1(f)$, *then there exists* $n \geq 1$ *such that* $f^n(x) \notin B(\mathrm{Per}_1(f), \delta)$.

Recall that a continuous map $S : X \to X$ of a compact metric space X is *expansive* if there exists a positive η (an *expansive constant* for f) such that for all distinct $x, y \in X$ there exists $n \geq 0$ such that $\mathrm{dist}(S^n(x), S^n(y)) \geq \eta$. Using Lemma 1.5 one proves the following.

THEOREM 1.6. *The map* $f : J \to J$ *is open and expansive*, *and any positive number* $\eta \leq \delta_1$ *is an expansive constant for* f.

As an immediate consequence of this theorem, Lemma 2.2 of [DU2] and [Ru, p. 128] (see also [PU]), we get the following.

COROLLARY 1.7 (Closing Lemma). *For every* $\varepsilon > 0$ *there exists* $\widetilde{\varepsilon} > 0$ *such that if* $n \geq 0$ *is an integer*, $x \in J$, *and* $|f^n(x) - x| < \widetilde{\varepsilon}$, *then there exists a point* $y \in J$ *such that*

$$f^n(y) = y \quad \textit{and} \quad |f^j(y) - f^j(x)| < \varepsilon \quad \textit{for all } j = 0, 1, \ldots, n-1. \ \blacksquare$$

2. Bounded distortion. This section is of somewhat technical character and for further reading it is not necessary to become familiar with all the proofs included here. It is devoted to the distortion properties of iterates of f. First observe that for every $\omega \in \Omega$ there is a continuous inverse branch $f_\omega^{-1} : B(\omega, \delta_1) \to S^1$ of f such that $f_\omega^{-1}(\omega) = \omega$. By (1.3), $f_\omega^{-1}(B(\omega, \delta_1)) \subset B(\omega, \delta_1)$ and therefore all iterates $f_\omega^{-n}(B(\omega, \delta_1)) \subset B(\omega, \delta_1)$, $n \geq 1$, are well defined. Moreover, by Lemma 1.3(d) the map f_ω^{-1} preserves one-sided neighborhoods of ω. Therefore, since δ_1 is an expansive constant, ω is the only fixed point in $B(\omega, \delta_1)$ and $\lim_{n\to\infty} f_\omega^{-n}(x) = \omega$ for all $\omega \in \Omega$ and all $x \in B(\omega, \delta_1)$.

LEMMA 2.1. *For all* $\omega \in \Omega$ *and all* $x \in B(\omega, \delta_1\|f\|^{-1}) \setminus \{\omega\}$ *we have*

$$\frac{|x-\omega|}{|f(x)-x|} \le \sum_{n=1}^{\infty} |(f_\omega^{-n})'(x)| \le \frac{|f_\omega^{-1}(x)-\omega|}{|x-f_\omega^{-1}(x)|}.$$

The proof is obtained by integrating partial sums of the series $\sum_{n=1}^{\infty} |(f_\omega^{-n})'(x)|$.

Sending neighborhoods of neutral points to infinity via the mapping $1/(x-\omega)$ one can fairly easily prove the following local results.

COROLLARY 2.2. $\forall \omega \in \Omega\ \forall 0 < R \le \delta\ \exists L_1(R) \ge 1\ \forall z \in B(\omega,\delta) \setminus B(\omega, R)$ $\forall n \ge 1$,

$$L_1(R)^{-1} \le |f_\omega^{-n}(z) - \omega| n^{1/\beta} \le L_1(R).$$

LEMMA 2.3. $\forall \omega \in \Omega\ \ \forall 0 < R \le \delta\ \ \exists L_2(R) \ge 2\ \ \forall z \in B(\omega,\delta) \setminus B(\omega, R)$ $\forall n \ge 1$,

$$L_2(R)^{-1} \le |(f_\omega^{-n})'(z)| n^{(\beta+1)/\beta} \le L_2(R).$$

Despite their very technical character we now provide detailed proofs of the distortion results as they form essential tools in Sections 8–10. Since $\beta < \theta/(1-\theta)$, it follows from Lemma 2.3 that for every $\omega \in \Omega$ and every $x \in B(\omega,\delta)$,

$$\sum_{n=1}^{\infty} |(f_\omega^{-n})'(x)|^\theta < \infty \tag{2.1}$$

and the convergence is uniform on compact subsets of $B(\Omega,\delta) \setminus \Omega$.

Now observe that for every $x \in S^1$ and every $n \ge 1$, if $f^n(x)$ is well defined, then there exists a continuous inverse branch $f_x^{-n} : B(f^n(x), \delta_2) \to S^1$ of f^n sending $f^n(x)$ to x, where $\delta_2 = \min\{l(f(\Delta_i)) : i \in I\}$. We shall prove the following.

LEMMA 2.4. $\forall t > 0\ \forall 0 < s < 1\ \exists K_1(t,s) > 0\ \exists M(t,s) > 0$ *such that if* $x \in S^1$, $n \ge 0$, $f^n(x)$ *is well defined*, *and* $\operatorname{dist}(f^n(x), \Omega) \ge t$, *then* $\sum_{j=1}^{n-1} |(f^j)'(x)|^{-\theta} \le M(t,s)$ *and* $|(f_x^{-n})'(y)| \le K_1(t,s)|(f_x^{-n})'(z)|$ *for all* $y, z \in B(f^n(x), \min\{\delta, st\})$. *Moreover*, *for every* $t > 0$, $\lim_{s\to 0} K_1(t,s) = 1$.

Proof. Set $r = \min\{\delta, (1-s)t\}$, $\lambda = \lambda(t,s) = \inf\{|f'(z)| : z \notin B(\Omega, r\|f'\|^{-1})\}$ and let $K = K(t,s) > 0$ be the supremum of the series appearing in (2.1) taken over the set $B(\Omega, r) \setminus B(\Omega, r\|f'\|^{-1})$. Fix $y \in B(f^n(x), \min\{\delta, st\})$, for every $0 \le j \le n$ put $y_j = f^j(f_x^{-n}(y))$ and let $p(j)$ be the number of integers $0 \le i \le n-1-j$ such that $f^i(y) \notin B(\Omega, r\|f'\|^{-1})$. Define also increasing sequences $0 \le k_j \le l_j \le n$ determined by the requirements that

(a) $\{y_{k_j}, y_{k_j+1}, \ldots, y_{l_j}\} \subset B(\Omega, r)$ and
(b) if $i \notin G = \bigcup_j \{k_j, k_j+1, \ldots, l_j\}$, then $y_i \notin B(\Omega, r)$.

Since $y = y_n \notin B(\Omega, r)$, we conclude that $y_{l_j} \in B(\Omega, r) \setminus B(\Omega, r\|f'\|^{-1})$ for all j. Thus

$$\sum_{i=k_j}^{l_j} |(f^{n-i})'(y_i)|^{-\theta} \le (K+1)|(f^{n-l_j})'(y_{l_j})|^{-\theta} \le (K+1)\lambda^{-\theta p(l_j)}$$

and then

$$\begin{aligned}
\sum_{i=0}^{n-1} |(f^{n-i})'(y_i)|^{-\theta} &\le \sum_j (K+1)\lambda^{-\theta p(l_j)} + \sum_{i \notin G} \lambda^{-\theta p(i)} \\
&\le (K+1)\sum_{i=0}^{n-1} \lambda^{-\theta i} \\
&= (K+1)\frac{\lambda^\theta}{\lambda^\theta - 1},
\end{aligned} \tag{2.2}$$

where the second inequality is due to the fact that all the numbers $p(l_j)$ and $p(i)$, $i \notin G$, are mutually distinct. So, the first claim of the lemma is proven. As a matter of fact, in the proof of the two other claims we will use a slightly stronger version of this estimate where we let the point y vary in $B(f^n(x), \min\{\delta, st\})$ with i. Let now z be another point in $B(f^n(x), \min\{\delta, st\})$. Then using (1.2) and the mean value theorem we see that for every j there exists $w^{(j)} \in [z_j, y_j]$ such that

$$\begin{aligned}
|\log|f'(z_j)| - \log|f'(y_j)|| \le ||f'(z_j)| - |f'(y_j)|| &\le Q|z_j - y_j|^\theta \\
&= Q|(f^{n-j})'(w_j^{(j)})|^{-\theta}|z-y|^\theta \\
&\le Q(2st)^\theta|(f^{n-j})'(w_j^{(j)})|^{-\theta}.
\end{aligned}$$

Hence applying (2.2), in fact its stronger version discussed above, we get

$$\begin{aligned}
|\log|(f_x^{-n})'(y)| - \log|(f_x^{-n})'(z)|| &\le \sum_{j=0}^{n-1} |\log|f'(z_j)| - \log|f'(y_j)|| \\
&\le (2st)^\theta Q \sum_{j=0}^{n-1} |(f^{n-j})'(w_j^{(j)})|^{-\theta} \\
&\le (2st)^\theta Q(K+1)\frac{\lambda^\theta}{\lambda^\theta - 1}.
\end{aligned}$$

So, the first part of the proof is finished by setting

$$K_1(t,s) = \exp\left((2st)^\theta Q(K+1)\frac{\lambda^\theta}{\lambda^\theta - 1}\right).$$

In order to see that $\lim_{s\to 0} K_1(t,s)=1$ it suffices to notice that

$$\lim_{s\to 0}\lambda(t,s)=\inf\{|f'(z)| : z\notin B(\Omega,\min\{\delta,t\})\}>1$$

and $\lim_{s\to 0} K(t,s)$ is finite as the supremum of the series appearing in (2.1) over the set $B(\Omega,\min\{\delta,t\})\setminus B(\Omega,\min\{\delta,t\}/\|f'\|)$. The proof is finished. ∎

Observe that given $\omega\in\Omega$ and $0<t<\delta$, partitioning separately both connected components of $B(\omega,\delta)\setminus B(\omega,t)$ into finitely many segments of length $\le t/2$, and increasing $K_1(t,t/2)$ if necessary, we derive from Lemma 2.4 the following.

COROLLARY 2.5. *For every $0<t<\delta$ there exists $K_1(t)>0$ such that if $x\in S^1$, $n\ge 0$, $f^n(x)$ is well defined and belongs to $B(\omega,\delta)\setminus B(\omega,t)$, then $|(f_x^{-n})'(y)|\le K_1(t)|(f_x^{-n})'(z)|$ for all y,z lying in the same connected component of $B(\omega,\delta)\setminus B(\omega,t)$ as $f^n(x)$.*

LEMMA 2.6. *For every $0<s<1$ there exists $K_2(s)>1$ such that if $x\in S^1$, $n\ge 0$, and $f^n(x)$ is well defined, then $|(f_x^{-n})'(y)|\le K_2(s)|(f_x^{-n})'(z)|$ for all $y,z\in B(f^n(x),\min\{s\operatorname{dist}(f^n(x),\Omega),\delta/4\})$.*

Before starting the proof let us give a few words of comment on this lemma. First of all this is a substantial improvement of Lemma 2.4 since now the distortion constant $K_2(s)$ is independent of the distance from $f^n(x)$ to Ω; it depends only on the ratio of the radius of the ball around $f^n(x)$ and $\operatorname{dist}(f^n(x),\Omega)$. Note also that the lemma is vacuous if $f^n(x)\in\Omega$.

Proof of Lemma 2.6. If $\operatorname{dist}(f^n(x),\Omega)\ge\delta/2$, then

$$s\operatorname{dist}(f^n(x),\Omega)=\frac{s}{\delta/2}\operatorname{dist}(f^n(x),\Omega)\frac{\delta}{2}\le\frac{s}{\delta/2}\operatorname{diam}(S^1)\frac{\delta}{2}=\frac{2s}{\delta}\frac{\delta}{2}$$

and therefore it follows from Lemma 2.4 that any constant $K_2(s)\le K_1(\delta/2,2s/\delta)$ works in this case. So, we can suppose that $\operatorname{dist}(f^n(x),\Omega)<\delta/2$ and let $\omega\in\Omega$ be the only point such that $|f^n(x)-\omega|<\delta/2$. Denote the ball $B(f^n(x),\min\{s\operatorname{dist}(f^n(x),\Omega),\delta/4\})$ by $B(f^n(x))$. Since $B(f^n(x))\subset B(f^n(x),s|f^n(x)-\omega|)\subset B(\omega,\delta)$, for every $y\in B(f^n(x))$ there exists a unique integer $k=k(y)$ such that $f^k(y)\in B(\omega,\delta)\setminus B(\omega,\delta/\|f'\|)$.

Suppose now additionally that $f_x^{-n}=f_\omega^{-n}$. Then for every $y\in B(f^n(x))$ we have $f_x^{-n}(y)=f_\omega^{-(n+k)}(f^k(y))$, thus by Lemma 2.3,

$$L_2^{-1}(n+k)^{-(\beta+1)/\beta}\le|(f_x^{-n})'(y)|\le L_2(n+k)^{-(\beta+1)/\beta},$$

where $L_2=L_2(\delta/\|f'\|)$ is the constant of Lemma 2.3. Since

$$(1-s)|f^n(x)-\omega|\le|y-\omega|\le(1+s)|f^n(x)-\omega|,$$

it follows from Corollary 2.2 that we have $(1-s)|f^n(x)-\omega|\le L_1k^{-1/\beta}$ and

$(1+s)|f^n(x)-\omega| \ge L_1^{-1}k^{-1/\beta}$, where $L_1 = L_1(\delta/\|f'\|)$. Thus

$$\frac{\max\{k(y) : y \in B(f^n(x))\}}{\min\{k(y) : y \in B(f^n(x))\}} \le \left(L_1^2 \frac{1+s}{1-s}\right)^{\beta}.$$

Denote the number on the right-hand side of this inequality by $a(s)^\beta \ge 1$. We then have for all $y, z \in B(f^n(x))$,

$$\begin{aligned}\frac{|(f_x^{-n})'(y)|}{|(f_x^{-n})'(z)|} &\le \frac{L_2(n+k(y))^{-(\beta+1)/\beta}}{L_2^{-1}(n+k(z))^{-(\beta+1)/\beta}}\\ &= L_2^2\left(\frac{n+k(y)}{n+k(z)}\right)^{-(\beta+1)/\beta} \le L_2^2 a(s)^{\beta+1}\end{aligned}$$

and therefore we are done in this case.

In the general case let $0 \le j \le n$ be the least integer such that $f^i(x) \in B(\Omega, \delta/2)$ for all $j \le i \le n$. Then $f^i(x) = f_\omega^{-(n-i)}(f^n(x))$ and $f_x^{-n} = f_x^{-(i-1)} \circ g \circ f_\omega^{-(n-i)}$, where g is the inverse branch of f sending $f^i(x)$ to $f^{i-1}(x)$ and $f_x^{-(i-1)}$ is the inverse branch of f^{i-1} sending $f^{i-1}(x)$ to x. Now, we have just proved that $f_\omega^{-(n-i)}$ has distortion bounded by a number depending only on s, uniform boundedness of distortion of g is obvious, and since the point $f^{(i-1)}(x)$ is far away from Ω (at least at distance $\ge \delta/2$), a uniform bound on the distortion of $f_x^{-(i-1)}$ follows from the first part of the proof. We are done. ∎

As an immediate consequence of Lemma 2.6 we get the following.

COROLLARY 2.7. *For all* $0 < \gamma < 1$ *sufficiently small*, *for all* $x \in S^1$, *and* $n \ge 0$ *such that* $f^n(x)$ *is well defined*,

$$|(f_x^{-n})'(y)| \le K_2(\gamma)|(f_x^{-n})'(z)|$$

for all $y, z \in B(f^n(x), \gamma \operatorname{dist}(f^n(x), \Omega))$.

Our last result in this section is in some sense a partial improvement of Lemma 2.6 in an attempt to have $\lim_{s\to 0} K_2(s) = 1$.

LEMMA 2.8. *For every integer* $q \ge 1$ *there exists an increasing function* $Q_q : (0, \delta) \to [1, \infty]$ *such that* $\lim_{t\to 0} Q_q(t) = 1$ *and* $|(f_x^{-n})'(y)| \le Q_q(t)|(f_x^{-n})'(z)|$ *for all* $y, z \in \Delta$, *where* $\Delta \subset B(\Omega, t)$ *is an arbitrary subarc of* S^1 *such that* $\#(\Delta \cap \{f_\omega^{-j}(\partial B(\omega, \delta)) : j \ge 0\}) \le q$ *and* x *is any point in* S^1 *such that* $f^n(x)$ *is well defined and* $f^n(x) \in B(\Delta, t)$.

Proof. Observe that without loosing generality one can assume $q = 1$. Take $w \in \partial B(\omega, \delta)$ such that $\Delta \subset [\omega, w]$. Suppose first that $x = \omega$ is a parabolic point. Take any $v \in B(\omega, t)$. In view of (1.4) we have $|(f_\omega^{-n})'(v)| \le$

$|(f_\omega^{-n})'(f_\omega^{-1}(v))|$ for all $n \ge 1$. On the other hand,

$$|(f_\omega^{-n})'(f_\omega^{-1}(v))| = |(f_\omega^{-n})'(v)| \cdot \frac{|f'(f_\omega^{-1}(v))|}{|f'(f_\omega^{(-n+1)}(v))|} \le |(f_\omega^{-n})'(v)| \cdot |f'(f_\omega^{-1}(v))|.$$

Hence

$$1 \le \frac{|(f_\omega^{-n})'(f_\omega^{-1}(v))|}{|(f_\omega^{-n})'(v)|} \le |f'(f_\omega^{-1}(v))|$$

for all $n \ge 1$. Since, by continuity of f', we have $\lim_{v\to\omega} |f'(f_\omega^{-1}(v))| = |f'(\omega)| = 1$, from (1.4) (monotonicity of f') we get the existence of a function $Q_1(t)$ claimed in the lemma as long as only the inverse branches of the form f_ω^{-n}, $\omega \in \Omega$, are involved.

In the general case, using what has been proved above, one repeats the argument described in the last part of the proof of Lemma 2.6. ∎

Frequently in the sequel, if there are no specific requirements on how small $\gamma > 0$ is to be we will drop the dependence of $K_2(\gamma)$ on γ writing K_2 for $K_2(\gamma)$. We end up this section fixing the notation $R(\omega) = B(\omega,\delta) \setminus B(\omega, \delta/\|f'\|)$.

3. Pressure and dimensions. This section is somewhat sketchy, of rather general character and overlaps [DU1] as regards the content as well as the methods used. Given $f : J \to J$, we recall first that the *Lyapunov exponent* $\chi_\mu(f)$ of f with respect to an ergodic f-invariant measure μ is defined as $\chi_\mu(f) = \int \log|f'|\,d\mu$ and the *pressure function* $\mathrm{P}(t)$, $t \in [0,\infty)$, is the topological pressure (see [Bo], [Wa]) of the map $f : J \to J$ and potential $-t\log|f'|$. We have the following.

Proposition 3.1. *If μ is an ergodic f-invariant measure*, *then* $\chi_\mu(f) \ge 0$. *Additionally*, $\chi_\mu(f) = 0 \Leftrightarrow \mu(\Omega) = 1 \Leftrightarrow \mu(\Omega) > 0 \Leftrightarrow \mu(\{\omega\}) = 1$ *for some* $\omega \in \Omega$.

Proposition 3.2. *The function* $t \mapsto \mathrm{P}(t)$, $t \in \mathbb{R}$, *is convex*, *continuous*, *nonincreasing*, *and nonnegative if* $\Omega \ne \emptyset$.

Recall that $\mathrm{HD}(\mu)$, the *Hausdorff dimension* of the measure μ, is defined to be $\inf\{\mathrm{HD}(Y) : \mu(Y) = 1\}$. By definition, $\mathrm{HD}(\mu) \le \mathrm{HD}(J) \le 1$ and hence $\sup\{\mathrm{HD}(\mu)\} \le 1$, where the supremum, denoted by $\mathrm{DD}(J)$, is taken over all ergodic f-invariant measures μ of positive entropy. We have the following.

Lemma 3.3. (a) $\mathrm{P}(t) > 0$ *for every* $t \in [0, \mathrm{DD}(J))$.

(b) *If* $\Omega = \emptyset$, *then* $\mathrm{P}(t) < 0$ *for every* $t \in (\mathrm{DD}(J), \infty)$. *If* $\Omega \ne \emptyset$, *then* $\mathrm{P}(t) = 0$ *for every* $t \in [\mathrm{DD}(J), \infty)$.

(c) $\mathrm{P}|_{[0,\mathrm{DD}(J)]}$ *is decreasing.*

It follows from this lemma that if $\Omega \ne \emptyset$, then the pressure function has a phase transition at the point $s = \mathrm{DD}(J)$. An intriguing problem arises of what kind this phase transition is. Is for example $\mathrm{P}(t)$ differentiable at s or not? A partial answer to such problems is contained in the following.

THEOREM 3.4. *The function* $\mathrm{P}(t)$ *is differentiable at* $t = \mathrm{DD}(J)$ *if and only if there is no equilibrium state of positive entropy for the potential* $-\mathrm{DD}(J)\log|f'|$.

4. Conformal measures and dimensions. This section constitutes a natural extension of the previous one enriching its results by employing the method of conformal measures along the lines worked out in [DU1], [DU5], and [U1] (see also [PU]). Let $t \geq 0$ be a real number. A Borel probability measure m on the Cantor set J is called *t-conformal* for f if and only if

$$m(f(A)) = \int_A |f'|^t\, dm$$

for every *special set* $A \subset J$, that is, a Borel subset of J such that $f|_A$ is injective. Notice that if m is t-conformal, then $m(f(A)) \leq \int_A |f'|^t\, dm$ for every Borel set $A \subset J$. From (1.3) and primitiveness of the incidence matrix A we conclude that any conformal measure for f is positive on nonempty open subsets of J.

LEMMA 4.1. *Let* $x \in J \setminus \bigcup_{n=0}^{\infty} f^{-n}(\Omega)$. *Then there exist an increasing sequence* $\{n_j = n_j(x) : j \geq 1\}$ *of positive integers, a sequence* $\{r_j(x)\}_{j=1}^{\infty}$ *of positive reals decreasing to* 0, *and an element* $y \in \omega(x) \setminus B(\Omega, \delta)$ *with the following properties*:

(a) $y = \lim_{j\to\infty} f^{n_j}(x)$.

(b) $f^{n_j}(x) \notin B(\Omega, \delta)$.

(c) *If* m *is a* t-*conformal measure for* f, *then there exists a constant* $B(m) \geq 1$ *such that for all* $j \geq 1$,

$$B(m)^{-1} \leq m(B(x, r_j(x)))/r_j(x)^t \leq B(m).$$

The idea of the proof of this lemma is to iterate the point x forward to be infinitely often far away from neutral points and to define the balls $B(x, r_j(x))$ as the preimages of balls of some fixed radius. As a consequence of this lemma and Besicovitch type covering results we get the following.

LEMMA 4.2. *If* H_t *is the* t-*dimensional Hausdorff measure on* J, Π_t *is the* t-*dimensional packing measure on* J, *and* m *is a* t-*conformal measure for* $f : J \to J$, *then* H_t *is absolutely continuous with respect to* m *and the Radon–Nikodym derivative is bounded from above. Consequently*, $t \geq \mathrm{HD}(J)$ *and there is no* t-*conformal measure for* $t < \mathrm{HD}(J)$. *If additionally* m *is atom free*, *then* $m \ll \Pi_t$ *and*, *in particular*, $\Pi_t(J) > 0$.

Let $e(J)$ be the infimum of all exponents $t \geq 0$ such that a t-conformal measure exists, and let $\delta(J)$ be the first zero of the pressure function $\mathrm{P}(t)$.

The concluding result of this section is the following.

THEOREM 4.3. *We have* $\mathrm{DD}(J) = \delta(J) = e(J) = \mathrm{HD}(J)$ *and an* h-*conformal measure exists*, *where* h *denotes the common value of these four numbers.*

Proof. That $\delta(J) = \mathrm{DD}(J) \le \mathrm{HD}(J) \le e(J)$ can be seen from Lemmas 3.3 and 4.2. So, in order to complete the proof it suffices to find a $\delta(J)$-conformal measure on J. But since by Theorem 1.6 the mapping $f : J \to J$ is open and expansive, and since $\mathrm{P}(\delta(J)) = 0$, the existence of such a measure follows from Theorem 3.12 of [DU6]. ∎

5. Local behavior around parabolic points. In this section we collect some results about the local behavior of conformal measures around parabolic points. For every $\omega \in \Omega$ let

$$\alpha(\omega) = h + \beta(\omega)(h-1).$$

We begin with the following.

LEMMA 5.1. *If* m *is an* h-*conformal measure for* $f : J \to J$, *then* $\exists C_1 \ge 1\ \forall \omega \in \Omega\ \forall 0 < r \le 1$,

$$C_1^{-1} \le m(B(\omega, r) \setminus \{\omega\})/r^{\alpha(\omega)} \le C_1.$$

LEMMA 5.2. $\forall \zeta > 0\ \exists C_2 = C_2(\zeta) \ge 1\ \forall \omega \in \Omega\ \forall z \in J$,

$$C_2^{-1}|z-\omega|^{\alpha(\omega)} \le m(B(z, \zeta|z-\omega|) \setminus \{\omega\}) \le C_2|z-\omega|^{\alpha(\omega)}.$$

THEOREM 5.3. *We have* $h = \mathrm{HD}(J) > \max\{\beta(\omega)/(\beta(\omega)+1) : \omega \in \Omega\}$.

Proof. Fix $\omega \in \Omega$. Since δ is an expansive constant for f, the interior of at least one of the two connected components of $R(\delta)$ has a nonempty intersection with the set J. Call it $R(\omega)$. Since by Theorem 4.3 there exists an h-conformal measure m for $f : J \to J$, it follows from Lemma 2.3 that

$$1 \ge \sum_{n=1}^{\infty} m(f_\omega^{-n}(R(\omega))) \ge L_2(\delta/\|f'\|)^{-h} m(R(\omega)) \sum_{n=1}^{\infty} n^{-h(\beta+1)/\beta}.$$

Since $m(R(\omega)) > 0$, this formula implies that the last series converges. Therefore, $h > \beta(\omega)/(\beta(\omega)+1)$. The proof is finished. ∎

Although the next theorem is of global character we place it in this section since the most important ingredient of its proof is Lemma 2.3 which is obviously of local flavor.

THEOREM 5.4. *There exists a unique* (*up to equivalence of measures*) h-*conformal measure. Moreover*, *this measure is continuous.*

In Section 8 we shall show more: there is only one such measure.

6. Geometric measures. In this section following the ideas of [DU3], [DU4], and [U2] we deal with geometric properties of the set J. We define X to be $J \setminus \bigcup_{n=0}^{\infty} f^{-n}(\Omega)$.

LEMMA 6.1. *For every $C_3 > 0$ there exists $C_4 > 0$ such that if $n \ge 0$, $f^n(z) \in B(\omega,\delta)$, $\omega \in \Omega$, and $f^{n-1}(z) \notin B(\omega,\delta)$ (in case $n \ge 1$), then for every $r > 0$ satisfying $r|(f^n)'(z)| \le \gamma\delta K_2^{-1}$ and $r|(f^n)'(z)| \ge C_3|f^n(z)-\omega|$ we have*

$$C_4^{-1}(r|(f^n)'(z)|)^{\beta(\omega)(h-1)} \le m(B(z,r))/r^h \le C_4(r|(f^n)'(z)|)^{\beta(\omega)(h-1)}.$$

In the proof of Theorem 6.2 below we shall construct (positive) integer-valued functions $n = n(z,r)$, $k = k(z,r)$ and $u = u(z,r)$ ($z \in J$, $0 < r < 1$). Although $n = n(z,r)$ and $k = k(z,r)$ do not appear in the formulation of Theorem 6.2 we will use them and their properties several times in the sequel.

THEOREM 6.2. *There exists $Q \ge 1$ such that for every pair (z,r), $z \in J$, $0 < r < 1$, there exists a number $\beta(z,r) \in \{\beta(\omega) : \omega \in \Omega\} \cup \{0\}$ such that*

$$Q^{-1}(r|(f^u)'(z)|)^{\beta(z,r)(h-1)} \le m(B(z,r))/r^h \le Q(r|(f^u)'(z)|)^{\beta(z,r)(h-1)}.$$

Moreover, $\gamma\delta(K_2\|f'\|)^{-1}|f^u(z)-\omega| \le r|(f^u)'(z)| \le \gamma\delta K_2^{-1}$ and there is a continuous inverse branch $f_z^{-u} : B(f^u(z), r|(f^u)'(z)|) \to S^1$ sending $f^u(z)$ to z.

Proof. Suppose first that $\sup_{n\ge0}\{r|(f^n)'(z)|\} > \gamma\delta(K_2\|f'\|)^{-1}$ and let $n = n(z,r) \ge 0$ be a minimal integer such that $r|(f^n)'(z)| > \gamma\delta(K_2\|f'\|)^{-1}$. Then also $r|(f^n)'(z)| \le \gamma\delta K_2^{-1}$. We say that the pair (z,r) belongs to the family $\Re$ if $f^n(z) \notin B(\Omega,\delta)$. Since the conformal measure m is positive on nonempty open sets, $\inf\{m(B(x,\gamma\delta K_2^{-2}\|f'\|^{-1})) : x \in J\} > 0$. Therefore, using Corollary 2.7 we deduce the existence of a constant $C_5 > 0$ independent of $(z,r) \in \Re$ and such that

$$C_5^{-1} \le m(B(z,r))/r^h \le C_5.$$

So, in this case our theorem is proved by setting $u(z,r) = n(z,r)$.

Let $\omega \in \Omega$. Given $(z,r) \in \Re$ suppose first that $f^n(z) \in B(\omega,\delta)$. Let $0 \le k = k(z,r) \le n$ be the least integer such that $f^j(z) \in B(\omega,\delta)$ for every $j = k, k+1, \ldots, n$. Consider all the numbers $r_i = |f^i(z)-\omega| \cdot |(f^i)'(z)|^{-1}$, where $i = k, k+1, \ldots, n$. From the definition of $n(z,r)$ it follows that $r_n = |f^n(z)-\omega| \cdot |(f^n)'(z)|^{-1} \le K_2\|f'\|(\gamma\delta)^{-1}r$ and therefore there exists a minimal $k \le u = u(z,r) \le n$ such that $r_u \le K_2\|f'\|(\gamma\delta)^{-1}r$. Then

$$\gamma\delta(K_2\|f'\|)^{-1}|f^u(z)-\omega| \le r|(f^u)'(z)| \le \gamma\delta K_2^{-1}. \tag{6.1}$$

Thus, if $u = k$, then it follows from Lemma 6.1 with $C_3 = \gamma\delta(K_2\|f'\|)^{-1}$ that there exists a constant $C_6 > 0$ such that

$$(6.2)\quad C_6^{-1}(r|(f^u)'(z)|)^{\beta(\omega)(h-1)} \le m(B(z,r))/r^h \le C_6(r|(f^u)'(z)|)^{\beta(\omega)(h-1)}.$$

So, we are done in this case. If $u > k$ then $r_{u-1} > K_2\|f'\|(\gamma\delta)^{-1}r$ and therefore, using (1.3) and (1.4), we get

$$r_u = \frac{|f^u(z)-\omega|}{|f^{u-1}(z)-\omega|}|f'(f^{u-1}(z))|^{-1}r_{u-1} \ge \|f\|^{-1}r_{u-1} \ge K_2(\gamma\delta)^{-1}r.$$

Thus

$$(6.3)\qquad r|(f^u)'(z)| \le \gamma\delta K_2^{-1}|f^u(z)-\omega|.$$

Let $f_z^{-u} : B(f^u(z), \gamma|f^u(z)-\omega|) \to S^1$ be the continuous inverse branch of f^u which sends $f^u(z)$ to z. By Lemma 5.2, it follows from formulas (6.3), (6.1), and Corollary 2.7 that formula (6.2) continues to hold in case $u > k$, with a possibly bigger constant C_6.

It remains to deal with the case of $\sup_{n\ge 0}\{r|(f^n)'(z)|\} \le \gamma\delta(K_2\|f'\|)^{-1}$. Then by (1.3), $z \in J \setminus \bigcup_{j=1}^\infty f^{-j}(\Omega)$. Let $u = u(z,r) \ge 0$ be the minimal integer such that $f^u(z) \in \Omega$ and let $f_z^{-u} : B(f^u(z), K_2r|(f^u)'(z)|) \to S^1$ be a continuous inverse branch sending $f^u(z)$ to z. Applying Corollary 2.7 we therefore obtain

$$K_2^{-h}|(f^u)'(z)|^{-h}m(B(f^u(z), K_2^{-1}r|(f^u)'(z)|)) \le m(B(z,r)) \le K_2^h|(f^u)'(z)|^{-h}m(B(f^n(z), K_2r|(f^u)'(z)|)).$$

and employing Lemma 5.1 finishes the proof. ∎

It is not difficult to prove the following result used in the proof of Theorem 6.4.

Lemma 6.3. *There exists $\xi > 0$ sufficiently small such that if $x \in X$, q is a positive integer, $f^q(x) \in B(\omega,\xi)$, $\omega \in \Omega$, and $f^{q-1}(x) \notin B(\Omega,\delta)$, then*

$$u(x, \gamma\delta(K\|f'\|)^{-1}|f^q(x)-\omega|\cdot|(f^q)'(x)|^{-1}) = q.$$

Theorem 6.4. *We have $0 < \Pi_h(J) < \infty$ and $\mathrm{H}_h(J) < \infty$. Additionally, $\mathrm{H}_h(J) = 0$ if and only if $h < 1$. Moreover, Π_h is equivalent to m with Radon–Nikodym derivative bounded away from zero and infinity.*

Proof. The inequalities $\mathrm{H}_h(J) < \infty$, $0 < \Pi_h(J)$, and uniform boundedness of $dm/d\Pi_h$ follow from Lemma 4.2. Let $\alpha = \max\{\alpha(\omega) : \omega \in \Omega\}$. Since $h \le 1$, it follows from Theorem 6.2 that $\liminf_{r\to 0}(m(B(z,r)/r^h) \ge Q^{-1}(\gamma\delta K_2^{-1})^{\alpha(h-1)}$ for all $z \in J$. Therefore the well-known results from geometric measure theory (see [TT], comp. [Ma] and [DU3]) imply that $d\Pi_h/dm \le \text{const}\cdot Q(\gamma\delta K_2^{-1})^{\alpha(1-h)}$ and $\Pi_h(J) < \infty$.

Now it remains to show that $\mathrm{H}_h(J) = 0$ if $h < 1$. Let $J_0 = \{z \in J : \omega(z) \cap \Omega = \emptyset\}$. Since it can be proved that the set of transitive points of f has m-measure zero, it follows from Lemma 4.2 that $\mathrm{H}_h(J_0) = 0$, whence we only need to show that $\mathrm{H}_h(X \setminus J_0) = 0$, but this follows immediately from Lemma 6.3, Theorem 6.2, and well-known results from geometric measure theory (see [TT], comp. [Ma] and [DU3]). The proof is finished. ∎

The next result is a combined consequence of Theorem 6.4 and the observation (based on geometrical consequences of bounded distortion) that if J is disconnected, then its Lebesgue measure is zero.

THEOREM 6.5. *If J is disconnected, then $h = \mathrm{HD}(J) < 1$. In particular, the Lebesgue measure of J is equal to* 0.

Remark 6.6. We end up this section with the remark that making use of the concept of the jump transformation (see the next section) one could prove, essentially as in [DU4], that the box counting dimension of J exists and coincides with $\mathrm{HD}(J)$.

7. Jump transformation and invariant measures. Using the existence of Markov partitions of arbitrarily small diameters (for example cylinders of length n, where n increases to infinity) and the distortion results of Section 2, and then proceeding similarly to [DU2], one can equip the dynamical system $(f, I; \Delta_j, j \in I)$ with the structure of a Markov fibered system (for the background about Markov fibered systems and Schweiger formalism see [ADU], [DU2], and [Sc] for example). All the results obtained in this theory apply to the h-conformal measure m and our map $f : J \to J$. In particular, fixing the Markov partition given by cylinders of length $k \geq 3$, one defines the *jump transformation* $f^* : J \setminus \Omega \to J$ by setting

$$f^*(x) = f^{n(x)+1}(x),$$

where $n(x) \geq 0$ is the least integer $n \geq 0$ such that $f^n(x) \notin \bigcup_{\omega \in \Omega} f_\omega^{-k}(\Delta_\omega)$ and Δ_ω is the union of all Δ_i, $i \in I$, that contain ω.

The two basic results concerning Schweiger fromalism formulated in our setting of parabolic Cantor sets are the following.

THEOREM 7.1. *There exists a unique, ergodic, f^*-invariant probability measure μ^* equivalent to m. Moreover, the Radon–Nikodym derivative $\psi^* = d\mu^*/dm$ satisfies $D^{-1} \leq \psi^* \leq D$ for some constant $D > 0$.*

THEOREM 7.2. *The map f admits a unique (up to a multiplicative constant), σ-finite, invariant measure μ equivalent to m with Radon–Nikodym derivative $d\mu/dm$ given by the formula*

$$\frac{d\mu}{dm}(x) = \psi^*(x) + \sum_{n=1}^{\infty} \psi^*(f_\omega^{-n}(x))|(f_\omega^{-n})'(x)|^h$$

if $x \in f_\omega^{-k}(\Delta_\omega)$ for some $\omega \in \Omega$. If $x \notin \bigcup_{\omega \in \Omega} f_\omega^{-k}(\Delta_\omega)$, then $(d\mu/dm)(x) = \psi^*(x)$. The measure μ is ergodic and conservative.

Notice that in particular the measure μ is (up to a multiplicative constant) independent of the jump transformation used in its construction.

Since the Radon–Nikodym derivatives of h-conformal measures are constant along orbits of f, combining Theorems 5.4 and 7.2 we get the following.

THEOREM 7.3. *There exists a unique h-conformal measure m for the map $f : J \to J$. Moreover, this measure is continuous.*

In the sequel we will need the following technical result.

LEMMA 7.4. *If F is a Borel subset of J and $\overline{F} \cap \Omega = \emptyset$, then $\mu(F) < \infty$.*

Proof. This lemma follows immediately from Theorems 7.2 and 7.1 taking in the definition of the jump transformation k so large that $\bigcup_{\omega\in\Omega} f_\omega^{-k}(\Delta_\omega) \cap F = \emptyset$. ∎

In the context of dynamical Cantor sets the following criterion for the finiteness of the invariant measure μ can be proved.

THEOREM 7.5. *The f-invariant σ-finite measure μ, equivalent to the conformal measure m, is finite if and only if $h > 2\max\{\beta(\omega)/(\beta(\omega)+1) : \omega \in \Omega\}$.*

Sections 8–10, the last three sections of this paper, are devoted to the study of the rigidity problem for parabolic Cantor sets. To be more precise, we explore the problem of what are necessary and sufficient conditions for two parabolic Cantor sets which are topologically conjugate to be conjugate in a smoother manner like bi-Lipschitz continuous or real-analytic. In Section 8 we resolve this problem (see Theorem 8.1) in terms of the spectra of moduli of multipliers of periodic points as well as in terms of the measure classes of packing measures and Hausdorff dimensions.

In Section 9 dealing with real-analytic systems we prove (see Theorem 9.9) a much stronger rigidity result that absolute continuity with respect to packing measures (the equality of Hausdorff dimensions is not required!) implies that the conjugating homeomorphism is real-analytic.

In the last section, Section 10, we undertake the most geometrical approach defining and proving the existence of the scaling function. We then express a partial solution of the rigidity problem in terms of these functions.

Our approach to the rigidity problem of parabolic Cantor sets is motivated by the results and ideas used in the setting of hyperbolic systems. See for example [Su1], [Su2], [Pr2], [Pr3], [PT], [LS], and [Be], where also a more complete collection of literature can be found.

8. Rigidity of dynamical Cantor sets. In this section we deal with two dynamical systems $(f, I; \Delta_{f,j}, j \in I)$ and $(g, I; \Delta_{g,j}, j \in I)$ assuming that these are set-theoretically equivalent, that is, that $\Delta_{f,i} \cap \Delta_{f,j} \neq \emptyset$ if and only if $\Delta_{g,i} \cap \Delta_{g,j} \neq \emptyset$. Then the map $\phi : J_f \to J_g$ given by the formula

$$\phi(\pi_f(\tau)) = \pi_g(\tau)$$

is well defined (that is, for all $x \in J_f$ it does not depend on the choice of $\tau \in \pi_f^{-1}(x)$) and moreover it can be easily checked that ϕ is a homeomorphism. Our main aim in this section is to prove the following rigidity theorem.

THEOREM 8.1. *The following three conditions are equivalent.*

(a) *If* $z \in \mathrm{Per}_n(f)$, *then* $|(g^n)'(\phi(z))| = |(f^n)'(z)|$.

(b) *The dimensions* $h_f = \mathrm{HD}(J_f)$ *and* $h_g = \mathrm{HD}(J_g)$ *are equal and the homeomorphism* ϕ *transports the measure class of the packing measure* Π_{h_f} *on* J_f *onto the measure class of the packing measure* Π_{h_g} *on* J_g.

(c) *Both homeomorphisms* ϕ *and* ϕ^{-1} *are Lipschitz continuous.*

We shall also provide the proof of the following theorem which sheds some light on what is going on in the general case.

THEOREM 8.2. *The conjugacy* $\phi : J_f \to J_g$ *is Hölder continuous if and only if* $\phi(\Omega_f) = \Omega_g$.

Since the proofs of Theorem 8.2 and the implication (b)⇒(c) have a considerable overlap, we partially proceed with them simultaneously. In fact, we begin with two general technical lemmas, then we prove the implication (c)⇒(a) of Theorem 8.1 and we begin the proof that (b)⇒(c) including there the proof of Theorem 8.2. We end the section with the implication (a)⇒(b).

The definition we intend to give now and the lemma following it involve only one single dynamical system $(f, I; \Delta_j, j \in I)$ and therefore formulating these and proving Lemma 8.4 we skip the subscript "f" when dealing with objects associated with this dynamical system.

DEFINITION 8.3. Suppose that a positive number $\zeta \le \delta$ is given. If $\omega \in \Omega$, we set $R_\zeta(\omega) = B(\omega, \zeta) \setminus B(\omega, \zeta/\|f'\|)$. If x and y (not necessarily different) are in the closure of the same connected component of $B(\omega, \delta) \setminus \{\omega\}$, then we let $z \in \{x, y\}$ be the point lying farther from ω. We denote by $0 \le q = q(x, y) \le \infty$ the largest integer such that $f_\omega^{-q}(z) \in [x, y]$, and if $x, y \in B(\omega, \zeta)$, we denote by $p = p(\zeta, x, y) \ge 0$ the least integer such that $f^p(z) \in R_\zeta(\omega)$. If $x = y$ we write $q(x)$ and $p(\zeta, x)$ instead of $q(x, x)$ and $p(\zeta, x, x)$.

LEMMA 8.4. $\forall 0 < \zeta \le \delta$ $\forall 0 < \xi \le \zeta$ $\forall \omega \in \Omega$ $\forall x \ne y \in S^1$ $\exists C(\zeta, \xi)$ *such that if* x *and* y *belong to the closure of the same connected component of* $B(\omega, \delta) \setminus \{\omega\}$ *and* $|f^p(y) - f^p(x)| \ge \xi$, *then*

$$C(\zeta, \xi)^{-1} \sum_{j=0}^{q} (p + j)^{-(\beta(\omega)+1)/\beta(\omega)} \le |y - x| \le C(\zeta, \xi) \sum_{j=0}^{q} (p + j)^{-(\beta(\omega)+1)/\beta(\omega)},$$

where we assume $0^{-1} = 1$.

P r o o f. Without loosing generality we may assume that $z = y$, where z is described in Definition 8.3. Suppose first that $q \ge 1$. Then by the definitions of q and p we have

$$\bigcup_{j=0}^{q-1} f_\omega^{-(p+j)}([f_\omega^{-1}(f^p(y)), f^p(y)]) \subset [x, y]$$

and

$$\bigcup_{j=0}^{q} f_\omega^{-(p+j)}([f_\omega^{-1}(f^p(y)), f^p(y)]) \supset [x, y].$$

Since $f^p(y) \in R_\zeta(\omega)$, we have $|f_\omega^{-1}(f^p(y)) - \omega| \ge r(\zeta)$, where $r(\zeta) = \zeta/\|f'\| - L(\beta(\omega)+1)^{-1}\zeta^{\beta(\omega)+1}$. Hence $[f_\omega^1(f^p(y)), f^p(y)] \subset B(\omega, \delta) \setminus B(\omega, r(\zeta))$ and applying Lemma 2.3 we get

$$\begin{aligned} r(\zeta) L_2(r(\zeta))^{-1} & \sum_{j=0}^{q-1} (p+j)^{-(\beta(\omega)+1)/\beta} \\ & \le |f_\omega^{-1}(f^p(y)) - f^p(y)| L_2(r(\zeta))^{-1} \sum_{j=0}^{q-1} (p+j)^{-(\beta(\omega)+1)/\beta} \\ & \le |x - y| \end{aligned}$$

and

$$\begin{aligned} |x - y| & \le |f_\omega^{-1}(f^p(y)) - f^p(y)| L_2(r(\zeta)) \sum_{j=0}^{q} (p+j)^{-(\beta(\omega)+1)/\beta} \\ & \le \delta L_2(r(\zeta)) \sum_{j=0}^{q} (p+j)^{-(\beta(\omega)+1)/\beta}. \end{aligned}$$

Since $(p+q)^{-(\beta(\omega)+1)/\beta} \le (p+q-1)^{-(\beta(\omega)+1)/\beta}$, combining the last two displays we get

$$\frac{1}{2} L_2(r(\zeta))^{-1} \sum_{j=0}^{q} (p+j)^{-(\beta(\omega)+1)/\beta} \le |x-y| \le \delta L_2(r(\zeta)) \sum_{j=0}^{q} (p+j)^{-(\beta(\omega)+1)/\beta}$$

and we are done in the case $q \ge 1$.

If $q = 0$, then we have $[x, y] \subset f_\omega^{-p}([f_\omega^{-1}(y), y])$ and similarly to the above we get $|x - y| \le \delta L_2(r(\zeta)) p^{-(\beta(\omega)+1)/\beta}$. On the other hand, in this case $[x, y] = f_\omega^{-p}([f_\omega^{-1}(y), y])$. Since $[f^p(x), f^p(y)] \subset [f_\omega^{-1}(f^p(y)), f^p(y)]$, as before we get $[f^p(x), f^p(y)] \subset B(\omega, \delta) \setminus B(\omega, r(\zeta))$. Therefore, in view of Lemma 2.3,

$$|x-y| \ge |f^p(x) - f^p(y)| L_2(r(\zeta))^{-1} p^{-(\beta(\omega)+1))/\beta} \ge \xi L_2(r(\zeta))^{-1} p^{-(\beta(\omega)+1)/\beta}.$$

The proof is finished. ∎

LEMMA 8.5. *If J_f and J_g are two dynamical Cantor sets and $\phi : J_f \to J_g$ is the canonical topological conjugacy between them, then $\phi(\Omega_f) = \Omega_g$ if and only if* $\exists \kappa \geq 1\ \forall x \in J_f\ \forall n \geq 1$,

$$\kappa^{-1} \log |(f^n)'(x)| \leq \log |(g^n)'(\phi(x))| \leq \kappa \log |(f^n)'(x)|.$$

Proof. The "if" part is proved by a straightforward computation. In order to prove the converse, it is of course sufficient to show only one of these two inequalities, say the second. Toward this end fix first $\omega \in \Omega_f$ and notice that by (2.2) there exists a constant $B \geq 1$ (even independent of $\omega \in \Omega_f$) such that

$$B^{-1}|x - \omega|^{\beta(\omega)} \leq \log |f'(x)| \leq B|x - \omega|^{\beta(\omega)}$$

and

$$B^{-1}|\phi(x) - \phi(\omega)|^{\beta(\phi(\omega))} \leq \log |g'(\phi(x))| \leq B|\phi(x) - \phi(\omega)|^{\beta(\phi(\omega))}$$

for all x sufficiently close to ω, say if $|x-\omega| \leq \zeta_1(\omega)$ for some $0 < \zeta_1(\omega) < \delta_f$. It also follows from Corollary 2.2 and the definition of the number $p(x, \zeta_1(\omega))$ that

$$W^{-1} p(x, \zeta_1(\omega))^{-1/\beta(\omega)} \leq |x - \omega| \leq W p(x, \zeta_1(\omega))^{-1/\beta(\omega)}$$

and

$$W^{-1} p(x, \zeta_1(\omega))^{-1/\beta(\phi(\omega))} \leq |\phi(x) - \phi(\omega)| \leq W p(x, \zeta_1(\omega))^{-1/\beta(\phi(\omega))}$$

for some constant $W \geq 1$ (independent of $\omega \in \Omega_f$) and all x with $|x - \omega| \leq \zeta_1(\omega)$. Then

$$\begin{aligned} \log |g'(\phi(x))| &\leq B|\phi(x) - \phi(\omega)|^{\beta(\phi(\omega))} \leq BW^{\beta(\phi(\omega))} p(x, \zeta_1(\omega))^{-1} \\ &\leq BW^{\beta(\phi(\omega))} W^{\beta(\omega)} |x - \omega|^{\beta(\omega)} \\ &\leq B^2 W^{\beta(\phi(\omega)) + \beta(\omega)} \log |f'(x)|. \end{aligned}$$

Thus $\log |g'(\phi(x))| \leq \kappa_1 \log |f'(x)|$ for all $x \in B(\Omega_f, \zeta_1)$, where $\zeta_1 = \min\{\zeta_1(\omega) : \omega \in \Omega_f\} > 0$ and $\kappa_1 = B^2 W^{\beta(\phi(\omega)) + \beta(\omega)}$. Hence setting $a = \inf\{\log |f'(x)| : x \in J_f \setminus B(\Omega_f, \zeta_1)\}$ we get $\log |g'(\phi(x))| \leq \max\{\kappa_1, \log \|g'\|/a\} \log |f'(x)|$ for all $x \in J_f$. Now the straightforward application of the chain rule completes the proof. ∎

Proof that (c)⇒(a). Indeed, suppose that there is a periodic point z of period n such that $|(g^n)'(\phi(z))| \neq |(f^n)'(z)|$. Then without loosing generality we can suppose that $|(g^n)'(\phi(z))| < |(f^n)'(z)|$. Fix $\lambda_1, \lambda_2 > 1$ such that $|(g^n)'(\phi(z))| < \lambda_2 < \lambda_1 < |(f^n)'(z)|$ and take $0 < \varepsilon_f < \delta_f$ and $0 < \varepsilon_g < \delta_g$ so small that $|(f^n)'(x)| \geq \lambda_1$ for all $x \in B(z, \varepsilon_f)$, $|(g^n)'(y)| \leq \lambda_2$ for all $y \in B(\phi(z), \varepsilon_g)$, and $\phi(J_f \cap B(z, \varepsilon_f)) \subset J_g \cap B(\phi(z), \varepsilon_g)$. Fix $x \in J_f \cap B(z, \varepsilon_f) \setminus \{z\}$. Then for all $k \geq 1$ we have $|f_z^{-nk}(x) - z| \leq \lambda_1^{-k}|x - z|$

and $|g_{\phi(z)}^{-nk}(\phi(x)) - \phi(z)| \geq \lambda_2^{-k}|\phi(x) - \phi(z)|$. Therefore

$$\lim_{k\to\infty} \frac{|g_{\phi(z)}^{-nk}(\phi(x)) - \phi(z)|}{|f_z^{-nk}(x) - z|} \geq \lim_{k\to\infty} \left(\frac{\lambda_1}{\lambda_2}\right)^k \frac{|\phi(x) - \phi(z)|}{|x - z|} = \infty$$

and since $g_{\phi(z)}^{-nk}(\phi(x)) = \phi(f_z^{-nk}(x))$, this shows that ϕ is not Lipschitz continuous. ■

Proof that (b)⇒(c). Since the two measures m_g and $m_f \circ \phi^{-1}$ are equivalent, the measures μ_g^* and $\mu_f^* \circ \phi^{-1}$ are also equivalent, whence, in view of Proposition 7.1 these are equal as equivalent ergodic probability g^*-invariant measures. Therefore, it follows from the last part of that proposition that there exists $M \geq 1$ such that

$$M^{-1} \leq m_g(\phi(A))/m_f(A) \leq M \tag{8.1}$$

for all Borel subsets A of J_f. In order to continue the proof we need the following.

LEMMA 8.6. *If* (b) *is satisfied and* $\omega \in \Omega_f$, *then* $\phi(\omega) \in \Omega_g$ *and* $\beta(\phi(\omega)) = \beta(\omega)$.

Proof. Take $\varepsilon_f, \varepsilon_g > 0$ so small that $\phi(J_f \cap B(\omega, \varepsilon_f)) \subset B(\phi(\omega), \varepsilon_g)$. Suppose now that $|g'(\phi(\omega))| > 1$ and fix $1 < \lambda < |g'(\phi(\omega))|$. Take $0 < \varepsilon \leq \varepsilon_g$ so small that $|g'(z)| \geq \lambda$ for all $z \in B(\phi(\omega), \varepsilon)$. Fix $y \in J_g \cap B(\phi(\omega), \varepsilon)$. By conformality of m_g we have, for all $n \geq 0$,

$$m_g([g_{\phi(\omega)}^{-(n+1)}(y), g_{\phi(\omega)}^{-n}(y)]) \leq \lambda^{-n} m_g([g_{\phi(\omega)}^{-1}(y), y]) \leq \lambda^{-n}.$$

On the other hand, in view of Lemma 2.3, for all $n \geq 0$ we get

$$\begin{aligned} &m_f([f_\omega^{-(n+1)}(\phi^{-1}(y)), f_\omega^{-n}(\phi^{-1}(y))]) \\ &\qquad \geq L_{2,f}(R) n^{-h_f(\beta(\omega)+1)/\beta(\omega)} m_f([f_\omega^{-1}(\phi^{-1}(y)), \phi^{-1}(y)]), \end{aligned}$$

where $R = |\omega - \phi^{-1}(y)|$. Therefore

$$\begin{aligned} &\frac{m_g([g_{\phi(\omega)}^{-(n+1)}(y), g_{\phi(\omega)}^{-n}(y)])}{m_f([f_\omega^{-(n+1)}(\phi^{-1}(y)), f_\omega^{-n}(\phi^{-1}(y))])} \\ &\qquad \leq (L_{2,f}(R) m_f([f_\omega^{-1}(\phi^{-1}(y)), \phi^{-1}(y)]))^{-1} \lambda^{-n} n^{h_f(\beta(\omega)+1)/\beta(\omega)}. \end{aligned}$$

Since $\lim_{n\to\infty} \lambda^{-n} n^{h_f(\beta(\omega)+1)/\beta(\omega)} = 0$ and $m_f([f_\omega^{-1}(\phi^{-1}(y)), \phi^{-1}(y)]) > 0$ we arrive at a contradiction with (8.1) and the proof of the first part of Lemma 8.6 is finished.

In order to prove the second part of the lemma we apply Lemma 2.3 again, this time to both the maps f and g, obtaining as a result the existence of a constant $M > 0$ such that for all $n \geq 1$,

$$M^{-1} \leq n^{-h_g(\beta(\phi(\omega))+1)/\beta(\phi(\omega)) + h_f(\beta(\omega)+1)/\beta(\omega)} \leq M.$$

Thus $h_g(\beta(\phi(\omega))+1)/\beta(\phi(\omega)) = h_f(\beta(\omega)+1)/\beta(\omega)$. Since the dimensions h_g and h_f are equal, we get $\beta(\phi(\omega)) = \beta(\omega)$, which finishes the proof of Lemma 8.6. ∎

Now, let us continue the proof of the implication (b)⇒(c) including the proof of Theorem 8.2. Fix $0 < \eta < \delta_f/4$ so small that if $x, y \in J_f$ with $|x-y| \le \eta$, then $|\phi(x)-\phi(y)| < \delta_g/4$. Let $\tau > 0$ be so small that $|x-y| \le \tau$ implies $|\phi^{-1}(x)-\phi^{-1}(y)| < \eta/\|f'\|$ and let $\eta_1 > 0$ be so small that $|x-y| \le \eta_1$ implies that $|\phi(x)-\phi(y)| < \tau/2$. Finally, let $\tau_1 > 0$ be so small that if $|x-y| \le \tau_1$, then $|\phi^{-1}(x)-\phi^{-1}(y)| < \eta_1/\|f'\|$.

Consider now an arbitrary pair of points $x \ne y \in J_f$ with $|x-y| < \eta_1/\|f'\|$. Since by Lemma 2.4, J_f has no isolated points, in order to prove the Lipschitz continuity of ϕ we may assume that $m_f([x,y]) > 0$. Then also $m_g([x,y]) > 0$. Let $n = n(x,y) \ge 1$ be the least integer such that $|f^n(y)-f^n(x)| \ge \eta_1/\|f'\|$. Then $|f^n(y)-f^n(x)| \le \eta_1$. We will consider several cases.

Case 1: $\{f^n(y), f^n(x)\} \cap (J_f \setminus B(\Omega_f, \eta/\|f'\|)) \ne \emptyset$. Without loosing generality we may assume that $f^n(x) \in J_f \setminus B(\Omega_f, \eta/\|f'\|)$, whence in view of Lemma 8.6 and the choice of τ we have $g^n(x) \in J_g \setminus B(\Omega_g, \tau)$. Thus, applying Lemma 2.6 we get

$$K_{f,2}(1/2)^{-1}|(f^n)'(x)| \le \frac{|f^n(y)-f^n(x)|}{|y-x|} \le K_{f,2}(1/2)|(f^n)'(x)|,$$

$$K_{g,2}(1/2)^{-1}|(g^n)'(\phi(x))| \le \frac{|g^n(\phi(y))-g^n(\phi(x))|}{|\phi(y)-\phi(x)|} \le K_{g,2}(1/2)|(g^n)'(\phi(x))|.$$

Using these two formulas and applying also Lemma 8.5 we now get

$$|\phi(y)-\phi(x)| \le \tfrac{1}{2}K_{g,2}(1/2)K_{f,2}(1/2)(\|f'\|\eta_1^{-1})^{1/\kappa}|y-x|^{1/\kappa},$$

which ends the proof of Hölder continuity in this case.

To continue the proof of the implication (b)⇒(c) notice that we get two similar inequalities for conformal measures

$$K_{f,2}(1/2)^{-h_f}|(f^n)'(x)|^{h_f} \le \frac{m_f([f^n(x), f^n(y)])}{m_f([x,y])} \le K_{f,2}(1/2)^{h_f}|(f^n)'(x)|^{h_f},$$

and

$$\begin{aligned} &K_{g,2}(1/2)^{-h_g}|(g^n)'(\phi(x))|^{h_g} \\ &\qquad \le \frac{m_g([g^n(\phi(x)), g^n(\phi(y))])}{m_g([\phi(x),\phi(y)])} \le K_{g,2}(1/2)^{h_g}|(g^n)'(\phi(x))|^{h_g}. \end{aligned}$$

It now follows from the above inequalities for measures, from (8.1) and since $h_f = h_g$ that

$$(K_{g,2}^h M^2 K_{f,2}^h(1/2))^{-1} \le \frac{|(f^n)'(x)|^h}{|(g^n)'(\phi(x))|^h} \le K_{g,2}^h M^2 K_{f,2}^h(1/2).$$

Hence applying the inequalities involving distances we get

$$\begin{aligned}\frac{|\phi(y)-\phi(x)|}{|y-x|} &\ge K_{g,2}^{-1}K_{f,2}^{-1}(1/2)\frac{|g^n(\phi(y))-g^n(\phi(x))|}{|f^n(y)-f^n(x)|}\\ &\ge (K_{g,2}K_{f,2}(1/2))^{-2}M^{-2/h}\frac{\tau_1}{\eta_1}\end{aligned}$$

and

$$\begin{aligned}\frac{|\phi(y)-\phi(x)|}{|y-x|} &\le K_{g,2}(1/2)K_{f,2}(1/2)\frac{|g^n(\phi(y))-g^n(\phi(x))|}{|f^n(y)-f^n(x)|}\cdot\frac{|(f^n)'(x)|}{|(g^n)'(\phi(x))|}\\ &\le (K_{g,2}K_{f,2}(1/2))^2M^{2/h}\frac{\tau\|f'\|}{2\eta_1}.\end{aligned}$$

So, we are done in this case.

Case 2: $\{f^n(y),f^n(x)\}\subset B(\Omega_f,\eta/\|f'\|)$. Since $|\phi(y)-\phi(x)|\le\eta_1\le\eta/2\le\delta_f/2$ there is $\omega\in\Omega_f$ such that $f^n(x),f^n(y)\in B(\omega,\eta/\|f'\|)$. Let us consider

Case 2.1: *$f^n(y)$ and $f^n(x)$ are in the same connected component of* $B(\omega,\eta/\|f'\|)\setminus\{\omega\}$. Let $0\le k=k(x,y)\le n$ be the least integer such that $[f^j(x),f^j(y)]\subset B(\omega,\eta/\|f'\|)$ for all $k\le j\le n$. Finally, let $q=q(f^k(x),f^k(y))$ and $p=p(\eta,f^k(x),f^k(y))$. Since $p\ge n-k$, we get $|f^{p+k}(y)-f^{p+k}(x)|\ge|f^n(y)-f^n(x)|\ge\eta_1/\|f'\|$. Since $\eta_1/\|f'\|\le\eta$, it follows from Lemma 2.2 that with the constant $C_f=C(\eta,\eta_1/\|f'\|)>0$ and $\beta=\beta(\omega)$ we have

$$C_f^{-1}\sum_{j=0}^{q}(p+j)^{-(\beta+1)/\beta}\le|f^k(y)-f^k(x)|\le C_f\sum_{j=0}^{q}(p+j)^{-(\beta+1)/\beta}. \tag{8.2}$$

Now, since ϕ is a topological conjugacy between f and g, we have $q(g^k(\phi(x)),g^k(\phi(y)))=q(f^k(x),f^k(y))$. Let S be the closure of the connected component of $B(\omega,\eta/\|f'\|)\setminus\{\omega\}$ that has nonempty intersection with $\{f^k(x),f^k(y)\}$ and let $\kappa=\kappa(\omega)>0$ be the diameter of $\phi(S\cap J_f)$. Note that then $p(\kappa\|g'\|,g^k(\phi(x)),g^k(\phi(y)))=p(\eta,f^k(x),f^k(y))$, and as $|g^k(\phi(x))-g^k(\phi(y))|\ge\tau_1$, using Lemma 8.6 and applying Lemma 8.4 for the map g, we have

$$C_{g,\omega}^{-1}\sum_{j=0}^{q}(p+j)^{-(\beta+1)/\beta}\le|g^k(\phi(y))-g^k(\phi(x))|\le C_{g,\omega}\sum_{j=0}^{q}(p+j)^{-(\beta+1)/\beta},$$

where $C_{g,\omega}=C(\kappa(\omega)\|g'\|,\min\{\tau_1,\kappa(\omega)\|g'\|\})$ is the constant produced in Lemma 8.4 associated with the map g. Combining this formula and (8.2) we get

$$(C_fC_g)^{-1}\le\frac{|g^k(\phi(y))-g^k(\phi(x))|}{|f^k(y)-f^k(x)|}\le C_fC_g, \tag{8.3}$$

where $C_g = \max\{C_{g,\omega} : \omega \in \Omega_f\}$. Observe now that by the definition of n and k we have $|f^{k-1}(y) - f^{k-1}(x)| \le \eta_1/\|f'\|$ and $\operatorname{dist}(\Omega_f, \{f^{k-1}(y), f^{k-1}(x)\}) \ge \delta_f/\|f'\|$. Hence $|g^{k-1}(\phi(y)) - g^{k-1}(\phi(x))| \le \tau/2$ and $\operatorname{dist}(\Omega_g, \{g^{k-1}(\phi(y)), g^{k-1}(\phi(x))\}) \ge \tau$. So, by representing the inverse branches f_x^{-k} and $g_{\phi(x)}^{-k}$ respectively as the compositions $f_x^{-(k-1)} \circ f^{-1}_{f^{k-1}(x)}$ and $g_{\phi(x)}^{-(k-1)} \circ g^{-1}_{g^{k-1}(\phi(x))}$, it follows from Lemma 2.4 that

$$(K_{f,1}(1/2)\|f'\|)^{-1}|(f^k)'(x)| \le \frac{|f^k(y) - f^k(x)|}{|y-x|} \le K_{f,1}(1/2)\|f'\||(f^k)'(x)|$$

and

$$\begin{aligned}(K_{g,1}(1/2)\|g'\|)^{-1}&|(g^k)'(\phi(x))| \\ &\le \frac{|g^k(\phi(y)) - g^k(\phi(x))|}{|\phi(y)-\phi(x)|} \le K_{g,1}(1/2)\|g'\||(g^k)'(\phi(x))|,\end{aligned}$$

So similarly to Case 1, applying Lemma 8.5 and (8.3), we get $|\phi(y) - \phi(x)| \le C|y-x|^{1/\kappa}$, where C is a universal constant, which finishes the proof of Hölder continuity in this case.

Similarly for conformal measures

$$\begin{aligned}(K_{f,1}(1/2)\|f'\|)^{-h}|(f^k)'(x)|^h &\le \frac{m_f([f^k(y), f^k(x)])}{m_f([y,x])} \\ &\le (K_{f,1}(1/2)\|f'\|)^h|(f^k)'(x)|^h\end{aligned}$$

and

$$\begin{aligned}(K_{g,1}(1/2)\|g'\|)^{-h}|(g^k)'(\phi(x))|^h &\le \frac{m_g([g^k(\phi(y)), g^k(\phi(x))])}{m_g([\phi(y), \phi(x)])} \\ &\le (K_{g,1}(1/2)\|g'\|)^h|(g^k)'(\phi(x))|^h.\end{aligned}$$

From the last two inequalities (involving measures) and from (8.1) we derive

$$\begin{aligned}((K_{f,1}(1/2)\|f'\|)^h &M^2(K_{g,1}(1/2)\|g'\|)^h)^{-1} \\ &\le \frac{|(f^k)'(x)|^h}{|(g^k)'(\phi(x))|^h} \le (K_{f,1}(1/2)\|f'\|)^h M^2 (K_{g,1}(1/2)\|g'\|)^h.\end{aligned}$$

Hence, applying the estimates for distances and (8.3), we get

$$\begin{aligned}&\frac{|\phi(y) - \phi(x)|}{|y-x|} \\ &\le K_{g,1}(1/2)\|g'\|K_{f,1}(1/2)\|f'\|\frac{|g^k(\phi(y)) - g^k(\phi(x))|}{|f^k(y) - f^k(x)|} \cdot \frac{|(f^k)'(x)|^h}{|(g^k)'(\phi(x))|^h} \\ &\le (K_{g,1}(1/2)\|g'\|K_{f,1}(1/2)\|f'\|)^2 M^{2/h} C_f C_g\end{aligned}$$

and

$$\frac{|\phi(y)-\phi(x)|}{|y-x|}$$
$$\ge (K_{g,1}(1/2)\|g'\|K_{f,1}(1/2)\|f'\|)^{-1}\frac{|g^k(\phi(y))-g^k(\phi(x))|}{|f^k(y)-f^k(x)|}\cdot\frac{|(f^k)'(x)|^h}{|(g^k)'(\phi(x))|^h}$$
$$\ge (K_{g,1}(1/2)\|g'\|K_{f,1}(1/2)\|f'\|)^{-2}M^{-2/h}(C_fC_g)^{-1}.$$

Therefore the proof is also finished in this case.

Case 2.2: *$f^n(y)$ and $f^n(x)$ are in different connected components of $B(\omega,\eta/\|f'\|)\setminus\{\omega\}$.* Then also $f^k(y)$ and $f^k(x)$ are in different connected components of $B(\omega,\eta/\|f'\|)\setminus\{\omega\}$. Since the map $f^k|_{[x,y]}$ (even more, the map $f^n|_{[x,y]}$) is well defined there exists a (unique) point $v\in(x,y)$ such that $f^k(v)=w$, in particular $v\in J_f$. Now, note that since $n(x,v),n(y,v)\ge n(x,y)$, both pairs (x,v) and (y,v) fall in Case 2.1 (although it would not hurt us, Case 1 is forbidden for the pairs (x,v) and (y,v) since, by the choice of η_1 and η_2 the nth iterates of both points must then be outside Ω_f and therefore the numbers $|\phi(x)-\phi(y)|$ and $|x-v|$ are comparable, as are the distances $|\phi(y)-\phi(v)|$ and $|y-v|$. Combining these together finishes the proof of Theorem 8.3 and the implication (b)⇒(c). ∎

In order to prove the implication (a)⇒(b) let us introduce the following notation. For every $x\in J_f$ let

$$\eta(x)=\log|g'(\phi(x))|-\log|f'(x)|.$$

Fix $x\in J_f$, a transitive point of f, where transitivity means that $\overline{\{f^n(x):n\ge0\}}$, the closure of the forward trajectory of x, is equal to J_f. Then for every $n\ge0$ set

$$u(f^n(x))=\sum_{j=0}^{n-1}\eta(f^j(x)), \tag{8.4}$$

which is well defined since x, being transitive, is not eventually periodic. We first prove the following technical result whose idea, common in hyperbolic dynamics, is taken from [Bo].

LEMMA 8.7. *If x is a transitive point of f, then for every $0<t<\delta/2$ the function u restricted to the set $(J_f\setminus B(\Omega_f,t))\cap\{f^n(x):n\ge0\}$ is uniformly continuous.*

Proof. Fix $0<\varepsilon<1/2$ and let $0<\zeta<\varepsilon t$ be a number less than the number produced in Corollary 1.7 associated with εt. Consider two points $f^m(x),f^n(x)\in J_f\setminus B(\Omega_f,t)$ with $|f^n(x)-f^m(x)|<\zeta$. Without loosing generality we may assume that $m\le n$. Then in view of Corollary 1.7 there exists a point $y\in J_f$ such that $f^{n-m}(f^m(y))=f^m(y)$ and

$|f^{m+j}(x)-f^{m+j}(y)| < \varepsilon t$ for all $j = 0,1,\ldots,n-m$. Since by the assumption $\sum_{j=m}^{n-1}\eta(f^j(y)) = 0$, we therefore get

$$\begin{aligned}
u(f^n(x)) - u(f^m(x)) &= \sum_{j=m}^{n-1}\eta(f^j(x)) = \sum_{j=m}^{n-1}(\eta(f^j(x)) - \eta(f^j(y)))\\
&= \sum_{j=m}^{n-1}((\log|g'(\phi(g^j(x)))| - \log|g'(\phi(g^j(y)))|)\\
&\quad - (\log|f'(f^j(x))| - \log|f'(f^j(y))|))\\
&= \log\left|\frac{(g^{n-m})'(\phi(g^m(x)))}{(g^{n-m})'(\phi(g^m(y)))}\right| - \log\left|\frac{(f^{n-m})'(f^m(x))}{(f^{n-m})'(f^m(y))}\right|.
\end{aligned}$$

Thus, in order to show that $|u(f^n(x)) - u(f^m(x))|$ is small if $\zeta > 0$ is small it suffices to prove that both terms on the right hand side are small in absolute value. Since ϕ is a homeomorphism it is enough to establish this for the second term. And indeed, since $\varepsilon t < \delta/4 < \delta$, it follows from the properties of y that $f^m(y) = f_{f^m(x)}^{-(n-m)}(f^n(y))$, where $f_{f^m(x)}^{-(n-m)}$, the continuous inverse branch of f^{n-m} sending $f^n(x)$ to $f^m(x)$, is defined on $B(f^n(x),\delta)$. Therefore, since $|f^n(x) - f^m(x)| < \zeta \le \delta/4$, since $|f^n(x) - f^m(x)| < \varepsilon t$, and since $\mathrm{dist}(f^n(x), \Omega_f) \ge t$, it follows from Lemma 2.4 that

$$\left|\log\left|\frac{(f^{n-m})'(f^m(x))}{(f^{n-m})'(f^m(y))}\right|\right| \le |\log K_1(t,\varepsilon)|$$

and $\lim_{\varepsilon\to 0}|\log K_1(t,\varepsilon)| = 0$. The proof is finished. ∎

Proceeding with the proof of the implication (a)⇒(b) we show the following.

LEMMA 8.8. *The functions* $\log|f'(x)|$ *and* $\log|g'(\phi(x))|$ *are cohomologous in the class of continuous functions on* J_f, *that is*, *there exists a continuous function* $u : J_f \to \mathbb{R}$ *such that*

$$\log|g'(\phi(z))| - \log|f'(z)| = u(f(z)) - u(z)$$

for all $z \in J_f$.

Proof. Since the matrix A is primitive, it follows from Lemma 1.2(d) that there exists a transitive point $x \in J_f$. We shall show that u defined by (8.4) on the forward trajectory of x extends continuously to J_f and satisfies the cohomological equation required in Lemma 8.8. First note that by (8.4),

$$\eta(z) = u(f(z)) - u(z) \tag{8.5}$$

for all $z \in \{f^n(x) : n \ge 1\}$ and in view of Lemma 8.7, u extends continuously to the set $J_f \setminus \Omega_f$. Therefore (8.5) holds for all $z \in J_f \setminus (\Omega_f \cup f^{-1}(\Omega))$. Using these two facts we now show that u extends continuously to J_f and that then (8.5) holds for all $z \in \Omega_f$. Indeed, let $\omega \in \Omega_f$. Take $x \in f^{-1}\{\omega\} \setminus \{\omega\}$ and define $u(\omega)$ by the formula $u(\omega) = \eta(x) + u(x)$. We first show that u is continuous at ω. So, let $y_n \to \omega$, $y_n \neq \omega$. Since by Theorem 1.6 the map $f : J_f \to J_f$ is open there exists a sequence $x_n \to x$ such that $f(x_n) = y_n$ and therefore

$$\lim_{n\to\infty} u(y_n) = \lim_{n\to\infty} (\eta(x_n) + u(x_n)) = \eta(x) + u(x) = u(\omega).$$

The continuity of u at ω is therefore proven. Notice also that we have simultaneously shown that $u(\omega)$ is independent of the choice of $x \in f^{-1}\{\omega\} \setminus \{\omega\}$. Therefore u extends continuously to J_f and (8.5) holds for all $z \in J_f \setminus \Omega_f$. But since the functions appearing in (8.5) are continuous and $J_f \setminus \Omega_f$ is dense in J_f, we conclude that (8.5) is true for all $z \in J_f$. ∎

Proof that (a)⇒(b). The proof we present here is similar to the proof of Lemma 4.1. From Lemma 8.8 we deduce the existence of a constant $Q \geq 1$ such that for all $z \in J_f$ and all $n \geq 1$ we have

$$Q^{-1} \leq |(g^n)'(\phi(z))|/|(f^n)'(z)| \leq Q. \tag{8.6}$$

We show that the measure $m_g \circ \phi$ is absolutely continuous with respect to m_f. So, take $\eta > 0$ so small that if $|x - y| \leq \eta$, then $|\phi^{-1}(x) - \phi^{-1}(y)| < \delta$. Fix $\gamma_g > 0$ so small as required in Corollary 2.7 for the map g and then take $\gamma_f > 0$ so small as required in Corollary 2.7 for f and moreover so small that if $|x - y| \leq \gamma_f\delta$, then $|\phi(x) - \phi(y)| < \gamma_g\eta$. As in the proof of Lemma 4.1 it follows from Theorem 1.6 that for every $x \in J_f \setminus \bigcup_{n=0}^\infty f^{-n}(\Omega)$, there exists a sequence $\{n_j = n_j(x) : j \geq 1\}$ such that $f^{n_j(x)} \notin B(\Omega, \delta)$. Let $f_x^{-n_j} : B(f^{n_j}(x), \gamma\delta) \to S^1$ be the continuous inverse branch of f^{n_j} sending $f^{n_j}(x)$ to x. Then it follows from Corollary 2.7 that $f^{n_j}(B(x, r_j)) \supset B(f^{n_j}(x), (K_2(\gamma_f))^{-2}\gamma_f\delta)$ and

$$m_f(B(x, r_j)) \geq K_2(\gamma_f)^{-h} P |(f^{n_j})'(x)|^{-h}, \tag{8.7}$$

where $P = \inf\{m(B(z, K_2(\gamma_f)^{-2}\gamma_f\delta)) : z \in J\} > 0$ and

$$r_j = r_j(x) = K_2(\gamma_f)^{-1} |(f_x^{-n_j})'(f^{n_j}(x))| \gamma\delta = K_2(\gamma_f)^{-1}\gamma_f\delta |(f^{n_j})'(x)|^{-1}.$$

Since also $B(x, r_j) \subset f_x^{-n_j}(B(f^{n_j}(x), \gamma_f\delta))$, by the choice of γ_f we get

$$\phi(B(x, r_j)) \subset \phi(f_x^{-n_j}(B(f^{n_j}(x), \gamma_f\delta))) \subset g_{\phi(x)}^{-n_j}(B(g^{n_j}(\phi(x)), \gamma_g\eta)).$$

Since by the property (a), $\phi(\Omega_f) = \Omega_g$, and since $\operatorname{dist}(f_j^n(x), \Omega_f) \geq \delta_f$, it follows from the choice of η that $\operatorname{dist}(g^{n_j}(\phi(x)), \Omega_g) > \eta$. Hence, applying Corollary 2.7 for g, using (8.6) and (8.7) we get

$$
\begin{aligned}
&m_g(\phi(B(x,r_j(x)))\\
&\qquad \le m_g(g_{\phi(x)}^{-n_j}(B(g^{n_j}(\phi(x)),\gamma_g\eta)))\\
&\qquad \le K_{g,2}(\gamma_g)^h m_g(B(g^{n_j}(\phi(x)),\gamma_g\eta))|(g^{n_j})'(\phi(x))|^{-h}\\
&\qquad \le K_{g,2}(\gamma_g)^h|(g^{n_j})'(\phi(x))|^{-h} \le K_{g,2}(\gamma_g)^h Q^h|(f^{n_j})'(x)|^{-h}\\
&\qquad \le K_{f,2}(\gamma_f)^h K_{g,2}^h(\gamma_g) Q^h P^{-1} m_f(\phi(B(x,r_j(x))).
\end{aligned}
$$

So, applying Lemma 4.3 finishes the proof. ■

Remark. The hyperbolic situation is technically simpler and one usually proves first that (a) implies (c) (see e.g. [Pr4]). Then (c)⇒(b) is immediate and (b)⇒(a) is a consequence of the theory of Gibbs states for hyperbolic systems and characterizations of Hausdorff measures and dimensions in terms of thermodynamic formalism. In the parabolic case the implication (a)⇒(c) splits into two steps: First one shows that $\operatorname{diam}(\phi(B(x,r_j(x)))) \asymp \operatorname{diam}(B(x,r_j(x)))$ for some sequence of radii $r_j(x)$ tending to zero. This gives (a)⇒(b). Showing this for all radii, which is much harder, gives us the implication (b)⇒(c).

9. Real-analytic systems. In this section we consider parabolic Cantor sets generated by dynamical systems $(f, I; \Delta_j, j \in I)$ with f being real-analytic on each set Δ_j. It turns out that then the rigidity theorem, Theorem 8.1, takes on a much stronger form, namely in the condition (b) the assumption of equality of Hausdorff dimensions can be dropped. In order to show this we work first with complex-analytic extensions of f to get analyticity of the Radon–Nikodym derivative $d\mu/dm$. This in turn, with the help of complex-analytic methods, implies real analyticity of the Jacobian of the map $f: J \to J$ with respect to the measure μ. The last step indirectly employing the concept of nonlinearity of expanding dynamical Cantor sets due to Sullivan shows that the Jacobian is not everywhere locally constant, which constitutes the last major ingredient of the proof of real analyticity of the conjugacy ϕ. We begin with the following.

DEFINITION 9.1. A dynamical system $(f, I; \Delta_j, j \in I)$ is said to be *real-analytic* if the map $f: \bigcup_{j\in I} \Delta_j \to S^1$ has a real-analytic extension onto an open neighborhood of $\bigcup_{j\in I} \Delta_j$ in S^1.

The remark that enables us to take advantage of the theory of complex-analytic functions is that for any real analytic dynamical system there exists an open neighborhood H of $\bigcup_{j\in I} \Delta_j$ in $\mathbb{C}$ and a $\mathbb{C}$-analytic function on H whose restriction to $\bigcup_{j\in I} \Delta_j$ coincides with f. We call this function the (*complex*) *analytic extension* of f and we keep for it the same symbol f. Our exposition begins with citing the following improved version of the

Koebe Distortion Theorem proven in [Pr1] (for the classical version and some discussion of the subject see [Po] for example).

LEMMA 9.2 (The Koebe Distortion Theorem). *Given an open bounded subset G of the complex plane $\mathbb{C}$ there exists a constant $K > 1$ such that if $B(z,\delta) \subset G$ and $H : B(z,\delta) \to G$ is a holomorphic univalent map, then for every $0 < \lambda < 1$ and every $x \in B(z,\delta)$ we have* $\max\{|H'(x)|/|H'(z)|, |H'(z)|/|H'(x)|\} < K(1-\lambda)^{-1}$.

Switching to the setting of parabolic Cantor sets and using the idea from [PUZ, p. 198] we shall prove the following.

LEMMA 9.3. *Let $V \subset J$ be an open neighborhood of Ω. Then there exists an $r > 0$ such that for every $x \in J\setminus V$, every $n \ge 0$ and every $z \in J \cap f^{-n}(x)$ there is an inverse $\mathbb{C}$-analytic branch $f_z^{-n} : B_{\mathbb{C}}(x, 2r) \to \mathbb{C}$ of f^n sending x to z. Additionally*, $\lim_{n\to\infty} \sup_{x\in J\setminus V} \max_{z\in f^{-n}(x)}\{\mathrm{diam}(f_z^{-n}(B_{\mathbb{C}}(z,r)))\} = 0$.

Proof. Since $f : H \to \mathbb{C}$, being analytic, is open and since J is compact, we have $\eta = \mathrm{dist}(J, \partial(H \cap f(H))) > 0$. Hence, using compactness of J again we see that there exists $s > 0$ such that all the inverse branches of f are well defined on the balls $B(x,s)$, $x \in J$. Suppose now additionally that $x \notin V$ and consider an arbitrary infinite sequence $x_n \in J$, $n \ge 0$, such that $f(x_{n+1}) = x_n$ and $x_0 = x$. Set

$$b_n = \tfrac{1}{2} M(t,1/2)^{-1} |(f_{x_{n+1}}^{-(n+1)})'(x)|,$$

where $t = \mathrm{dist}(\Omega, J\setminus V)$ and $M(t,1/2)$ is taken from Lemma 2.4. In view of Lemma 2.4, $\sum_{n=0}^{\infty} b_n \le 1/2$ and therefore the product $\prod_{n\ge0}(1-b_n)^{-1}$ converges. In fact, it lies between 1 and e. Hence there exists $r > 0$ independent of x so small that

$$2r \prod_{n\ge0}(1-b_n)^{-1} \le \min\{s, \delta, t/2, s(2KM(t,1/2))^{-1}\}. \tag{9.1}$$

We show by induction that for every $n \ge 1$ there is an analytic inverse branch $f_{x_n}^{-n} : B(x, 2r\prod_{k\ge n}(1-b_k)^{-1}) \to \mathbb{C}$ sending x to x_n and

$$f_{x_n}^{-n}\Big(B\Big(x, 2r\prod_{k\ge n}(1-b_k)^{-1}\Big)\Big) \subset B(x_n, s).$$

Indeed, for $n = 0$, $f_{x_0}^{-0}$ is the identity map and our assertion follows from (9.1). So, fix some $n \ge 0$ and suppose the assertion is true for this n. Then by the definition of s the inverse branch $f_{x_{n+1}}^{-(n+1)} : B(x, 2r\prod_{k\ge n}(1-b_k)^{-1}) \to \mathbb{C}$ is also well defined and by Lemma 9.2 (the Koebe Distortion Theorem), the definition of b_n's and (9.1),

$$f_{x_{n+1}}^{-(n+1)}\Big(B\Big(x, 2r\prod_{k\ge n+1}(1-b_k)^{-1}\Big)\Big)$$
$$\subset B\Big(x_{n+1}, 2r\prod_{k\ge n+1}(1-b_k)^{-1}Kb_n^{-1}|(f_{x_{n+1}}^{-(n+1)})'(x)|\Big)$$
$$\subset B\Big(x_{n+1}, 2r\prod_{k\ge 0}(1-b_k)^{-1}K2M(t,1/2)\Big) \subset B(x_{n+1}, s).$$

Thus, the inductive reasoning is complete and as $\prod_{k\ge n}(1-b_k)^{-1}\ge 1$ for every n, the first part of the lemma is proven. The second part now follows immediately from Lemmas 1.1 and 9.2. ∎

As an immediate consequence of Lemmas 9.3 and 9.2 we get the following.

Corollary 9.4. *$\forall\lambda>1\ \exists q\ \forall n\ge q\ \forall z\in J\setminus V$, if $f_\nu^{-n}: B(z,2r)\to\overline{\mathbb{C}}$ is an inverse branch of f^n then $|(f_\nu^{-n})'(x)|<\lambda^{-1}$ for every $x\in B(z,r)$.*

Our next goal is to show that the Radon–Nikodym derivative $d\mu/dm$ produced in Theorem 7.2 allows a real-analytic extension, that is in fact even a complex analytic extension. First, using Lemma 9.3 and proceeding essentially in the same way as in the proof of Lemma 4.3 of [U1] (comp. also [Su1] for a hyperbolic case) one can prove the following.

Lemma 9.5. *If (f, J_f) is real-analytic, then there exists a $\mathbb{C}$-analytic extension of $\psi^* = d\mu^*/dm$ onto an open neighborhood of $\bigcup_{j\in J}\Delta_j$.*

Now, as an immediate consequence of Lemma 9.5 and Theorem 7.2, along with real analyticity of $1/|f'|^h$, and Lemma 2.3, we get the following.

Lemma 9.6. *The Radon–Nikodym derivative $\psi = d\mu/dm$ has a real-analytic extension to the set $\bigcup_{j\in J}\Delta_j\setminus\Omega$.*

Let now ϱ_μ denote the Jacobian of the map f with respect to the measure μ. Since $\varrho_\mu(x) = |f'(x)|^h\psi(f(x))/\psi(x)$, we derive from Lemma 9.6 the following main technical result about real analyticity.

Lemma 9.7. *The Jacobian ϱ_μ has a real-analytic extension to the set $\bigcup_{j\in J}\Delta_j\setminus\Omega$.*

Notice that although μ is determined only up to a multiplicative constant, the Jacobian ϱ_μ is unique. Our first consequence of Lemma 9.7 is the following.

Lemma 9.8. *If $(f, I; \Delta_j, j\in I)$ is a real-analytic parabolic system, then there is $i\in I$ such that the Jacobian ϱ_μ of f with respect to the invariant measure μ is not locally constant at any point of Δ_i.*

Proof. Suppose to the contrary that every interval Δ_j contains a point (not necessarily lying in J) around which the Jacobian ϱ_μ is constant. Then Lemma 9.7 yields that ϱ_μ is constant on each whole interval Δ_j, $j \in I$. Denote this common value by ϱ_j. Since μ is invariant, $\sum_{y\in f^{-1}(x)} \varrho_i^{-1}(y) = 1$ for μ-almost every $x \in J$. Therefore, since all the points in sufficiently small neighborhoods of all fixed points of f have at least two distinct preimages under f (one of which lying again in a small neighborhood of fixed points) and since μ is positive on nonempty open sets, it follows that $\varrho_\mu(y) > 1$ for all y in a sufficiently small neighborhood of fixed points. Let $\lambda > 1$ be the minimum of such Jacobians ϱ_j. Take now an arbitrary point $\omega \in \Omega$ and choose one point $z \in J \cap B(\omega,\delta)\setminus\{\omega\}$. In view of Lemma 7.4, $\mu([f_\omega^{-1}(z), z)) < \infty$. Thus

$$\begin{aligned}\mu([\omega,z)) &= \sum_{n\ge 0}\mu(f_\omega^{-n}([f_\omega^{-1}(z),z))) \le \sum_{n\ge 0}\lambda^{-n}\mu([f_\omega^{-1}(z),z)) \\ &= \frac{1}{1-\lambda}\mu([f_\omega^{-1}(z),z)) < \infty.\end{aligned}$$

Choosing if necessary one point in $J\cap B(\omega,\delta)\setminus\{\omega\}$ located on the other side of ω, we therefore conclude that ω has a neighborhood of finite μ-measure. Since Ω is finite the same continues to be true for the whole set Ω. Combining this fact and Lemma 7.4 we deduce that $\mu(J) < \infty$. But this contradicts Theorem 7.5 as in our situation $\beta(\omega) = 1$ for all $\omega \in \Omega$, and finishes the proof of the lemma. ■

Let us now prove the main result of this section. To the best of our knowledge the already classical idea of its proof goes back to [Pr4] and [SS] and since then its modifications have been used a number of times.

THEOREM 9.9. *Let (J_f, f) and (J_g, g) be two real-analytic parabolic systems and let $\phi : J_f \to J_g$ be the corresponding canonical topological conjugacy. If the homeomorphism ϕ transports the measure class of the packing measure Π_{h_f} on J_f onto the measure class of the packing measure Π_{h_g} on J_g, then ϕ and ϕ^{-1} extend to real-analytic maps on open neighborhoods in S^1 respectively of J_f and J_g. In particular*, $\mathrm{HD}(J_f) = \mathrm{HD}(J_g)$.

Proof. Fix an f-invariant measure μ_f equivalent to the conformal measure m_f. Since ϕ transports the measure class of m_f to the measure class of the conformal measure m_g, the measure $\mu_g = \mu_f \circ \phi^{-1}$ is g-invariant and equivalent to m_g. Thus Lemmas 9.6–9.8 also apply to the system (g, μ_g). Since ϕ is invertible the equality $\mu_g = \mu_f \circ \phi^{-1}$ equivalently means that ϱ_ϕ, the Jacobian of ϕ with respect to the measures μ_f and μ_g, is equal to 1. The formula $g\circ\phi = \phi\circ f$ combined with the chain rule therefore gives $\varrho_g \circ \phi = \varrho_f$ μ_f-a.e., where ϱ_g and ϱ_f denote respectively the Jacobians of the maps g and f with respect to the measures μ_g and μ_f. Since μ_f is positive on nonempty open subsets of J_f and since by Lemma 9.7, both sides of the last equality are continuous on $J_f \setminus (\Omega_f \cup \phi^{-1}(\Omega_g))$, we get

$$\varrho_g \circ \phi(x) = \varrho_f(x) \tag{9.5}$$

for all $x \in J_f \setminus (\Omega_f \cup \phi^{-1}(\Omega_g))$. Now Lemma 9.8 applied to the real-analytic system (g, J_g) produces an open arc $V \subset S^1$ such that $V \cap J_g \neq \emptyset$ and $\varrho_g|_V$ is injective. Let $W = \phi^{-1}(V \cap J_g)$. Since W is a nonempty subset of J_f and since $\varrho_g(V)$ is an open subset of $\mathbb{R}$, using (9.5), we deduce the existence of an open subset U of $S^1 \setminus (\Omega_f \cup \phi^{-1}(\Omega_g))$ such that $\emptyset \neq U \cap J_f \subset W$, $\varrho_f(U) \subset \varrho_g(V)$ and

$$\phi(x) = (\varrho_g|_V)^{-1} \circ \varrho_f(x) \tag{9.6}$$

for all $x \in J_f \cap U$. In particular, $\phi|_{J_f \cap U}$ has a real-analytic extension on U. Take now an arbitrary point $z \in J_f$. In view of Lemma 1.2(d) there exist $y \in J_f \cap U$ and $n \geq 0$ such that $f^n(y) = z$. Take $r > 0$, depending on y and n, so small that there exists $f_y^{-n} : B(z,r) \to S^1$, a continuous inverse branch of f^n sending z to y. We may additionally require $r > 0$ to be so small that $f_y^{-n}(B(z,r)) \subset U$ and $g^n(\varrho_g|_V)^{-1} \circ \varrho_f \circ f_y^{-n}(B(z,r))$ is well defined. From $\phi \circ f^n = g^n \circ \phi$ (on J_f) we deduce that $\phi = g^n \circ \phi \circ f_y^{-n}$ on $J_f \cap B(z,r)$. So, since f_y^{-n} on $B(z,r)$ is real-analytic and since g^n is real-analytic on any arc where it is well defined, using (9.6) we deduce that $g^n \circ (\varrho_g|_V)^{-1} \circ \varrho_f \circ f_y^{-n} : B(z,r) \to S^1$ gives a real-analytic extension of $\phi|_{J_f \cap B(z,r)}$ to the ball $B(z,r)$. Thus we have proved that every point of J_f has an open connected neighborhood in S^1 to which ϕ can be extended in a real-analytic fashion. Now, to conclude the proof, it suffices to remark that any two such real-analytic extensions, defined on overlapping intervals, coincide on their intersection. ∎

10. The scaling function. In this section we collect some basic properties of the scaling function associated with a cookie-cutter Cantor set construction, stressing differences between the parabolic and hyperbolic cases. Next we formulate a rigidity theorem in terms of scaling functions. Throughout the section we assume that the basic sets Δ_j, $j \in I$, are mutually disjoint, which implies that $\Sigma_A^\infty = \Sigma^\infty$ is the full shift space over $d = \#I$ elements, $\pi : \Sigma^\infty \to J$ is a homeomorphism, and J is a topological Cantor set. Moreover, we require that for all $j \in I$,

$$f(\Delta_j) \supset \bigcup_{i \in I} \Delta_i \tag{10.1}$$

and the endpoints of the interval $f(\Delta_j)$ are contained in $\bigcup_{i \in I} \Delta_i$, hence are the same for all $j \in I$.

Recall that in Section 1 we have denoted by $\Delta(\tau)$, $\tau \in \Sigma^n$, the interval $\Delta_{\tau_0} \cap f^{-1}(\Delta_{\tau_1}) \cap \ldots \cap f^{-n}(\Delta_{\tau_n})$. Now we want to extend this definition letting τ be of the form $\varrho\gamma$, where $\varrho \in \Sigma^*$ and γ ranges over the set $\mathcal{G}$ (consisting of $d-1$ elements) of gaps between the elements Δ_j, $j \in I$. We set

$$\Delta(\varrho\gamma) = \Delta(\varrho) \cap f^{-(|\varrho|+1)}(\gamma)$$

and now we are in a position to define the function $S : \Sigma^* \to [0,1]^{2d-1}$ putting for all $\tau \in \Sigma^*$ and $j \in I \cup \mathcal{G}$,

$$S(\tau)(j) = S_j(\tau) = |\Delta(\tau j)|/|\Delta(\tau)|.$$

Note that $\sum_j S(\tau)(j) = 1$. We will also consider functions S defined on the dual shift space $\widetilde{\Sigma}^*$ consisting of all left-infinite words $\ldots\tau_n\tau_{n-1}\ldots\tau_0$, $\tau_i \in I$. Given $n \geq 0$ and $\tau \in \widetilde{\Sigma}^*$ we define $S_n(\tau) = S(\tau_n\tau_{n-1}\ldots\tau_0)$. So, $S_n : \widetilde{\Sigma}^* \to [0,1]^{2d-1}$. Our first aim is to prove the following.

THEOREM 10.1. *The sequence* $\{S_n : \widetilde{\Sigma}^* \to [0,1]^{2d-1} : n \geq 1\}$ *converges uniformly. The limit function* $S : \widetilde{\Sigma}^* \to [0,1]^{2d-1}$, *called the scaling function*, *is continuous.*

Proof. Take $j \in I \cap \mathcal{G}$. Fix also integers $k, n \geq 0$. Take an auxiliary $x \in \Delta(\tau|_{n+k})$. In view of the Mean Value Theorem there exist $y \in \Delta(\tau_k j)$ and $z \in \Delta(\tau|_k)$ such that $|\Delta(\tau|_{n+k}j)| = |(f_x^{-n})'(y)| \cdot |\Delta(\tau_k j)|$ and $|\Delta(\tau|_{n+k})| = |(f_x^{-n})'(z)| \cdot |\Delta(\tau_k)|$. Therefore

$$\begin{aligned}(10.2)\quad & |S_{n+k}(\tau)(j) - S_k(\tau)(j)| \\ &= \left| \frac{|\Delta(\tau|_{n+k}j)|}{|\Delta(\tau|_{n+k})|} - \frac{|\Delta(\tau_k j)|}{|\Delta(\tau_k)|} \right| \\ &= \left| \frac{|(f_x^{-n})'(y)| \cdot |\Delta(\tau_k j)|}{|(f_x^{-n})'(z)| \cdot |\Delta(\tau_k)|} - \frac{|\Delta(\tau_k j)|}{|\Delta(\tau_k)|} \right| \\ &= \frac{|\Delta(\tau_k j)|}{|\Delta(\tau_k)|} \left| \frac{|(f_x^{-n})'(y)|}{|(f_x^{-n})'(z)|} - 1 \right| \leq \left| \frac{|(f_x^{-n})'(y)|}{|(f_x^{-n})'(z)|} - 1 \right|.\end{aligned}$$

With the help of (10.2) we shall prove that all the sequences $S_n(\cdot)(j)$, $j \in I \cup G$, satisfy the uniform Cauchy condition. Indeed, fix again $j \in I \cup G$ and $\varepsilon > 0$. Take $\psi > 0$ so small that $\max\{Q_1(2\psi) - 1, 1 - Q_1(2\psi)^{-1}\} < \varepsilon$, where Q_1 is the function produced in Lemma 2.8. Now fix $A(\varepsilon) > 0$ so small that setting

$$K_1 = K_1(\delta/\|f'\|, L_1(\delta/\|f'\|)^{\beta+1}) L_2(\delta/\|f'\|)\delta^{-1}\|f'\|\psi^{-(\beta+1)}A(\varepsilon),$$

where the function K_1 is produced in Lemma 2.4, we have $\max\{K_1 - 1, 1 - K_1^{-1}\} < \varepsilon/2$. Finally, by Lemma 1.1 we can fix $k \geq 1$ so large that

$$\operatorname{diam}(\Delta(\tau|_k)) < A(\varepsilon) \tag{10.3}$$

for all $\tau \in \widetilde{\Sigma}$. Take now an arbitrary $\tau \in \widetilde{\Sigma}$ and suppose that $\operatorname{dist}(\Omega, \Delta(\tau|_k)) \geq \psi$. Let $t \geq 0$ be the least integer such that $\Delta(\tau|_k) = f_\omega^{-(k-t)}(\Delta(\tau|_t))$

for some $\omega \in \Omega$. Since ψ is positive, $\operatorname{dist}(\Omega, \Delta(\tau|_t)) \ge \delta/\|f'\|$. If $t = k$, then $\operatorname{diam}(\Delta(\tau|_t)) < A(\varepsilon)$. Otherwise, using Corollary 2.2 we conclude that $\operatorname{dist}(\Omega, \Delta(\tau|_k)) \le L_1(\delta/\|f'\|)(k-t)^{-1/\beta}$. Hence $L_1(\delta/\|f'\|)(k-t)^{-1/\beta} \ge \psi$ and therefore $k - t \le (L_1(\delta/\|f'\|)\psi^{-1})^{\beta}$. Thus by Lemma 2.3 we get

$$\begin{aligned}\operatorname{diam}(\Delta(\tau|_k)) &\ge L_2(\delta/\|f'\|)^{-1}(k-t)^{-(\beta+1)/\beta}\operatorname{diam}(\Delta(\tau|_t))\\ &\ge L_2(\delta/\|f'\|)^{-1}L_1(\delta/\|f'\|)^{-(\beta+1)}\psi^{\beta+1}\operatorname{diam}(\Delta(\tau|_t)),\end{aligned}$$

which implies that

$$\begin{aligned}\operatorname{diam}(\Delta(\tau|_t)) &\le L_2(\delta/\|f'\|)L_1(\delta/\|f'\|)^{\beta+1}\psi^{-(\beta+1)}\operatorname{diam}(\Delta(\tau|_k))\\ &\le L_2(\delta/\|f'\|)L_1(\delta/\|f'\|)^{\beta+1}\psi^{-(\beta+1)}A(\varepsilon).\end{aligned}$$

Hence applying (10.2) and Lemma 2.4, it follows from the choice of k and ψ that for every $n \ge 0$ we have

$$\begin{aligned}(10.4)\quad |S_{n+k}(\tau)(j) - S_k(\tau)(j)|&\\ &\le |S_{n+k}(\tau)(j) - S_t(\tau)(j)| + |S_t(\tau)(j) - S_k(\tau)(j)|\\ &\le 2\max\{|K_1 - 1|, |1 - K_1^{-1}|\} < \varepsilon.\end{aligned}$$

So, we can assume that

$$\operatorname{dist}(\Omega, \Delta(\tau|_k)) < \psi.$$

Then $\Delta(\tau|_k) \subset B(\Omega, 2\psi)$. Therefore if $\tau|_k$ does not consist only of indices corresponding to one parabolic point (so the assumptions of Lemma 2.8 are satisfied with $q = 1$), then it follows from (10.2), Lemma 2.8, and the choice of ψ that for every $n \ge 0$,

$$|S_{n+k}(\tau)(j) - S_k(\tau)(j)| \le \max\{Q_1(2\psi) - 1, 1 - Q_1^{-1}(2\psi)\} < \varepsilon.$$

Now, the only case left is when $\tau|_k$ consists of indices j_ω only for some $\omega \in \Omega$, where $j_\omega \in I$ is determined by the requirement that $\omega \in \Delta_{j_\omega}$. Since by the Mean Value Theorem $\lim_{x\to\omega} |f_\omega^{-1}(x) - \omega|/|x - \omega| = 1$ and in view of Corollary 2.2 and Lemma 2.3 we deduce that $\lim_{n\to\infty} S_n(j_\omega^n)(j)$ is equal to 1 if $j = j_\omega$ and 0 otherwise. Hence taking k sufficiently large, larger than required in (10.3) perhaps, we see that $|S_{n+k}(\tau)(j) - S_k(\tau)(j)| < \varepsilon$ if $\tau|_{n+k} = j_\omega^{n+k}$. Otherwise look at the largest number q such that $\tau|_q = j_\omega^q$. Then $k \le q < n + k$ and

$$\begin{aligned}|S_{n+k}(\tau)(j) - S_k(\tau)(j)|&\\ &\le |S_{n+k}(\tau)(j) - S_{q+1}(\tau)(j)|\\ &\quad + |S_{q+1}(\tau)(j) - S_q(\tau)(j)| + |S_q(\tau)(j) - S_k(\tau)(j)|.\end{aligned}$$

As above, $|S_q(\tau)(j) - S_k(\tau)(j)| < \varepsilon$. Moreover, the first summand $|S_{n+k}(\tau)(j) - S_{q+1}(\tau)(j)|$ is estimated from above by ε similarly to the two summands in (10.4) ($q + 1$ corresponds to t) and in view of (10.2) applied with $n = 1$

the second summand $|S_{q+1}(\tau)(j) - S_q(\tau)(j)|$ is less than ε if and only if $\mathrm{diam}(\Delta(\tau|_k))$, and consequently also $\mathrm{diam}(\Delta(\tau|_k))$ is sufficiently small. Then $|S_{n+k}(\tau)(j) - S_k(\tau)(j)| < 3\varepsilon$, which completes the proof of the uniform convergence of the sequence S_n. Since all the functions S_n are obviously continuous the limit function is also continuous and the proof is finished. ∎

Now we shall prove the fact, actually already proven in the course of the proof of Theorem 10.1, which describes some differences between parabolic and hyperbolic dynamical Cantor sets in the language of scaling functions.

LEMMA 10.2. *$S(\tau)(j) = 0$ if and only if for all $n \ge 0$, $\Delta(\tau_n)$ is the (only) element containing some $\omega \in \Omega$ and Δ_j does not contain ω.*

Proof. Suppose first that for all $\omega \in \Omega$ not all the elements $\Delta(\tau_n)$, $n \ge 0$, contain ω. If $\Delta(\tau_0) \cap \Omega \ne \emptyset$, set $q = 0$. Otherwise there exists a least $q \ge 1$ such that $\tau_q \ne \tau_0$. In any case $\mathrm{dist}(\Omega, \Delta(\tau|_q)) \ge \delta/2$. In view of the Mean Value Theorem there exist $y \in \Delta(\tau|_q j) \subset \Delta(\tau|_q)$ and $z \in \Delta(\tau|_q)$ such that $|\Delta(\tau|_{q+n} j)| = |(f_t^{-n})'(y)| \cdot |\Delta(\tau|_q j)|$ and $|\Delta(\tau|_{q+n})| = |(f_t^{-n})'(z)| \cdot |\Delta(\tau|_q)|$, where f_t^{-n} denotes the inverse branch of f^n sending $\Delta(\tau|_q)$ to $\Delta(\tau|_{q+n})$. Therefore

$$S_{q+n}(\tau)(j) = \frac{|\Delta(\tau|_{q+n} j)|}{|\Delta(\tau|_{q+n})|} = \frac{|(f_t^{-n})'(y)|}{|(f_t^{-n})'(z)|} S_q(\tau)(j)$$

and applying Corollary 2.5 we get $S_{q+n}(\tau)(j) \ge K_1(\delta/2)^{-1} S_q(\tau)(j)$. So, letting $n \to \infty$ (and employing Theorem 10.1 of course), we get $S(\tau)(j) \ge K_1(\delta/2)^{-1} S_q(\tau)(j) > 0$.

Now suppose that $\Delta(\tau|_n) = f_\omega^{-n}(\Delta_{\tau_0})$ for all $n \ge 0$ and some $\omega \in \Omega$. If j is taken such that $\omega \notin \Delta_j$, then in view of Lemma 2.3 and Corollary 2.2,

$$S_n(\tau)(j) = \frac{|\Delta(\tau|_n j)|}{|\Delta(\tau|_n)|} \le \frac{L_2(\delta/2) n^{-(\beta+1)/\beta}}{L_1(\delta/2)^{-1} n^{-1/\beta}} = L_1(\delta/2) L_2(\delta/2) n^{-1}.$$

Hence $S(\tau)(j) = 0$. Since $\sum_j S(\tau)(j) = 1$, the proof is complete. ∎

COROLLARY 10.3. *If two dynamical Cantor sets J_f and J_g generated respectively by dynamical systems $(f, I; \Delta_{f,j}, j \in I)$ and $(g, I; \Delta_{g,j}, j \in I)$ have the same scaling functions, then the topological conjugacy $\phi : J_f \to J_g$ sends the set of parabolic points of f onto the set of parabolic points of g.*

THEOREM 10.4. *If two dynamical Cantor sets J_f and J_g generated respectively by dynamical systems $(f, I; \Delta_{f,j}, j \in I)$ and $(g, I; \Delta_{g,j}, j \in I)$ have the same scaling functions, then the topological conjugacy $\phi : J_f \to J_g$ is Lipschitz continuous. Conversely, if the conjugacy $\phi : J_f \to J_g$ is a C^1 diffeomorphism, then the Cantor sets J_f and J_g have the same scaling functions.*

Proof. Let us prove first the second part of this theorem. Indeed, keep the same notation ϕ for a C^1 extension of ϕ to an open neighborhood of J_f. Decreasing this neighborhood if necessary we can assume that ϕ', the derivative of ϕ, nowhere vanishes. Therefore, for every $n \geq 0$ sufficiently large and every $\tau \in \Sigma^n$, the map $\phi|_{\Delta_f(\tau)}$ is well defined and $\phi(\Delta_f(\tau)) = \Delta_g(\tau)$. Now, in view of the Mean Value Theorem, for every $\tau \in \widetilde{\Sigma}$, every $j \in I$ and every sufficiently large $n \geq 0$, there are $y \in \Delta_f(\tau|_n j) \subset \Delta_f(\tau|_n)$ and $z \in \Delta_f(\tau|_n)$ such that $|\Delta_g(\tau|_n j)| = |\phi'(y)| \cdot |\Delta_f(\tau|_n j)|$ and $|\Delta_g(\tau|_n)| = |\phi'(z)| \cdot |\Delta_f(\tau|_n)|$. Thus

$$S_{g,n}(\tau)(j) = \frac{|\phi'(y)|}{|\phi'(z)|} S_{f,n}(\tau)(j).$$

Since $\lim_{n\to\infty} |\Delta(\tau|_n)| = 0$, it follows from positiveness and continuity of ϕ' that

$$S_g(\tau)(j) = \lim_{n\to\infty} S_{g,n}(\tau)(j) = \lim_{n\to\infty} S_{f,n}(\tau)(j) = S_f(\tau)(j),$$

finishing the proof of the second part of the theorem.

In order to prove the first part of this theorem we will show that condition (a) of Theorem 8.1 is satisfied, that is, that the spectra of moduli of periodic points of f and g are the same. So, let z be an arbitrary periodic point of f, say of period $q \geq 1$. For $0 \leq j \leq q-1$ let $f^j(z) \in \Delta_f(\tau_j)$ and let $\tau = \tau_0\tau_1\ldots\tau_{q-1}$. Our aim is to show that $|(g^q)'(z)| = |(f^q)'(z)|$. In view of Corollary 10.3 we may assume that neither z nor $\phi(z)$ are parabolic. Denoting by τ^n the concatenation of n words τ, we get

$$\begin{aligned}
&\frac{|\Delta_g(\tau^{n+1}\tau_0)|}{|\Delta_g(\tau^n\tau_0)|} \Big/ \frac{|\Delta_f(\tau^{n+1}\tau_0)|}{|\Delta_f(\tau^n\tau_0)|} \\
&= \frac{|\Delta_g(\tau^n\tau_1\ldots\tau_{q-1}\tau_0)|}{|\Delta_g(\tau^n\tau_1\ldots\tau_{q-1})|} \cdot \frac{|\Delta_g(\tau^n\tau_1\ldots\tau_{q-1})|}{|\Delta_g(\tau^n\tau_1\ldots\tau_{q-2})|} \cdot \ldots \cdot \frac{|\Delta_g(\tau^{n+1}\tau_0\tau_1)|}{|\Delta_g(\tau^{n+1}\tau_0)|} \\
&\quad \times \left(\frac{|\Delta_f(\tau^n\tau_1\ldots\tau_{q-1}\tau_0)|}{|\Delta_f(\tau^n\tau_1\ldots\tau_{q-1})|} \cdot \frac{|\Delta_f(\tau^n\tau_1\ldots\tau_{q-1})|}{|\Delta_f(\tau^n\tau_1\ldots\tau_{q-2})|} \cdot \ldots \cdot \frac{|\Delta_f(\tau^{n+1}\tau_0\tau_1)|}{|\Delta_f(\tau^{n+1}\tau_0)|} \right)^{-1} \\
&= S_g(\tau^n\tau_1\ldots\tau_{q-1})(\tau_0) S_g(\tau^n\tau_1\ldots\tau_{q-2})(\tau_{q-1}) \ldots S_g(\tau^n\tau_1\ldots\tau_{q-2})(\tau_{q-1}) \\
&\quad \times S_f^{-1}(\tau^n\tau_1\ldots\tau_{q-1})(\tau_0) S_f^{-1}(\tau^n\tau_1\ldots\tau_{q-2})(\tau_{q-1}) \ldots S_f^{-1}(\tau^n\tau_0)(\tau_1)
\end{aligned}$$

Thus, denoting by $\tau^\infty \in \widetilde{\Sigma}$ the infinite concatenation of τ's we obtain

$$\begin{aligned}
(10.5)\qquad &\lim_{n\to\infty} \left(\frac{|\Delta_g(\tau^{n+1}\tau_0)|}{|\Delta_g(\tau^n\tau_0)|} \Big/ \frac{|\Delta_f(\tau^{n+1}\tau_0)|}{|\Delta_f(\tau^n\tau_0)|} \right) \\
&= \frac{S_g(\tau^\infty\tau_1\ldots\tau_{q-1})(\tau_0)}{S_f(\tau^\infty\tau_1\ldots\tau_{q-1})(\tau_0)} \cdot \frac{S_g(\tau^\infty\tau_1\ldots\tau_{q-2})(\tau_{q-1})}{S_f(\tau^\infty\tau_1\ldots\tau_{q-2})(\tau_{q-1})} \cdot \ldots \cdot \frac{S_g(\tau^\infty\tau_0)(\tau_1)}{S_f(\tau^\infty\tau_0)(\tau_1)} = 1.
\end{aligned}$$

On the other hand, since $\Delta_f(\tau^n\tau_0) = f^q(\Delta_f(\tau^{n+1}\tau_0))$ and $\Delta_g(\tau^n\tau_0) =$

$g^q(\Delta_g(\tau^{n+1}\tau_0))$, by the Mean Value Theorem there are two points $x_n \in \Delta_f(\tau^{n+1}\tau_0)$ and $y_n \in \Delta_g(\tau^{n+1}\tau_0)$ such that $|\Delta_f(\tau^n\tau_0)| = |(f^q)'(x_n)| \cdot |\Delta_f(\tau^{n+1}\tau_0)|$ and $|\Delta_g(\tau^n\tau_0)| = |(g^q)'(y_n)| \cdot |\Delta_g(\tau^{n+1}\tau_0)|$. Combining these equalities and (10.5) we get

$$\frac{|(g^q)'(\phi(z))|}{|(f^q)'(z)|} = \lim_{n\to\infty} \frac{|(g^q)'(y_n)|}{|(f^q)'(x_n)|} = 1.$$

Applying now Theorem 8.1 completes the proof. ■

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