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Categoricity of theories in $L_{\kappa\omega}$, when κ is a measurable cardinal. Part 1

by

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Abstract. We assume a theory T in the logic $L_{\kappa\omega}$ is categorical in a cardinal $\lambda \geq \kappa$, and κ is a measurable cardinal. We prove that the class of models of T of cardinality $< \lambda$ (but $\geq |T| + \kappa$) has the amalgamation property; this is a step toward understanding the character of such classes of models.

Annotated content

0. Introduction

1. Preliminaries. We review material on fragments $\mathcal{F}$ of $L_{\kappa\aleph_0}$ (including the theory T) and basic model theoretic properties (Tarski–Vaught property and L.S.), define amalgamation, indiscernibles and E.M. models, then limit ultrapowers which are *suitable* (for $L_{\kappa\omega}$) and in particular ultralimits. We then introduce a notion basic for this paper: $M \underset{\text{nice}}{\preceq_{\mathcal{F}}} N$ if there is an $\preceq_{\mathcal{F}}$-embedding of N into a suitable ultralimit of M extending the canonical one.

2. The amalgamation property for regular categoricity. We first get amalgamation in $(K_\lambda, \preceq_{\mathcal{F}})$ when one of the extensions is nice (2.1). We prove that if T is categorical in the regular $\lambda > |\mathcal{F}| + \kappa$, then $(K_{<\lambda}, \preceq_{\mathcal{F}})$ has the amalgamation property. For this we show that nice extensions (in $\mathcal{K}_{<\lambda}$) preserve being non-amalgamation basis. We also start investigating (in 2.5) the connection between extending the linear order I and the model EM(I): $I \underset{\text{nice}}{\subseteq} J \Rightarrow \text{EM}(I) \underset{\text{nice}}{\preceq} \text{EM}(J)$; and give sufficient condition for $I \underset{\text{nice}}{\subseteq} J$ (in 2.6). From this we get in K_λ a model such that any submodel of an expansion is a $\underset{\text{nice}}{\preceq}$-submodel (in 2.7, 2.10(2)), and conclude the amalgamation property in $(K_{<\lambda}, \preceq_{\mathcal{F}})$ when λ is regular (in 2.9) and something for singulars (2.10).

3. Towards removing the assumption of regularity from the existence of universal extensions. The problem is that EM(λ) has many models which "sit" well in

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it and many which are amalgamation bases but we need to get this simultaneously. First (3.1) we show that if $\langle M_i : i < \theta^+ \rangle$ is an $\prec_{\mathcal{F}}$-increasing continuous sequence of models of $K_\theta \subseteq K = \mathrm{Mod}(T)$ then for a club of $i < \theta^+$, $M_i \underset{\text{nice}}{\preceq} \bigcup\{M_j : j < \theta^+\}$. We define nice models (Def. 3.2; essentially, every reasonable extension is nice), show a variant is equivalent (3.4), and implies being an amalgamation base (3.5), and we prove that in K_θ the nice models are dense (3.3). Then we define a universal extension of $M \in K_\theta$ in K_σ (Def. 3.6), prove existence inside a model (3.7), and after preparation (3.8) prove existence (3.9, 3.10, 3.11).

4. (θ, σ)-saturated models. If $M_i \in K_\theta$ for $i \leq \sigma$ is increasing continuous, with M_{i+1} universal over M_i, and each M_i nice, then M_σ is (θ, σ)-saturated over M_0. We show existence (and uniqueness). We connect this to a more usual saturation and prove that (θ, σ)-saturation implies niceness (in 4.10).

5. The amalgamation property for $\mathcal{K}_{<\lambda}$. After preliminaries we prove that for $\theta \leq \lambda$ (and $\theta \geq |\mathcal{F}| + \kappa$ of course) every member of K_θ can be extended to one with many nice submodels; this is done by induction on θ using the niceness of (θ_1, σ_1)-saturated models. Lastly, we conclude that every $M \in K_{<\lambda}$ is nice hence $\mathcal{K}_{<\lambda}$ has the amalgamation property.

0. Introduction. The main result of this paper is a proof of the following theorem:

THEOREM. *Suppose that T is a theory in a fragment of $L_{\kappa\omega}$, where κ is a measurable cardinal. If T is categorical in the cardinal $\lambda > \kappa + |T|$, then $\mathcal{K}_{<\lambda}$, the class of models of T of power strictly less than λ (but $\geq \kappa + |T|$), has the amalgamation property* (see Definition 1.3).

The interest of this theorem stems in part from its connection with the study of categoricity spectra. For a theory T in a logic $\underline{\mathbf{L}}$ let us define $\mathrm{Cat}(T)$, the *categoricity spectrum* of T, to be the collection of those cardinals λ in which T is categorical. In the 1950's Łoś conjectured that if T is a countable theory in first-order logic, then $\mathrm{Cat}(T)$ contains every uncountable cardinal or no uncountable cardinal. This conjecture, based on the example of algebraically closed fields of fixed characteristic, was verified by Morley [M], who proved that if a countable first-order theory is categorical in some uncountable cardinal, then it is categorical in every uncountable cardinal. Following advances made by Rowbottom [Ro], Ressayre [Re] and Shelah [Sh1], Shelah [Sh31] proved the Łoś conjecture for uncountable first-order theories: if T is a first-order theory categorical in some cardinal $\lambda > |T| + \aleph_0$, then T is categorical in every cardinal $\lambda > |T| + \aleph_0$.

It is natural to ask whether analogous results hold for theories in logics other than first-order logic. Perhaps the best-known extensions of first-order logic are the infinitary logics $L_{\kappa\lambda}$. As regards theories in $L_{\omega_1\omega}$, Shelah [Sh87] continuing work begun in [Sh48] introduced the concept of excellent classes: these have models in all cardinalities, have the amalgamation property and satisfy the Łoś conjecture. In particular, if φ is an excellent sentence of $L_{\omega_1\omega}$, then the Łoś conjecture holds for φ. Furthermore, under some set-theoretic assumptions (weaker than the Generalized Continuum Hypothesis), if φ is a sentence in $L_{\omega_1\omega}$ which is categorical in $\aleph_n$ for every natural number n (or even just if φ is a sentence in $L_{\omega_1\omega}$ with at least one uncountable model not having too many models in each $\aleph_n$), then φ is excellent.

Now [Sh300], [Sh-h] try to develop classification theory in some non-elementary classes. We cannot expect much for $L_{\kappa\lambda}$ for $\lambda > \aleph_0$. Shelah conjectured that if φ is a sentence in $L_{\omega_1\omega}$ categorical in some $\lambda \geq \beth_{\omega_1}$, then φ is categorical in every $\lambda \geq \beth_{\omega_1}$. (Recall that the Hanf number of $L_{\omega_1\omega}$ is $\beth_{\omega_1}$, so if ψ is a sentence in $L_{\omega_1\omega}$ and ψ has a model of power $\lambda \geq \beth_{\omega_1}$, then ψ has a model in every power $\lambda \geq \beth_{\omega_1}$. See [K].) There were some who asked why so tardy a beginning. Recent work of Hart and Shelah [HaSh323] showed that for every natural number k greater than 1 there is a sentence ψ_k in $L_{\omega_1\omega}$ which is categorical in the cardinals $\aleph_0, \ldots, \aleph_{k-1}$, but which has many models of power λ for every cardinal $\lambda \geq 2^{\aleph_{k-1}}$. The general conjecture for $L_{\omega_1\omega}$ remains open nevertheless.

As regards theories in $L_{\kappa\omega}$, progress has been recorded under the assumption that κ is a strongly compact cardinal. Under this assumption Shelah and Makkai [MaSh285] have established the following results for a λ-categorical theory T in a fragment $\mathcal{F}$ of $L_{\kappa\omega}$: (1) if λ is a successor cardinal and $\lambda > ((\kappa')^{\kappa})^+$, where $\kappa' = \max(\kappa, |\mathcal{F}|)$, then T is categorical in every cardinal greater than or equal to $\min(\lambda, \beth_{(2^{\kappa'})^+})$, (2) if $\lambda > \beth_{\kappa+1}(\kappa')$, then T is categorical in every cardinal of the form $\beth_\delta$ with δ divisible by $(2^{\kappa'})^+$ (i.e. for some ordinal $\alpha > 0$, $\delta = (2^{\kappa'})^+ \cdot \alpha$ (ordinal multiplication)).

In proving theorems of this kind, one has recourse to the amalgamation property which makes possible the construction of analogues of saturated models. In turn, these are of major importance in categoricity arguments. The amalgamation property holds for theories in first-order logic [CK] and in $L_{\kappa\kappa}$ when κ is a strongly compact cardinal (see [MaSh285]: although $\prec_{L_{\kappa\kappa}}$ fails the Tarski–Vaught property for unions of chains of length κ (whereas $\prec_{L_{\kappa\omega}}$ has it), under a categoricity assumption it can be shown that $\prec_{L_{\kappa\omega}}$ and $\prec_{L_{\kappa\kappa}}$ coincide). However, it is not known in general for theories in $L_{\kappa\omega}$ or $L_{\kappa\kappa}$ when one weakens the assumption on κ, in particular when κ is just a measurable cardinal. Nevertheless, categoricity does imply the existence of reasonably saturated models in an appropriate sense, and it is possible to begin classification theory. This is why the main theorem of the present paper is of relevance regarding the categoricity spectra of theories in $L_{\kappa\omega}$ when κ is measurable.

A sequel to this paper under preparation tries to provide a characterization of $\mathrm{Cat}(T)$ at least parallel to that in [MaSh285] and we hope to deal with the corresponding classification theory later. This division of labor both respects historical precedent and is suggested by the increasing complexity of the material. Another sequel deals with abstract elementary classes (in the sense of [Sh88]) (see [Sh472], [Sh394] respectively). On later work see [Sh576], [Sh600].

The paper is divided into five sections. Section 1 is preliminary and notational. In Section 2 it is shown that if T is categorical in the regular cardinal $\lambda > \kappa + |T|$, then $\mathcal{K}_{<\lambda}$ has the amalgamation property. Section 3 deals with weakly universal models, Section 4 with (θ, σ)-saturated and $\bar{\theta}$-saturated models. In Section 5 the amalgamation property for $\mathcal{K}_{<\lambda}$ is established.

All the results in this paper (other than those explicitly credited) are due to Saharon Shelah.

1. Preliminaries. To start things off in this section, let us fix notation, provide basic definitions and well-known facts, and formulate our working assumptions.

The working assumptions in force throughout the paper are these.

Assumption 1. The cardinal κ is an uncountable measurable cardinal, and so there is a κ-complete nonprincipal ultrafilter on κ.

Assumption 2. The theory T is a theory in the infinitary logic $L_{\kappa\omega}$.

From these assumptions follow certain facts, of which the most important are these.

Fact 1. *For each model M of T, κ-complete ultrafilter D over I and suitable set G of equivalence relations on $I \times I$* (see 1.7.4) *the limit ultrapower* $\mathrm{Op}(M) = \mathrm{Op}(M, I, D, G)$ *is a model of T.*

Fact 2. *For each linear order $I = (I, \leq)$ there exists a generalized Ehrenfeucht–Mostowski model* $\mathrm{EM}(I)$ *of T.*

The remainder of this section provides more detailed explanations and references.

Relevant set-theoretic and model-theoretic information on measurable cardinals can be found in [J], [CK] and [D]. L denotes a language, i.e. a set of finitary relation and function symbols, including equality. $|L|$ is the cardinality of the language L. For a cardinal $\lambda \leq \kappa$, $L_{\kappa\lambda}$ is the smallest set of (infinitary) formulas in the language L which contains all first-order formulas and which is closed under (1) the formation of conjunctions (disjunctions) of any set of formulas of power less than κ, provided that the set of free variables in the conjunctions (disjunctions) has power less than λ; (2) the formation of $\forall \overline{x} \varphi$, $\exists \overline{x} \varphi$, where $\overline{x} = \langle x_\alpha : \alpha < \lambda' \rangle$ is a sequence of variables of length $\lambda' < \lambda$. ([K] and [D] are comprehensive references for $L_{\omega_1\omega}$ and

$L_{\kappa\lambda}$ respectively.) Whenever we use the notation $\varphi(\overline{x})$ to denote a formula in $L_{\kappa\lambda}$, we mean that $\overline{x}$ is a sequence $\langle x_\alpha : \alpha < \lambda' \rangle$ of variables of length $\lambda' < \lambda$, and all the free variables of $\varphi(\overline{x})$ are among $\overline{x} = \langle x_\alpha : \alpha < \lambda' \rangle$. So if $\varphi(\overline{x})$ is a formula in $L_{\kappa\omega}$, then $\overline{x}$ is a finite sequence of variables.

$\mathcal{F}$ denotes a fragment of $L_{\kappa\omega}$, i.e. a set of formulas of $L_{\kappa\omega}$ which contains all atomic formulas of L, and which is closed under negations, finite conjunctions (finite disjunctions), and the formation of subformulas. An $\mathcal{F}$-formula is just an element of $\mathcal{F}$.

T is a theory in $L_{\kappa\omega}$, so there is a fragment $\mathcal{F}$ of $L_{\kappa\omega}$ such that $T \subset \mathcal{F}$ and $|\mathcal{F}| < |T|^+ + \kappa$.

Models of T (invariably referred to as models) are L-structures which satisfy the sentences of T. They are generally denoted $M, N, \ldots$; $|M|$ is the universe of the L-structure M; $\|M\|$ is the cardinality of $|M|$. For a set A, $|A|$ is the cardinality of A. ${}^{<\omega}A$ is the set of finite sequences in A and for $\overline{a} = \langle a_1, \ldots, a_n \rangle \in {}^{<\omega}A$, $\lg(\overline{a}) = n$ is the length of $\overline{a}$. Similarly, if $\overline{a} = \langle a_\zeta : \zeta < \delta \rangle$, we write $\lg(\overline{a}) = \delta$, where δ is an ordinal. For an element R of L, $\mathrm{val}(M, R)$, or R^M, is the interpretation of R in the L-structure M.

We ignore models of power less than κ. K is the class of all models of T;

$$K_\lambda = \{M \in K : \|M\| = \lambda\},$$
$$K_{<\lambda} = \bigcup_{\mu<\lambda} K_\mu, \quad K_{\le\lambda} = \bigcup_{\mu\le\lambda} K_\mu, \quad K_{[\mu,\lambda)} = \bigcup_{\mu\le\chi<\lambda} K_\chi.$$

We write $f : M \to_{\mathcal{F}} N$ (abbreviated $f : M \to N$) to mean that f is an *$\mathcal{F}$-elementary embedding* (briefly, an *embedding*) of M into N, i.e. f is a function with domain $|M|$ into $|N|$ such that for every $\mathcal{F}$-formula $\varphi(\overline{x})$, and $\overline{a} \in {}^{<\omega}|M|$ with $\lg(\overline{a}) = \lg(\overline{x})$, $M \vDash \varphi[\overline{a}]$ iff $N \vDash \varphi[f(\overline{a})]$, where if $\overline{a} = \langle a_i : i < n \rangle$, then $f(\overline{a}) := \langle f(a_i) : i < n \rangle$. In the special case where the embedding f is a set-inclusion (so that $|M| \subset |N|$), we write $M \prec_{\mathcal{F}} N$ (briefly $M \prec N$) instead of $f : M \to_{\mathcal{F}} N$ and we say that M is an *$\mathcal{F}$-elementary submodel* of N, or N is an *$\mathcal{F}$-elementary extension* of M.

$(I, \le_I), (J, \le_J), \ldots$ are partial orders; we will not bother to subscript the order relation unless really necessary; we write I for $(I, \le)$. $(I, \le)$ is *directed* iff for every i_1 and i_2 in I, there is $i \in I$ such that $i_1 \le i$ and $i_2 \le i$. $(I, <)^*$ is the (reverse) linear order $(I^*, <^*)$, where $I^* = I$ and $s <^* t$ iff $t < s$.

A set $\langle M_i : i \in I \rangle$ of models indexed by I is a *$\prec_{\mathcal{F}}$-directed system* iff $(I, \le)$ is a directed partial order and for $i \le j$ in I, $M_i \prec_{\mathcal{F}} M_j$. The union $\bigcup_{i\in I} M_i$ of a $\prec_{\mathcal{F}}$-directed system $\langle M_i : i \in I \rangle$ of L-structures is an L-structure. In fact, more is true.

Fact 1.1 (Tarski–Vaught property). (1) *The union of a $\prec_{\mathcal{F}}$-directed system $\langle M_i : i \in I \rangle$ of models of T is a model of T, and for every $j \in I$, $M_j \prec_{\mathcal{F}} \bigcup_{i\in I} M_i$.*

(2) *If M is a fixed model of T such that for every $i \in I$ there is $f_i : M_i \to_{\mathcal{F}} M$ and for all $i \le j$ in I, $f_i \subseteq f_j$, then $\bigcup_{i\in I} f_i : \bigcup_{i\in I} M_i \to_{\mathcal{F}} M$. In particular, if $M_i \prec_{\mathcal{F}} M$ for every $i \in I$, then $\bigcup_{i\in I} M_i \prec_{\mathcal{F}} M$.*

Let α be an ordinal. A $\prec_{\mathcal{F}}$-*chain* of models of length α is a sequence $\langle M_\beta : \beta < \alpha\rangle$ of models such that if $\beta < \gamma < \alpha$, then $M_\beta \prec_{\mathcal{F}} M_\gamma$. The chain is *continuous* if for every limit ordinal $\beta < \alpha$, $M_\beta = \bigcup_{\gamma<\beta} M_\gamma$.

Fact 1.2 (Downward Loewenheim–Skolem property). *Suppose that M is a model of T, $A \subset |M|$ and* $\max(\kappa + |T|, |A|) \le \lambda \le \|M\|$. *Then there is a model N such that $A \subset |N|$, $\|N\| = \lambda$ and $N \prec_{\mathcal{F}} M$.*

Finally, $\lambda > \kappa + |T|$ usually denotes a power in which T is categorical.

Now we turn from the rather standard model-theoretic background to the more specific concepts which are central in our investigation.

Definition 1.3. (1) Suppose that $<$ is a binary relation on a class K of models. $\mathcal{K} = \langle K, <\rangle$ has the *amalgamation property* (AP) iff for every $M, M_1, M_2 \in K$, if f_i is an isomorphism from M onto $\mathrm{rng}(f_i)$ and $\mathrm{rng}(f_i) < M_i$ for $i = 1, 2$, then there exist $N \in K$ and isomorphisms g_i from M_i onto $\mathrm{rng}(g_i)$ for $i = 1, 2$ such that $\mathrm{rng}(g_i) < N$ and $g_1 f_1 = g_2 f_2$. The model N is called an *amalgam* of M_1, M_2 over M with respect to f_1, f_2.

(2) An L-structure M is an *amalgamation base* (a.b.) for $\mathcal{K} = \langle K, <\rangle$ iff $M \in K$ and whenever for $i = 1, 2$, $M_i \in K$ and f_i is an isomorphism from M onto $\mathrm{rng}(f_i)$ with $\mathrm{rng}(f_i) < M_i$, then there exist $N \in K$ and isomorphisms g_i $(i = 1, 2)$ from M_i onto $\mathrm{rng}(g_i)$ such that $\mathrm{rng}(g_i) < N$ and $g_1 f_1 = g_2 f_2$.

So $\mathcal{K} = \langle K, <\rangle$ has AP iff every model in K is an a.b. for $\mathcal{K}$.

Example 1.3A. Suppose that T is a theory in first-order logic having an infinite model. Define, for M, N in the class $K_{\le|T|+\aleph_0}$ of models of T of power at most $|T| + \aleph_0$, $M < N$ iff the identity is an embedding of M into an elementary submodel of N. Then $\mathcal{K}_{\le|T|+\aleph_0} = \langle K_{\le|T|+\aleph_0}, <\rangle$ has AP (see [CK]).

Example 1.3B. Suppose that T is a theory in $L_{\kappa\omega}$ and $\mathcal{F}$ is a fragment of $L_{\kappa\omega}$ containing T with $|\mathcal{F}| < |T|^+ + \kappa$. Let $<$ be the binary relation $\prec_{\mathcal{F}}$ defined on the class K of all models of T. $M \in K$ is an a.b. for $\mathcal{K}$ iff whenever for $i = 1, 2$, $M_i \in K$ and f_i is an $\prec_{\mathcal{F}}$-elementary embedding of M into M_i, there exist $N \in K$ and $\mathcal{F}$-elementary embeddings g_i $(i = 1, 2)$ of M_i into N such that $g_1 f_1 = g_2 f_2$.

Definition 1.4. Suppose that $<$ is a binary relation on a class K of models. Let μ be a cardinal. $M \in K_{\le\mu}$ is a μ-*counter amalgamation basis* (μ-c.a.b.) of $\mathcal{K} = \langle K, <\rangle$ iff there are $M_1, M_2 \in K_{\le\mu}$ and isomorphisms f_i from M into M_i such that

(A) $\mathrm{rng}(f_i) < M_i$ $(i = 1, 2)$,

(B) there is no amalgam $N \in K_{\le\mu}$ of M_1, M_2 over M with respect to f_1, f_2.

Observation 1.5. Suppose that $T, \mathcal{F}$ and $<$ are as in 1.3B and $\kappa + |T| \le \mu < \lambda$. Note that if there is an amalgam N' of M_1, M_2 over M (for M_1, M_2, M in $K_{\le\mu}$), then by 1.2 there is an amalgam $N \in K_{\le\mu}$ of M_1, M_2 over M.

Indiscernibles and Ehrenfeucht–Mostowski structures. The basic results on generalized Ehrenfeucht–Mostowski models can be found in [Sh-a] or [Sh-c, Ch. VII]. We recall here some notation. Let $\mathbf{I}$ be a class of models which we call the *index models*. Denote the members of $\mathbf{I}$ by $I, J, \ldots$ For $I \in \mathbf{I}$ we say that $\langle \overline{a}_s : s \in I\rangle$ is *indiscernible* in M iff for every $\overline{s}, \overline{t} \in {}^{<\omega}I$ realizing the same atomic type in I, $\overline{a}_{\overline{s}}$ and $\overline{a}_{\overline{t}}$ realize the same type in M (where $\overline{a}_{\langle s_0,\ldots,s_n\rangle} = \overline{a}_{s_0}{}^\wedge \ldots {}^\wedge\overline{a}_{s_n}$). If $L \subseteq L'$ are languages and Φ is a function with domain including $\{\mathrm{tp}_{\mathrm{at}}(\overline{s}, \emptyset, I) : \overline{s} \in {}^{<\omega}I\}$ and $I \in \mathbf{I}$, we let $\mathrm{EM}'(I, \Phi)$ be an L'-model generated by $\bigcup_{s\in I} \overline{a}_s$ such that $\mathrm{tp}_{\mathrm{at}}(\overline{a}_{\overline{s}}, \emptyset, M) = \Phi(\mathrm{tp}_{\mathrm{at}}(\overline{s}, \emptyset, I))$. We say that Φ is *proper* for $\mathbf{I}$ if for every $I \in \mathbf{I}$, $\mathrm{EM}'(I, \Phi)$ is well defined.

Let $\mathrm{EM}(I, \Phi)$ be the L-reduct of $\mathrm{EM}'(I, \Phi)$. For the purposes of this paper we will let $\mathbf{I}$ be the class $\mathbf{LO}$ of linear orders and Φ will be proper for $\mathbf{LO}$. For $I \in \mathbf{LO}$ we abbreviate $\mathrm{EM}'(I, \Phi)$ by $\mathrm{EM}'(I)$ and $\mathrm{EM}(I, \Phi)$ by $\mathrm{EM}(I)$.

Claim 1.6A. *For each linear order $I = (I, \le)$ there exists a generalized Ehrenfeucht–Mostowski model* $\mathrm{EM}(I)$ *of* T (see Nadel [N] and Dickmann [D1] or [Sh-c, VII, §5]; there are "large" models by using limit ultrapowers, see 1.12).

Let $\mathcal{F}$ be a fragment of $L_{\kappa\omega}$. Recall that a theory $T \subset \mathcal{F}$ is called a *universal theory* in $L_{\kappa\omega}$ iff the axioms of T are sentences of the form $\forall\overline{x}\varphi(\overline{x})$, where $\varphi(\overline{x})$ is a quantifier-free formula in $L_{\kappa\omega}$.

Definition and Proposition 1.6. *Suppose that T is a theory such that $T \subset \mathcal{F}$, where $\mathcal{F}$ is a fragment of $L_{\kappa\omega}$. There are a* (*canonically constructed*) *finitary language L_{sk} and a universal theory T_{sk} in $L_{\kappa\omega}$ such that*:

(0) *$L \subset L_{\mathrm{sk}}$ and $|L_{\mathrm{sk}}| \le |\mathcal{F}| + \aleph_0$*;

(1) *the L-reduct of any L_{sk}-model of T_{sk} is a model of T*;

(2) *whenever N_{sk} is an L_{sk}-model of T_{sk} and M_{sk} is a substructure of N_{sk}, then $M_{\mathrm{sk}}\restriction L \prec_{\mathcal{F}} N_{\mathrm{sk}}\restriction L$*;

(3) *any L-model of T can be expanded to an L_{sk}-model of T_{sk}*;

(4) *if $M \prec_{\mathcal{F}} N$, then there are L_{sk}-expansions $M_{\mathrm{sk}}, N_{\mathrm{sk}}$ of M, N respectively such that M_{sk} is a substructure of N_{sk} and N_{sk} is a model of T_{sk}*;

(5) *to any $\mathcal{F}$-formula $\varphi(\overline{x})$, there corresponds a quantifier-free formula $\varphi^{\mathrm{qf}}(\overline{x})$ of $(L_{\mathrm{sk}})_{\kappa\omega}$ such that*

$$T_{\mathrm{sk}} \models \forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \varphi^{\mathrm{qf}}(\overline{x})).$$

Limit ultrapowers, iterated ultrapowers and nice extensions. An important technique we shall use in studying the categoricity spectrum of a theory in $L_{\kappa\omega}$ is the limit ultrapower. It is convenient to record here the well-known definitions and properties of limit and iterated ultrapowers (see Chang and Keisler [CK] and Hodges and Shelah [HoSh109]) and then to examine nice extensions of models.

Definition 1.7.1. Suppose that M is an L-structure, I is a nonempty set, D is an ultrafilter on I, and G is a filter on $I \times I$. For each $g \in {}^I|M|$, let $\mathrm{eq}(g) = \{\langle i,j\rangle \in I \times I : g(i) = g(j)\}$ and $g/D = \{f \in {}^I|M| : g = f \bmod D\}$, where $g = f \bmod D$ iff $\{i \in I : g(i) = f(i)\} \in D$. Let $\prod_{D/G}|M| = \{g/D : g \in {}^I|M| \text{ and } \mathrm{eq}(g) \in G\}$. Note that $\prod_{D/G}|M|$ is a nonempty subset of $\prod_D |M| = \{g/D : g \in {}^I|M|\}$ and is closed under the constants and functions of the ultrapower $\prod_D M$ of M modulo D. The limit ultrapower $\prod_{D/G} M$ of the L-structure M (with respect to I, D, G) is the substructure of $\prod_D M$ whose universe is the set $\prod_{D/G}|M|$. The canonical map d from M into $\prod_{D/G} M$ is defined by $d(a) = \langle a_i : i \in I\rangle/D$, where $a_i = a$ for every $i \in I$. Note that the limit ultrapower $\prod_{D/G} M$ depends only on the equivalence relations which are in G, i.e. if $\mathbf{E}$ is the set of all equivalence relations on I and $G \cap \mathbf{E} = G' \cap \mathbf{E}$, where G' is a filter on $I \times I$, then $\prod_{D/G} M = \prod_{D/G'} M$.

Definition 1.7.2. Let M be an L-structure, $\langle Y, <\rangle$ a linear order and, for each $y \in Y$, let D_y be an ultrafilter on a nonempty set I_y. Write $H = \prod_{y\in Y} I_y$. Let $\prod_{y\in Y} D_y$ be the set of $s \subset H$ for which there are $y_1 < \ldots < y_n$ in Y such that

(1) for all $i, j \in H$, if $i\restriction\{y_1,\ldots,y_n\} = j\restriction\{y_1,\ldots,y_n\}$ then $i \in s$ iff $j \in s$;
(2) $\{\langle i(y_1),\ldots,i(y_n)\rangle : i \in s\} \in D_{y_1} \times \ldots \times D_{y_n}$.

Write $E = \prod_{y\in Y} D_y$. The *iterated ultrapower* $\prod_E |M|$ *of the set* $|M|$ with respect to $\langle D_y : y \in Y\rangle$ is the set $\{f/E : f : H \to |M|$ and for some finite $Z_f \subset Y$ for all $i, j \in H$, if $i\restriction Z_f = j\restriction Z_f$, then $f(i) = f(j)\}$. The *iterated ultrapower* $\prod_E M$ *of the L-structure* M with respect to $\langle D_y : y \in Y\rangle$ is the L-structure whose universe is the set $\prod_E |M|$; for each n-ary predicate symbol R of L, $R^{\Pi_E M}(f_1/E,\ldots,f_n/E)$ iff $\{i \in H : R^M(f_1(i),\ldots,f_n(i))\} \in E$; for each n-ary function symbol F of L, $F^{\Pi_E M}(f_1/E,\ldots,f_n/E) = \langle F^M(f_1(i),\ldots,f_n(i)) : i \in H\rangle/E$. The canonical map $d : M \to \prod_E M$ is defined as usual by $d(a) = \langle a : i \in H\rangle/E$.

Remark 1.7.3. (1) Every ultrapower is a limit ultrapower: take $G = P(I \times I)$ and note that $\prod_D M = \prod_{D/G} M$.

(2) Every iterated ultrapower is a limit ultrapower. [Why? let the iterated ultrapower be defined by $\langle Y, <\rangle$ and $\langle (I_y, D_y) : y \in Y\rangle$ (see Definition 1.7.2). For $Z \in [Y]^{<\omega}$, let $A_Z = \{(i,j) \in H \times H : i{\restriction}Z = j{\restriction}Z\}$. Note that $\{A_Z : Z \in [Y]^{<\omega}\}$ has the finite intersection property and hence can be extended to a filter G on $H \times H$. Now for any model M we have $\prod_E M \cong \prod_{D/G} M$ for every filter D over H extending E under the map $f/E \to f/D$.]

Definition 1.7.4. Suppose that M is an L-structure, D is an ultrafilter on a nonempty set I, and G is a *suitable* set of equivalence relations on I, i.e.

(i) if $e \in G$ and e' is an equivalence relation on I coarser than e, then $e' \in G$;

(ii) G is closed under finite intersections;

(iii) if $e \in G$, then $D/e = \{A \subset I/e : \bigcup_{x\in A} x \in D\}$ is a κ-complete ultrafilter on I/e which, for simplicity, has cardinality κ; we state this as "(I, D, G) is κ-complete".

(We may say (I, D, G) is suitable.)

Then $\mathrm{Op}(M, I, D, G)$ is the limit ultrapower $\prod_{D/\widehat{G}} M$, where $\widehat{G}$ is the filter on $I \times I$ generated by G. One abbreviates $\mathrm{Op}(M, I, D, G)$ by $\mathrm{Op}(M)$, and one writes f_{Op} for the canonical map $d : M \to \mathrm{Op}(M)$.

Note that

Observation/Convention 1.7.4A. 1) For any L-structure N, f_{Op} is an $L_{\kappa\omega}$-elementary embedding of N into $\mathrm{Op}(N)$ and in particular $f_{\mathrm{Op}} : N \to_{\mathcal{F}} \mathrm{Op}(N)$.

2) Since f_{Op} is canonical, one very often identifies N with the L-structure $\mathrm{rng}(f_{\mathrm{Op}})$ which is an $\mathcal{F}$-elementary substructure of $\mathrm{Op}(N)$, and one writes $N \prec_{\mathcal{F}} \mathrm{Op}(N)$. In particular, for any model M of $T \subset \mathcal{F}$ and Op, $f_{\mathrm{Op}} : M \to_{\mathcal{F}} \mathrm{Op}(M)$ (briefly written, $M \prec_{\mathcal{F}} \mathrm{Op}(M)$ and sometimes even $M \prec \mathrm{Op}(M)$) so that $\mathrm{Op}(M)$ is a model of T too.

3) Remark that if D is a κ-complete ultrafilter on I and G is a filter on $I \times I$, then $\mathrm{Op}(M, I, D, G)$ is well defined.

4) "Suitable limit ultrapower" means one using a suitable triple.

More information on limit and iterated ultrapowers can be found in [CK] and [HoSh109].

Observation 1.7.5. Suppose that M is a model of a theory $T \subset \mathcal{F}$, where $\mathcal{F}$ is a fragment of $L_{\kappa\omega}$. Given θ-complete ultrafilters D_1 on I_1, D_2 on I_2 and suitable filters G_1 on $I_1 \times I_1, G_2$ on $I_2 \times I_2$ respectively, there exist a θ-complete ultrafilter D on a set I and a suitable filter G on $I \times I$ such that

$$\mathrm{Op}(M, I, D, G) = \mathrm{Op}(\mathrm{Op}(M, I_1, D_1, G_1), I_2, D_2, G_2)$$

and (D, G, I) is κ-complete. Also iterated ultrapower (along any linear order) with each iterand being an ultrapower by a κ-complete ultrafilter, gives a suitable triple (in fact, even iteration of suitable limit ultrapowers is a suitable ultrapower).

DEFINITION 1.8. Suppose that K is a class of L-structures and $<$ is a binary relation on K. For $M, N \in K$, write $f : M \underset{\text{nice},<}{\to} N$ to mean

(1) f is an isomorphism from M into N and $\text{rng}(f) < N$;

(2) there are a set I, an ultrafilter D on I, a suitable set G of equivalence relations on I (so Definition 1.7.4(i)–(iii) holds), and an isomorphism g from N into $\text{Op}(M, I, D, G)$ such that $\text{rng}(g) < \text{Op}(M, I, D, G)$ and $gf = f_{\text{Op}}$, where f_{Op} is the canonical embedding of M into $\text{Op}(M, I, D, G)$. f is called a $<$-*nice embedding* of M into N. Of course one writes $f : M \underset{\text{nice}}{\to} N$ and says that f is a *nice embedding* of M into N when $<$ is clear from the context.

EXAMPLE 1.9.1. Consider T, $\mathcal{F}$ and $\mathcal{K} = \langle K, <\rangle$ as set up in 1.3B. In this case $f : M \underset{\text{nice},<}{\to} N$ holds iff $f : M \to_{\mathcal{F}} N$ and for some suitable $\langle I, D, G\rangle$ and some $g : N \to_{\mathcal{F}} \text{Op}(M, I, D, G)$, $gf = f_{\text{Op}}$.

Abusing notation one writes $M \underset{\text{nice}}{\to} N$ to mean that there are f, g and Op such that $f : M \underset{\text{nice},<}{\to} N$ using g and Op. *If not said otherwise*, $<$ is $<_{\mathcal{F}}$. We may also write $M \underset{\text{nice}}{\preceq} N$, and for linear orders we use $I \underset{\text{nice}}{\subseteq} J$.

EXAMPLE 1.9.2. Let **LO** be the class of linear orders and let $(I, \leq_I) < (J, \leq_J)$ mean that $(I, \leq_I) \subseteq (J, \leq_J)$, i.e. $(I, \leq_I)$ is a suborder of $(J, \leq_J)$. If $f : (I, \leq_I) \underset{\text{nice},<}{\to} (J, \leq_J)$, then for some Op, identifying isomorphic orders, one has $(I, \leq_I) \subseteq (J, \leq_J) \subseteq \text{Op}(I, \leq_I)$.

OBSERVATION 1.10. *Suppose that* $T, \mathcal{F}$ *and* $\mathcal{K}$ *are as in* 1.3B *and* 1.9.1. *Suppose further that* $M \underset{\text{nice}}{<} N$ *and* $M \preceq_{\mathcal{F}} M' \preceq_{\mathcal{F}} N$ *for* $M, M', N \in K$. *Then* $M \underset{\text{nice}}{<} M'$.

Proof. For some f, g and Op, $f : M \to_{\mathcal{F}} N$, $g : N \to_{\mathcal{F}} \text{Op}(M)$ and $gf = f_{\text{Op}}$. Now $g : M' \to_{\mathcal{F}} \text{Op}(M)$ (since $M' \preceq_{\mathcal{F}} N$) and $gf = f_{\text{Op}}$ so that $M \underset{\text{nice}}{<} M'$.

OBSERVATION 1.11. *Suppose that* $\langle M_i : i \leq \delta\rangle$ *is a continuous increasing chain and for each* $i < \delta$, $M_i \underset{\text{nice}}{<} M_{i+1}$. *Then for every* $i < \delta$, $M_i \underset{\text{nice}}{<} M_\delta$.

Proof (like the proof of 1.7.3(2)). For each $i < \delta$, there are (I_i, D_i, G_i) as in Definition 1.7.4 which witness $M_i \underset{\text{nice}}{\leq} M_{i+1}$. Let $I := \prod_{i<\delta} I_i$ and $G := \{e : e \subseteq I \times I$ and for some $n < \omega$ and $\alpha_1 < \ldots < \alpha_n < \delta$ and $e_1 \in G_{\alpha_1}, \ldots, e_n \in G_{\alpha_n}$, we have: for every $x, y \in I$ such that $(x(\alpha_l), y(\alpha_l)) \in e_l$

for $l = 1, \ldots, n$, we have $(x, y) \in e\}$. D will be any ultrafilter on I such that: if $n < \omega$ and $\alpha_1 < \ldots < \alpha_n < \delta$, $e_1 \in G_{\alpha_1}, \ldots, e_n \in G_{\alpha_n}$, e_l is an equivalence relation on I_{α_l} for $l = 1, \ldots, n$ and $A \in (D_{\alpha_1}/e_{\alpha_1}) \times \ldots \times (D_{\alpha_n}/e_{\alpha_n})$, then the set $\{x \in I : \langle x(\alpha_1)/e_{\alpha_1}, \ldots, x(\alpha_n)/e_{\alpha_n}\rangle \in A\}$ belongs to D. We leave the rest to the reader.

CLAIM 1.12. *For every model M and $\lambda \geq \kappa + |\mathcal{F}| + \|M\|$ there is N such that $M \preceq_{\mathcal{F}}^{\text{nice}} N$, $M \neq N$ and $\|N\| = \lambda$.*

Proof. As κ is measurable.

2. The amalgamation property for regular categoricity. The main aim in this section is to show that if T is categorical in the regular cardinal $\lambda > \kappa + |T|$, then $\mathcal{K}_{<\lambda} = \langle K_{<\lambda}, \preceq_{\mathcal{F}}\rangle$ has the amalgamation property (AP) (Definition 1.3(1)). Categoricity is not presumed if not required.

LEMMA 2.1. *Suppose that $\kappa + |T| \leq \mu \leq \lambda$, $M, M_1, M_2 \in K_{\leq\mu}$, $f_1 : M \underset{\text{nice}}{\rightarrow} M_1$, $f_2 : M \rightarrow_{\mathcal{F}} M_2$. Then there is an amalgam $N \in K_{\leq\mu}$ of M_1, M_2 over M with respect to f_1, f_2. Moreover, there are $g_l : M_l \rightarrow_{\mathcal{F}} N$ for $l = 1, 2$ such that $g_1 f_1 = g_2 f_2$ and* $\text{rng}(g_2) \cap \text{rng}(g_1) = \text{rng}(g_1 f_1)$.

Proof. There are g and Op such that $g : M_1 \rightarrow_{\mathcal{F}} \text{Op}(M)$ and $g f_1 = f_{\text{Op}}$. Then f_2 induces an $\mathcal{F}$-elementary embedding f_2^* of $\text{Op}(M)$ into $\text{Op}(M_2)$ such that $f_2^* f_{\text{Op}} = f_{\text{Op}} f_2$. Let $g_1 = f_2^* g$ and $g_2 = f_{\text{Op}} \restriction M_2$. By 1.2 one finds $N \in K_{\leq\mu}$ such that $\text{rng}(g_1) \cup \text{rng}(g_2) \subset N \prec_{\mathcal{F}} \text{Op}(M_2)$. Now N is an amalgam of M_1, M_2 over M with respect to f_1, f_2 since $g_1 f_1 = f_2^* g f_1 = f_2^* f_{\text{Op}} = f_{\text{Op}} f_2 = g_2 f_2$. The last phrase in the lemma is easy by properties of Op.

LEMMA 2.2. *Suppose that $M \in K_{\leq\mu}$ is a μ-c.a.b. and $\kappa + |T| \leq \mu < \lambda$. Then $N \in K_{<\lambda}$ is a $\|N\|$-c.a.b. whenever $f : M \underset{\text{nice}}{\rightarrow} N$.*

Proof. Suppose that $g : N \rightarrow_{\mathcal{F}} \text{Op}(M)$ and $g f = f_{\text{Op}}$. Since M is a μ-c.a.b., for some $M_i \in K_{\leq\mu}$ and $f_i : M \rightarrow_{\mathcal{F}} M_i$ $(i = 1, 2)$ there is no amalgam of M_1, M_2 over M with respect to f_1, f_2. Let f_i^* be the $\mathcal{F}$-elementary embedding from $\text{Op}(M)$ into $\text{Op}(M_i)$ defined by f_i (note that $f_i^* f_{\text{Op}} = f_{\text{Op}} f_i$, $i = 1, 2$). Choose N_i of power $\|N\|$ such that $M_i \cup \text{rng}(f_i^* g) \subset N_i \prec_{\mathcal{F}} \text{Op}(M_i)$. Note that $f_i^* f : N \rightarrow_{\mathcal{F}} N_i$. It suffices to show that there is no amalgam of N_1, N_2 over N with respect to $f_1^* g, f_2^* g$.

Well, suppose that one could find an amalgam N^* and $h_i : N_i \rightarrow_{\mathcal{F}} N^*$, $i = 1, 2$, with $h_1(f_1^* g) = h_2(f_2^* g)$. Using 1.2 choose M^* such that $\|M^*\| \leq \mu$, $M^* \preceq_{\mathcal{F}} N^*$ and $\text{rng}(h_1 f_{\text{Op}} \restriction M_1) \cup \text{rng}(h_2 f_{\text{Op}} \restriction M_2) \subset |M^*|$. Set $g_i = h_i f_{\text{Op}} \restriction M_i$, for $i = 1, 2$, and note that

$$\begin{aligned} g_1 f_1 = h_1 f_{\text{Op}} f_1 = h_1 f_1^* f_{\text{Op}} = h_1 f_1^* g f = h_2 f_2^* g f \\ = h_2 f_2^* f_{\text{Op}} = h_2 f_{\text{Op}} f_2 = g_2 f_2. \end{aligned}$$

In other words, M^* is an amalgam of M_1, M_2 over M with respect to f_1, f_2—contradiction. It follows that N is a $\|N\|$-c.a.b.

COROLLARY 2.3. *Suppose that* $\kappa+|T| \le \mu < \lambda$. *If* $M \in K_\mu$ *is a* μ*-c.a.b., then there exists* $M^* \in K_\lambda$ *such that*

(∗) $M \preceq_{\mathcal{F}} M^*$ *and for every* $M' \in K_{<\lambda}$, *if* $M \preceq_{\mathcal{F}} M' \preceq_{\mathcal{F}} M^*$, *then* M' *is a* $\|M'\|$*-c.a.b.*

Proof. As $\|M\| \ge \kappa$, for some appropriate Op one has $\|\mathrm{Op}(M)\| \ge \lambda$, and by 1.2 one finds $M^* \in K_\lambda$ such that $M \subset M^* \preceq_{\mathcal{F}} \mathrm{Op}(M)$. Let us check that M^* works in (∗). Take $M' \in K_{<\lambda}$ with $M \preceq_{\mathcal{F}} M' \preceq_{\mathcal{F}} M^*$; so $M \underset{\text{nice}}{\preceq} M'$ since $M^* \preceq_{\mathcal{F}} \mathrm{Op}(M)$; hence by 2.2, M' is a $\|M'\|$-c.a.b.

THEOREM 2.4. *Suppose that* T *is* λ*-categorical and* $\lambda = \mathrm{cf}(\lambda) > \kappa + |T|$. *If* $\mathcal{K}_{<\lambda}$ *fails AP, then there is* $N^* \in K_\lambda$ *such that for some continuous increasing* $\prec_{\mathcal{F}}$*-chain* $\langle N_i \in K_{<\lambda} : i < \lambda\rangle$ *of models*,

(1) $N^* = \bigcup_{i<\lambda} N_i$;
(2) *for every* $i < \lambda$, $N_i \underset{\text{nice}}{\not\preceq} N_{i+1}$ (*and so* $N_i \underset{\text{nice}}{\not\preceq} N^*$).

Proof. $\mathcal{K}_{<\lambda}$ fails AP, so for some $\mu < \lambda$ and $M \in K_{\le\mu}$, M is a μ-c.a.b. By 2.2 and 1.12, without loss of generality, $M \in K_\mu$. Choose by induction a continuous strictly increasing $\prec_{\mathcal{F}}$-chain $\langle N_i \in K_{<\lambda} : i < \lambda\rangle$ as follows: $N_0 = M$; at a limit ordinal i, take the union; at a successor ordinal $i = j+1$, if there is $N \in K_{<\lambda}$ such that $N_j \preceq N$ and $N_j \underset{\text{nice}}{\not\preceq} N$, choose $N_i = N$, otherwise choose for N_i any nontrivial $\mathcal{F}$-elementary extension of N_j of power less than λ.

CLAIM. $(\exists j_0 < \lambda)(\forall j \in (j_0, \lambda))(N_j$ *is a* $\|N_j\|$*-c.a.b.*).

Proof. Suppose not. So one has a strictly increasing sequence $\langle j_i : i < \lambda\rangle$ such that for each $i < \lambda$, N_{j_i} is not a $\|N_{j_i}\|$-c.a.b. Let $N_* = \bigcup_{i<\lambda} N_{j_i}$. So $\|N_*\| = \lambda$. Applying 2.3 one can find $M^* \in K_\lambda$ such that whenever $M' \in K_{<\lambda}$ and $M \preceq M' \preceq M^*$, then M' is a $\|M'\|$-c.a.b.

Since T is λ-categorical, there is an isomorphism g of N_* onto M^*. Let $N = g^{-1}(M)$ and $M_i = g(N_i)$ for $i < \lambda$. Since $\|N\| = \mu < \mathrm{cf}(\lambda) = \lambda$, there is $i_0 < \lambda$ such that $N \subset N_{j_{i_0}}$.

In fact, $N_{j_{i_0}}$ is a $\|N_{j_{i_0}}\|$-c.a.b. [Otherwise, consider $M_{j_{i_0}}$. Since $M \preceq_{\mathcal{F}} M_{j_{i_0}} \preceq_{\mathcal{F}} M^*$ and $\|M_{j_{i_0}}\| < \lambda$, $M_{j_{i_0}}$ is a $\|M_{j_{i_0}}\|$-c.a.b., so there are $f_l : M_{j_{i_0}} \to_{\mathcal{F}} M'_l$ $(l = 1, 2)$ with no amalgam of M'_1, M'_2 over $M_{j_{i_0}}$ with respect to f_1, f_2. If $N_{j_{i_0}}$ is *not* a $\|N_{j_{i_0}}\|$-c.a.b., then one can find an amalgam $N^+ \in K_{\le\|N_{j_{i_0}}\|}$ of M'_1, M'_2 over $N_{j_{i_0}}$ with respect to $f_1 g, f_2 g$ such that $h_l : M'_l \to_{\mathcal{F}} N^+$ and $h_1(f_1 g) = h_2(f_2 g)$; so $h_1 f_1 = h_2 f_2$ and N^+ is thus an amalgam of M'_1, M'_2 over $M_{j_{i_0}}$ with respect to f_1, f_2, and $\|N^+\| \le \|N_{j_{i_0}}\| =$

$\|M_{j_{i_0}}\|$—contradiction.] This contradicts the choice of $N_{j_{i_0}}$. So the claim is correct.

It follows that for each $j \in (j_0, \lambda)$ there are N_j^1, N_j^2 in $K_{<\lambda}$ and $f_l : N_j \to_{\mathcal{F}} N_j^l$ such that no amalgam of N_j^1, N_j^2 over N_j with respect to f_1, f_2 exists. By 2.2 for some $l \in \{1,2\}, N_j \underset{\text{nice}}{\not\preceq} N_{j+1}^l$. So by the inductive choice of $\langle N_{j+1} : j < \lambda\rangle$, $(\forall j \in (j_0,\lambda))(N_j \underset{\text{nice}}{\not\preceq} N_{j+1})$. Taking $N^* = \bigcup_{j_0<j<\lambda} N_j$, one completes the proof. (Of course for $j_0 < j < \lambda$, $N_j \underset{\text{nice}}{\not\preceq} N^*$: if $N_j \preceq_{\mathcal{F}} N^* \preceq_{\mathcal{F}} \mathrm{Op}(N_j)$, then by 1.10, $N_j \underset{\text{nice}}{\preceq} N_{j+1}$—contradiction).

THEOREM 2.5. *Suppose that $(I,<_I),(J,<_J)$ are linear orders and I is a suborder of J. If $(I,<_I) \underset{\text{nice}}{\subseteq} (J,<_J)$, then* $\mathrm{EM}(I) \underset{\text{nice}}{\preceq} \mathrm{EM}(J)$.

P r o o f. Without loss of generality, for some cardinal μ, ultrafilter D on μ and suitable set G, a filter on $\mu\times\mu$, $(I,<_I) \subseteq (J,<_J) \subseteq \mathrm{Op}((I,<_I),\mu,D,G) = \mathrm{Op}(I,<)$, and $|\mathrm{Op}(I,<)| = \{f/D : f \in {}^{\mu}I, \mathrm{eq}(f) \in G\}$, where $\mathrm{eq}(f) = \{(i,j) \in \mu\times\mu : f(i) = f(j)\}$. So for each $t \in J$, there exists $f_t \in {}^{\mu}I$ such that $t = f_t/D$. Note that if $t \in I$, then $f_t/D = f_{\mathrm{Op}}(t)$ so that without loss of generality, for all $i < \mu, f_t(i) = t$. Define a map h from $\mathrm{EM}(J)$ into $\mathrm{Op}(\mathrm{EM}(I))$ as follows. An element of $\mathrm{EM}(J)$ has the form

$$\tau^{\mathrm{EM}'(J)}(x_{t_1},\ldots,x_{t_n}),$$

where $t_1,\ldots,t_n \in J$ and τ is an L-term. Define, for $t \in J$, $g_t \in {}^{\mu}\mathrm{EM}(I)$ by $g_t(i) = x_{f_t(i)}$. Note that $f_t(i) \in I$, so that $x_{f_t(i)} \in \mathrm{EM}(I)$ and so $g_t/D \in \mathrm{Op}(\mathrm{EM}(I))$. Let $h(\tau^{\mathrm{EM}'(J)}(x_{t_1},\ldots,x_{t_n})) = \tau^{\mathrm{Op}(\mathrm{EM}'(I))}(g_{t_1}/D,\ldots,g_{t_n}/D)$, which is an element in $\mathrm{Op}(\mathrm{EM}(I))$. The reader is invited to check that h is an $\mathcal{F}$-elementary embedding of $\mathrm{EM}(J)$ into $\mathrm{Op}(\mathrm{EM}(I))$. So $\mathrm{EM}(I) \preceq_{\mathcal{F}} \mathrm{EM}(J)$.

Finally, note that if $\overline{\tau} = \tau^{\mathrm{EM}'(I)}(x_{t_1},\ldots,x_{t_n}) \in \mathrm{EM}(I)$ with $t_1,\ldots,t_n \in I$, then

$$\begin{aligned} h(\overline{\tau}) &= \tau^{\mathrm{Op}(\mathrm{EM}'(I))}(g_{t_1}/D,\ldots,g_{t_n}/D) \\ &= \tau^{\mathrm{Op}(\mathrm{EM}'(I))}(\langle x_{t_1} : i<\mu\rangle/D,\ldots,\langle x_{t_n} : i<\mu\rangle/D) \\ &= f_{\mathrm{Op}}(\tau^{\mathrm{EM}'(I)}(x_{t_1},\ldots,x_{t_n})) = f_{\mathrm{Op}}(\overline{\tau}). \end{aligned}$$

Thus $\mathrm{EM}(I) \underset{\text{nice}}{\preceq} \mathrm{EM}(J)$.

CRITERION 2.6. *Suppose that $(I,<)$ is a suborder of the linear order $(J,<)$. If*

$(*)$ *for every $t \in J \setminus I$,*

$(\aleph)$ $$\mathrm{cf}((I,<)\restriction\{s \in I : (J,<) \models s < t\}) = \kappa$$

or

$$(\beth)\qquad \mathrm{cf}((I,<)^*{\restriction}\{s\in I:(J,<)^*\vDash s<^* t\})=\kappa$$

then $(I,<)\subseteq_{\text{nice}}(J,<)$. [Notation: $(I,<)^*$ is the (reverse) linear order $(I^*,<^*)$, where $I^*=I$ and $(I^*,<^*)\vDash s<^* t$ iff $(I,<)\vDash t<s$.]

P r o o f. Let us list some general facts which facilitate the proof.

FACT (A). *Let* $\underline{\kappa}$ *denote the linear order* $(\kappa,<)$, *where* $<$ *is the usual order* $\in{\restriction}\kappa\times\kappa$. *If* $J_1=\underline{\kappa}+J_0$, *then* $\underline{\kappa}\subseteq_{\text{nice}}J_1$ (+ *is addition of linear orders in which all elements in the first order precede those in the second*).

FACT (B). *If* $\underline{\kappa}\subseteq(I,<)$, $\underline{\kappa}$ *is unbounded in* $(I,<)$ *and* $J_1=I+J_0$, *then* $I\subseteq_{\text{nice}}J_1$.

FACT (C). *If* $I\subseteq_{\text{nice}}J$, *then* $I+J_1\subseteq_{\text{nice}}J+J_1$.

FACT (D). $I\subseteq_{\text{nice}}J$ *iff* $(J,<)^*\subseteq_{\text{nice}}(I,<)^*$.

FACT (E). *If* $\langle I_\alpha:\alpha\le\delta\rangle$ *is a continuous increasing sequence of linear orders and for* $\alpha<\delta$, $I_\alpha\subseteq_{\text{nice}}I_{\alpha+1}$, *then* $I_\alpha\subseteq_{\text{nice}}I_\delta$.

Now using these facts, let us prove the criterion. Define an equivalence relation E on $J\setminus I$ as follows: tEs iff t and s define the same Dedekind cut in $(I,<)$. Let $\{t_\alpha:\alpha<\delta\}$ be a set of representatives of the E-equivalence classes. For each $\beta\le\delta$, define

$$I_\beta=J{\restriction}\{t:t\in I\vee(\exists\alpha<\beta)(tEt_\alpha)\}$$

so $I_0=I$, $I_\delta=J$ and $\langle I_\alpha:\alpha\le\delta\rangle$ is a continuous increasing sequence of linear orders. By Fact (E), to show that $I\subseteq_{\text{nice}}J$, it suffices to show that $I_\alpha\subseteq_{\text{nice}}I_{\alpha+1}$ for each $\alpha<\delta$.

Fix $\alpha<\delta$. Now t_α belongs to $J\setminus I$, so $(\aleph)$ or $(\beth)$ holds. By Fact (D), it is enough to treat the case $(\aleph)$. So, without loss of generality, $\mathrm{cf}((I,<){\restriction}\{s\in I:(J,<)\models s<t_s\})=\kappa$.

Let

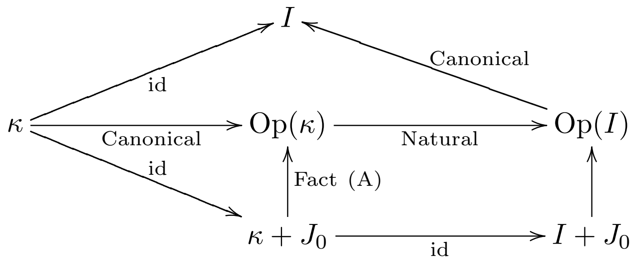
$$I_\alpha^{\mathrm{a}}=\{t\in I_\alpha:t<t_\alpha\},$$
$$I_\alpha^{\mathrm{b}}=\{t\in I_{\alpha+1}:t\in I_\alpha^{\mathrm{a}}\vee tEt_\alpha\},$$
$$I_\alpha^{\mathrm{c}}=\{t\in I_\alpha:t>t_\alpha\}.$$

Note that $I_\alpha=I_\alpha^{\mathrm{a}}+I_\alpha^{\mathrm{c}}$ and $I_{\alpha+1}=I_\alpha^{\mathrm{b}}+I_\alpha^{\mathrm{c}}$. Recalling Fact (C), it is now enough to show that $I_\alpha^{\mathrm{a}}\subseteq_{\text{nice}}I_\alpha^{\mathrm{b}}$. Identifying isomorphic orders and using $(\aleph)$, one deduces that $\underline{\kappa}$ is unbounded in I_α^{a} and $I_\alpha^{\mathrm{b}}=I_\alpha^{\mathrm{a}}+(I_\alpha^{\mathrm{b}}\setminus I_\alpha^{\mathrm{a}})$ so by Fact (B), $I_\alpha^{\mathrm{a}}\subseteq_{\text{nice}}I_\alpha^{\mathrm{b}}$ as required.

Of the five facts, we prove (A), (B) and (E), as (C) and (D) are obvious.

P r o o f o f F a c t (A). Since κ is measurable, there is a κ-complete uniform ultrafilter D on κ (see [J]). For every linear order J_0 (or J_0^*) there is $\mathrm{Op}_{I,D}(-)$, the iteration of I ultrapowers $(-)^\kappa/D$, ordered in the order J_0 (or J_0^*), giving the required embedding (use 1.7.5).

P r o o f o f F a c t (B). Since $\underline{\kappa} \subseteq I$ and using Fact (A), we know that there is an operation Op such that the following diagram commutes:



Chasing through the diagram, we obtain the required embedding.

P r o o f o f F a c t (E). Apply 1.11 to the chain $\langle I_\alpha : \alpha \le \delta\rangle$.

Fact 2.7. *Suppose that $\lambda \ge \kappa$. There exist a linear order $(I, <_I)$ of power λ and a sequence $\langle A_i \subset I : i \le \lambda\rangle$ of pairwise disjoint subsets of I, each of power κ, such that $I = \bigcup_{i\le\lambda} A_i$ and*

$$(*) \qquad \text{if } \lambda \in X \subset \lambda \dotplus 1, \text{ then } I\restriction \bigcup_{i\in X} A_i \underset{\text{nice}}{\subseteq} I.$$

P r o o f. Let $I = (\lambda \dotplus 1) \times \kappa$ and define $<_I$ on I by $(i_1, \alpha_1) <_I (i_2, \alpha_2)$ iff $i_1 < i_2$ or ($i_1 = i_2$ and $\alpha_1 > \alpha_2$). For each $i \le \lambda$, let $A_i = \{i\} \times \kappa$. Check $(*)$ of 2.6: suppose that $\lambda \in X \subset \lambda + 1$. Write $I_X = I\restriction\bigcup_{i\in X} A_i$. To show that $I_X \underset{\text{nice}}{\subseteq} I$, one deploys Criterion 2.6. Consider $t \in I - I_X$, say $t = (i, \alpha)$ (note that $\alpha < \kappa$ and $i < \lambda$, since $\lambda \in X$) and $i \notin X$. Let $j = \min(X - i)$; note that j is well defined, since $\lambda \in X - i$, and $j \ne i$. For every $\beta < \kappa$, one has $t <_I (j, \beta)$ and $(j, \beta) \in I_X$. Also if $s \in I_X$ and $t <_I s$, then for some $\beta < \kappa$, $(j, \beta) <_I s$. Thus $\langle (j, \beta) : \beta < \kappa\rangle$ is a cofinal sequence in $(I_X\restriction\{s \in I : t <_I s\})^*$. By the criterion, $I_X \underset{\text{nice}}{\subseteq} I$.

Theorem 2.8. *Suppose that $\kappa = \mathrm{cf}(\delta) \le \delta < \lambda$. Then* $\mathrm{EM}(\delta) \underset{\text{nice}}{\preceq} \mathrm{EM}(\lambda)$.

P r o o f. By Fact (B) of 2.6, one has $\delta \underset{\text{nice}}{\subseteq} \lambda$; so by 2.5, $\mathrm{EM}(\delta) \underset{\text{nice}}{\preceq} \mathrm{EM}(\lambda)$.

Now let us turn to the main theorem of this section.

Theorem 2.9. *Suppose that T is categorical in the regular cardinal $\lambda > \kappa + |T|$. Then $\mathcal{K}_{<\lambda}$ has the amalgamation property.*

P r o o f. Suppose that $\mathcal{K}_{<\lambda}$ fails AP. Note that $\|\mathrm{EM}(\lambda)\| = \lambda$. Apply 2.4 to find $M^* \in K_\lambda$ and $\langle M_i : i < \lambda\rangle$ satisfying 2.4(1), (2). Since T is λ-categorical, $M^* \cong \mathrm{EM}(\lambda)$, so without loss of generality, $\mathrm{EM}(\lambda) = \bigcup_{i<\lambda} M_i$. Now $C = \{i < \lambda : M_i = \mathrm{EM}(i)\}$ is a club of λ. Choose $\delta \in C$ with $\mathrm{cf}(\delta) = \kappa$. By 2.8, $\mathrm{EM}(\delta) \underset{\text{nice}}{\preceq} \mathrm{EM}(\lambda)$, so $M_\delta \underset{\text{nice}}{\preceq} M^*$. But of course by 2.4(2), $M_\delta \underset{\text{nice}}{\not\preceq} M^*$—contradiction.

THEOREM 2.10. *Suppose that T is categorical in $\lambda > \kappa + |T|$. Then*:

(1) *T has a model M of power λ such that if $N \preceq_{\mathcal{F}} M$ and $\|N\| < \lambda$, then there exists N' such that*

(α) *$N \preceq_{\mathcal{F}} N' \preceq_{\mathcal{F}} M$;*
(β) *$\|N'\| = \|N\| + \kappa + |T|$;*
(γ) *$N' \underset{\text{nice}}{\preceq} M$.*

(2) *T has a model M of power λ and an expansion M^+ of M by at most $\kappa + |T|$ functions such that if $N^+ \subseteq M^+$, then $N^+ \restriction L \underset{\text{nice}}{\preceq} M$.*

P r o o f. Let $\langle I, \langle A_i : i \le \lambda\rangle\rangle$ be as in 2.7. Let $M = \mathrm{EM}(I)$. Suppose that $N \preceq M$ and $\|N\| < \lambda$. Then there exists $J \subset I$ with $|J| < \lambda$ such that $N \subset \mathrm{EM}(J)$, hence $N \preceq_{\mathcal{F}} \mathrm{EM}(J) \preceq_{\mathcal{F}} \mathrm{EM}(I)$. So there is $X \subset \lambda + 1$ with $\lambda \in X$ and $|X| < \lambda$ such that $J \subset \bigcup_{i\in X} A_i$. Note that $|\bigcup_{i\in X} A_i| \le |X|\kappa < \lambda$. Now $N' = \mathrm{EM}(I \restriction \bigcup_{i\in X} A_i)$ is as required, since $I \restriction \bigcup_{i\in X} A_i \underset{\text{nice}}{\subseteq} I$ and so by 2.5, $\mathrm{EM}(I \restriction \bigcup_{i\in X} A_i) \underset{\text{nice}}{\preceq} \mathrm{EM}(I)$. This proves (1).

(2) We expand $M = \mathrm{EM}(I)$ as follows:

(a) By all functions of $\mathrm{EM}'(I)$.

(b) By the unary functions f_l ($l < \omega$) which are chosen as follows: we know that for each $b \in M$ there is an L_1-term τ_b (L_1 is the vocabulary of $\mathrm{EM}'(I)$) and $t(b,0) < t(b,1) < \ldots < t(b, n_{\tau_b} - 1)$ from I such that $b = \tau_b(x_{t(b,0)}, x_{t(b,1)}, \ldots, x_{t(b,n_{\tau_b}-1)})$ (it is not unique, but we can choose one; really if we choose it with n_b minimal it is almost unique). We let

$$f_l(b) = \begin{cases} x_{t(b,l)} & \text{if } l < n_{\tau_b}, \\ b & \text{if } l \ge n_{\tau_b}. \end{cases}$$

(c) By unary functions g_α, g^α for $\alpha < \kappa$ such that if $t < s$ are in I and $\alpha = \mathrm{otp}[(t,s)^*_I]$ then $g^\alpha(x_t) = x_s$, $g_\beta(x_s) = x_t$ for some $\beta < \kappa$ (more formally, $g^\alpha(x_{(i,\beta)}) = x_{(i,\beta+\alpha)}$ and $g_\alpha(x_{(i,\beta)}) = x_{(i,\alpha)}$) and in the other cases $g^\alpha(b) = b$, $g_\alpha(b) = b$.

(d) By individual constants $c_\alpha = x_{(\lambda,\alpha)}$ for $\alpha < \kappa$.

Now suppose N^+ is a submodel of M^+ and N its L-reduct. Let $J := \{t \in I : x_t \in N\}$. Now J is a subset of I of cardinality $\le \|N\|$ as for $t \neq s$ from J, $x_t \neq x_s$. Also if $b \in N$ then by (b), $x_{t(b,l)} \in N$, hence $b \in \mathrm{EM}(J)$; on the other hand, if $b \in \mathrm{EM}(J)$ then by (a) we have $b \in N$; so we can conclude $N = \mathrm{EM}(J)$. So far this holds for any linear order I.

By (c), $J = \bigcup_{i\in X} A_i$ for some $X \subseteq \lambda + 1$, and by (d), $\lambda \in X$.

Now $\mathrm{EM}(J) \underset{\text{nice}}{\preceq} \mathrm{EM}(I) \neq M$ by 2.7.

3. Towards removing the assumption of regularity from the existence of universal extensions. In Section 2 we showed that $\mathcal{K}_{<\lambda}$ has the amalgamation property when T is categorical in the regular cardinal $\lambda > \kappa + |T|$. We now study the situation in which λ is not assumed to be regular.

Our problem is that while we know that most submodels of $N \in K_\lambda$ sit well in N (see 2.10(2)) and that there are quite many $N \in K_{<\lambda}$ which are amalgamation bases, our difficulty is to get those things together: constructing $N \in K_\lambda$ as $\bigcup_{i<\lambda} N_i$ with $N_i \in K_{<\lambda}$ means N has $\preceq_{\mathcal{F}}$-submodels not included in any N_i.

THEOREM 3.1. *Suppose that T is categorical in λ and $\kappa + |T| \leq \theta < \lambda$. If $\langle M_i \in K_\theta : i < \theta^+\rangle$ is an increasing continuous $\prec_{\mathcal{F}}$-chain, then*

$$\Big\{i < \theta^+ : M_i \underset{\text{nice}}{\preceq} \bigcup_{j<\theta^+} M_j\Big\} \in D_{\theta^+}.$$

Remark 3.1A. (1) We cannot use 2.10(1) e.g. as possibly λ has cofinality $< \kappa + |T|$.

(2) Recall that D_{θ^+} is the closed unbounded filter on θ^+.

Proof of Theorem 3.1. Write $M_{\theta^+} = \bigcup_{i<\theta^+} M_i$. Choose an operation Op such that for all $i < \theta^+$, $\|\mathrm{Op}(M_i)\| \geq \lambda$. Let $M_i^* = \mathrm{Op}(M_i)$. Applying 1.2 for nonlimit ordinals, and 1.1 for limit ordinals, one finds inductively an increasing continuous $\prec_{\mathcal{F}}$-chain $\langle N_i : i \leq \theta^+\rangle$ such that $M_i \preceq N_i \preceq M_i^*$ and $\|N_i\| = \lambda$ for $i < \theta^+$, and $N_{\theta^+} = \bigcup_{i<\theta^+} N_i$. Note that $\|N_{\theta^+}\| = \theta^+ \cdot \lambda = \lambda$.

Since T is λ-categorical, $N_{\theta^+} \cong \mathrm{EM}(I)$, where 2.7 furnishes I of power λ. By 2.10(2), there is an expansion $N^+_{\theta^+}$ of N_{θ^+} by at most $\kappa + |T|$ functions such that if $A \subset |N^+_{\theta^+}|$ is closed under the functions of $N^+_{\theta^+}$, then $(N^+_{\theta^+}{\restriction}L){\restriction}A \underset{\text{nice}}{\preceq} N_{\theta^+}$.

Choose a set A_i and an ordinal j_i, by induction on $i < \theta^+$, satisfying

(1) $A_i \subset |N_{\theta^+}|$, $|A_i| \leq \theta$; $\langle A_i : i < \theta^+\rangle$ is continuous increasing;
(2) $\langle j_i : i < \theta^+\rangle$ is continuous increasing;
(3) A_i is closed under the functions of $N^+_{\theta^+}$;
(4) $A_i \subset |N_{j_{i+1}}|$;
(5) $|M_i| \subset A_{i+1}$.

This is possible: for zero or limit ordinals unions work; for $i+1$ choose j_{i+1} to satisfy (2) and (4), and A_{i+1} to satisfy (1), (3) and (5).

By (2), $C = \{i < \theta^+ : i \text{ is a limit ordinal and } j_i = i\}$ is a club of θ^+, i.e. $C \in D_{\theta^+}$.

Fix $i \in C$. Note that $|M_i| \subset A_i$ and $A_i \subset |N_i|$ (since $|M_i| = \bigcup_{j<i} |M_j| \subseteq \bigcup_{j<i} A_{j+1} = A_i = \bigcup_{i'<i} A_{i'} \subset \bigcup_{i'<i} |N_{j_{i'+1}}| = N_{j_i} = N_i$ (using (5), (1), (4), (2) and $j_i = i$)) and so $M_i \preceq_{\mathcal{F}} (N^+_{\theta^+} \restriction L) \restriction A_i \preceq_{\mathcal{F}} N_i \preceq_{\mathcal{F}} M^*_i = \mathrm{Op}(M_i)$, so that $M_i \underset{\text{nice}}{\preceq} (N^+_{\theta^+} \restriction L) \restriction A_i$. However, by (3) and the choice of N_{θ^+} and $N^+_{\theta^+}$ one also has $(N^+_{\theta^+} \restriction L) \restriction A_i \underset{\text{nice}}{\preceq} N_{\theta^+}$. So by transitivity of $\underset{\text{nice}}{\preceq}$, one obtains $M_i \underset{\text{nice}}{\preceq} N_{\theta^+}$.

Finally, remark that $M_{\theta^+} \preceq_{\mathcal{F}} N_{\theta^+}$ since $M_i \underset{\text{nice}}{\preceq} N_i \preceq_{\mathcal{F}} N_{\theta^+}$ for every $i < \theta^+$. Hence $C \subset \{i < \theta^+ : M_i \underset{\text{nice}}{\preceq} M_{\theta^+}\} \in D_{\theta^+}$.

DEFINITION 3.2. Suppose that $\theta \in [\kappa + |T|, \lambda)$ and $M \in K_\theta$. Then M is *nice* iff whenever $M \preceq_{\mathcal{F}} N \in K_\theta$, then $M \underset{\text{nice}}{\preceq} N$. (The analogous $\mathcal{F}$-elementary embedding definition runs: M is nice iff whenever $f : M \to_{\mathcal{F}} N \in K_\theta$ then $f : M \underset{\text{nice}}{\to} N$.)

THEOREM 3.3. *Suppose that T is categorical in λ and $M \in K_\theta$ with $\theta \in [\kappa + |T|, \lambda)$. Then there exists $N \in K_\theta$ such that $M \preceq_{\mathcal{F}} N$ and N is nice.*

Proof. Suppose otherwise. We will define a continuous increasing $\prec_{\mathcal{F}}$-chain $\langle M_i \in K_\theta : i < \theta^+\rangle$ such that for $j < \theta^+$,

$$(*)_j \qquad M_j \underset{\text{nice}}{\not\preceq} M_{j+1}.$$

For $i = 0$, put $M_0 = M$; if i is a limit ordinal, put $M_i = \bigcup_{j<i} M_j$; if $i = j+1$, then, since 3.3 is assumed to fail, M_{j+1} exists as required in $(*)_j$ (otherwise M_i works as N in 3.3). But now $\langle M_i : i < \theta^+\rangle$ yields a contradiction to 3.1, since $C = \{i < \theta^+ : M_i \underset{\text{nice}}{\preceq} \bigcup_{j<\theta^+} M_j\} \in D_{\theta^+}$ by 3.1 so that choosing j from C one has $M_j \underset{\text{nice}}{\preceq} M_{j+1}$ by 1.10, contradicting $(*)_j$.

THEOREM 3.4. *Suppose that T is categorical in λ and $\theta \in [\kappa + |T|, \lambda)$. If $M \in K_\theta$ is nice and $f : M \to_{\mathcal{F}} N \in K_{\le\lambda}$, then $f : M \underset{\text{nice}}{\to} N$.*

Proof. Choosing an appropriate Op and using 1.2 one finds N_1 such that $N \preceq_{\mathcal{F}} N_1$ and $\|N_1\| = \lambda$. Find $M'_1 \underset{\text{nice}}{\preceq} N_1$ by 2.10(2) such that $\mathrm{rng}(f) \subset |M'_1|$ and $\|M'_1\| = \theta$. So $M'_1 \preceq_{\mathcal{F}} N_1$ and so $\mathrm{rng}(f) \preceq_{\mathcal{F}} M'_1$. Since M is nice, we have $f : M \underset{\text{nice}}{\to} M'_1$. Now $M'_1 \underset{\text{nice}}{\preceq} N_1$, so $f : M \underset{\text{nice}}{\to} N_1$. So there are Op and $g : N_1 \to_{\mathcal{F}} \mathrm{Op}(M)$ satisfying $gf = f_{\mathrm{Op}}$. Since $N \preceq_{\mathcal{F}} N_1$ it follows that $f : M \underset{\text{nice}}{\to} N$ as required.

Corollary 3.5. *Suppose that $M \in K_\theta$ is nice with $\theta \in [\kappa + |T|, \lambda)$. Then M is an a.b. in $\mathcal{K}_{\le\lambda}$, i.e. if $f_i : M \to_{\mathcal{F}} M_i$ and $M_i \in K_{\le\lambda}$ $(i = 1, 2)$, then there exists an amalgam $N \in K_{\le\lambda}$ of M_1, M_2 over M with respect to f_1, f_2.*

Proof. By 3.4, $f_i : M \underset{\text{nice}}{\to} M_i$ $(i = 1, 2)$. Hence by 2.1 there is an amalgam $N \in K_{\le\lambda}$ of M_1, M_2 over M with respect to f_1, f_2.

Definition 3.6. Suppose that $\theta \in [\kappa + |T|, \lambda)$ and σ is a cardinal.

(1) A model $M \in K_\theta$ is *σ-universal* iff for every $N \in K_\sigma$, there exists an $\mathcal{F}$-elementary embedding $f : N \to_{\mathcal{F}} M$. M is *universal* iff M is $\|M\|$-universal.

(2) A model $M_2 \in K_{\ge\sigma}$ is *σ-universal over the model* M_1 (and one writes $M_1 \underset{\sigma\text{-univ}}{\preceq} M_2$) iff $M_1 \preceq_{\mathcal{F}} M_2$ and whenever $M_1 \preceq_{\mathcal{F}} M_2' \in K_\sigma$, then there exists an $\mathcal{F}$-elementary embedding $f : M_2' \to_{\mathcal{F}} M_2$ such that $f{\restriction}M_1$ is the identity. (The embedding version runs: there exists $h : M_1 \to_{\mathcal{F}} M_2$ such that whenever $g : M_1 \to_{\mathcal{F}} M_2' \in K_{\|\sigma\|}$, then there exists $f : M_2' \to_{\mathcal{F}} M_2$ with $fg = h$.) M_2 is *universal over* M_1 ($M_1 \underset{\text{univ}}{\preceq} M_2$) iff M_2 is $\|M_2\|$-universal over M_1.

(3) M_2 is *σ-universal over M_1 in M* iff $M_1 \preceq_{\mathcal{F}} M_2 \preceq_{\mathcal{F}} M$, $\|M_1\| \le \sigma$ and whenever $M_2' \in K_\sigma$ and $M_1 \preceq_{\mathcal{F}} M_2' \preceq_{\mathcal{F}} M$, then there exists an $\mathcal{F}$-elementary embedding $f : M_2' \to M_2$ such that $f{\restriction}M_1$ is the identity. M_2 is *universal over M_1 in M* iff M_2 is $\|M_2\|$-universal over M_1 in M.

(4) M_2 is *weakly σ-universal over* M_1 (written $M_1 \underset{\sigma\text{-wu}}{\prec} M_2$) iff $M_1 \preceq_{\mathcal{F}} M_2 \in K_\sigma$ and whenever $M_2 \prec_{\mathcal{F}} M_2' \in K_\sigma$, then there exists an $\mathcal{F}$-elementary embedding $f : M_2' \to_{\mathcal{F}} M_2$ such that $f{\restriction}M_1$ is the identity. (The embedding version is: there exists $h : M_1 \to_{\mathcal{F}} M_2$ such that whenever $g : M_2 \to_{\mathcal{F}} M_2' \in K_\sigma$, then there exists $f : M_2' \to_{\mathcal{F}} M_2$ such that $h = fgh$ (written $h : M_1 \underset{\sigma\text{-wu}}{\to} M_2$).). M_2 is *weakly universal over* M_1 ($M_1 \underset{\text{wu}}{\preceq} M_2$) iff M_2 is $\|M_2\|$-weakly universal over M_1.

(Note that "M_2 is σ-universal over M_1" does not necessarily imply "M_2 is weakly σ-universal over M_1" as possibly $\|M_2\| > \sigma$.)

Remark 3.6A. Observe that

(5) In $K_{<\lambda}$, if M_1 is an a.b., then weak universality over M_1 is equivalent to universality over M_1.

Proof. Suppose that $h : M_1 \underset{\text{wu}}{\to} M_2$ and $g : M_1 \to_{\mathcal{F}} M_2' \in K_{\|M_2\|}$. Since M_1 is an a.b. there exist a model N and $h' : M_2 \to_{\mathcal{F}} N$, $g' : M_2' \to_{\mathcal{F}} N$ satisfying $h'h = g'g$. By 1.2 we can assume that $\|N\| = \|M_2\|$. Since M_2 is weakly universal over M_1, there exists $h'' : N \to_{\mathcal{F}} M_2$ with $h = h''h'h$. Let

$f = h''g' : M_2' \to M_2$, and note that $fg\restriction M_1 = h''g'g = h''h'h = h$, so that M_2 is universal over M_1.

R e m a r k 3.6B. Conversely,

(6) For any model M, universality over M implies weak universality over M.

LEMMA 3.7. *Suppose that T is categorical in λ and $\theta \in [\kappa + |T|, \lambda)$. If $M \in K_\theta$ and $M \preceq_{\mathcal{F}} N \in K_\lambda$, then there exists $M^+ \in K_\theta$ such that*

(1) $M \preceq_{\mathcal{F}} M^+ \preceq_{\mathcal{F}} N$;
(2) M^+ *is universal over* M *in* N.

P r o o f. Now choose I such that

$(*)_\lambda[I]$ (i) I is a linear order of cardinality λ;
(ii) if $\theta \in [\aleph_0, \lambda)$ and $J_0 \subseteq I$ with $|J_0| = \theta$ then there is J_1 satisfying $J_0 \subseteq J_1 \subseteq I$, $|J_1| = \theta$, and such that for every $J^* \subseteq I$ of cardinality $\le \theta$ there is an order preserving (one-to-one) mapping from $J_0 \cup J^*$ into $J_0 \cup J_1$ which is the identity on J_0.

Essentially the construction follows Laver [L] and [Sh220, Appendix]; but for our present purpose let $I = ({}^{\omega>}\lambda, <_{\mathrm{lex}})$; given θ and J_0 we can increase J_0 so without loss of generality, $J_0 = {}^{\omega>}A$, $A \subseteq \lambda$, $|A| = \theta$. Define an equivalence relation E on $I \setminus J_0$: $\eta E \nu \Leftrightarrow (\forall \varrho \in J_0)(\varrho <_{\mathrm{lex}} \eta \equiv \varrho <_{\mathrm{lex}} \nu)$; clearly it has $\le \theta$ equivalence classes. Let $\{\eta_i^* : i < i^* \le \theta\}$ be a set of representatives, each of minimal length, so $\eta^* \restriction (\lg \eta_i^* - 1) \in J_0$ and $\eta_i^*(\lg \eta_i^* - 1) \in \lambda \setminus A$.

Let $J_1 = I \cup \{\eta_i^{*\wedge}\nu : \nu \in {}^{\omega>}\theta \text{ and } i < i^*\}$. Then clearly $J_0 \subseteq J_1 \subseteq I$ and $|J_1| = \theta$. Suppose $J_0 \subseteq J \subseteq I$, $|J| \le \theta$, and we should find the required embedding h. As before we can assume that $J = {}^{\omega>}B$, $|B| = \theta$ and $A \subseteq B$. Now $h\restriction J_0 = \mathrm{id}_{J_0}$ so it is enough to define $h\restriction(J_1 \cap (\eta_i^*/E))$, hence it is enough to embed $J_1 \cap (\eta_i^*/E)$ into $\{\eta_1^{*\wedge}\nu : \nu \in {}^{\omega>}\theta\}$ (under $<_{\mathrm{lex}}$).

Let $\gamma = \mathrm{otp}B$. It is enough to show $({}^{<\omega}\gamma, <_{\mathrm{lex}})$ can be embedded into ${}^{\omega>}\theta$, where of course $|\gamma| \le \theta$. This is proved by induction on γ.

Since T is λ-categorical and $\mathrm{EM}(I)$ is a model of T of power λ, there is an isomorphism g from $\mathrm{EM}(I)$ onto N. It follows from $(*)_\lambda[I]$ that $M^+ = g''\mathrm{EM}(J) \in K_\theta$ satisfies (1) and (2). (Analogues of (1) and (2) are checked also in the course of the proof of 3.11.)

LEMMA 3.8. *Suppose that T is categorical in λ, $\theta \in [\kappa + |T|, \lambda)$ and $\langle M_i \in K_\theta : i < \theta^+\rangle$ and $\langle N_i \in K_\lambda : i < \theta^+\rangle$ are continuous $\prec_{\mathcal{F}}$-chains such that for every $i < \theta^+$, $M_i \preceq_{\mathcal{F}} N_i$. Then there exists $i(*) < \theta^+$ such that $(i(*), \theta^+) \subset C = \{i < \theta^+ : M_{i+1}$ can be $\mathcal{F}$-elementarily embedded into N_i over $M_0\}$.*

Proof. Apply 3.7 for $M_0 \in K_\theta$ and $N_{\theta^+} = \bigcup_{i<\theta^+} N_i \in K_\lambda$ (noting that $M_0 \preceq_{\mathcal{F}} N_0 \preceq_{\mathcal{F}} N_{\theta^+}$) to find $M^+ \in K_\theta$ such that $M_0 \preceq_{\mathcal{F}} M^+ \preceq_{\mathcal{F}} N_{\theta^+}$ and M^+ is universal over M_0 in N_{θ^+}.

For some $i(*) < \theta^+$, $M^+ \subseteq N_{i(*)}$ and so $M^+ \preceq_{\mathcal{F}} N_{i(*)}$. If $i \in (i(*), \theta^+)$, then $M_{i+1} \in K_\theta$ and $M_0 \preceq_{\mathcal{F}} M_{i+1} \preceq_{\mathcal{F}} N_{i+1} \preceq_{\mathcal{F}} N_{\theta^+}$, so there is an $\mathcal{F}$-elementary embedding $f : M_{i+1} \to_{\mathcal{F}} M^+$ and $f \restriction M_0$ is the identity. Now $M^+ \preceq_{\mathcal{F}} N_{i(*)} \preceq_{\mathcal{F}} N_i$, so $f : M_{i+1} \to_{\mathcal{F}} N_i$. Hence $(i(*),\ \theta^+) \subset C$ as required.

THEOREM 3.9. *Suppose that T is categorical in λ, $\theta \in [\kappa + |T|, \lambda)$, and $M \in K_\theta$. Then there exists $M^+ \in K_\theta$ such that*

(ℵ) *$M \preceq_{\mathcal{F}} M^+$ and M^+ is nice*;

(ℶ) *M^+ is weakly universal over M.*

Proof. Define by induction on $i < \theta^+$ continuous $\prec_{\mathcal{F}}$-chains $\langle M_i \in K_\theta : i < \theta^+ \rangle$ and $\langle N_i \in K_\lambda : i < \theta^+ \rangle$ such that

(0) $M_0 = M$;

(1) $M_i \preceq_{\mathcal{F}} N_i$;

(2) if $(*)_i$ holds, then M_{i+1} cannot be $\mathcal{F}$-elementarily embedded into N_i over M_0, where $(*)_i$ is the statement: there are $M' \in K_\theta$ and $N' \in K_\lambda$ such that $M_i \preceq_{\mathcal{F}} M'$, $N_i \preceq_{\mathcal{F}} N'$, $M' \preceq_{\mathcal{F}} N'$ and M' cannot be $\mathcal{F}$-elementarily embedded into N_i over M_0;

(3) $M_{i+1} \underset{\text{nice}}{\preceq} N_{i+1}$.

This is possible. N_0 is obtained by an application of 1.2 to an appropriate $\mathrm{Op}(M_0)$ of power at least λ. At limit stages, continuity dictates that one take unions. Suppose that M_i has been defined. If $(*)_i$ does not hold, by 2.10(2) there is $M'' \in K_\theta$ with $M_i \preceq_{\mathcal{F}} M'' \underset{\text{nice}}{\preceq} N_i$. Let $M_{i+1} = M''$ and $N_{i+1} = N_i$. If $(*)_i$ does hold for M', N', let $N_{i+1} = N'$; note that by 2.10(2) there exists $M'' \in K_\theta$ such that $M' \preceq_{\mathcal{F}} M'' \underset{\text{nice}}{\preceq} N'$; now let $M_{i+1} = M''$. Note that in each case, (3) is satisfied.

Find $i(*) < \theta^+$ and C as in 3.8 and choose $i \in C$. By (1), $M_{i+1} \preceq_{\mathcal{F}} N_{i+1}$ so by 3.7 there exists $M^- \in K_\theta$ such that $M_{i+1} \preceq_{\mathcal{F}} M^- \preceq_{\mathcal{F}} N_{i+1}$ and M^- is weakly universal over M_{i+1} in N_{i+1}. By 3.3 one can find $M^+ \in K_\theta$ such that $M^- \preceq_{\mathcal{F}} M^+$ and M^+ is nice. So M^+ satisfies (ℵ). It remains to show that M^+ is weakly universal over M. Suppose not and let $g : M^+ \to_{\mathcal{F}} M^* \in K_\theta$, where M^* cannot be $\mathcal{F}$-elementarily embedded in M^+ over M, hence cannot be $\preceq_{\mathcal{F}}$-elementarily embeddable in M^- over M, hence in N_{i+1} over M. Since $M_{i+1} \preceq_{\mathcal{F}} M^* \in K_\theta$ and by (3), $M_{i+1} \underset{\text{nice}}{\preceq} N_{i+1} \in K_\lambda$, by 2.1 there is an amalgam $N^* \in K_\lambda$ of M^*, N_{i+1}. The existence of M^*, N^* implies that $(*)_{i+1}$ holds since M^* cannot be $\mathcal{F}$-elementarily embedded into N_{i+1} over M_0, hence M_{i+2} cannot be $\mathcal{F}$-elementarily embedded into N_{i+1} in contradiction to the choice of i as by 3.7, $i+1$ is in C.

COROLLARY 3.10. *If T is categorical in λ, $\theta \in [\kappa + |T|, \lambda)$ and $M \in K_\theta$ is an a.b.* (e.g. M is nice—see 2.1), *then there exists $M^+ \in K_\theta$ such that*

($\aleph$) *$M \preceq_{\mathcal{F}} M^+$ and M^+ is nice*;
(β) *M^+ is universal over M.*

Proof. 3.9 and 3.6A(5).

COROLLARY 3.11. *Suppose that T is categorical in λ and $\theta \in [\kappa+|T|, \lambda)$. Then there is a nice universal model $M \in K_\theta$.*

Proof. By 3.3 it suffices to find a universal model of power θ, noting that universality is preserved under $\mathcal{F}$-elementary extensions in the same power.

As in the proof of 3.7, there is a linear order $(I, <_I)$ of power λ and $J \subset I$ with $|J| = \theta$ such that

(∗) $(\forall J' \subset I)$ (if $|J'| \le \theta$, then there is an order-preserving injective map g from J' into J).

CLAIM. $\mathrm{EM}(J) \in K_\theta$ *is universal.*

Proof. $\mathrm{EM}(J)$ is a model of power θ since $\max(|J|, \kappa + |T|) \le \theta$ and $\theta = |J| \le \|\mathrm{EM}(J)\|$. Suppose that $N \in K_\theta$. Applying 1.2 to a suitably large $\mathrm{Op}(N)$ find $M \in K_\lambda$ with $N \preceq_{\mathcal{F}} M$ so that by λ-categoricity of T, $M \cong \mathrm{EM}(I)$. There is a surjective $\mathcal{F}$-elementary embedding $h : N \to_{\mathcal{F}} N' \preceq_{\mathcal{F}} \mathrm{EM}(I)$ and there exists $J' \subset I$ with $|J'| \le \|N'\| + \kappa + |T| = \theta$ such that $N' \subseteq \mathrm{EM}(J')$. So by (∗) there is an order-preserving injective map g from J' into J. Then g induces an $\mathcal{F}$-elementary embedding $\widehat{g}$ from $\mathrm{EM}(J')$ into $\mathrm{EM}(J)$. Let $f = \widehat{g}h$. Then $f : N \to_{\mathcal{F}} \mathrm{EM}(J)$ is as required.

THEOREM 3.12. *Suppose that T is categorical in λ, $\theta \in [\kappa + |T|, \lambda)$, $N \in K_{<\lambda}$ is nice, $M \in K_\theta$ and $M \underset{\text{nice}}{\preceq} N$. Then M is nice.*

Proof. Let $B \in K_\theta$ with $M \preceq_{\mathcal{F}} B$. Show that $M \underset{\text{nice}}{\preceq} B$. Well, since $M \underset{\text{nice}}{\preceq} N$ and $M \preceq_{\mathcal{F}} B$, by 2.1 there exists an amalgam $M^* \in K_{<\lambda}$ of N, B over M. Without loss of generality, by 1.5, $\|M^*\| = \|N\|$. Now N is nice, hence $N \underset{\text{nice}}{\preceq} M^*$. Since $M \underset{\text{nice}}{\preceq} N$, it follows by 1.7.5 that $M \underset{\text{nice}}{\preceq} M^*$. Since $M \preceq_{\mathcal{F}} B \preceq_{\mathcal{F}} M^*$, it follows by 1.10 that $M \underset{\text{nice}}{\preceq} B$.

4. (θ, σ)-saturated models. In this section we define notions of saturation which will be of use in proving amalgamation for $\mathcal{K}_\lambda$.

DEFINITION 4.1. Suppose that σ is a limit ordinal with $1 \leq \sigma \leq \theta \in [\kappa + |T|, \lambda)$.

(1) An L-structure M is (θ, σ)-*saturated* iff

(a) $\|M\| = \theta$;

(b) there exists a continuous $\prec_{\mathcal{F}}$-chain $\langle M_i \in K_\theta : i < \sigma \rangle$ such that (i) M_0 is nice and universal, (ii) M_{i+1} is universal over M_i, (iii) M_i is nice, and (iv) $M = \bigcup_{i<\sigma} M_i$.

(2) M is θ-*saturated* iff M is $(\theta, \mathrm{cf}(\theta))$-saturated.

(3) M is (θ, σ)-*saturated over* N iff M is (θ, σ)-saturated as witnessed by a chain $\langle M_i : i < \sigma \rangle$ such that $N \subseteq M_0$.

The principal facts established in this section connect the existence, uniqueness and niceness of (θ, σ)-saturated models.

THEOREM 4.2. *Suppose that T is categorical in λ and $\sigma \leq \theta \in [\kappa + |T|, \lambda)$. Then*

(1) *there exists a (θ, σ)-saturated model M*;

(2) *M is unique up to isomorphism*;

(3) *M is nice.*

Proof. One proves (1), (2) and (3) simultaneously by induction on σ.

(1) Choose a continuous $\prec_{\mathcal{F}}$-chain $\langle M_i \in K_\theta : i < \sigma \rangle$ of nice models by induction on i as follows. For $i = 0$, apply 3.11 to find a nice universal model $M_0 \in K_\theta$. For $i = j + 1$, note that M_j is an a.b. by 3.5 (since M_j is nice), hence by 3.10 there exists a nice model $M_i \in K_\theta$ such that $M_j \preceq_{\mathcal{F}} M_i$ and M_i is universal over M_j. For limit i, let $M_i = \bigcup_{j<i} M_j$. Note that by the inductive hypothesis (3) on σ for $i < \sigma$, since M_i is (θ, i)-saturated, M_i is nice. Thus $M = \bigcup_{i<\sigma} M_i$ is (θ, σ)-saturated (witnessed by $\langle M_i : i < \sigma \rangle$). Note that M is universal, since $\langle M_i : i < \sigma \rangle$ is $\preceq_{\mathcal{F}}$-increasing and M_0 is universal.

(2) As σ is a limit ordinal, a standard back-and-forth argument shows that if M and N are (θ, σ)-saturated models, then M and N are isomorphic.

(3) By the uniqueness (i.e. by (2)) it suffices to prove that some (θ, σ)-saturated model is nice. Suppose that M is (θ, σ)-saturated. We will show that M is nice.

If $\mathrm{cf}(\sigma) < \sigma$, then M is also $(\theta, \mathrm{cf}(\sigma))$-saturated and hence by the inductive hypothesis (3) on σ for $\mathrm{cf}(\sigma)$, M is nice. So we will assume that $\mathrm{cf}(\sigma) = \sigma$. Choose a continuous $\prec_{\mathcal{F}}$-chain $\langle M_i \in K_\theta : i < \theta^+ \rangle$ such that: M_0 is nice and universal (possible by 3.11); if M_i is nice, then $M_{i+1} \in K_\theta$ is nice and universal over M_i (possible by 3.5 and 3.10); if M_i is not nice (so necessarily i is a limit ordinal), then $M_{i+1} \in K_\theta$, $M_i \preceq_{\mathcal{F}} M_{i+1}$ and $M_i \not\preceq_{\text{nice}} M_{i+1}$. By 3.1 and 1.10 there is a club C of θ^+ such that if $i \in C$, then $M_i \preceq_{\text{nice}} M_{i+1}$. So by the choice of $\langle M_i : i < \theta^+\rangle$, if $i \in C$, then M_i is nice. Choose $i \in C$ with $i = \sup(i \cap C)$ and $\text{cf}(i) = \sigma$. It suffices to show that M_i is (θ,σ)-saturated (for then, by (2), M_i is isomorphic to M and so M is nice). Choose a continuous increasing sequence $\langle \alpha_\zeta : \zeta < \sigma\rangle \subset C$ such that $i = \sup_{\zeta<\sigma} \alpha_\zeta$ (recall that $i = \sup(i \cap C)$ and $\text{cf}(i) = \sigma$). Now $M_i = \bigcup_{\zeta<\sigma} M_{\alpha_\zeta}$. Of course M_{α_0} is universal (since M_0 is universal and $M_0 \preceq_{\mathcal{F}} M_{\alpha_0}$), $M_{\alpha_{\zeta+1}}$ is universal over M_{α_ζ} since $M_{\alpha_\zeta+1}$ is universal over M_{α_ζ} and $M_{\alpha_\zeta} \preceq_{\mathcal{F}} M_{\alpha_\zeta+1} \preceq_{\mathcal{F}} M_{\alpha_{\zeta+1}}$. Also M_{α_ζ} is nice for each $\zeta < \sigma$ since $\alpha_\zeta \in C$. Hence M_i is (θ,σ)-saturated.

Remark 4.3. Remember that by 3.12, if T is categorical in λ, $\theta \in [\kappa + |T|, \lambda)$, $N \in K_{<\lambda}$ is nice, $M \in K_\theta$ and $M \preceq_{\text{nice}} N$, then M is nice.

THEOREM 4.4. *Suppose that T is categorical in λ and $\kappa + |T| \le \theta < \theta^+ < \lambda$. If $\langle M_i \in K_\theta : i < \theta^+\rangle$ is a continuous $\prec_{\mathcal{F}}$-chain of nice models such that M_{i+1} is universal over M_i for $i < \theta^+$, then $\bigcup_{i<\theta^+} M_i$ is (θ^+,θ^+)-saturated.*

Proof. Write $M = \bigcup_{i<\theta^+} M_i$. Note that if $\langle M_i' \in K_\theta : i < \theta^+\rangle$ is any other continuous $\prec_{\mathcal{F}}$-chain of nice models such that M_{i+1}' is universal over M_i', then $\bigcup_{i<\theta^+} M_i' \cong M$ (use again the back-and-forth argument).

By 4.2 there exists a (θ^+,θ^+)-saturated model N which is unique and nice. In particular, $\|N\| = \theta^+$ and there exists a continuous $\prec_{\mathcal{F}}$-chain $\langle N_i \in K_{\theta^+} : i < \theta^+\rangle$ such that (i) N_0 is nice and universal, (ii) N_{i+1} is universal over N_i, (iii) N_i is nice, and (iv) $N = \bigcup_{i<\theta^+} N_i$. It suffices to prove that M and N are isomorphic models.

Without loss of generality, $|N| = \theta^+$. By 1.2, $C_1 = \{\delta < \theta^+ : N{\restriction}\delta \preceq_{\mathcal{F}} N\}$ contains a club of θ^+. By 3.1 there exists a club C_2 of θ^+ such that for every $\delta \in C_2$, $N{\restriction}\delta \preceq_{\text{nice}} N$. Since $\{|N_i| : i < \theta^+\}$ is a continuous increasing sequence of subsets of θ^+, it follows that $C_3 = \{\delta < \theta^+ : \delta \subseteq |N_\delta|\}$ is a club of θ^+. Hence there is a club C_4 of θ^+ such that $C_4 \subset C_1 \cap C_2 \cap C_3 \cap [\theta, \theta^+)$. Note that for $\delta \in C_4$ one has $N{\restriction}\delta \preceq_{\text{nice}} N$, $|N{\restriction}\delta| = \delta \subseteq |N_\delta|$ and $N_\delta \preceq_{\mathcal{F}} N$, so that $N{\restriction}\delta \preceq_{\mathcal{F}} N_\delta \preceq_{\mathcal{F}} N$ and so by 1.10, $N{\restriction}\delta \preceq_{\text{nice}} N_\delta$. $\langle N_\delta : \delta \in C_4\rangle$ is a continuous increasing $\prec_{\mathcal{F}}$-chain, $N_\delta \in K_{\theta^+}$ and $N{\restriction}\delta \in K_\theta$.

By 3.12, $N{\restriction}\delta$ is nice since N_δ is nice (by (iii)). So by 3.10, $N{\restriction}\delta$ has a nice $\prec_{\mathcal{F}}$-extension $B_\delta \in K_\theta$ which is universal over $N{\restriction}\delta$. Without loss of generality, $N{\restriction}\delta \preceq_{\mathcal{F}} B_\delta \preceq N$. [Why? since $N{\restriction}\delta \preceq_{\mathcal{F}} B_\delta$ (in fact $N{\restriction}\delta \preceq_{\text{nice}} B_\delta$) and $N{\restriction}\delta \preceq_{\text{nice}} N_\delta$, by 2.1 there exists an amalgam $A_\delta \in K_{\le\theta^+}$ of B_δ, N_δ

over $N{\restriction}\delta$. Let $f_\delta : B_\delta \to_{\mathcal{F}} A_\delta$ be a witness. But $N_{\delta+1}$ is universal over N_δ (by (ii)), so A_δ can be $\prec_{\mathcal{F}}$-elementarily embedded into $N_{\delta+1}$ over N_δ (say by g_δ), hence B_δ can be $\prec_{\mathcal{F}}$-elementarily embedded into N (using $g_\delta f_\delta$).]

Let $C_5 = \{\delta \in C_4 : \text{if } \alpha \in C_4 \cap \delta, \text{ then } |B_\alpha| \subset \delta\}$. Note that C_5 is a club of θ^+ since $\|B_\alpha\| = \theta$. [Let $E_\alpha = (\sup |B_\alpha|, \theta^+) \cap C_4$ for $\alpha \in C_4$, $E_\alpha = \theta^+$ for $\alpha \notin C_4$, and let E be the diagonal intersection of $\langle E_\alpha : \alpha < \theta^+\rangle$, i.e. $E = \{\delta < \theta^+ : (\forall \alpha < \delta)(\delta \in E_\alpha)\}$. Note that E is club of θ^+ and $C_5 \supseteq E \cap C_4$, which is a club of θ^+.] Thus $\langle N{\restriction}\delta : \delta \in C_5\rangle$ is a continuous $\prec_{\mathcal{F}}$-chain of nice models, each of power θ. If $\delta_1 \in C_5$ and $\delta_2 = \min(C_5 \setminus (\delta_1 + 1))$, then $N{\restriction}\delta_1 \preceq_{\mathcal{F}} B_{\delta_1} \preceq_{\mathcal{F}} N{\restriction}\delta_2$. Hence $N{\restriction}\delta_2$ is universal over $N{\restriction}\delta_1$ (since B_{δ_1} is universal over $N{\restriction}\delta_1$). Let $\{\delta_i : i < \theta^+\}$ enumerate C_5 and set $M'_i = N{\restriction}\delta_i$. Note that $N = \bigcup_{i<\theta^+} M'_i$. Then $\langle M'_i \in K_\theta : i < \theta^+\rangle$ is a continuous $\prec_{\mathcal{F}}$-chain of nice models, and M'_{i+1} is universal over M'_i. Therefore N and M are isomorphic (as said at the beginning of the proof), as required.

Notation 4.5. $\Theta = \{\overline{\theta} : \overline{\theta} = \langle \theta_i : i < \delta\rangle$ is a continuous (strictly) increasing sequence of cardinals, $\kappa + |T| < \theta_0$, $\delta < \theta_0$ (a limit ordinal), $\bigcup_{i<\delta} \theta_i \le \lambda\}$ and $\Theta^- = \{\overline{\theta} \in \Theta : \sup \theta_i < \lambda\}$.

Remark 4.6. Let $\theta = \sup \operatorname{rng}(\overline{\theta})$ for $\overline{\theta} \in \Theta$. Then θ is singular, since $\operatorname{cf}(\theta) \le \delta < \theta_0 \le \theta$.

Definition 4.7. Let $\overline{\theta} \in \Theta$. A model M is *$\overline{\theta}$-saturated* iff there is a continuous $\prec_{\mathcal{F}}$-chain $\langle M_i \in K_{\theta_i} : i < \delta\rangle$ such that $M = \bigcup_{i<\delta} M_i$, M_i is nice and M_{i+1} is θ_{i+1}-universal over M_i.

Definition 4.8. Suppose that $\overline{\theta} \in \Theta$. $\Pr(\overline{\theta})$ holds iff every $\overline{\theta}$-saturated model is nice.

Remark 4.9. (1) If $\overline{\theta}_1, \overline{\theta}_2 \in \Theta$, $\operatorname{rng}(\overline{\theta}_1) \subseteq \operatorname{rng}(\overline{\theta}_2)$, $\sup \operatorname{rng}(\overline{\theta}_1) = \sup \operatorname{rng}(\overline{\theta}_2)$, and M is $\overline{\theta}_2$-saturated, then M is $\overline{\theta}_1$-saturated.

(2) For $\overline{\theta} \in \Theta$ and $\Pr(\overline{\theta}')$ whenever $\overline{\theta}' \in \Theta$ is a proper initial segment of $\overline{\theta}$, there is a $\overline{\theta}$-saturated model and it is unique.

Theorem 4.10. *Suppose that T is categorical in λ, $\overline{\theta} \in \Theta^-$ and for every limit ordinal $\alpha < \lg(\overline{\theta})$, $\Pr(\overline{\theta}{\restriction}\alpha)$. Then* $\Pr(\overline{\theta})$.

Proof. By 4.9(1) and the uniqueness of $\overline{\theta}$-saturated models (4.9(2)), without loss of generality one may assume that $\operatorname{otp}(\overline{\theta}) = \operatorname{cf}(\sup \operatorname{rng}(\overline{\theta}))$. Let $\theta = \sup \operatorname{rng}(\overline{\theta})$. Now by 4.6, $(\operatorname{cf}(\theta))^+ < \theta$, so by [Sh420, 1.5 + 1.2(1)] there exists $\langle S, \langle C_\alpha : \alpha \in S\rangle\rangle$ such that

(α) $S \subset \theta^+$ is set of ordinals; $0 \notin S$;

(β) $S_1 = \{\alpha \in S : \operatorname{cf}(\alpha) = \operatorname{cf}(\theta)\}$ is a stationary subset of θ^+;

(γ) if α ($\in S$) is a limit ordinal then $\alpha = \sup C_\alpha$ and if $\alpha \in S$ then $\operatorname{otp}(C_\alpha) \le \operatorname{cf}(\theta)$;

(δ) if $\beta \in C_\alpha$, then $\beta \in S$ and $C_\beta = C_\alpha \cap \beta$;
(ε) C_α is a set of successor ordinals.

[Note that the existence of $\langle S, \langle C_\alpha : \alpha \in S \rangle \rangle$ is provable in ZFC.]

Without loss of generality, $S \setminus S_1 = \bigcup\{C_\alpha : \alpha \in S_1\}$. We shall construct the required model by induction, using $\langle C_\alpha : \alpha \in S \rangle$. Remember $\overline{\theta} = \langle \theta_\zeta : \zeta < \mathrm{cf}(\theta) \rangle$. Let us start by defining by induction on $\alpha < \theta^+$ the following entities: M_α, $M_{\alpha\xi}$ (for $\alpha < \theta^+$, $\xi < \mathrm{cf}(\theta)$), and N_α (only when $\alpha \in \bigcup_{\beta \in S} C_\beta$) such that

(Aא) $M_\alpha \in K_\theta$;
(Aב) $\langle M_\alpha : \alpha < \theta^+ \rangle$ is a continuous increasing $\prec_{\mathcal{F}}$-chain of models;
(Aג) $M_{\alpha+1}$ is nice, and if M_α is not nice, then $M_\alpha \underset{\text{nice}}{\not\preceq} M_{\alpha+1}$;
(Aד) $M_\alpha \neq M_{\alpha+1}$;
(Aה) $M_{\alpha+1}$ is weakly universal over M_α;
(Bא) $M_\alpha = \bigcup_{\xi < \mathrm{cf}(\theta)} M_{\alpha\xi}$, $\|M_{\alpha\xi}\| = \theta_\xi$;

if $\alpha \in S_1$, $\beta \in C_\alpha$, $\gamma \in C_\alpha$, $\beta < \gamma$, then

(Bב) $N_\beta \preceq_{\mathcal{F}} M_\beta$;
(Bג) $\|N_\beta\| = \theta_{\mathrm{otp}(C_\beta)}$;
(Bד) $(\forall \xi < \mathrm{otp}(C_\gamma))(M_{\beta\xi} \subseteq N_\gamma)$;
(Bה) N_β is nice;
(Bו) N_γ is $\theta_{\mathrm{otp}(C_\gamma)}$-universal over N_β.

There are now two tasks at hand. First of all, we shall explain how to construct these entities (The Construction). Then we shall use them to build a nice $\overline{\theta}$-saturated model (Proving $\Pr(\overline{\theta})$). From the uniqueness of $\overline{\theta}$-saturated models it will thus follow that $\Pr(\overline{\theta})$ holds.

The Construction.

Case (i): $\beta = 0$. Choose $M_0 \in K_\theta$ and $\langle M_{0\xi} \in K_\theta : \xi < \mathrm{cf}(\theta) \rangle$ with $M_0 = \bigcup_{\xi < \mathrm{cf}(\theta)} M_{0\xi}$ using 1.2. There is no need to define N_0 since $0 \notin C_\alpha$.

Case (ii): *β is a limit ordinal*. Let $M_\beta = \bigcup_{\gamma < \beta} M_\gamma$ and choose $\langle M_{\beta\xi} : \xi < \mathrm{cf}(\theta) \rangle$ using 1.2. Again there is no call to define N_β since C_α is always a set of successor ordinals.

Case (iii): *β is a successor ordinal*, $\beta = \gamma + 1$. Choose $M'_\gamma \in K_\theta$ such that $M_\gamma \preceq_{\mathcal{F}} M'_\gamma$ and if possible $M_\gamma \underset{\text{nice}}{\not\preceq} M'_\gamma$; without loss of generality, M'_γ is weakly universal over M_γ and is nice. If $\beta \notin S$, then define things as above, taking into account (Aג). The definitions of M_β, $M_{\beta\xi}$ present no special difficulties. Now suppose that $\beta \in S$. The problematic entity to define is N_β.

If $C_\beta = \emptyset$, choose for N_β any nice $\preceq_{\mathcal{F}}$-submodel (of power $\theta_{\mathrm{otp}(C_\beta)}$) of M_γ.

If $C_\beta \neq \emptyset$, then first define $N^-_\beta = \bigcup_{\gamma\in C_\beta} N_\gamma$. Note that N^-_β is nice. [If C_β has a last element β', then $N^-_\beta = N_{\beta'}$, which is nice; if C_β has no last element, then $N^-_\beta = \bigcup_{\gamma\in C_\beta} N_\gamma$ is $\overline{\theta}\restriction\mathrm{otp}(C_\beta)$-saturated, and, by the hypothesis of the theorem, $\Pr(\overline{\theta}\restriction\mathrm{otp}(C_\beta))$, so N^-_β is nice.] Also $N^-_\beta \preceq_{\mathcal F} M_\gamma$. If $\mathrm{otp}(C_\beta)$ is a limit ordinal we let $N_\beta = N^-_\beta$ and $M_\beta = M'_\gamma$, so we have finished, so assume $\mathrm{otp}(C_\beta)$ is a successor ordinal. To complete the definition of N_β, one requires a Lemma (the proof of which is similar to 3.9, 3.10):

(∗) *if $A \subset M \in K_\theta$ and $|A| \le \theta_j < \theta$, then there exist a nice $M^+ \in K_\theta$ with $M \preceq_{\mathcal F} M^+$, and nice models $N^*, N^+ \in K_{\theta_j}$ such that $A \subseteq N^* \preceq_{\mathcal F} N^+ \preceq_{\mathcal F} M^+$ and N^+ is universal over N^*.*

Why is this enough? Use the Lemma with $M = M'_\beta$ and $A = N^-_\beta \cup \bigcup_{\xi<\mathrm{otp}(C_\beta),\,\alpha\in C_\beta} M_{\alpha\xi}$ to find N^*, N^+, M^+ and choose N^+, M^+ as N_β, M_β respectively.

Why does (∗) hold? The proof of (∗) is easy.

Proving $\Pr(\overline{\theta})$. For $\alpha \in S_1$, consider $\langle N_\beta : \beta \in C_\alpha\rangle$. For $\beta,\gamma \in C_\alpha$ with $\beta<\gamma$, one has $\bigcup_{\xi<\mathrm{otp}(C_\beta)} M_{\beta\xi} \subseteq N_\gamma$ by (B⌉). Therefore $M_\beta \subseteq \bigcup_{\gamma\in C_\alpha} N_\gamma$ ($M_\beta = \bigcup_{\xi<\mathrm{cf}(\theta)} M_{\beta\xi} = \bigcup_{\xi<\mathrm{cf}(\alpha)} M_{\beta\xi}$ ($\alpha\in S_1$); for $\xi<\mathrm{cf}(\alpha)$, choose $\gamma \in C_\alpha$ with $\xi,\beta<\gamma$; so $M_{\beta\xi} \subseteq N_\gamma$ and $M_\beta \subseteq \bigcup_{\gamma\in C_\alpha} N_\gamma$).

Thus $M_\beta \subseteq \bigcup_{\gamma\in C_\alpha} N_\gamma$ for every $\beta\in C_\alpha$, hence $M_\alpha = \bigcup_{\beta\in C_\alpha} M_\beta \subseteq \bigcup_{\gamma\in C_\alpha} N_\gamma$ (remember $\alpha = \sup C_\alpha$ as $\alpha\in S_1$). If $\gamma \in C_\alpha$, then $N_\gamma \preceq_{\mathcal F} M_\gamma$ (by (B⊐)), and so $\bigcup_{\gamma\in C_\alpha} N_\gamma \subseteq \bigcup_{\beta\in C_\alpha} M_\beta = M_\alpha$ by continuity. So $M_\alpha = \bigcup_{\beta\in C_\alpha} N_\beta$, hence $\langle N_\beta : \beta\in C_\alpha\rangle$ exemplifies M_α is $\overline{\theta}$-saturated (remember $\Pr(\overline{\theta}\restriction\delta)$ for every limit $\delta<\lg(\overline{\theta})$; more exactly, we use $\langle N'_i : i<\theta\rangle$, $N'_i = \bigcup\{N_\beta : \beta\in C_\alpha$ and [i limit $\Rightarrow \mathrm{otp}(\beta\cap C_\alpha) < i$], [$i$ nonlimit $\Rightarrow \mathrm{otp}(\beta\cap C_\alpha) \le i$]$\}$). So M_α is $\overline{\theta}$-saturated for every $\alpha\in S_1$. In other words, $\{\alpha<\theta^+ : M_\alpha$ is $\overline{\theta}$-saturated$\} \supseteq S_1$ and is stationary, so, applying 3.1, there exists $\alpha<\theta^+$ such that M_α is $\overline{\theta}$-saturated and $M_\alpha \underset{\text{nice}}{\preceq} \bigcup_{\beta<\theta^+} M_\beta$. Hence by 1.10, $M_\alpha \underset{\text{nice}}{\preceq} M_{\alpha+1}$ and so, since $M_{\alpha+1}$ is nice (A⅃), M_α is nice (by 3.12).

We conclude that $\Pr(\overline{\theta})$ holds.

To round off this section of the paper, let us make the connection between $\overline{\theta}$-saturation and $(\theta,\mathrm{cf}(\theta))$-saturation (notation follows 4.5–4.10).

Theorem 4.11. *Assume that T is categorical in λ. Let $\overline{\theta} \in \Theta^-$ and $\theta = \sup\theta_i$. Then every $\overline{\theta}$-saturated model is $(\theta,\mathrm{cf}(\theta))$-saturated.*

Proof. Let $\langle M_\alpha : \alpha<\theta^+\rangle$ be as in the proof of 4.10. By 3.1 there exists a club C of θ^+ such that $M_\alpha \underset{\text{nice}}{\preceq} \bigcup_{\beta<\theta^+} M_\beta$ for every $\alpha\in C$, hence by the construction M_α is nice. So if $\alpha,\beta\in C$ and $\alpha<\beta$, then M_β is a universal extension of M_α and for $\gamma = \sup(\gamma \cap C)$, $\gamma \in C$, one sees that M_γ is $(\theta, \mathrm{cf}(\gamma))$-saturated. Choose $\gamma \in S_1 \cap C$ and $\sup(\gamma \cap C) = \gamma$. So M_γ is $(\theta, \mathrm{cf}(\theta))$-saturated and also $\overline{\theta}$-saturated (see proof of 4.10). Together we finish.

5. The amalgamation property for $\mathcal{K}_{<\lambda}$. Corollaries 5.4 and 5.5 are the goal of this section, showing that every element of $K_{<\lambda}$ is nice (5.4) and $\mathcal{K}_{<\lambda}$ has the amalgamation property (5.5).

LEMMA 5.1. *Suppose that $\langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ is a continuous strictly increasing sequence of ordinals, $\mu = \sup_{i<\mathrm{cf}(\mu)} \mu_i$, and $\kappa + |T| \le \mu_0 < \mu \le \lambda$. Then there exist a linear order I of power μ and a continuous increasing sequence $\langle I_i : i < \mathrm{cf}(\mu)\rangle$ of linear orders such that*

(1) $\kappa + |T| \le |I_i| \le \mu_i$;

(2) $\bigcup_{i<\mathrm{cf}(\mu)} I_i = I$;

(3) *every $t \in I_{i+1} \setminus I_i$ defines a Dedekind cut of I_i in which* (*at least*) *one side of the cut has cofinality κ.*

Proof. Let $I = (\{0\} \times \mu) \cup (\{1\} \times \kappa)$, $I_i = (\{0\} \times \mu_i) \cup (\{1\} \times \kappa)$ ordered by $(i, \alpha) <_I (j, \beta)$ iff $i < j$ or $0 = i = j$ and $\alpha < \beta$, or $1 = i = j$ and $\alpha > \beta$.

LEMMA 5.2. *Suppose that T is categorical in $\lambda > \mathrm{cf}(\lambda)$ and $\kappa + |T| < \mu \le \lambda$. If $M \in K_\lambda$, then there exists a continuous increasing $\prec_{\mathcal{F}}$-chain $\langle M_i : i < \mathrm{cf}(\lambda)\rangle$ of models such that*

(1) $M \preceq_{\mathcal{F}} \bigcup_{i<\mathrm{cf}(\lambda)} M_i$;

(2) $\|\bigcup_{i<\mathrm{cf}(\lambda)} M_i\| = \lambda$;

(3) $\kappa + |T| \le \|M_i\| < \|M_{i+1}\| < \lambda$;

(4) *for each $i < \mathrm{cf}(\lambda)$, $M_i \underset{\text{nice}}{\preceq} \bigcup_{j<\mathrm{cf}(\lambda)} M_j$.*

Proof. If λ is a limit cardinal, choose a continuous increasing sequence $\langle \mu_i : i < \mathrm{cf}(\lambda)\rangle$ with $\lambda = \sup_{i<\mathrm{cf}(\lambda)} \mu_i$ and $\kappa + |T| \le \mu_0 < \lambda$. If λ is a successor let $\mu_i = 1 + i$. Let $\langle I, \langle I_i : i < \mathrm{cf}(\lambda)\rangle\rangle$ be as in 5.1. By λ-categoricity of T, without loss of generality, $M = \mathrm{EM}(\lambda)$. Let $M_i = \mathrm{EM}(I_i)$ for $i < \mathrm{cf}(\lambda)$. Clearly (1), (2) and (3) hold. To obtain (4), observe that by 2.6 and 3.5 it suffices to remark that by demand (3) from 5.1 on $\langle I_i : i < \mathrm{cf}(\lambda)\rangle$ clause ($\aleph$) or ($\beth$) in 2.6 holds for each $t \in I \setminus I_i$.

THEOREM 5.3. *For every $\mu \in [\kappa + |T|, \lambda]$ and $M \in K_\mu$, there exists $M' \in K_\mu$ with $M \preceq_{\mathcal{F}} M'$ such that*

$(*)_{M'}$ *for every $A \subset |M'|$ with $|A| < \lambda$ and $|A| \le \mu$, there is $N \in K_{\kappa+|T|+|A|}$ such that $A \subset N \preceq_{\mathcal{F}} M'$ and N is nice.*

P r o o f. The proof is by induction on μ.

C a s e 1: $\mu = \kappa + |T|$. By 3.3 there is $M' \in K_\mu$ such that $M \preceq_{\mathcal{F}} M'$ and M' is nice. Given $A \subset |M'|$ let $N = M'$ and note that N is as required in $(*)_{M'}$.

C a s e 2: $\kappa + |T| < \mu$. Without loss of generality, one can replace M by any $\prec_{\mathcal{F}}$-extension in K_μ. Choose a continuous increasing sequence $\langle \mu_i : i < \mathrm{cf}(\mu)\rangle$ such that if μ is a limit cardinal it is a strictly increasing sequence with limit μ; if μ is a successor, use $\mu_i^+ = \mu$ and in both cases $\kappa + |T| \le \mu_i < \mu$. Find $\overline{M} = \langle M_i : i < \mathrm{cf}(\mu)\rangle$ such that

(a) $M \preceq_{\mathcal{F}} \bigcup_{i<\mathrm{cf}(\mu)} M_i$;
(b) $\|\bigcup_{i<\mathrm{cf}(\mu)} M_i\| = \mu$;
(c) $\|M_i\| = \mu_i$;
(d) $M_i \underset{\text{nice}}{\preceq} \bigcup_{j<\mathrm{cf}(\mu)} M_j$.

Why does $\overline{M}$ exist? If $\mu = \lambda$ by 5.2, otherwise by 4.4 (μ regular) and 4.11 (μ singular).

Choose by induction on $i < \mathrm{cf}(\mu)$ models L_i^0, L_i^1, L_i^2 in that order such that

(א) $M_i \preceq_{\mathcal{F}} L_i^0 \preceq_{\mathcal{F}} L_i^1 \preceq_{\mathcal{F}} L_i^2 \in K_{\mu_i}$;
(ב) $j < i \Rightarrow L_j^2 \preceq_{\mathcal{F}} L_i^0$;
(ג) $(*)_{L_i^1}$ holds, i.e. for each $A \subset |L_i^1|$, there is $N \in K_{\le \kappa+|T|+|A|}$ such that $A \subset N \preceq_{\mathcal{F}} L_i^1$ and N is nice (so in particular L_i^1 is nice, letting $A = |L_i^1|$);
(ד) L_i^2 is nice and μ_i-universal over L_i^1;
(ה) L_i^0 is increasing continuous;
(ו) $L_i^l \cap \bigcup_{j<\mathrm{cf}(\mu)} M_j = M_i$ (or use a system of $\preceq_{\mathcal{F}}$-embeddings).

For $i = 0$, let $L_i^0 = M_0$. For $i = j+1$, note that by 2.1 there is an amalgam $L_i^0 \in K_{\mu_i}$ of M_i, L_j^2 over M_j since $M_j \underset{\text{nice}}{\preceq} M_i$ and $M_j \preceq_{\mathcal{F}} L_j^2$ (use the last phrase of 2.1 for clause (ו)); actually not really needed. For limit i, continuity necessitates choosing $L_i^0 = \bigcup_{j<i} L_j^0$ (note that in this case $L_i^0 = \bigcup_{j<i} L_j^2$). To choose L_i^1 apply the inductive hypothesis with respect to μ_i, L_i^0 to find L_i^1 so that $L_i^0 \preceq_{\mathcal{F}} L_i^1$ and (ג)$(*)_{(L_i^1)}$ holds. To choose L_i^2 apply 3.10 to $L_i^1 \in K_{\mu_i}$ giving $L_i^1 \preceq_{\mathcal{F}} L_i^2$, L_i^2 is nice and μ_i-universal over L_i^1 (so (ד) holds).

Let $L = \bigcup_{i<\mathrm{cf}(\mu)} L_i^0 = \bigcup_{i<\mathrm{cf}(\mu)} L_i^1 = \bigcup_{i<\mathrm{cf}(\mu)} L_i^2$, and let $L_i = L_i^0$ if i is a limit, L_i^1 otherwise. Now show by induction L_i is nice. [Why? show by induction on i for $i = 0$ or i successor that $L_i = L_i^1$, hence use clause (ג); if i is limit then L_i is $(\overline{\theta}\restriction i)$-saturated, hence L_i is nice by 4.8, 4.10.] Now $\langle L_i : i < \mathrm{cf}(\mu)\rangle$ witnesses that if μ is regular, then L is (μ,μ)-saturated by 4.4 and if μ is singular, then L is $\overline{\mu}$-saturated; in all cases L is $\overline{\mu}$-saturated of power μ, hence by the results of Section 4 (i.e. 4.8, 4.10), if $\mu < \lambda$ then L is nice.

Claim. *$M' = L$ is as required.*

Proof. $M \preceq_{\mathcal{F}} \bigcup_{i<\mathrm{cf}(\mu)} M_i \preceq_{\mathcal{F}} \bigcup_{i<\mathrm{cf}(\mu)} L_i^0 = L \in K_\mu$. Suppose that $A \subset |L|$. If $|A| = \mu$, then necessarily $\mu < \lambda$ and we take $N = L$. So without loss of generality, $|A| < \mu$. If $\mu = \mathrm{cf}(\mu)$ or $|A| < \mathrm{cf}(\mu)$, then there is $i < \mathrm{cf}(\mu)$ such that $A \subset L_i^1$ and, by $(\beth), (*)_{L_i^1}$ holds, so there is $N \in K_{\kappa+|T|+|A|}$ such that $A \subset N \preceq_{\mathcal{F}} L_i^1, N$ is nice and $N \preceq_{\mathcal{F}} L$ as required. So suppose that $\mathrm{cf}(\mu) \le |A| < \mu$. Choose by induction on $i < \mathrm{cf}(\mu)$ models N_i^0, N_i^1, N_i^2 in that order such that

(α) $N_i^0 \preceq_{\mathcal{F}} N_i^1 \preceq_{\mathcal{F}} N_i^2$;
(β) $N_i^2 \preceq_{\mathcal{F}} N_{i+1}^0$;
(γ) $A \cap L_i^0 \subseteq N_i^0 \preceq_{\mathcal{F}} L_i^0$;
(δ) $N_i^1 \preceq_{\mathcal{F}} L_i^1$ and N_i^1 is nice;
(ε) $N_i^2 \preceq_{\mathcal{F}} L_i^2$, N_i^2 is nice and universal over N_i^1;
(ζ) N_i^0, N_i^1, N_i^2 have power at most $\min\{|T| + \kappa + |A|, \mu_i\}$.

For $i = 0$, apply 1.2 for $A \cap L_0^0, L_0^0$; for $i = j + 1$, apply 1.2 to find $N_i^0 \in K_{\mu_i}$ such that $(A \cap L_i^0) \cup N_j^2 \subset N_i^0 \preceq_{\mathcal{F}} L_i^0$ (in particular, $N_j^2 \preceq_{\mathcal{F}} N_i^0$); for limit i, $N_i^0 = \bigcup_{j<i} N_j^0$. To choose N_i^1, use $(*)_{L_i^1}$ for the set $A_i = N_i^0$ to find a nice $N_i^1 \in K_{\le\kappa+|T|+|A|}$ with $N_i^0 \preceq_{\mathcal{F}} N_i^1 \preceq_{\mathcal{F}} L_i^1$. Note that $\|N_i^1\| \le \mu_i$. Finally, to choose N_i^2 note that by 3.9 the model N_i^1 has a nice extension N_i^+ (of power $\|N_i^1\|$) weakly universal over N_i^1. Now N_i^1 is nice, hence N_i^2 is universal over N_i^1 (by 3.6A(5)) and by 2.1 there is an amalgam N_i of N_i^+, L_i^1 over N_i^1 such that $\|N_i\| \le \mu_i$. Since L_i^2 is universal over L_i^1 one can find an $\mathcal{F}$-elementary submodel N_i^2 of L_i^2 isomorphic to N_i. Let N_i be N_i^0 if i is a limit, N_i^1 otherwise; prove by induction on i that N_i is nice (by 4.2).

Now $\bigcup_{i<\mathrm{cf}(\mu)} N_i^0$ is an $\mathcal{F}$-elementary submodel of L of power at most $\kappa + |T| + |A|$, including A (by (γ)) and $\bigcup_{i<\mathrm{cf}(\mu)} N_i^0$ is $(\kappa + |T| + |A|, \mathrm{cf}(\mu))$-saturated, hence (by 4.2) nice, as required.

Corollary 5.4. *Suppose that T is categorical in λ. Then every element of $K_{<\lambda}$ is nice.*

Proof. Suppose otherwise and let $N_0 \in K_{<\lambda}$ be a model which is not nice. Choose a suitable Op such that $\|\mathrm{Op}(N_0)\| \ge \lambda$ and by 1.2 find $M_0 \in K_\lambda$ with $N_0 \preceq_{\mathcal{F}} M_0 \preceq_{\mathcal{F}} \mathrm{Op}(N_0)$, i.e. $N_0 \underset{\text{nice}}{\preceq} M_0$. It follows that

(+) if $N_0 \preceq_{\mathcal{F}} N \preceq_{\mathcal{F}} M_0$ and $N \in K_{<\lambda}$ then N is not nice.

[Why? By 4.3; alternatively suppose by contradiction that N is nice. So there is $N_1 \in K_{<\lambda}$ such that $N_0 \preceq_{\mathcal{F}} N_1$, $N_0 \underset{\text{nice}}{\not\preceq} N_1 \cdot N_0 \underset{\text{nice}}{\preceq} N$ since $N_0 \underset{\text{nice}}{\preceq} M_0$ and

$N \preceq_{\mathcal{F}} M_0$, hence there is an amalgam $N' \in K_{<\lambda}$ of N_1, N over N_0. Since N is nice, $N \underset{\text{nice}}{\preceq} N'; N_0 \underset{\text{nice}}{\preceq} N$, $N_0 \underset{\text{nice}}{\preceq} N'$ and so $N_0 \underset{\text{nice}}{\preceq} N_1$, a contradiction.] On the other hand, applying 5.3 for $\mu = \lambda$ there exists $M' \in K_\lambda$ satisfying $(*)_{M'}$. By λ-categoricity of T, without loss of generality, $(*)_{M_0}$ holds (see 5.3) and for $A = |N_0|$ yields a nice model $N \in K_{\kappa+|T|+\|N_0\|}$ such that $N_0 \preceq_{\mathcal{F}} N \preceq_{\mathcal{F}} M_0$, contradicting (+).

COROLLARY 5.5. *Suppose that T is categorical in λ. Then $\mathcal{K}_{<\lambda}$ has the amalgamation property.*

Proof. 2.1 and the previous corollary.

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