Categoricity of theories in $L_{\kappa\omega}$, when
$\kappa$ is a measurable cardinal. Part 1

by

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Abstract. We assume a theory $T$ in the logic $L_{\kappa\omega}$ is categorical in a cardinal $\lambda \geq \kappa$, and $\kappa$ is a measurable cardinal. We prove that the class of models of $T$ of cardinality $< \lambda$ (but $\geq |T| + \kappa$) has the amalgamation property; this is a step toward understanding the character of such classes of models.

Annotated content

0. Introduction

1. Preliminaries. We review material on fragments $\mathcal{F}$ of $L_{\kappa\aleph_0}$ (including the theory $T$) and basic model theoretic properties (Tarski–Vaught property and L.S.), define amalgamation, indiscernibles and E.M. models, then limit ultrapowers which are suitable (for $L_{\kappa\omega}$) and in particular ultralimits. We then introduce a notion basic for this paper: $M \preceq F N$ if there is an $\preceq F$-embedding of $N$ into a suitable ultralimit of $M$ extending the canonical one.

2. The amalgamation property for regular categoricity. We first get amalgamation in $(K_{\lambda}, \preceq F)$ when one of the extensions is nice (2.1). We prove that if $T$ is categorical in the regular $\lambda > |\mathcal{F}| + \kappa$, then $(K_{<\lambda}, \preceq F)$ has the amalgamation property. For this we show that nice extensions (in $K_{<\lambda}$) preserve being non-amalgamation basis. We also start investigating (in 2.5) the connection between extending the linear order $I$ and the model $EM(I)$. $I \subseteq J \Rightarrow EM(I) \preceq EM(J)$; and give sufficient condition for $I \subseteq J$ (2.6). From this we get in $K_\lambda$ a model such that any submodel of an expansion is a $\preceq F$-submodel (in 2.7, 2.10(2)), and conclude the amalgamation property in $(K_{<\lambda}, \preceq F)$ nice when $\lambda$ is regular (in 2.9) and something for singulars (2.10).

3. Towards removing the assumption of regularity from the existence of universal extensions. The problem is that $EM(\lambda)$ has many models which “sit” well in

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it and many which are amalgamation bases but we need to get this simultaneously. First (3.1) we show that if \( \langle M_i : i < \theta^+ \rangle \) is an \(<\mathcal{F}\)-increasing continuous sequence of models of \( K_\theta \subseteq K = \text{Mod}(T) \) then for a club of \( i < \theta^+ \), \( M_i \preceq \bigcup_{\text{nice}} \{ M_j : j < \theta^+ \} \). We define nice models (Def. 3.2; essentially, every reasonable extension is nice), show a variant is equivalent (3.4), and implies being an amalgamation base (3.5), and we prove that in \( K_\theta \) the nice models are dense (3.3). Then we define a universal extension of \( M \in K_\theta \) in \( K_\sigma \) (Def. 3.6), prove existence inside a model (3.7), and after preparation (3.8) prove existence (3.9, 3.10, 3.11).

4. \((\theta, \sigma)\)-saturated models. If \( M_i \in K_\theta \) for \( i \leq \sigma \) is increasing continuous, with \( M_{i+1} \) universal over \( M_i \), and each \( M_i \) nice, then \( M_\sigma \) is \((\theta, \sigma)\)-saturated over \( M_0 \). We show existence (and uniqueness). We connect this to a more usual saturation and prove that \((\theta, \sigma)\)-saturation implies niceness (in 4.10).

5. The amalgamation property for \( K_{< \lambda} \). After preliminaries we prove that for \( \theta \leq \lambda \) (and \( \theta \geq |\mathcal{F}| + \kappa \) of course) every member of \( K_\theta \) can be extended to one with many nice submodels; this is done by induction on \( \theta \) using the niceness of \((\theta_1, \sigma_1)\)-saturated models. Lastly, we conclude that every \( M \in K_{< \lambda} \) is nice hence \( K_{< \lambda} \) has the amalgamation property.

0. Introduction. The main result of this paper is a proof of the following theorem:

**Theorem.** Suppose that \( T \) is a theory in a fragment of \( L_{\kappa \omega} \), where \( \kappa \) is a measurable cardinal. If \( T \) is categorical in the cardinal \( \lambda > \kappa + |T| \), then \( K_{< \lambda} \), the class of models of \( T \) of power strictly less than \( \lambda \) (but \( \geq \kappa + |T| \)), has the amalgamation property (see Definition 1.3).

The interest of this theorem stems in part from its connection with the study of categoricity spectra. For a theory \( T \) in a logic \( \mathcal{L} \) let us define \( \text{Cat}(T) \), the categoricity spectrum of \( T \), to be the collection of those cardinals \( \lambda \) in which \( T \) is categorical. In the 1950’s Łoś conjectured that if \( T \) is a countable theory in first-order logic, then \( \text{Cat}(T) \) contains every uncountable cardinal or no uncountable cardinal. This conjecture, based on the example of algebraically closed fields of fixed characteristic, was verified by Morley [M], who proved that if a countable first-order theory is categorical in some uncountable cardinal, then it is categorical in every uncountable cardinal. Following advances made by Rowbottom [Ro], Ressayre [Re] and Shelah [Sh1], Shelah [Sh31] proved the Łoś conjecture for uncountable first-order theories: if \( T \) is a first-order theory categorical in some cardinal \( \lambda > |T| + \aleph_0 \), then \( T \) is categorical in every cardinal \( \lambda > |T| + \aleph_0 \).

It is natural to ask whether analogous results hold for theories in logics other than first-order logic. Perhaps the best-known extensions of first-order logic are the infinitary logics \( L_{\kappa \lambda} \). As regards theories in \( L_{\omega_1 \omega} \), Shelah [Sh87] continuing work begun in [Sh48] introduced the concept of excellent classes: these have models in all cardinalities, have the amalgamation property and
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satisfy the Łoś conjecture. In particular, if $\varphi$ is an excellent sentence of $L_{\omega_1}$, then the Łoś conjecture holds for $\varphi$. Furthermore, under some set-theoretic assumptions (weaker than the Generalized Continuum Hypothesis), if $\varphi$ is a sentence in $L_{\omega_1}$ which is categorical in $\aleph_n$ for every natural number $n$ (or even just if $\varphi$ is a sentence in $L_{\omega_1}$ with at least one uncountable model not having too many models in each $\aleph_n$), then $\varphi$ is excellent.

Now [Sh300], [Sh-h] try to develop classification theory in some non-elementary classes. We cannot expect much for $L_{\kappa_\lambda}$ for $\lambda > \aleph_0$. Shelah conjectured that if $\varphi$ is a sentence in $L_{\omega_1}$ categorical in some $\lambda \geq \beth_1$, then $\varphi$ is categorical in every $\lambda \geq \beth_1$. (Recall that the Hanf number of $L_{\omega_1}$ is $\beth_1$, so if $\psi$ is a sentence in $L_{\omega_1}$ and $\psi$ has a model of power $\lambda \geq \beth_1$, then $\psi$ has a model in every power $\lambda \geq \beth_1$. See [K].) There were some who asked why so tardy a beginning. Recent work of Hart and Shelah [HaSh323] showed that for every natural number $k$ greater than 1 there is a sentence $\psi_k$ in $L_{\omega_1}$ which is categorical in the cardinals $\aleph_0, \ldots, \aleph_{k-1}$, but which has many models of power $\lambda$ for every cardinal $\lambda \geq 2^{\aleph_k-1}$.

As regards theories in $L_{\kappa_\omega}$, progress has been recorded under the assumption that $\kappa$ is a strongly compact cardinal. Under this assumption Shelah and Makkai [MaSh285] have established the following results for a $\lambda$-categorical theory $T$ in a fragment $\mathcal{F}$ of $L_{\kappa_\omega}$: (1) if $\lambda$ is a successor cardinal and $\lambda > ((\kappa')^+)$, where $\kappa' = \max(\kappa, |\mathcal{F}|)$, then $T$ is categorical in every cardinal greater than or equal to $\min(\lambda, \beth_{(2^{\kappa')^+})}$, (2) if $\lambda > \beth_{\kappa+1}(\kappa')$, then $T$ is categorical in every cardinal of the form $\beth_\delta$ with $\delta$ divisible by $(2^{\kappa'})^+$ (i.e. for some ordinal $\alpha > 0$, $\delta = (2^{\kappa'})^+ \cdot \alpha$ (ordinal multiplication)).

In proving theorems of this kind, one has recourse to the amalgamation property which makes possible the construction of analogues of saturated models. In turn, these are of major importance in categoricity arguments. The amalgamation property holds for theories in first-order logic [CK] and in $L_{\kappa\kappa}$ when $\kappa$ is a strongly compact cardinal (see [MaSh285]: although $\prec_{L_{\kappa_\omega}}$ fails the Tarski–Vaught property for unions of chains of length $\kappa$ (whereas $\prec_{L_{\omega_1}}$ has it), under a categoricity assumption it can be shown that $\prec_{L_{\kappa_\omega}}$ and $\prec_{L_{\kappa_\kappa}}$ coincide). However, it is not known in general for theories in $L_{\kappa_\omega}$ or $L_{\kappa\kappa}$ when one weakens the assumption on $\kappa$, in particular when $\kappa$ is just a measurable cardinal. Nevertheless, categoricity does imply the existence of reasonably saturated models in an appropriate sense, and it is possible to begin classification theory. This is why the main theorem of the present paper is of relevance regarding the categoricity spectra of theories in $L_{\kappa_\omega}$ when $\kappa$ is measurable.

A sequel to this paper under preparation tries to provide a characterization of $\text{Cat}(T)$ at least parallel to that in [MaSh285] and we hope to deal
with the corresponding classification theory later. This division of labor both
respects historical precedent and is suggested by the increasing complexity
of the material. Another sequel deals with abstract elementary classes (in
the sense of [Sh88]) (see [Sh472], [Sh394] respectively). On later work see
[Sh576], [Sh600].

The paper is divided into five sections. Section 1 is preliminary and no-
tional. In Section 2 it is shown that if \( T \) is categorical in the regular
cardinal \( \lambda > \kappa + |T| \), then \( K_{<\lambda} \) has the amalgamation property. Section 3
deals with weakly universal models, Section 4 with \((\theta, \sigma)\)-saturated and
\( \bar{\theta} \)-saturated models. In Section 5 the amalgamation property for \( K_{<\lambda} \) is
established.

All the results in this paper (other than those explicitly credited) are
due to Saharon Shelah.

1. Preliminaries. To start things off in this section, let us fix notation,
provide basic definitions and well-known facts, and formulate our working
assumptions.

The working assumptions in force throughout the paper are these.

Assumption 1. The cardinal \( \kappa \) is an uncountable measurable cardinal,
and so there is a \( \kappa \)-complete nonprincipal ultrafilter on \( \kappa \).

Assumption 2. The theory \( T \) is a theory in the infinitary logic \( L_{\kappa\omega} \).

From these assumptions follow certain facts, of which the most important
are these.

Fact 1. For each model \( M \) of \( T \), \( \kappa \)-complete ultrafilter \( D \) over \( I \) and
suitable set \( G \) of equivalence relations on \( I \times I \) (see 1.7.4) the limit ultrapower
\( \text{Op}(M) = \text{Op}(M,I,D,G) \) is a model of \( T \).

Fact 2. For each linear order \( I = (I, \leq) \) there exists a generalized
Ehrenfeucht–Mostowski model \( \text{EM}(I) \) of \( T \).

The remainder of this section provides more detailed explanations and
references.

Relevant set-theoretic and model-theoretic information on measurable
cardinals can be found in [J], [CK] and [D]. \( L \) denotes a language, i.e. a
set of finitary relation and function symbols, including equality. \( |L| \) is the
cardinality of the language \( L \). For a cardinal \( \lambda \leq \kappa, L_{\kappa\lambda} \) is the smallest set of
(infinitary) formulas in the language \( L \) which contains all first-order formulas
and which is closed under (1) the formation of conjunctions (disjunctions)
of any set of formulas of power less than \( \kappa \), provided that the set of free
variables in the conjunctions (disjunctions) has power less than \( \lambda \); (2) the
formation of \( \forall \varphi \), \( \exists \varphi \), where \( \varphi = (x_\alpha : \alpha < \lambda') \) is a sequence of variables
of length \( \lambda' < \lambda \), \(|K|\) and \( |D| \) are comprehensive references for \( L_{\omega_1\omega} \) and
$L\kappa\lambda$ respectively.) Whenever we use the notation $\varphi(\overline{x})$ to denote a formula in $L\kappa\lambda$, we mean that $\overline{x}$ is a sequence $\langle x_\alpha : \alpha < \lambda' \rangle$ of variables of length $\lambda' < \lambda$, and all the free variables of $\varphi(\overline{x})$ are among $\overline{x} = \langle x_\alpha : \alpha < \lambda' \rangle$. So if $\varphi(\overline{x})$ is a formula in $L\kappa\omega$, then $\overline{x}$ is a finite sequence of variables.

$F$ denotes a fragment of $L\kappa\omega$, i.e. a set of formulas of $L\kappa\omega$, which contains all atomic formulas of $L$, and which is closed under negations, finite conjunctions (finite disjunctions), and the formation of subformulas. An $F$-formula is just an element of $F$.

$T$ is a theory in $L\kappa\omega$, so there is a fragment $F$ of $L\kappa\omega$ such that $T \subset F$ and $|F| < |T|^+ + \kappa$.

Models of $T$ (invariably referred to as models) are $L$-structures which satisfy the sentences of $T$. They are generally denoted $M, N, \ldots$; $|M|$ is the universe of the $L$-structure $M$; $|M|$ is the cardinality of $|M|$. For a set $A$, $|A|$ is the cardinality of $A$. $<^\omega A$ is the set of finite sequences in $A$ and for $\overline{a} = \langle a_1, \ldots, a_n \rangle \in <^\omega A$, $\lg(\overline{a}) = n$ is the length of $\overline{a}$. Similarly, if $\overline{a} = \langle a_\zeta : \zeta < \delta \rangle$, we write $\lg(\overline{a}) = \delta$, where $\delta$ is an ordinal. For an element $R$ of $L$, $\text{val}(M, R)$, or $R^M$, is the interpretation of $R$ in the $L$-structure $M$.

We ignore models of power less than $\kappa$. $K$ is the class of all models of $T$;

$$K_\kappa = \{ M \in K : |M| = \kappa \},$$

$$K_{<\lambda} = \bigcup_{\mu < \lambda} K_\mu, \quad K_{\leq \lambda} = \bigcup_{\mu \leq \lambda} K_\mu, \quad K_{[\mu, \lambda)} = \bigcup_{\mu \leq \chi < \lambda} K_\chi.$$

We write $f : M \to_N N$ (abbreviated $f : M \to N$) to mean that $f$ is an $F$-elementary embedding (briefly, an embedding) of $M$ into $N$, i.e. $f$ is a function with domain $|M|$ into $|N|$ such that for every $F$-formula $\varphi(\overline{x})$, and $\overline{a} \in <^\omega |M|$ with $\lg(\overline{a}) = \lg(\overline{x})$, $M \models \varphi[\overline{a}]$ iff $N \models \varphi[f(\overline{a})]$, where if $\overline{a} = \langle a_i : i < n \rangle$, then $f(\overline{a}) := (f(a_i) : i < n)$. In the special case where the embedding $f$ is a set-inclusion (so that $|M| \subset |N|$), we write $M \prec_N N$ (briefly $M \preceq_N N$) instead of $f : M \to_N N$ and we say that $M$ is an $F$-elementary submodel of $N$, or $N$ is an $F$-elementary extension of $M$.

$(I, \leq_I, (J, \leq_J), \ldots$ are partial orders; we will not bother to subscript the order relation unless really necessary; we write $I$ for $(I, \leq_I)$. $(I, \leq_I)$ is directed iff for every $i_1$ and $i_2$ in $I$, there is $i \in I$ such that $i_1 \leq i$ and $i_2 \leq i$. $(I, \leq_I)$ is the (reverse) linear order $(I^*, \leq^*)$, where $I^* = I$ and $s \leq^* t$ iff $t \leq s$.

A set $\{ M_i : i \in I \}$ of models indexed by $I$ is a $\prec_F$-directed system iff $(I, \leq_I)$ is a directed partial order and for $i \leq j$ in $I$, $M_i \prec_N M_j$. The union $\bigcup_{i \in I} M_i$ of a $\prec_F$-directed system $\{ M_i : i \in I \}$ of $L$-structures is an $L$-structure. In fact, more is true.

**Fact 1.1** (Tarski–Vaught property). (1) The union of a $\prec_F$-directed system $\{ M_i : i \in I \}$ of models of $T$ is a model of $T$, and for every $j \in I$, $M_j \prec_N \bigcup_{i \in I} M_i$.\*
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(2) If $M$ is a fixed model of $T$ such that for every $i \in I$ there is $f_i : M_i \rightarrow M$ and for all $i \leq j$ in $I$, $f_i \subseteq f_j$, then $\bigcup_{i \in I} f_i : \bigcup_{i \in I} M_i \rightarrow M$. In particular, if $M_i \prec M$ for every $i \in I$, then $\bigcup_{i \in I} M_i \prec M$.

Let $\alpha$ be an ordinal. A $\prec_f$-chain of models of length $\alpha$ is a sequence $(M_\beta : \beta < \alpha)$ of models such that if $\beta < \gamma < \alpha$, then $M_\beta \prec_f M_\gamma$. The chain is continuous if for every limit ordinal $\beta < \alpha$, $M_\beta = \bigcup_{\gamma < \beta} M_\gamma$.

**Fact 1.2** (Downward Loewenheim–Skolem property). Suppose that $M$ is a model of $T$, $A \subseteq |M|$ and $\max(\kappa + |T|, |A|) \leq \lambda \leq \|M\|$. Then there is a model $N$ such that $A \subseteq |N|$, $\|N\| = \lambda$ and $N \prec_f M$.

Finally, $\lambda > \kappa + |T|$ usually denotes a power in which $T$ is categorical.

Now we turn from the rather standard model-theoretic background to the more specific concepts which are central in our investigation.

**Definition 1.3.** (1) Suppose that $<$ is a binary relation on a class $K$ of models. $K = (K, <)$ has the amalgamation property (AP) iff for every $M, M_1, M_2 \in K$, if $f_i$ is an isomorphism from $M$ onto $\text{rng}(f_i)$ and $\text{rng}(f_i) < M_i$ for $i = 1, 2$, then there exist $N \in K$ and isomorphisms $g_i$ from $M_i$ onto $\text{rng}(g_i)$ for $i = 1, 2$ such that $\text{rng}(g_i) < N$ and $g_1 f_1 = g_2 f_2$. The model $N$ is called an amalgam of $M_1, M_2$ over $M$ with respect to $f_1, f_2$.

(2) An $L$-structure $M$ is an amalgamation base (a.b.) for $K = (K, <)$ iff $M \in K$ and whenever for $i = 1, 2, M_i \in K$ and $f_i$ is an isomorphism from $M$ onto $\text{rng}(f_i)$ with $\text{rng}(f_i) < M_i$, then there exist $N \in K$ and isomorphisms $g_i$ for $i = 1, 2$ from $M_i$ onto $\text{rng}(g_i)$ such that $\text{rng}(g_i) < N$ and $g_1 f_1 = g_2 f_2$.

So $K = (K, <)$ has AP iff every model in $K$ is an a.b. for $K$.

**Example 1.3A.** Suppose that $T$ is a theory in first-order logic having an infinite model. Define, for $M, N$ in the class $K_{\leq |T| + \aleph_0}$ of models of $T$ of power at most $|T| + \aleph_0$, $M < N$ iff the identity is an embedding of $M$ into an elementary submodel of $N$. Then $K_{\leq |T| + \aleph_0} = (K_{\leq |T| + \aleph_0}, <)$ has AP (see [CK]).

**Example 1.3B.** Suppose that $T$ is a theory in $L_{\kappa, \omega}$ and $\mathcal{F}$ is a fragment of $L_{\kappa, \omega}$ containing $T$ with $|\mathcal{F}| < |T|^+ + \kappa$. Let $<$ be the binary relation $\prec_f$ defined on the class $K$ of all models of $T$. $M \in K$ is an a.b. for $K$ iff whenever for $i = 1, 2, M_i \in K$ and $f_i$ is an $\prec_f$-elementary embedding of $M$ into $M_i$, there exist $N \in K$ and $\mathcal{F}$-elementary embeddings $g_i$ $(i = 1, 2)$ of $M_i$ into $N$ such that $g_1 f_1 = g_2 f_2$.

**Definition 1.4.** Suppose that $<$ is a binary relation on a class $K$ of models. Let $\mu$ be a cardinal. $M \in K_{\leq \mu}$ is a $\mu$-counter amalgamation basis ($\mu$-c.a.b.) of $K = (K, <)$ iff there are $M_1, M_2 \in K_{\leq \mu}$ and isomorphisms $f_i$ from $M$ into $M_i$ such that
(A) \( \text{rng}(f_i) < M_i \) \( (i = 1, 2) \),
(B) there is no amalgam \( N \in K_{\leq \mu} \) of \( M_1, M_2 \) over \( M \) with respect to \( f_1, f_2 \).

Observation 1.5. Suppose that \( T, F \) and \( < \) are as in 1.3B and \( \kappa + |T| \leq \mu < \lambda \). Note that if there is an amalgam \( N' \) of \( M_1, M_2 \) over \( M \) (for \( M_1, M_2, M \) in \( K_{\leq \mu} \)), then by 1.2 there is an amalgam \( N \in K_{\leq \mu} \) of \( M_1, M_2 \) over \( M \).

\textbf{Indiscernibles and Ehrenfeucht–Mostowski structures.} The basic results on generalized Ehrenfeucht–Mostowski models can be found in [Sh-a] or [Sh-c, Ch. VII]. We recall here some notation. Let \( I \) be a class of models which we call the \textit{index models}. Denote the members of \( I \) by \( I, J, \ldots \). For \( I \in I \)
we say that \( \langle \pi_s : s \in I \rangle \) is \textit{indiscernible} in \( M \) iff for every \( \bar{s}, \bar{t} \in \text{<}_I \) realizing the same atomic type in \( I, \pi_t \) and \( \pi_t \) realize the same type in \( M \) (where \( \pi_{s_0, \ldots, s_n} = \pi_{s_0} \land \ldots \land \pi_{s_n} \)). If \( L \subseteq L' \) are languages and \( \Phi \) is a function with domain including \( \{ \text{tp}_{at}(\pi, \emptyset, I) : \pi \in \text{<}_I \} \) and \( I \in I \), we let \( EM'(I, \Phi) \) be an \( L' \)-model generated by \( \bigcup_{s \in I} \pi_s \) such that \( \text{tp}_{at}(\pi_s, \emptyset, M) = \Phi(\text{tp}_{at}(\pi, \emptyset, I)) \). We say that \( \Phi \) is \textit{proper} for \( I \) if for every \( I \in I \), \( EM'(I, \Phi) \) is well defined.

Let \( EM(I, \Phi) \) be the \( L \)-reduct of \( EM'(I, \Phi) \). For the purposes of this paper we will let \( I \) be the class \( \mathbf{LO} \) of linear orders and \( \Phi \) will be proper for \( \mathbf{LO} \). For \( I \in \mathbf{LO} \) we abbreviate \( EM'(I, \Phi) \) by \( EM'(I) \) and \( EM(I, \Phi) \) by \( EM(I) \).

Claim 1.6A. For each linear order \( I = (I, \leq) \) there exists a generalized Ehrenfeucht–Mostowski model \( EM(I) \) of \( T \) (see Nadel [N] and Dickmann [D1] or [Sh-c, VII, §5]; there are “large” models by using limit ultrapowers, see 1.12).

Let \( F \) be a fragment of \( L_{\kappa\omega} \). Recall that a theory \( T \subset F \) is called a \textit{universal theory} in \( L_{\kappa\omega} \) iff the axioms of \( T \) are sentences of the form \( \forall \bar{x} \varphi(\bar{x}) \), where \( \varphi(\bar{x}) \) is a quantifier-free formula in \( L_{\kappa\omega} \).

\textbf{Definition and Proposition 1.6.} Suppose that \( T \) is a theory such that \( T \subset F \), where \( F \) is a fragment of \( L_{\kappa\omega} \). There are a (canonically constructed) finitary language \( L_{sk} \) and a universal theory \( T_{sk} \) in \( L_{\kappa\omega} \) such that:

(0) \( L \subset L_{sk} \) and \( |L_{sk}| \leq |F| + \aleph_0 \);
(1) the \( L \)-reduct of any \( L_{sk} \)-model of \( T_{sk} \) is a model of \( T \);
(2) whenever \( N_{sk} \) is an \( L_{sk} \)-model of \( T_{sk} \) and \( M_{sk} \) is a substructure of \( N_{sk} \), then \( M_{sk} \upharpoonright L \preceq_{F} N_{sk} \upharpoonright L \);
(3) any \( L \)-model of \( T \) can be expanded to an \( L_{sk} \)-model of \( T_{sk} \);
(4) if \( M \preceq_{F} N \), then there are \( L_{sk} \)-expansions \( M_{sk}, N_{sk} \) of \( M, N \) respectively such that \( M_{sk} \) is a substructure of \( N_{sk} \) and \( N_{sk} \) is a model of \( T_{sk} \).
(5) to any $\mathcal{F}$-formula $\varphi(\bar{x})$, there corresponds a quantifier-free formula $\varphi^{sf}(\bar{x})$ of $(L_{sk})_{K\omega}$ such that 

\[ T_{sk} \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \varphi^{sf}(\bar{x})). \]

**Limit ultrapowers, iterated ultrapowers and nice extensions.** An important technique we shall use in studying the categoricity spectrum of a theory in $L_{K\omega}$ is the limit ultrapower. It is convenient to record here the well-known definitions and properties of limit and iterated ultrapowers (see Chang and Keisler [CK] and Hodges and Shelah [HoSh109]) and then to examine nice extensions of models.

**Definition 1.7.1.** Suppose that $M$ is an $L$-structure, $I$ is a nonempty set, $D$ is an ultrafilter on $I$, and $G$ is a filter on $I \times I$. For each $g \in I\{M\}$, let $eq(g) = \{(i, j) \in I \times I : g(i) = g(j)\}$ and $g/D = \{f \in I\{M\} : g = f \mod D\}$, where $g = f \mod D$ iff $\{i \in I : g(i) = f(i)\} \in D$. Let $\prod_{D/G}\{M\} = \{g/D : g \in I\{M\} \text{ and } eq(g) \in G\}$. Note that $\prod_{D/G}\{M\}$ is a nonempty subset of $\prod_D\{M\} = \{g/D : g \in I\{M\}\}$ and is closed under the constants and functions of the ultrapower $\prod_D\{M\}$ of $M$ modulo $D$. The limit ultrapower $\prod_{D/G}\{M\}$ of the $L$-structure $M$ (with respect to $I, D, G$) is the substructure of $\prod_D\{M\}$ whose universe is the set $\prod_{D/G}\{M\}$. The canonical map $d$ from $M$ into $\prod_{D/G}\{M\}$ is defined by $d(a) = \langle a_i : i \in I \rangle/D$, where $a_i = a$ for every $i \in I$. Note that the limit ultrapower $\prod_{D/G}\{M\}$ depends only on the equivalence relations which are in $G$, i.e. if $E$ is the set of all equivalence relations on $I$ and $G \cap E = G' \cap E$, where $G'$ is a filter on $I \times I$, then $\prod_{D/G'}\{M\} = \prod_{D/G'}\{M\}$.

**Definition 1.7.2.** Let $M$ be an $L$-structure, $(Y, <)$ a linear order and, for each $y \in Y$, let $D_y$ be an ultrafilter on a nonempty set $I_y$. Write $H = \prod_{y \in Y}I_y$. Let $\prod_{y \in Y}D_y$ be the set of $s \subset H$ for which there are $y_1 < \ldots < y_n$ in $Y$ such that

1. for all $i, j \in H$, if $i\{y_1, \ldots, y_n\} = j\{y_1, \ldots, y_n\}$ then $i \in s$ iff $j \in s$;
2. $\langle i(y_1), \ldots, i(y_n) \rangle : i \in \prod_{y \in Y}D_y$.

Write $E = \prod_{y \in Y}D_y$. The *iterated ultrapower* $\prod_E\{M\}$ of the set $\{M\}$ with respect to $\langle \prod_{y \in Y}I_y : y \in Y \rangle$ is the set $\{f/E : f : H \rightarrow \{M\}\}$ and for some finite $Z_f \subset Y$ for all $i, j \in H$, if $i/Z_f = j/Z_f$, then $f(i) = f(j)$. The *iterated ultrapower* $\prod_E\{M\}$ of the $L$-structure $M$ with respect to $\langle \prod_{y \in Y}I_y : y \in Y \rangle$ is the $L$-structure whose universe is the set $\prod_E\{M\}$; for each $n$-ary predicate symbol $R$ of $L$, $R^{\prod_E\{M\}}(f_1/E, \ldots, f_n/E$) iff $\langle i \in H : R^{M}(f_1(i), \ldots, f_n(i)) \rangle \in E$; for each $n$-ary function symbol $F$ of $L$, $F^{\prod_E\{M\}}(f_1/E, \ldots, f_n/E$) $= \langle F^{M}(f_1(i), \ldots, f_n(i)) : i \in H\rangle/E$. The canonical map $d : M \rightarrow \prod_E\{M\}$ is defined as usual by $d(a) = \langle a : i \in H\rangle/E$.

**Remark 1.7.3.** (1) Every ultrapower is a limit ultrapower: take $G = P(I \times I)$ and note that $\prod_D\{M\} = \prod_{D/G}\{M\}$.
(2) Every iterated ultrapower is a limit ultrapower. [Why? let the iterated ultrapower be defined by \((Y, <)\) and \(\langle (I_y, D_y) : y \in Y \rangle\) (see Definition 1.7.2). For \(Z \in [Y]^{<\omega}\), let \(A_Z = \{(i, j) \in H \times H : i|Z = j|Z\}\). Note that \(\{A_Z : Z \in [Y]^{<\omega}\}\) has the finite intersection property and hence can be extended to a filter \(G\) on \(H \times H\). Now for any model \(M\) we have \(\prod_E M \cong \prod_{D/G} M\) for every filter \(D\) over \(H\) extending \(E\) under the map \(f/E \rightarrow f/D\).]

**Definition 1.7.4.** Suppose that \(M\) is an \(L\)-structure, \(D\) is an ultrafilter on a nonempty set \(I\), and \(G\) is a suitable set of equivalence relations on \(I\), i.e.

(i) if \(e \in G\) and \(e'\) is an equivalence relation on \(I\) coarser than \(e\), then \(e' \in G\);

(ii) \(G\) is closed under finite intersections;

(iii) if \(e \in G\), then \(D/e = \{A \subset I/e : \bigcup_{x \in A} x \in D\}\) is a \(\kappa\)-complete ultrafilter on \(I/e\) which, for simplicity, has cardinality \(\kappa\); we state this as “\((I, D, G)\) is \(\kappa\)-complete”.

(We may say \((I, D, G)\) is suitable.)

Then \(\text{Op}(M, I, D, G)\) is the limit ultrapower \(\prod_{D/G} M\), where \(\hat{G}\) is the filter on \(I \times I\) generated by \(G\). One abbreviates \(\text{Op}(M, I, D, G)\) by \(\text{Op}(M)\), and one writes \(f_{\text{Op}}\) for the canonical map \(d : M \rightarrow \text{Op}(M)\).

Note that

**Observation/Convention 1.7.4A.** 1) For any \(L\)-structure \(N\), \(f_{\text{Op}}\) is an \(L_{\kappa\omega}\)-elementary embedding of \(N\) into \(\text{Op}(N)\) and in particular \(f_{\text{Op}} : N \rightarrow \text{Op}(N)\).

2) Since \(f_{\text{Op}}\) is canonical, one very often identifies \(N\) with the \(L\)-structure \(\text{rng}(f_{\text{Op}})\), which is an \(\mathcal{F}\)-elementary substructure of \(\text{Op}(N)\), and one writes \(N \prec \mathcal{F} \text{Op}(N)\). In particular, for any model \(M\) of \(T \subset \mathcal{F}\) and \(\text{Op}\), \(f_{\text{Op}} : M \rightarrow \mathcal{F} \text{Op}(M)\) (briefly written, \(M \prec \mathcal{F} \text{Op}(M)\)) so that \(\text{Op}(M)\) is a model of \(T\) too.

3) Remark that if \(D\) is a \(\kappa\)-complete ultrafilter on \(I\) and \(G\) is a filter on \(I \times I\), then \(\text{Op}(M, I, D, G)\) is well defined.

4) “Suitable limit ultrapower” means one using a suitable triple.

More information on limit and iterated ultrapowers can be found in [CK] and [HoSh109].

**Observation 1.7.5.** Suppose that \(M\) is a model of a theory \(T \subset \mathcal{F}\), where \(\mathcal{F}\) is a fragment of \(L_{\kappa\omega}\). Given \(\theta\)-complete ultrafilters \(D_1\) on \(I_1\), \(D_2\) on \(I_2\) and suitable filters \(G_1\) on \(I_1 \times I_1\), \(G_2\) on \(I_2 \times I_2\) respectively, there exist a \(\theta\)-complete ultrafilter \(D\) on a set \(I\) and a suitable filter \(G\) on \(I \times I\) such that \(\text{Op}(M, I, D, G) = \text{Op}(\text{Op}(M, I_1, D_1, G_1), I_2, D_2, G_2)\).
and \((D, G, I)\) is \(\kappa\)-complete. Also iterated ultrapower (along any linear order) with each iterand being an ultrapower by a \(\kappa\)-complete ultrafilter, gives a suitable triple (in fact, even iteration of suitable limit ultrapowers is a suitable ultrapower).

**Definition 1.8.** Suppose that \(K\) is a class of \(L\)-structures and \(<\) is a binary relation on \(K\). For \(M, N \in K\), write \(f : M \prec_n N\) to mean

1. \(f\) is an isomorphism from \(M\) into \(N\) and \(\text{rng}(f) < N\);
2. there are a set \(I\), an ultrafilter \(D\) on \(I\), a suitable set \(G\) of equivalence relations on \(I\) (so Definition 1.7.4(i)–(iii) holds), and an isomorphism \(g\) from \(N\) into \(\text{Op}(M, I, D, G)\) such that \(\text{rng}(g) < \text{Op}(M, I, D, G)\) and \(gf = f_{\text{Op}}\), where \(f_{\text{Op}}\) is the canonical embedding of \(M\) into \(\text{Op}(M, I, D, G)\). \(f\) is called a \(\prec\)-nice embedding of \(M\) into \(N\). Of course one writes \(f : M \rightarrow N\) and says that \(f\) is a \(\prec\)-nice embedding of \(M\) into \(N\) when \(<\) is clear from the context.

**Example 1.9.1.** Consider \(T, F\) and \(K = \langle K, <\rangle\) as set up in 1.3B. In this case \(f : M \prec_n N\) holds iff \(f : M \rightarrow F N\) and for some suitable \((I, D, G)\) and some \(g : N \rightarrow F \text{Op}(M, I, D, G)\), \(gf = f_{\text{Op}}\).

Abusing notation one writes \(M \rightarrow_n N\) to mean that there are \(f, g\) and \(\text{Op}\) such that \(f : M \prec_n N\) using \(g\) and \(\text{Op}\). If not said otherwise, \(<\) is \(\prec\).

We may also write \(M \preceq_n N\), and for linear orders we use \(I \subseteq J\).

**Example 1.9.2.** Let \(\mathsf{LO}\) be the class of linear orders and let \((I, \leq_I) < (J, \leq_J)\) mean that \((I, \leq_I) \subset (J, \leq_J)\), i.e. \((I, \leq_I)\) is a suborder of \((J, \leq_J)\). If \(f : (I, \leq_I) \rightarrow_n (J, \leq_J)\), then for some \(\text{Op}\), identifying isomorphic orders, one has \((I, \leq_I) \subset (J, \leq_J) \subset \text{Op}(I, \leq_I)\).

**Observation 1.10.** Suppose that \(T, F\) and \(K\) are as in 1.3B and 1.9.1. Suppose further that \(M \prec_n \mathcal{N}\) and \(M \preceq_n M' \preceq_n \mathcal{N}\) for \(M, M', \mathcal{N} \in K\). Then \(M \prec_n \mathcal{M}'\).

**Proof.** For some \(f, g\) and \(\text{Op}\), \(f : M \rightarrow_n \mathcal{N}, g : N \rightarrow_n \text{Op}(M)\) and \(gf = f_{\text{Op}}\). Now \(g : M' \rightarrow_n \text{Op}(M)\) (since \(M' \preceq_n \mathcal{N}\)) and \(gf = f_{\text{Op}}\) so that \(M \prec_n \mathcal{M}'\).

**Observation 1.11.** Suppose that \((M_i : i \leq \delta)\) is a continuous increasing chain and for each \(i < \delta\), \(M_i < M_{i+1}\). Then for every \(i < \delta\), \(M_i < M_{\delta}\).

**Proof** (like the proof of 1.7.3(2)). For each \(i < \delta\), there are \((I_i, D_i, G_i)\) as in Definition 1.7.4 which witness \(M_i \leq M_{i+1}\). Let \(I := \prod_{i \leq \delta} I_i\) and \(G := \{e : e \subseteq I \times I\) and for some \(n < \omega\) and \(\alpha_1 < \ldots < \alpha_n < \delta\) and \(e_1 \in G_{\alpha_1}, \ldots, e_n \in G_{\alpha_n},\) we have: for every \(x, y \in I\) such that \((x(\alpha_l), y(\alpha_l)) \in e_l\)
for \(l = 1, \ldots, n\), we have \((x, y) \in e\). \(D\) will be any ultrafilter on \(I\) such that:
if \(n < \omega\) and \(\alpha_1 < \ldots < \alpha_n < \delta, e_1 \in G_{\alpha_1}, \ldots, e_n \in G_{\alpha_n}, e_l\) is an equivalence relation on \(I_{\alpha_l}\) for \(l = 1, \ldots, n\) and \(A \in (D_{\alpha_1}/e_{\alpha_1}) \times \ldots \times (D_{\alpha_n}/e_{\alpha_n})\), then the set \(\{x \in I : (x(\alpha_1)/e_{\alpha_1}, \ldots, x(\alpha_n)/e_{\alpha_n}) \in A\}\) belongs to \(D\). We leave the rest to the reader.

**Claim 1.12.** For every model \(M\) and \(\lambda \geq \kappa + |\mathcal{F}| + \|M\|\) there is \(N\) such that \(M \preceq \mathcal{F} N\), \(M \neq N\) and \(\|N\| = \lambda\).

**Proof.** As \(\kappa\) is measurable.

### 2. The amalgamation property for regular categoricity.

The main aim in this section is to show that if \(T\) is categorical in the regular cardinal \(\lambda > \kappa + |T|\), then \(K_{<\lambda} = (K_{<\lambda}, \preceq_{\mathcal{F}})\) has the amalgamation property (AP) (Definition 1.3(1)). Categoricity is not presumed if not required.

**Lemma 2.1.** Suppose that \(\kappa + |T| \leq \mu \leq \lambda\), \(M, M_1, M_2 \in K_{\leq \mu}\), \(f_1 : M \rightarrow_{\nice} M_1, f_2 : M \rightarrow_{\mathcal{F}} M_2\). Then there is an amalgam \(N \in K_{\leq \mu}\) of \(M_1, M_2\) over \(M\) with respect to \(f_1, f_2\). Moreover, there are \(g_i : M_i \rightarrow_{\mathcal{F}} N\) for \(l = 1, 2\) such that \(g_1 f_1 = g_2 f_2\) and \(\text{rng}(g_2) \cap \text{rng}(g_1) = \text{rng}(g_1 f_1)\).

**Proof.** There are \(g\) and \(\text{Op}\) such that \(g : M_1 \rightarrow_{\mathcal{F}} \text{Op}(M)\) and \(g f_1 = f_\text{Op}\). Then \(f_2\) induces an \(\mathcal{F}\)-elementary embedding \(f_2^*\) of \(\text{Op}(M)\) into \(\text{Op}(M_2)\) such that \(f_2^* f_\text{Op} = f_\text{Op} f_2\). Let \(g_1 = f_2^* g\) and \(g_2 = f_\text{Op} f_2\). By 1.2 one finds \(N \in K_{\leq \mu}\) such that \(\text{rng}(g_1) \cup \text{rng}(g_2) \subset N \preceq_{\mathcal{F}} \text{Op}(M_2)\). Now \(N\) is an amalgam of \(M_1, M_2\) over \(M\) with respect to \(f_1, f_2\) since \(g_1 f_1 = f_2^* g f_1 = f_2^* f_\text{Op} = f_\text{Op} f_2 = g_2 f_2\). The last phrase in the lemma is easy by properties of \(\text{Op}\).

**Lemma 2.2.** Suppose that \(M \in K_{\leq \mu}\) is a \(\mu\)-c.a.b. and \(\kappa + |T| \leq \mu < \lambda\). Then \(N \in K_{<\lambda}\) is a \(\|N\|-\)c.a.b. whenever \(f : M \rightarrow_{\nice} N\).

**Proof.** Suppose that \(g : N \rightarrow_{\mathcal{F}} \text{Op}(M)\) and \(g f = f_\text{Op}\). Since \(M\) is a \(\mu\)-c.a.b., for some \(M_i \in K_{\leq \mu}\) and \(f_i : M_i \rightarrow_{\mathcal{F}} M_i\) (\(i = 1, 2\)) there is no amalgam of \(M_1, M_2\) over \(M\) with respect to \(f_1, f_2\). Let \(f_i^*\) be the \(\mathcal{F}\)-elementary embedding from \(\text{Op}(M)\) into \(\text{Op}(M_i)\) defined by \(f_i\) (note that \(f_i^* f_\text{Op} = f_\text{Op} f_i, i = 1, 2\)). Choose \(N_i\) of power \(\|N\|\) such that \(M_i \cup \text{rng}(f_i^* g) \subset N_i \preceq_{\mathcal{F}} \text{Op}(M_i)\). Note that \(f_i^* f : N \rightarrow_{\mathcal{F}} N_i\). It suffices to show that there is no amalgam of \(N_1, N_2\) over \(N\) with respect to \(f_1^* g, f_2^* g\).

Well, suppose that one could find an amalgam \(N^*\) and \(h_i : N_i \rightarrow_{\mathcal{F}} N^*\), \(i = 1, 2\), with \(h_1(f_1^* g) = h_2(f_2^* g)\). Using 1.2 choose \(M^*\) such that \(\|M^*\| \leq \mu, M^* \preceq_{\mathcal{F}} N^*\) and \(\text{rng}(h_1 f_\text{Op}|{M_i}) \cup \text{rng}(h_2 f_\text{Op}|{M_2}) \subset |M^*|\). Set \(g_i = h_i f_\text{Op}|{M_i}\) for \(i = 1, 2\), and note that
\[
\begin{align*}
g_1 f_1 &= h_1 f_\text{Op} f_1 = h_1 f_1^* f_\text{Op} = h_1 f_1^* g f = h_2 f_2^* g f = h_2 f_2^* f_\text{Op} = h_2 f_\text{Op} f_2 = g_2 f_2.
\end{align*}
\]
In other words, $M^*$ is an amalgam of $M_1, M_2$ over $M$ with respect to $f_1, f_2$—contradiction. It follows that $N$ is a $\|N\|$-c.a.b.

**Corollary 2.3.** Suppose that $\kappa + |T| \leq \mu < \lambda$. If $M \in K\mu$ is a $\mu$-c.a.b., then there exists $M^* \in K\lambda$ such that

(*) $M \preceq M^*$ and for every $M' \in K_{<\lambda}$, if $M \preceq M' \preceq M^*$, then $M'$ is a $\|M'\|$-c.a.b.

**Proof.** As $\|M\| \geq \kappa$, for some appropriate Op one has $\|\text{Op}(M)\| \geq \lambda$, and by 1.2 one finds $M^* \preceq M \preceq \text{Op}(M)$. Let us check that $M^*$ works in (*). Take $M' \in K_{<\lambda}$ with $M \preceq M' \preceq M^*$; so $M \preceq M'$ since $M^* \preceq \text{Op}(M)$; hence by 2.2, $M'$ is a $\|M'\|$-c.a.b.

**Theorem 2.4.** Suppose that $T$ is $\lambda$-categorical and $\lambda = \text{cf}(\lambda) > \kappa + |T|$. If $K_{<\lambda}$ fails AP, then there exists $N^* \in K\lambda$ such that for some continuous increasing $\prec$-chain $(N_i \in K_{<\lambda} : i < \lambda)$ of models,

1. $N^* = \bigcup_{i<\lambda} N_i$;
2. for every $i < \lambda$, $N_i \not\preceq N_{i+1}$ (and so $N_i \not\preceq N^*$).

**Proof.** $K_{<\lambda}$ fails AP, so for some $\mu < \lambda$ and $M \in K_{<\mu}$, $M$ is a $\mu$-c.a.b. By 2.2 and 1.12, without loss of generality, $M \in K_{<\mu}$. Choose by induction a continuous strictly increasing $\prec$-chain $(N_i \in K_{<\lambda} : i < \lambda)$ as follows: $N_0 = M$; at a limit ordinal $i$, take the union; at a successor ordinal $i = j + 1$, if there is $N \in K_{<\lambda}$ such that $N_j \preceq N$ and $N_j \not\preceq N$, choose $N_i = N$, otherwise choose for $N_i$ any nontrivial $\prec$-elementary extension of $N_j$ of power less than $\lambda$.

**Claim.** $(\exists j_0 < \lambda)(\forall j \in (j_0, \lambda))(N_j$ is a $\|N_j\|$-c.a.b.).

**Proof.** Suppose not. So one has a strictly increasing sequence $(j_i : i < \lambda)$ such that for each $i < \lambda$, $N_{j_i}$ is not a $\|N_{j_i}\|$-c.a.b. Let $N_* = \bigcup_{i<\lambda} N_{j_i}$. So $\|N_*\| = \lambda$. Applying 2.3 one can find $M^* \in K\lambda$ such that whenever $M' \in K_{<\lambda}$ and $M \preceq M' \preceq M^*$, then $M'$ is a $\|M'\|$-c.a.b.

Since $T$ is $\lambda$-categorical, there is an isomorphism $g$ of $N_*$ onto $M^*$. Let $N = g^{-1}(M)$ and $M_i = g(N_i)$ for $i < \lambda$. Since $\|N\| = \mu < \text{cf}(\lambda) = \lambda$, there is $i_0 < \lambda$ such that $N \subset N_{j_{i_0}}$.

In fact, $N_{j_{i_0}}$ is a $\|N_{j_{i_0}}\|$-c.a.b. (Otherwise, consider $M_{j_{i_0}}$. Since $M \preceq M_{j_{i_0}} \preceq M^*$ and $\|M_{j_{i_0}}\| < \lambda$, $M_{j_{i_0}}$ is a $\|M_{j_{i_0}}\|$-c.a.b., so there are $f_i : M_{j_{i_0}} \preceq M^*_i$ ($l = 1, 2$) with no amalgam of $M'_1, M'_2$ over $M_{j_{i_0}}$ with respect to $f_1, f_2$. If $N_{j_{i_0}}$ is not a $\|N_{j_{i_0}}\|$-c.a.b., then one can find an amalgam $N^+ \in K_{\leq \|N_{j_{i_0}}\|}$ of $M'_1, M'_2$ over $N_{j_{i_0}}$ with respect to $f_1, f_2$ and $N^+$ is thus an amalgam of $M'_1, M'_2$ over $M_{j_{i_0}}$ with respect to $f_1, f_2$, and $\|N^+\| \leq \|N_{j_{i_0}}\| =$...
\[\|M_{\lambda_0}\| - \text{contradiction.}\] This contradicts the choice of \(N_{\lambda_0}\). So the claim is correct.

It follows that for each \(j \in (j_0, \lambda)\) there are \(N_j^1, N_j^2\) in \(K_{<\lambda}\) and \(f_i : N_j \rightarrow_{\mathcal{F}} N_j^1\) such that no amalgam of \(N_j^1, N_j^2\) over \(N_j\) with respect to \(f_1, f_2\) exists. By 2.2 for some \(l \in \{1, 2\}, N_j \nsubseteq N_{j+1}^l\). So by the inductive choice of \(N_{j+1} : j < \lambda\), \((\forall j \in (j_0, \lambda))(N_j \nsubseteq N_{j+1})\). Taking \(N^* = \bigcup_{j_0 < j < \lambda} N_j\), one completes the proof. (Of course for \(j_0 < j < \lambda\), \(N_j \nsubseteq N_{\lambda+1}\) — contradiction).

**Theorem 2.5.** Suppose that \((I, <_I), (J, <_J)\) are linear orders and \(I\) is a suborder of \(J\). If \((I, <_I) \subseteq (J, <_J)\), then \(EM(I) \preceq_{\text{nice}} EM(J)\).

**Proof.** Without loss of generality, for some cardinal \(\mu\), ultrafilter \(D\) on \(\mu\) and suitable set \(G\), a filter on \(\mu \times \mu\), \((I, <_I) \subseteq (J, <_J) \subseteq Op((I, <_I), \mu, D, G) = Op(I, <_I)\), and \(|Op(I, <_I)| = \{|f/D : f \in \mu I, eq(f) \in G\}\), where \(eq(f) = \{(i, j) \in \mu \times \mu : f(i) = f(j)\}\). So for each \(t \in J\), there exists \(f_i \in \mu I\) such that \(t = f_i/D\). Note that if \(t \in I\), then \(f_i/D = f_{Op}(t)\) so that without loss of generality, for all \(i < \mu\), \(f_i(i) = t\). Define a map \(h\) from \(EM(J)\) into \(Op(EM(I))\) as follows. An element of \(EM(J)\) has the form
\[\tau^{EM(J)}(x_{t_1}, \ldots, x_{t_n}),\]
where \(t_1, \ldots, t_n \in J\) and \(\tau\) is an \(L\)-term. Define, for \(t \in J\), \(g_t \in \mu \text{EM}(I)\) by \(g_t(i) = x_{f_i(i)}\). Note that \(f_i(i) \in I\), so that \(x_{f_i(i)} \in \text{EM}(I)\) and so \(g_t/D \in Op(EM(I))\). Let \(h(\tau^{EM(J)}(x_{t_1}, \ldots, x_{t_n})) = \tau^{Op(EM(I))}(g_{t_1}/D, \ldots, g_{t_n}/D)\), which is an element in \(Op(EM(I))\). The reader is invited to check that \(h\) is an \(\mathcal{F}\)-elementary embedding of \(EM(J)\) into \(Op(EM(I))\). So \(EM(I) \preceq_{\mathcal{F}} EM(J)\).

Finally, note that if \(\tau = \tau^{EM(J)}(x_{t_1}, \ldots, x_{t_n}) \in EM(I)\) with \(t_1, \ldots, t_n \in I\), then
\[h(\tau) = \tau^{Op(EM(I))}(g_{t_1}/D, \ldots, g_{t_n}/D)\]
\[= \tau^{Op(EM(I))}(\langle x_{t_1} : i < \mu)/D, \ldots, \langle x_{t_n} : i < \mu\rangle/D)\]
\[= f_{Op}(\tau^{EM(I)}(x_{t_1}, \ldots, x_{t_n})) = f_{Op}(\tau).\]
Thus \(EM(I) \preceq_{\text{nice}} EM(J)\).

**Criterion 2.6.** Suppose that \((I, <)\) is a suborder of the linear order \((J, <)\). If
\[(*)\quad \text{for every } t \in J \setminus I,\]
\[(\forall)\quad \text{cf}((I, <)|\{s \in I : (J, <) \models s < t\}) = \kappa\]
or

\[
\text{cf}((I,<)^*|\{s \in I : (J,<)^* \models s <^* t\}) = \kappa
\]

then \((I,<) \subseteq\) \((J,<)\). [Notation: \((I,<)^*\) is the (reverse) linear order \((I^*,<^*)\), where \(I^* = I\) and \((I^*,<^*)\) \(\models s <^* t\) iff \((I,<) \models t < s\).]

\textbf{Proof.} Let us list some general facts which facilitate the proof.

\textbf{Fact (A).} Let \(\kappa\) denote the linear order \((\kappa,\prec)\), where \(\prec\) is the usual order \(\in|\kappa \times \kappa\). If \(J_1 = \kappa + J_0\), then \(\kappa \subseteq\) \(\text{nice}\) \(J_1\) (+ is addition of linear orders in which all elements in the first order precede those in the second).

\textbf{Fact (B).} If \(\kappa \subseteq\) \((I,<)\), \(\kappa\) is unbounded in \((I,<)\) and \(J_1 = I + J_0\), then \(I \subseteq\) \(\text{nice}\) \(J_1\).

\textbf{Fact (C).} If \(I \subseteq\) \(\text{nice}\) \(J\), then \(I + J_1 \subseteq\) \(\text{nice}\) \(J + J_1\).

\textbf{Fact (D).} \(I \subseteq\) \(\text{nice}\) \((J,<)^*\) iff \((I,<)\) \(\subseteq\) \(\text{nice}\) \((I,<)^*\).

\textbf{Fact (E).} If \(\{I_\alpha : \alpha \leq \delta\}\) is a continuous increasing sequence of linear orders and for \(\alpha < \delta\), \(I_\alpha \subseteq\) \(I_{\alpha + 1}\), then \(I_\alpha \subseteq\) \(\text{nice}\) \(I_\delta\).

Now using these facts, let us prove the criterion. Define an equivalence relation \(E\) on \(J \setminus I\) as follows: \(t E s\) iff \(t\) and \(s\) define the same Dedekind cut in \((I,<)\). Let \(\{t_\alpha : \alpha < \delta\}\) be a set of representatives of the \(E\)-equivalence classes. For each \(\beta \leq \delta\), define

\[
I_\beta = J \setminus \{t : t \in I \lor (\exists \alpha < \beta)(t E t_\alpha)\}
\]

so \(I_0 = I\), \(I_\delta = J\) and \(\{I_\alpha : \alpha \leq \delta\}\) is a continuous increasing sequence of linear orders. By Fact (E), to show that \(I \subseteq\) \(\text{nice}\) \(J\), it suffices to show that \(I_\alpha \subseteq I_{\alpha + 1}\) for each \(\alpha < \delta\).

Fix \(\alpha < \delta\). Now \(t_\alpha\) belongs to \(J \setminus I\), so \((8)\) or \((3)\) holds. By Fact (D), it is enough to treat the case \((8)\). So, without loss of generality, \(\text{cf}((I,<)|\{s \in I : (J,<) \models s < t_s\}) = \kappa\).

Let

\[
\begin{align*}
I_\alpha^a &= \{t \in I_\alpha : t < t_\alpha\}, \\
I_\alpha^b &= \{t \in I_{\alpha + 1} : t \in I_\alpha^a \lor t E t_\alpha\}, \\
I_\alpha^c &= \{t \in I_\alpha : t > t_\alpha\}.
\end{align*}
\]

Note that \(I_\alpha = I_\alpha^a + I_\alpha^c\) and \(I_{\alpha + 1} = I_\alpha^b + I_\alpha^c\). Recalling Fact (C), it is now enough to show that \(I_\alpha^a \subseteq I_\alpha^b\). Identifying isomorphic orders and using \((8)\), one deduces that \(\kappa\) is unbounded in \(I_\alpha^a\) and \(I_\alpha^b = I_\alpha^a + (I_\alpha^b \setminus I_\alpha^a)\) so by Fact (B), \(I_\alpha^a \subseteq I_\alpha^b\) as required.
Of the five facts, we prove (A), (B) and (E), as (C) and (D) are obvious.

Proof of Fact (A). Since \( \kappa \) is measurable, there is a \( \kappa \)-complete uniform ultrafilter \( D \) on \( \kappa \) (see [J]). For every linear order \( J_0 \) (or \( J_0^\ast \)) there is \( \text{Op}_{I,D}(\cdot) \), the iteration of \( I \) ultrapowers \( (\cdot)^\kappa/D \), ordered in the order \( J_0 \) (or \( J_0^\ast \)), giving the required embedding (use 1.7.5).

Proof of Fact (B). Since \( \kappa \subseteq I \) and using Fact (A), we know that there is an operation \( \text{Op} \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\kappa + J_0 & \xrightarrow{id} & I + J_0 \\
\downarrow \text{Op}(\cdot) & & \downarrow \text{Op}(\cdot) \\
\kappa & \xrightarrow{\text{Op}(\cdot)} & I \\
\end{array}
\]

Chasing through the diagram, we obtain the required embedding.

Proof of Fact (E). Apply 1.11 to the chain \( \langle I_\alpha : \alpha \leq \delta \rangle \).

FACT 2.7. Suppose that \( \lambda \geq \kappa \). There exist a linear order \( (I, <_I) \) of power \( \lambda \) and a sequence \( \langle A_i \subset I : i \leq \lambda \rangle \) of pairwise disjoint subsets of \( I \), each of power \( \kappa \), such that \( I = \bigcup_{i \leq \lambda} A_i \) and

\[(*) \quad \text{if } \lambda \in X \subset \lambda + 1, \text{ then } I \upharpoonright \bigcup_{i \in X} A_i \subseteq I.\]

Proof. Let \( I = (\lambda + 1) \times \kappa \) and define \( <_I \) on \( I \) by \( (i_1, \alpha_1) <_I (i_2, \alpha_2) \) iff \( i_1 < i_2 \) or \( (i_1 = i_2 \text{ and } \alpha_1 > \alpha_2) \). For each \( i \leq \lambda \), let \( A_i = \{i\} \times \kappa \). Check (\( \ast \)) of 2.6: suppose that \( \lambda \in X \subset \lambda + 1 \). Write \( I_X = I \upharpoonright \bigcup_{i \in X} A_i \). To show that \( I_X \subseteq I \), one deploys Criterion 2.6. Consider \( t \in I - I_X \), say \( t = (i, \alpha) \) (note that \( \alpha < \kappa \) and \( i < \lambda \), since \( \lambda \in X \) and \( i \notin X \). Let \( j = \min(X - i) \); note that \( j \) is well defined, since \( \lambda \in X - i \), and \( j \neq i \). For every \( \beta < \kappa \), one has \( t <_I (j, \beta) \) and \( (j, \beta) \in I_X \). Also if \( s \in I_X \) and \( t <_I s \), then for some \( \beta < \kappa \), \( (j, \beta) <_I s \). Thus \( \langle (j, \beta) : \beta < \kappa \rangle \) is a cofinal sequence in \( (I_X \upharpoonright \{s \in I : t <_I s\})^\ast \). By the criterion, \( I_X \subseteq I \).

THEOREM 2.8. Suppose that \( \kappa = \text{cf}(\delta) \leq \delta < \lambda \). Then \( \text{EM}(\delta) \leq \text{EM}(\lambda) \).

Proof. By Fact (B) of 2.6, one has \( \delta \subseteq \lambda \); so by 2.5, \( \text{EM}(\delta) \leq \text{EM}(\lambda) \).

Now let us turn to the main theorem of this section.

THEOREM 2.9. Suppose that \( T \) is categorical in the regular cardinal \( \lambda > \kappa + |T| \). Then \( K_{< \lambda} \) has the amalgamation property.
Proof. Suppose that $K_{<\lambda}$ fails AP. Note that $\|\text{EM}(\lambda)\| = \lambda$. Apply 2.4 to find $M^* \in K_{<\lambda}$ and $\{M_i : i < \lambda\}$ satisfying 2.4(1), (2). Since $T$ is $\lambda$-categorical, $M^* \cong \text{EM}(\lambda)$, so without loss of generality, $\text{EM}(\lambda) = \bigcup_{i<\lambda} M_i$. Now $C = \{i < \lambda : M_i = \text{EM}(i)\}$ is a club of $\lambda$. Choose $\delta \in C$ with $\text{cf}(\delta) = \kappa$. By 2.8, $\text{EM}(\delta) \preceq_{\text{nice}} \text{EM}(\lambda)$, so $M_\delta \preceq M^*$. But of course by 2.4(2), $M_\delta \not\preceq_{\text{nice}} M^*$—contradiction.

**Theorem 2.10.** Suppose that $T$ is categorical in $\lambda > \kappa + |T|$. Then:

1. $T$ has a model $M$ of power $\lambda$ such that if $N \preceq_{\text{nice}} M$ and $\|N\| < \lambda$, then there exists $N'$ such that
   
   - (a) $N \preceq_{\text{nice}} N' \preceq_{\text{nice}} M$;
   - (b) $\|N'\| = \|N\| + \kappa + |T|$;
   - (c) $N' \preceq_{\text{nice}} M$.

2. $T$ has a model $M$ of power $\lambda$ and an expansion $M^+$ of $M$ by at most $\kappa + |T|$ functions such that if $N^+ \preceq_{\text{nice}} M^+$, then $N^+ | L \preceq_{\text{nice}} M$. 

Proof. Let $\langle I, (A_i : i \leq \lambda) \rangle$ be as in 2.7. Let $M = \text{EM}(I)$. Suppose that $N \preceq M$ and $\|N\| < \lambda$. Then there exists $J \subseteq I$ with $|J| < \lambda$ such that $N \subseteq \text{EM}(J)$, hence $N \preceq_{\text{nice}} \text{EM}(J) \preceq_{\text{nice}} \text{EM}(I)$. So there is $X \subseteq \lambda + 1$ with $\lambda \in X$ and $|X| < \lambda$ such that $J \subseteq \bigcup_{i \in X} A_i$. Note that $|\bigcup_{i \in X} A_i| \leq |X|\kappa < \lambda$. Now $N' = \text{EM}(I|\bigcup_{i \in X} A_i)$ is as required, since $I|\bigcup_{i \in X} A_i \subseteq I$ and so by 2.5, $\text{EM}(I|\bigcup_{i \in X} A_i) \preceq_{\text{nice}} \text{EM}(I)$. This proves (1).

(2) We expand $M = \text{EM}(I)$ as follows:

- (a) By all functions of $\text{EM}'(I)$.

- (b) By the unary functions $f_l (l < \omega)$ which are chosen as follows: we know that for each $b \in M$ there is an $L_1$-term $\tau_b$ ($L_1$ is the vocabulary of $\text{EM}'(I)$) and $t(b,0) < t(b,1) < \ldots < t(b,n_{\tau_b} - 1)$ from $I$ such that $b = \tau_b(x_{i(0)}, x_{i(1)}, \ldots, x_{i(n_{\tau_b} - 1)})$ (it is not unique, but we can choose one; really if we choose it with $n_{\tau_b}$ minimal it is almost unique). We let $f_l(b) = \begin{cases} x_{i(l)} & \text{if } l < n_{\tau_b}, \\ b & \text{if } l \geq n_{\tau_b}. \end{cases}$

- (c) By unary functions $g_\alpha, g^\alpha$ for $\alpha < \kappa$ such that if $t < s$ are in $I$ and $\alpha = \text{otp}(\langle t, s \rangle)$ then $g^\alpha(x_t) = x_s, g_\beta(x_s) = x_t$ for some $\beta < \kappa$ (more formally, $g^\alpha(x_{i(\beta)}) = x_{i(\beta + \alpha)}$ and $g_\alpha(x_{i(\beta)}) = x_{i(\alpha)}$) and in the other cases $g^\alpha(b) = b, g_\alpha(b) = b$.

- (d) By individual constants $c_\alpha = x_{(\lambda, \alpha)}$ for $\alpha < \kappa$.

Now suppose $N^+$ is a submodel of $M^+$ and $N$ its $L$-reduct. Let $J := \{t \in I : x_t \in N\}$. Now $J$ is a subset of $I$ of cardinality $\leq \|N\|$ as
for \( t \neq s \) from \( J \), \( x_t \neq x_s \). Also if \( b \in N \) then by (b), \( x_t(b,t) \in N \), hence \( b \in EM(J) \); on the other hand, if \( b \in EM(J) \) then by (a) we have \( b \in N \); so we can conclude \( N = EM(J) \). So far this holds for any linear order \( I \).

By (c), \( J = \bigcup_{i \in \mathcal{X}} A_i \) for some \( \mathcal{X} \subseteq \lambda + 1 \), and by (d), \( \lambda \in X \).

Now \( EM(J) \preceq EM(I) \neq M \) by 2.7.

3. Towards removing the assumption of regularity from the existence of universal extensions. In Section 2 we showed that \( K_{<\lambda} \) has the amalgamation property when \( T \) is categorical in the regular cardinal \( \lambda > \kappa + |T| \). We now study the situation in which \( \lambda \) is not assumed to be regular.

Our problem is that while we know that most submodels of \( N \in K_{\lambda} \) sit well in \( N \) (see 2.10(2)) and that there are quite many \( N \in K_{<\lambda} \) which are amalgamation bases, our difficulty is to get those things together: constructing \( N \in K_{\lambda} \) as \( \bigcup_{i<\lambda} N_i \) with \( N_i \in K_{<\lambda} \) means \( N \) has \( \leq \mathcal{F} \)-submodels not included in any \( N_i \).

**Theorem 3.1.** Suppose that \( T \) is categorical in \( \lambda \) and \( \kappa + |T| \leq \theta < \lambda \). If \( \langle M_i : i < \theta^+ \rangle \) is an increasing continuous \( \mathcal{F} \)-chain, then
\[
\left\{ i < \theta^+ : M_i \succeq \bigcup_{i<\theta^+} M_j \right\} \in D_{\theta^+}.
\]

**Remark 3.1A.** (1) We cannot use 2.10(1) e.g. as possibly \( \lambda \) has cofinality \( < \kappa + |T| \).

(2) Recall that \( D_{\theta^+} \) is the closed unbounded filter on \( \theta^+ \).

**Proof of Theorem 3.1.** Write \( M_{\theta^+} = \bigcup_{i<\theta^+} M_i \). Choose an operation \( Op \) such that for all \( i < \theta^+ \), \( \| Op(M_i) \| \geq \lambda \). Let \( M_i^* = Op(M_i) \).

Applying 1.2 for nonlimit ordinals, and 1.1 for limit ordinals, one finds inductively an increasing continuous \( \mathcal{F} \)-chain \( \langle N_i : i \leq \theta^+ \rangle \) such that \( M_i \succeq N_i \succeq M_i^* \) and \( \| N_i \| = \lambda \) for \( i < \theta^+ \), and \( N_{\theta^+} = \bigcup_{i<\theta^+} N_i \). Note that
\[
\| N_{\theta^+} \| = \theta^+ \cdot \lambda = \lambda.
\]

Since \( T \) is \( \lambda \)-categorical, \( N_{\theta^+} \cong EM(I) \), where 2.7 furnishes \( I \) of power \( \lambda \). By 2.10(2), there is an expansion \( N_{\theta^+}^+ \) of \( N_{\theta^+} \) by at most \( \kappa + |T| \) functions such that if \( A \subset |N_{\theta^+}^+| \) is closed under the functions of \( N_{\theta^+}^+ \), then
\[
(N_{\theta^+}^+, |L|)A \succeq N_{\theta^+}^+.
\]

Choose a set \( A_i \) and an ordinal \( j_i \), by induction on \( i < \theta^+ \), satisfying
(1) \( A_i \subset |N_{\theta^+}^+|, |A_i| \leq \theta \); \( \langle A_i : i < \theta^+ \rangle \) is continuous increasing;
(2) \( \langle j_i : i < \theta^+ \rangle \) is continuous increasing;
(3) \( A_i \) is closed under the functions of \( N_{\theta^+}^+ \);
(4) \( A_i \subset |N_{j_i+1}^+| \);
(5) \( |M_i| \subset A_{i+1} \).
This is possible: for zero or limit ordinals unions work; for \( i + 1 \) choose \( j_i + 1 \) to satisfy (2) and (4), and \( A_{i+1} \) to satisfy (1), (3) and (5).

By (2), \( C = \{ i < \theta^+ : i \) is a limit ordinal and \( j_i = i \} \) is a club of \( \theta^+ \), i.e. \( C \in D_{\theta^+} \).

Fix \( i \in C \). Note that \( |M_i| \subset A_i \) and \( A_i \subset |N_i| \) (since \( |M_i| = \bigcup_{j<i} |M_j| \subset \bigcup_{j<i} A_j + 1 = A_i = \bigcup_{i' \subset i} A_{i'} \subset \bigcup_{i' \subset i} |N_{j_{i'}} + 1| = N_{j_i} = N_i \) (using (5), (1), (4), (2) and \( j_i = i \)) and so \( M_i \preceq \mathcal{F} (N_{\theta^+}^+ | L) \) \( |A_i| \leq \mathcal{F} N_i \preceq \mathcal{F} M^+_i = \text{Op}(M_i) \), so that \( M_i \preceq (N_{\theta^+}^+ | L) | A_i \). However, by (3) and the choice of \( N_{\theta^+} \) and \( N_{\theta^+}^+ \) one also has \( (N_{\theta^+}^+ | L) | A_i \preceq N_{\theta^+} \). So by transitivity of \( \preceq \), one obtains \( M_i \preceq N_{\theta^+} \).

Finally, remark that \( M_{\theta^+} \preceq \mathcal{F} N_{\theta^+} \) since \( M_i \preceq N_i \preceq \mathcal{F} N_{\theta^+} \) for every \( i < \theta^+ \). Hence \( C \subset \{ i < \theta^+ : M_i \preceq \mathcal{F} N_{\theta^+} \} \in D_{\theta^+} \).

**Definition 3.2.** Suppose that \( \theta \in [\kappa + |T|, \lambda) \) and \( M \in K_{\theta} \). Then \( M \) is nice iff whenever \( M \preceq \mathcal{F} N \in K_{\theta} \), then \( M \preceq N \). (The analogous \( \mathcal{F} \)-elementary embedding definition runs: \( M \) is nice iff whenever \( f : M \rightarrow \mathcal{F} N \in K_{\theta} \) then \( f : M \rightarrow N \).)

**Theorem 3.3.** Suppose that \( T \) is categorical in \( \lambda \) and \( M \in K_{\theta} \) with \( \theta \in [\kappa + |T|, \lambda) \). Then there exists \( N \in K_{\theta} \) such that \( M \preceq \mathcal{F} N \) and \( N \) is nice.

**Proof.** Suppose otherwise. We will define a continuous increasing \( \preceq \mathcal{F} \)-chain \( \langle M_i \in K_{\theta} : i < \theta^+ \rangle \) such that for \( j < \theta^+ \),

\[
(*)_j \quad M_j \not\preceq \mathcal{F} M_{j+1}.
\]

For \( i = 0 \), put \( M_0 = M \); if \( i \) is a limit ordinal, put \( M_i = \bigcup_{j<i} M_j \); if \( i = j + 1 \), then, since 3.3 is assumed to fail, \( M_{j+1} \) exists as required in \( (*)_j \) (otherwise \( M_i \) works as \( N \) in 3.3). But now \( \langle M_i : i < \theta^+ \rangle \) yields a contradiction to 3.1, since \( C = \{ i < \theta^+ : M_i \preceq \bigcup_{j<i} M_j \} \in D_{\theta^+} \) by 3.1 so that choosing \( j \) from \( C \) one has \( M_j \preceq M_{j+1} \) by 1.10, contradicting \( (*)_j \).

**Theorem 3.4.** Suppose that \( T \) is categorical in \( \lambda \) and \( \theta \in [\kappa + |T|, \lambda) \). If \( M \in K_{\theta} \) is nice and \( f : M \rightarrow \mathcal{F} N \in K_{\leq \lambda} \), then \( f : M \rightarrow N \).

**Proof.** Choosing an appropriate \( \text{Op} \) and using 1.2 one finds \( N_1 \) such that \( N \preceq \mathcal{F} N_1 \) and \( |N_1| = \lambda \). Find \( M'_1 \preceq N_1 \) by 2.10(2) such that \( \text{rng}(f) \subset |M'_1| \) and \( \|M'_1\| = \theta \). So \( M'_1 \preceq \mathcal{F} N_1 \) and so \( \text{rng}(f) \preceq \mathcal{F} M'_1 \). Since \( M \) is nice, we have \( f : M \rightarrow M'_1 \). Now \( M'_1 \preceq N_1 \), so \( f : M \rightarrow N_1 \). So there are \( \text{Op} \) and \( g : N_1 \rightarrow \mathcal{F} \text{Op}(M) \) satisfying \( gf = f_{\text{Op}} \). Since \( N \preceq \mathcal{F} N_1 \) it follows that \( f : M \rightarrow N \) as required.
Corollary 3.5. Suppose that $M \in K_{\theta}$ is nice with $\theta \in [\kappa + |T|, \lambda)$. Then $M$ is an a.b. in $K_{<\lambda}$, i.e. if $f_i : M \rightarrow M_i$ and $M_i \in K_{<\lambda}$ ($i = 1, 2$), then there exists an amalgam $N \in K_{<\lambda}$ of $M_1, M_2$ over $M$ with respect to $f_1, f_2$.

Proof. By 3.4, $f_i : M \rightarrow M_i$ ($i = 1, 2$). Hence by 2.1 there is an amalgam $N \in K_{<\lambda}$ of $M_1, M_2$ over $M$ with respect to $f_1, f_2$.

Definition 3.6. Suppose that $\theta \in [\kappa + |T|, \lambda)$ and $\sigma$ is a cardinal.

1. A model $M \in K_{\theta}$ is $\sigma$-universal iff for every $N \in K_{\sigma}$, there exists an $\mathcal{F}$-elementary embedding $f : N \rightarrow M$. $M$ is universal iff $M$ is $\|M\|$-universal.

2. A model $M_2 \in K_{<\sigma}$ is $\sigma$-universal over the model $M_1$ (and one writes $M_1 \preceq \sigma M_2$) iff $M_1 \preceq \sigma M_2$ and whenever $M_1 \preceq \sigma M_2' \in K_{\sigma}$, then there exists an $\mathcal{F}$-elementary embedding $f : M_2' \rightarrow \mathcal{F} M_2$ such that $f|M_1$ is the identity. (The embedding version runs: there exists $h : M_1 \rightarrow \mathcal{F} M_2$ such that whenever $g : M_1 \rightarrow \mathcal{F} M_2'$, then there exists $f : M_2' \rightarrow \mathcal{F} M_2$ with $fg = h$.) $M_2$ is universal over $M_1$ ($M_1 \preceq \sigma M_2$) iff $M_2$ is $\|M_2\|$-universal over $M_1$.

3. $M_2$ is $\sigma$-universal over $M_1$ in $M$ iff $M_1 \preceq \sigma M_2 \preceq \sigma M_1$, $\|M_1\| \preceq \sigma$ and whenever $M_2' \in K_\sigma$ and $M_1 \preceq \sigma M_2' \preceq \sigma M_1$, then there exists an $\mathcal{F}$-elementary embedding $f : M_2' \rightarrow M_2$ such that $f|M_1$ is the identity. $M_2$ is universal over $M_1$ in $M$ iff $M_2$ is $\|M_2\|$-universal over $M_1$ in $M$.

4. $M_2$ is weakly $\sigma$-universal over $M_1$ (written $M_1 \preceq \sigma M_2$) iff $M_1 \preceq \sigma M_2$ $M_2 \in K_\sigma$ and whenever $M_2' \preceq \sigma M_2'$ $M_2' \in K_\sigma$, then there exists an $\mathcal{F}$-elementary embedding $f : M_2' \rightarrow \mathcal{F} M_2$ such that $f|M_1$ is the identity. (The embedding version: there exists $h : M_1 \rightarrow \mathcal{F} M_2$ such that whenever $g : M_2 \rightarrow \mathcal{F} M_2'$, then there exists $f : M_2' \rightarrow \mathcal{F} M_2$ such that $h = fgh$ (written $h : M_1 \rightarrow M_2$)). $M_2$ is weakly universal over $M_1$ ($M_1 \preceq \sigma M_2$) iff $M_2$ is $\|M_2\|$-weakly universal over $M_1$.

(5) In $K_{<\lambda}$, if $M_1$ is an a.b., then weak universality over $M_1$ is equivalent to universality over $M_1$.

Proof. Suppose that $h : M_1 \rightarrow M_2$ and $g : M_1 \rightarrow \mathcal{F} M_2' \in K_{\|M_2\|}$. Since $M_1$ is an a.b. there exist a model $N$ and $h' : M_2 \rightarrow \mathcal{F} N$, $g' : M_2' \rightarrow \mathcal{F} N$ satisfying $h' h = g' g$. By 1.2 we can assume that $\|N\| = \|M_2\|$. Since $M_2$ is weakly universal over $M_1$, there exists $h'' : N \rightarrow \mathcal{F} M_2$ with $h = h'' h' h$. Let
\( f = h''g' : M_2' \to M_2 \), and note that \( fg|_M_1 = h''g'g = h''h'h = h \), so that \( M_2 \) is universal over \( M_1 \).

Remark 3.6B. Conversely,

(6) For any model \( M \), universality over \( M \) implies weak universality over \( M \).

Lemma 3.7. Suppose that \( T \) is categorical in \( \lambda \) and \( \theta \in [\kappa + |T|] \). If \( M \in K_\theta \) and \( M \preceq_\mathcal{F} N \in K_\lambda \), then there exists \( M^+ \in K_\theta \) such that

1. \( M \preceq_\mathcal{F} M^+ \preceq_\mathcal{F} N \);
2. \( M^+ \) is universal over \( M \) in \( N \).

Proof. Now choose \( I \) such that

\((*)_\lambda[I]\)

(i) \( I \) is a linear order of cardinality \( \lambda \);
(ii) if \( \theta \in [\kappa_0, \lambda) \) and \( J_0 \subseteq I \) with \( |J_0| = \theta \) then there is \( J_1 \) satisfying \( J_0 \subseteq J_1 \subseteq I \), \( |J_1| = \theta \), and such that for every \( J^* \subseteq I \) of cardinality \( \leq \theta \) there is an order preserving (one-to-one) mapping from \( J_0 \cup J^* \) into \( J_0 \cup J_1 \) which is the identity on \( J_0 \).

Essentially the construction follows Laver [L] and [Sh220, Appendix]; but for our present purpose let \( I = (\omega^\theta, <_{\text{lex}}) \); given \( \theta \) and \( J_0 \) we can increase \( J_0 \) so without loss of generality, \( J_0 = \omega^\theta A \), \( A \subseteq \lambda \), \( |A| = \theta \). Define an equivalence relation \( E \) on \( I \setminus J_0 \); \( \eta \in \mathcal{E} \nu \iff (\forall g \in J_0)(\eta <_{\text{lex}} \nu \equiv g <_{\text{lex}} \nu) \); clearly it has \( \leq \theta \) equivalence classes. Let \( \{ \eta_i^* : i < \iota^* \leq \theta \} \) be a set of representatives, each of minimal length, so \( \eta_i^* [\lg \eta_i^* - 1] \in J_0 \) and \( \eta_i^* [\lg \eta_i^* - 1] \in \lambda \setminus A \).

Let \( J_1 = I \cup \{ \eta_i^* \nu : \nu \in \omega^\theta \) \) and \( i < i^* \). Then clearly \( J_0 \subseteq J_1 \subseteq I \) and \( |J_1| = \theta \). Suppose \( J_0 \subseteq J \subseteq I \), \( |J| \leq \theta \), and we should find the required embedding \( h \). As before we can assume that \( J = \omega^\theta B \), \( |B| = \theta \) and \( A \subseteq B \). Now \( h|_{J_0} = \text{id}_{J_0} \) so it is enough to define \( h|(J_1 \cap (\eta_i^*/E)) \), hence it is enough to embed \( J_1 \cap (\eta_i^*/E) \) into \( \{ \eta_i^* \nu : \nu \in \omega^\theta \} \) (under \( <_{\text{lex}} \)).

Let \( \gamma = \text{otp}B \). It is enough to show \( (\omega^\omega, <_{\text{lex}}) \) can be embedded into \( \omega^\theta \), where of course \( |\gamma| \leq \theta \). This is proved by induction on \( \gamma \).

Since \( T \) is \( \lambda \)-categorical and \( \text{EM}(I) \) is a model of \( T \) of power \( \lambda \), there is an isomorphism \( g \) from \( \text{EM}(I) \) onto \( N \). It follows from \( (*)_\lambda[I] \) that \( M^+ = g^\text{EM}(J) \) in \( K_\theta \) satisfies (1) and (2). (Analogues of (1) and (2) are checked also in the course of the proof of 3.11.)

Lemma 3.8. Suppose that \( T \) is categorical in \( \lambda \), \( \theta \in [\kappa + |T|, \lambda] \) and \( \langle M_i \in K_\theta : i < \theta^+ \rangle \) and \( \langle N_i \in K_\lambda : i < \theta^+ \rangle \) are continuous \( \prec_\mathcal{F} \)-chains such that for every \( i < \theta^+ \), \( M_i \preceq_\mathcal{F} N_i \). Then there exists \( i(*) < \theta^+ \) such that \( (i(*), \theta^+) \subseteq C = \{ i < \theta^+ : M_{i+1} \text{ can be } \mathcal{F}-\text{elementarily embedded into } N_i \text{ over } M_0 \} \).
Proof. Apply 3.7 for $M_0 \in K_\theta$ and $N_\theta+ = \bigcup_{i < \theta^+} N_i \in K_\lambda$ (noting that $M_0 \preceq \mathcal{F} N_0 \preceq \mathcal{F} N_{\theta^+}$) to find $M^+ \in K_\theta$ such that $M_0 \preceq \mathcal{F} M^+ \preceq \mathcal{F} N_{\theta^+}$ and $M^+$ is universal over $M_0$ in $N_{\theta^+}$.

For some $i(*) < \theta^+$, $M^+ \subseteq N_{i(*)}$ and so $M^+ \preceq \mathcal{F} N_{i(*)}$. If $i \in (i(*), \theta^+)$, then $M_{i+1} \in K_\theta$ and $M_0 \preceq \mathcal{F} M_{i+1} \preceq \mathcal{F} N_{i+1} \preceq \mathcal{F} N_{\theta^+}$, so there is an $\mathcal{F}$-elementary embedding $f : M_{i+1} \rightarrow \mathcal{F} M^+$ and $f | M_0$ is the identity. Now $M^+ \preceq \mathcal{F} N_{i(*)} \preceq \mathcal{F} N_i$, so $f : M_{i+1} \rightarrow \mathcal{F} N_i$. Hence $(i(*), \theta^+) \subseteq C$ as required.

Theorem 3.9. Suppose that $T$ is categorical in $\lambda$, $\theta \in [\kappa + |T|, \lambda)$, and $M \in K_\theta$. Then there exists $M^+ \in K_\theta$ such that

(8) $M \preceq \mathcal{F} M^+$ and $M^+$ is nice;

(2) $M^+$ is weakly universal over $M$.

Proof. Define by induction on $i < \theta^+$ continuous $\preceq_{\mathcal{F}}$-chains $\langle M_i \in K_\theta : i < \theta^+ \rangle$ and $\langle N_i \in K_\lambda : i < \theta^+ \rangle$ such that

(0) $M_0 = M$;

(1) $M_i \preceq \mathcal{F} N_i$;

(2) if $(*)_i$ holds, then $M_{i+1}$ cannot be $\mathcal{F}$-elementarily embedded into $N_i$ over $M_0$, where $(*)_i$ is the statement: there are $M' \in K_\theta$ and $N' \in K_\lambda$ such that $M_i \preceq \mathcal{F} M', N_i \preceq \mathcal{F} N'$, $M' \preceq \mathcal{F} N'$ and $M'$ cannot be $\mathcal{F}$-elementarily embedded into $N_i$ over $M_0$;

(3) $M_{i+1} \preceq_{\text{nice}} N_{i+1}$.

This is possible. $N_0$ is obtained by an application of 1.2 to an appropriate Opt($M_0$) of power at least $\lambda$. At limit stages, continuity dictates that one take unions. Suppose that $M_i$ has been defined. If $(*)_i$ does not hold, by 2.10(2) there is $M'' \in K_\theta$ with $M_i \preceq \mathcal{F} M'' \preceq N_i$. Let $M_{i+1} = M''$ and $N_{i+1} = N_i$. If $(*)_i$ does hold for $M', N'$, let $N_{i+1} = N'$; note that by 2.10(2) there exists $M'' \in K_\theta$ such that $M' \preceq \mathcal{F} M'' \preceq N'$; now let $M_{i+1} = M''$.

Note that in each case, (3) is satisfied.

Find $i(*) < \theta^+$ and $C$ as in 3.8 and choose $i \in C$. By (1), $M_{i+1} \preceq \mathcal{F} N_{i+1}$ so by 3.7 there exists $M^- \in K_\theta$ such that $M_{i+1} \preceq \mathcal{F} M^- \preceq \mathcal{F} N_{i+1}$ and $M^-$ is weakly universal over $M_{i+1}$ in $N_{i+1}$. By 3.3 one can find $M^+ \in K_\theta$ such that $M^- \preceq \mathcal{F} M^+$ and $M^+$ is nice. So $M^+$ satisfies (8). It remains to show that $M^+$ is weakly universal over $M$. Suppose not and let $g : M^+ \rightarrow \mathcal{F} M^+ \in K_\theta$, where $M^+$ cannot be $\mathcal{F}$-elementarily embedded in $M^+$ over $M$, hence cannot be $\preceq_{\mathcal{F}}$-elementarily embeddable in $M^-$ over $M$, hence in $N_{i+1}$ over $M$. Since $M_{i+1} \preceq \mathcal{F} M^+ \in K_\theta$ and by (3), $M_{i+1} \preceq_{\text{nice}} N_{i+1} \in K_\lambda$, by 2.1 there is an amalgam $N^* \in K_\lambda$ of $M^*, N_{i+1}$. The existence of $M^*, N^*$ implies that $(*)_i$ holds since $M^*$ cannot be $\mathcal{F}$-elementarily embedded into $N_{i+1}$ over $M_0$, so by 3.3 one can find $M^+ \in K_{\lambda'}$ such that $M^+ \preceq \mathcal{F} M^+ \preceq N_{i+1}$ and $M^+$ is weakly universal over $M_{i+1}$ in $N_{i+1}$.
hence $M_{i+2}$ cannot be $\mathcal{F}$-elementarily embedded into $N_{i+1}$ in contradiction to the choice of $i$ as by 3.7, $i + 1$ is in $C$.

**Corollary 3.10.** If $T$ is categorical in $\lambda$, $\theta \in [\kappa + |T|, \lambda]$ and $M \in K_\theta$ is an a.b. (e.g. $M$ is nice—see 2.1), then there exists $M^+ \in K_\theta$ such that

1. $M^+ \preceq \mathcal{F} M$ and $M^+$ is nice;
2. $M^+$ is universal over $M$.

**Proof.** 3.9 and 3.6A(5).

**Corollary 3.11.** Suppose that $T$ is categorical in $\lambda$ and $\theta \in [\kappa + |T|, \lambda]$. Then there is a nice universal model $M \in K_\theta$.

**Proof.** By 3.3 it suffices to find a universal model of power $\theta$, noting that universality is preserved under $\mathcal{F}$-elementary extensions in the same power.

As in the proof of 3.7, there is a linear order $(I, <)$ of power $\lambda$ and $J \subset I$ with $|J| = \theta$ such that

1. $(\forall J' \subset I)$ (if $|J'| \leq \theta$, then there is an order-preserving injective map $g$ from $J'$ into $J$).

**Claim.** $\textbf{EM}(J) \in K_\theta$ is universal.

**Proof.** $\textbf{EM}(J)$ is a model of power $\theta$ since $\max(|J|, \kappa + |T|) \leq \theta$ and $\theta = |J| \leq \|\textbf{EM}(J)\|$. Suppose that $N \in K_\lambda$. Applying 1.2 to a suitably large $\text{Op}(N)$ find $M \in K_\lambda$ with $N \preceq \mathcal{F} M$ so that by $\lambda$-categoricity of $T$, $M \cong \textbf{EM}(I)$. There is a surjective $\mathcal{F}$-elementary embedding $h : N \rightarrow \mathcal{F} N' \preceq \mathcal{F} \textbf{EM}(I)$ and there exists $J' \subset I$ with $|J'| \leq \|N'|| + \kappa + |T| = \theta$ such that $N' \subset \textbf{EM}(J')$. So by $(\forall)$ there is an order-preserving injective map $g$ from $J'$ into $J$. Then $g$ induces an $\mathcal{F}$-elementary embedding $\hat{g}$ from $\textbf{EM}(J')$ into $\textbf{EM}(J)$. Let $f = \hat{g}h$. Then $f : N \rightarrow \mathcal{F} \textbf{EM}(J)$ is as required.

**Theorem 3.12.** Suppose that $T$ is categorical in $\lambda$, $\theta \in [\kappa + |T|, \lambda]$, $N \in K_\lambda$ is nice, $M \in K_\theta$ and $M \preceq N$. Then $M$ is nice.

**Proof.** Let $B \in K_\theta$ with $M \preceq \mathcal{F} B$. Show that $M \preceq B$. Well, since $M \preceq N$ and $M \preceq \mathcal{F} B$, by 2.1 there exists an amalgam $M^* \in K_{<\lambda}$ of $N, B$ over $M$. Without loss of generality, by 1.5, $\|M^*\| = \|N\|$. Now $N$ is nice, hence $N \preceq M^*$. Since $M \preceq N$, it follows by 1.7.5 that $M \preceq M^*$. Since $M \preceq \mathcal{F} B \preceq \mathcal{F} M^*$, it follows by 1.10 that $M \preceq B$.

### 4. $(\theta, \sigma)$-saturated models.

In this section we define notions of saturation which will be of use in proving amalgamation for $K_\lambda$. 
Definition 4.1. Suppose that $\sigma$ is a limit ordinal with $1 \leq \sigma \leq \theta \in [\kappa + |T|, \lambda)$.

1. An $L$-structure $M$ is $(\theta, \sigma)$-saturated iff
   (a) $|M| = \theta$;
   (b) there exists a continuous $\prec_F$-chain $\langle M_i \in K_\theta : i < \sigma \rangle$ such that
       (i) $M_0$ is nice and universal, (ii) $M_{i+1}$ is universal over $M_i$, (iii) $M_i$ is nice, and (iv) $M = \bigcup_{i < \sigma} M_i$.

2. $M$ is $\theta$-saturated iff $M$ is $(\theta, \text{cf}(\theta))$-saturated.

3. $M$ is $(\theta, \sigma)$-saturated over $N$ iff $M$ is $(\theta, \sigma)$-saturated as witnessed by a chain $\langle M_i : i < \sigma \rangle$ such that $N \subseteq M_0$.

The principal facts established in this section connect the existence, uniqueness and niceness of $(\theta, \sigma)$-saturated models.

Theorem 4.2. Suppose that $T$ is categorical in $\lambda$ and $\sigma \leq \theta \in [\kappa + |T|, \lambda)$. Then

1. there exists a $(\theta, \sigma)$-saturated model $M$;
2. $M$ is unique up to isomorphism;
3. $M$ is nice.

Proof. One proves (1), (2) and (3) simultaneously by induction on $\sigma$.

(1) Choose a continuous $\prec_F$-chain $\langle M_i \in K_\theta : i < \sigma \rangle$ of nice models by induction on $i$ as follows. For $i = 0$, apply 3.11 to find a nice universal model $M_0 \in K_\theta$. For $i = j + 1$, note that $M_j$ is an a.b. by 3.5 (since $M_j$ is nice), hence by 3.10 there exists a nice model $M_i \in K_\theta$ such that $M_j \prec_F M_i$ and $M_i$ is universal over $M_j$. For limit $i$, let $M_i = \bigcup_{j < i} M_j$.

(2) As $\sigma$ is a limit ordinal, a standard back-and-forth argument shows that if $M$ and $N$ are $(\theta, \sigma)$-saturated models, then $M$ and $N$ are isomorphic.

(3) By the uniqueness (i.e. by (2)) it suffices to prove that some $(\theta, \sigma)$-saturated model is nice. Suppose that $M$ is $(\theta, \sigma)$-saturated. We will show that $M$ is nice.

If $\text{cf}(\sigma) < \sigma$, then $M$ is also $(\theta, \text{cf}(\sigma))$-saturated and hence by the inductive hypothesis (3) on $\sigma$ for $\text{cf}(\sigma)$, $M$ is nice. So we will assume that $\text{cf}(\sigma) = \sigma$. Choose a continuous $\prec_F$-chain $\langle M_i \in K_\theta : i < \theta^+ \rangle$ such that: $M_0$ is nice and universal (possible by 3.11); if $M_i$ is nice, then $M_{i+1} \in K_\theta$ is nice and universal over $M_i$ (possible by 3.5 and 3.10); if $M_i$ is not nice
(so necessarily $i$ is a limit ordinal), then $M_{i+1} \in K_\theta$, $M_i \preceq_x M_{i+1}$ and $M_i \nsubseteq M_{i+1}$. By 3.1 and 1.10 there is a club $C$ of $\theta^+$ such that if $i \in C$, then $M_i \nsubseteq M_{i+1}$. So by the choice of $\langle M_i : i < \theta^+ \rangle$, if $i \in C$, then $M_i$ is nice. Choose $i \in C$ with $i = \sup(i \cap C)$ and $\text{cf}(i) = \sigma$. It suffices to show that $M_i$ is $(\sigma, \sigma)$-saturated (for then, by (2), $M_i$ is isomorphic to $M$ and so $M$ is nice). Choose a continuous increasing sequence $\langle \alpha_\zeta : \zeta < \sigma \rangle \subseteq C$ such that $i = \sup_\zeta \alpha_\zeta$ (recall that $i = \sup(i \cap C)$ and $\text{cf}(i) = \sigma$). Now $M_i = \bigcup_{\zeta < \sigma} M_{\alpha_\zeta}$. Of course $M_{\alpha_0}$ is universal (since $M_0$ is universal and $M_0 \preceq_x M_{\alpha_0}$). $M_{\alpha_0 \cup 1}$ is universal over $M_{\alpha_0}$ since $M_{\alpha_0 \cup 1}$ is universal over $M_{\alpha_0}$ and $M_{\alpha_0} \preceq_x M_{\alpha_0 \cup 1} \preceq_x M_{\alpha_0 \cup 1}$. Also $M_{\alpha_\zeta}$ is nice for each $\zeta < \sigma$ since $\alpha_\zeta \in C$. Hence $M_i$ is $(\sigma, \sigma)$-saturated.

Remark 4.3. Remember that by 3.12, if $T$ is categorical in $\lambda$, $\theta \in [\kappa + |T|, \lambda)$, $N \in K_{\kappa \lambda}$ is nice, $M \in K_\theta$ and $M \preceq N$, then $M$ is nice.

Theorem 4.4. Suppose that $T$ is categorical in $\lambda$ and $\kappa + |T| < \theta < \theta^+ < \lambda$. If $\langle M_i \in K_\theta : i < \theta^+ \rangle$ is a continuous $\prec_x$-chain of nice models such that $M_{i+1}$ is universal over $M_i$ for $i < \theta^+$, then $\bigcup_{i < \theta^+} M_i$ is $(\theta^+, \theta^+)$-saturated.

Proof. Write $M = \bigcup_{i < \theta^+} M_i$. Note that if $\langle M'_i \in K_\theta : i < \theta^+ \rangle$ is any other continuous $\prec_x$-chain of nice models such that $M_{i+1}'$ is universal over $M_i'$, then $\bigcup_{i < \theta^+} M'_i \equiv M$ (use again the back-and-forth argument).

By 4.2 there exists a $(\theta^+, \theta^+)$-saturated model $N$ which is unique and nice. In particular, $|N| = \theta^+$ and there exists a continuous $\prec_x$-chain $\langle N_i \in K_{\theta^+} : i < \theta^+ \rangle$ such that (i) $N_0$ is nice and universal, (ii) $N_{i+1}$ is universal over $N_i$, (iii) $N_i$ is nice, and (iv) $N = \bigcup_{i < \theta^+} N_i$. It suffices to prove that $M$ and $N$ are isomorphic models.

Without loss of generality, $|N| = \theta^+$. By 1.2, $C_1 = \{ \delta < \theta^+ : N|\delta \preceq_x N \}$ contains a club of $\theta^+$. By 3.1 there exists a club $C_2$ of $\theta^+$ such that for every $\delta \in C_2$, $N|\delta \preceq N$. Since $\{|N_i| : i < \theta^+\}$ is a continuous increasing sequence of subsets of $\theta^+$, it follows that $C_3 = \{ \delta < \theta^+ : \delta \subseteq |N_3| \}$ is a club of $\theta^+$. Hence there is a club $C_4$ of $\theta^+$ such that $C_4 \subseteq C_1 \cap C_2 \cap C_3 \cap [\theta, \theta^+)$.

Note that for $\delta \in C_4$ one has $N|\delta \preceq_x N$, $|N|\delta = \delta \subseteq |N_3|$ and $N_3 \preceq_x N$, so that $N|\delta \preceq_x N \wedge N_3 \preceq_x N$ and so by 1.10, $N|\delta \preceq N_3$. $\langle N_3 : \delta \in C_4 \rangle$ is a continuous increasing $\prec_x$-chain, $N_3 \in K_{\theta^+}$ and $N|\delta \in K_\theta$.

By 3.12, $N|\delta$ is nice since $N_3$ is nice (by (iii)). So by 3.10, $N|\delta$ has a nice $\prec_x$-extension $B_\delta \in K_\theta$ which is universal over $N|\delta$. Without loss of generality, $N|\delta \preceq_x B_\delta \preceq N$. [Why? since $N|\delta \preceq_x B_\delta$ (in fact $N|\delta \preceq_n B_\delta$) and $N|\delta \preceq N_3$, by 2.1 there exists an amalgam $A_\delta \in K_{\leq \theta^+}$ of $B_\delta$, $N_3$ nice.
over $N \models \delta$. Let $f_\delta : B_\delta \to \mathcal{A}_\delta$ be a witness. But $N_{\delta + 1}$ is universal over $N_{\delta}$ (by (ii)), so $A_\delta$ can be $<_\mathcal{A}$-elementarily embedded into $N_{\delta + 1}$ over $N_{\delta}$ (say by $g_\delta$), hence $B_\delta$ can be $<_\mathcal{A}$-elementarily embedded into $N$ (using $g_\delta f_\delta$).

Let $C_5 = \{ \delta \in C_4 : \alpha \in C_4 \cap \delta, \text{then } |B_\alpha| \subset \delta \}$. Note that $C_5$ is a club of $\theta^+$ since $|B_\alpha| = \theta$. Let $E_\alpha = (\sup |B_\alpha|, \theta^+) \cap C_4$ for $\alpha \in C_4$, $E_\alpha = \theta^+$ for $\alpha \notin C_4$, and let $E$ be the diagonal intersection of $\langle E_\alpha : \alpha < \theta^+ \rangle$, i.e. $E = \{ \delta < \theta^+ : (\forall \alpha < \delta)(\delta \in E_\alpha) \}$. Note that $E$ is club of $\theta^+$ and $C_5 \supseteq E \cap C_4$, which is a club of $\theta^+$. Thus $(N, \delta : \delta \in C_5)$ is a continuous $<_\mathcal{A}$-chain of nice models, each of power $\theta$. If $\delta_1 \in C_5$ and $\delta_2 = \min(C_5 \setminus (\delta_1 + 1))$, then $N_{\delta_1} \not\preceq \mathcal{A} \not\preceq N_{\delta_2}$. Hence $N_{\delta_2} \not\preceq \mathcal{A}$ is universal over $N_{\delta_1}$ (since $B_{\delta_1} \not\preceq \mathcal{A}$ is universal over $N_{\delta_1}$). Let $\{ \delta_i : i < \theta^+ \}$ enumerate $C_5$ and set $M_i = N_{\delta_i}$. Note that $N = \bigcup_{i < \theta^+} M_i$. Then $\langle M_i : i < \theta^+ \rangle$ is a continuous $<_\mathcal{A}$-chain of nice models, and $M_{i+1}$ is universal over $M_i$. Therefore $N$ and $M$ are isomorphic (as said at the beginning of the proof), as required.

**Notation 4.5.** $\Theta = \{ \overline{\theta} : \overline{\theta} = \langle \theta_i : i < \delta \rangle \}$ is a continuous (strictly) increasing sequence of cardinals, $\kappa + |T| < \theta_0$, $\delta < \theta_0$ (a limit ordinal), $\bigcup_{i < \delta} \theta_i \leq \lambda$ and $\Theta^- = \{ \overline{\theta} \in \Theta : \sup \theta_i < \lambda \}$.

**Remark 4.6.** Let $\theta = \sup \text{rng}(\overline{\theta})$ for $\overline{\theta} \in \Theta$. Then $\theta$ is singular, since $\text{cf}(\theta) \leq \delta < \theta_0 \leq \theta$.

**Definition 4.7.** Let $\overline{\theta} \in \Theta$. A model $M$ is $\overline{\theta}$-saturated iff there is a continuous $<_\mathcal{A}$-chain $\langle M_i \in K_{\theta_0} : i < \delta \rangle$ such that $M = \bigcup_{i < \delta} M_i$, $M_i$ is nice and $M_{i+1}$ is $\theta_i$-universal over $M_i$.

**Definition 4.8.** Suppose that $\overline{\theta} \in \Theta$. $\text{Pr}(\overline{\theta})$ holds iff every $\overline{\theta}$-saturated model is nice.

**Remark 4.9.** (1) If $\overline{\theta}_1, \overline{\theta}_2 \in \Theta$, $\text{rng}(\overline{\theta}_1) \subseteq \text{rng}(\overline{\theta}_2)$, $\sup \text{rng}(\overline{\theta}_1) = \sup \text{rng}(\overline{\theta}_2)$, and $M$ is $\overline{\theta}_2$-saturated, then $M$ is $\overline{\theta}_1$-saturated.

(2) For $\overline{\theta} \in \Theta$ and $\text{Pr}(\overline{\theta})$ whenever $\overline{\theta} \in \Theta$ is a proper initial segment of $\overline{\theta}$, there is a $\overline{\theta}$-saturated model and it is unique.

**Theorem 4.10.** Suppose that $T$ is categorical in $\lambda$, $\overline{\theta} \in \Theta$ and for every limit ordinal $\alpha < \text{lg}(\overline{\theta})$, $\text{Pr}(\overline{\theta}|\alpha)$, then $\text{Pr}(\overline{\theta})$.

**Proof.** By 4.9(1) and the uniqueness of $\overline{\theta}$-saturated models (4.9(2)), without loss of generality one may assume that $\text{otp}(\overline{\theta}) = \text{cf}(\text{sup \text{rng}(\overline{\theta})})$. Let $\theta = \sup \text{rng}(\overline{\theta})$. Now by 4.6, $(\text{cf}(\theta))^+ < \theta$, so by [Sh420, 1.5 + 1.2(1)] there exists $\langle S, \langle C_\alpha : \alpha \in S \rangle \rangle$ such that

1. $S \subset \theta^+$ is set of ordinals; $0 \notin S$;
2. $S_1 = \{ \alpha \in S : \text{cf}(\alpha) = \text{cf}(\theta) \}$ is a stationary subset of $\theta^+$;
3. if $\alpha \in S$ is a limit ordinal then $\alpha = \sup C_\alpha$ and if $\alpha \in S$ then $\text{otp}(C_\alpha) \leq \text{cf}(\theta)$;
(δ) if $\beta \in C_\alpha$, then $\beta \in S$ and $C_\beta = C_\alpha \cap \beta$;
(ε) $C_\alpha$ is a set of successor ordinals.

[Note that the existence of $\langle S, \langle C_\alpha : \alpha \in S \rangle \rangle$ is provable in ZFC.]
Without loss of generality, $S \setminus S_1 = \bigcup \{ C_\alpha : \alpha \in S_1 \}$. We shall construct the required model by induction, using $\langle C_\alpha : \alpha \in S \rangle$. Remember $\bar{\theta} = \langle \theta_\zeta : \zeta < \text{cf}(\theta) \rangle$. Let us start by defining by induction on $\alpha < \theta^+$ the following entities: $M_\alpha$, $M_\alpha \xi$ (for $\alpha < \theta^+$, $\xi < \text{cf}(\theta)$), and $N_\alpha$ (only when $\alpha \in \bigcup_{\beta \in S} C_\beta$) such that

(A) $M_\alpha \in K_{\bar{\theta}}$;
(A2) $\langle M_\alpha : \alpha < \theta^+ \rangle$ is a continuous increasing $\preceq_\mathcal{F}$-chain of models;
(A3) $M_{\alpha+1}$ is nice, and if $M_\alpha$ is not nice, then $M_\alpha \not\subseteq M_{\alpha+1}$;
(A4) $M_{\alpha+1}$ is weakly universal over $M_\alpha$;
(B) $M_\alpha = \bigcup_{\xi < \text{cf}(\theta)} M_\alpha \xi$, $\| M_\alpha \xi \| = \theta_\xi$;

if $\alpha \in S_1$, $\beta \in C_\alpha$, $\gamma \in C_\alpha$, $\beta < \gamma$, then

(B1) $N_\beta \preceq_\mathcal{F} M_\beta$;
(B2) $\| N_\beta \| = \text{otp}(C_\gamma)$;
(B3) $(\forall \xi < \text{otp}(C_\gamma))(M_{\beta \xi} \subseteq N_\gamma)$;
(B4) $N_\beta$ is nice;
(B5) $N_\gamma$ is $\text{otp}(C_\gamma)$-universal over $N_\beta$.

There are now two tasks at hand. First of all, we shall explain how to construct these entities (THE CONSTRUCTION). Then we shall use them to build a nice $\bar{\theta}$-saturated model (PROVING $\text{Pr}(\bar{\theta})$). From the uniqueness of $\bar{\theta}$-saturated models it will thus follow that $\text{Pr}(\bar{\theta})$ holds.

THE CONSTRUCTION.

Case (i): $\beta = 0$. Choose $M_0 \in K_{\bar{\theta}}$ and $\langle M_{0 \xi} : \xi < \text{cf}(\theta) \rangle$ with $M_0 = \bigcup_{\xi < \text{cf}(\theta)} M_{0 \xi}$ using 1.2. There is no need to define $N_0$ since $0 \notin C_\alpha$.

Case (ii): $\beta$ is a limit ordinal. Let $M_\beta = \bigcup_{\gamma < \beta} M_\gamma$ and choose $\langle M_{\beta \xi} : \xi < \text{cf}(\theta) \rangle$ using 1.2. Again there is no call to define $N_\beta$ since $C_\alpha$ is always a set of successor ordinals.

Case (iii): $\beta$ is a successor ordinal, $\beta = \gamma + 1$. Choose $M'_\gamma \in K_{\bar{\theta}}$ such that $M'_\gamma \preceq_\mathcal{F} M'_\gamma$, and if possible $M'_\gamma \not\subseteq M'_\gamma$; without loss of generality, $M'_\gamma$ is weakly universal over $M_\gamma$ and is nice. If $\beta \notin S$, then define things as above, taking into account (A2). The definitions of $M_\beta$, $M_{\beta \xi}$ present no special difficulties. Now suppose that $\beta \in S$. The problematic entity to define is $N_\beta$.

If $C_\beta = \emptyset$, choose for $N_\beta$ any nice $\preceq_\mathcal{F}$-submodel (of power $\text{otp}(C_\beta)$) of $M_\gamma$. 

If $C_{\beta} \neq \emptyset$, then first define $N_{\beta} = \bigcup_{\gamma \in C_{\beta}} N_{\gamma}$. Note that $N_{\beta}$ is nice. If $C_{\beta}$ has a last element $\beta'$, then $N_{\beta} = N_{\beta'}$, which is nice; if $C_{\beta}$ has no last element, then $N_{\beta} = \bigcup_{\gamma \in C_{\beta}} N_{\gamma}$ is $\bar{\theta}\otimes (C_{\beta})$-saturated, and, by the hypothesis of the theorem, $\Pr(\bar{\theta}\otimes (C_{\beta}))$, so $N_{\beta}$ is nice. Also $N_{\beta} \preceq_{F} M_{\gamma}$. If $\otimes (C_{\beta})$ is a limit ordinal we let $N_{\beta} = N_{\beta}^{\prime}$ and $M_{\beta} = M_{\beta}^{\prime}$, so we have finished, so assume $\otimes (C_{\beta})$ is a successor ordinal. To complete the definition of $N_{\beta}$, one requires a Lemma (the proof of which is similar to 3.9, 3.10):

\[ (*) \quad \text{if } A \subset M \subset K_{\theta} \text{ and } |A| \leq \theta_{j} < \theta, \text{ then there exist a nice } M^{+} \subset K_{\theta}, \text{ with } M \preceq_{F} M^{+}, \text{ and nice models } N^{*}, N^{+} \subset K_{\theta}, \text{ such that } A \subset N^{*} \preceq_{F} N^{+} \preceq_{F} M^{+} \text{ and } N^{+} \text{ is universal over } N^{*}. \]

Why is this enough? Use the Lemma with $M = M_{\beta}^{\prime}$ and $A = N_{\beta} \cup \bigcup_{\xi \in \otimes (C_{\beta}), \alpha \in C_{\beta}} M_{\alpha}$ to find $N^{*}, N^{+}, M^{+}$ and choose $N^{+}, M^{+}$ as $N_{\beta}, M_{\beta}$ respectively.

Why does $(*)$ hold? The proof of $(*)$ is easy.

**PROVING** $\Pr(\bar{\theta})$. For $\alpha \in S_{1}$, consider $(N_{\beta} : \beta \in C_{\alpha})$. For $\beta, \gamma \in C_{\alpha}$ with $\beta < \gamma$, one has $\bigcup_{\xi \in \otimes (C_{\beta})} M_{\beta \xi} \subseteq N_{\gamma}$ by (B$^{\wedge}$). Therefore $M_{\beta} \subseteq \bigcup_{\gamma \in C_{\alpha}} N_{\gamma}$ ($M_{\beta} = \bigcup_{\xi \in \otimes (\alpha)} M_{\beta \xi} = \bigcup_{\xi \in \otimes (\alpha)} M_{\beta \xi} (\alpha \in S_{1})$; for $\xi \in \otimes (\alpha)$, choose $\gamma \in C_{\alpha}$ with $\xi, \beta < \gamma$; so $M_{\beta \xi} \subseteq N_{\gamma}$ and $M_{\beta} \subseteq \bigcup_{\gamma \in C_{\alpha}} N_{\gamma}$).

Thus $M_{\beta} \subseteq \bigcup_{\gamma \in C_{\alpha}} N_{\gamma}$ for every $\beta \in C_{\alpha}$, hence $M_{\alpha} = \bigcup_{\beta \in C_{\alpha}} M_{\beta} \subseteq \bigcup_{\gamma \in C_{\alpha}} N_{\gamma}$ (remember $\alpha = \sup C_{\alpha}$ as $\alpha \in S_{1}$). If $\gamma \in C_{\alpha}$, then $N_{\gamma} \preceq_{F} M_{\gamma}$ (by (B$^{\wedge}$)), and so $\bigcup_{\gamma \in C_{\alpha}} N_{\gamma} \subseteq \bigcup_{\beta \in C_{\alpha}} M_{\beta} = M_{\alpha}$ by continuity. So $M_{\alpha} = \bigcup_{\beta \in C_{\alpha}} N_{\beta}$, hence $(N_{\beta} : \beta \in C_{\alpha})$ exemplifies $M_{\alpha}$ is $\bar{\theta}$-saturated (remember $\Pr(\bar{\theta})$) for every limit $\delta < \lg (\bar{\theta})$; more exactly, we use $(N_{i}^{\prime} : i < \theta)$, $N_{i}^{\prime} = \bigcup\{N_{\beta} : \beta \in C_{\alpha} \text{ and } [i \text{ limit } \Rightarrow \otimes (\beta \cap C_{\alpha}) < i], [i \text{ nonlimit } \Rightarrow \otimes (\beta \cap C_{\alpha}) \leq i]\})$. So $M_{\alpha}$ is $\bar{\theta}$-saturated for every $\alpha \in S_{1}$. In other words, $\{\alpha < \theta^{+} : M_{\alpha} \text{ is } \bar{\theta} \text{-saturated} \} \supseteq S_{1}$ and is stationary, so, applying 3.1, there exists $\alpha < \theta^{+}$ such that $M_{\alpha}$ is $\bar{\theta}$-saturated and $M_{\alpha} \preceq_{\bar{\theta} \text{-universal}} M_{\beta}$. Hence by 1.10, $M_{\alpha} \preceq_{\text{nice}} M_{\alpha+1}$ and so, since $M_{\alpha+1}$ is nice (A$^{\wedge}$), $M_{\alpha}$ is nice (by 3.12).

We conclude that $\Pr(\bar{\theta})$ holds.

To round off this section of the paper, let us make the connection between $\bar{\theta}$-saturation and $(\theta, \otimes (\theta))$-saturation (notation follows 4.5–4.10).

**THEOREM 4.11.** Assume that $T$ is categorical in $\lambda$. Let $\bar{\theta} \in \Theta^{\theta}$ and $\theta = \sup \theta_{i}$. Then every $\bar{\theta}$-saturated model is $(\theta, \otimes (\theta))$-saturated.

**Proof.** Let $\langle M_{\alpha} : \alpha < \theta^{+} \rangle$ be as in the proof of 4.10. By 3.1 there exists a club $C$ of $\theta^{+}$ such that $M_{\alpha} \preceq_{\text{nice}} \bigcup_{\beta < \theta^{+}} M_{\beta}$ for every $\alpha \in C$, hence by the construction $M_{\alpha}$ is nice. So if $\alpha, \beta \in C$ and $\alpha < \beta$, then $M_{\beta}$ is a
universal extension of $M_\kappa$ and for $\gamma = \sup(\gamma \cap C)$, $\gamma \in C$, one sees that $M_\gamma$ is $(\theta, \text{cf}(\gamma))$-saturated. Choose $\gamma \in S_1 \cap C$ and $\sup(\gamma \cap C) = \gamma$. So $M_\gamma$ is $(\theta, \text{cf}(\theta))$-saturated and also $\mathcal{B}$-saturated (see proof of 4.10). Together we finish.

5. The amalgamation property for $\mathcal{K}_{<\lambda}$. Corollaries 5.4 and 5.5 are the goal of this section, showing that every element of $\mathcal{K}_{<\lambda}$ is nice (5.4) and $\mathcal{K}_{<\lambda}$ has the amalgamation property (5.5).

**Lemma 5.1.** Suppose that $\langle \mu_i : i < \text{cf}(\mu) \rangle$ is a continuous strictly increasing sequence of ordinals, $\mu = \sup_{i < \text{cf}(\mu)} \mu_i$, and $\kappa + |T| \leq \mu_0 < \mu \leq \lambda$. Then there exist a linear order $I$ of power $\mu$ and a continuous increasing sequence $\langle I_i : i < \text{cf}(\mu) \rangle$ of linear orders such that

1. $\kappa + |T| \leq |I_i| \leq \mu_i$;
2. $\bigcup_{i < \text{cf}(\mu)} I_i = I$;
3. every $t \in I_{i+1} \setminus I_i$ defines a Dedekind cut of $I_i$ in which (at least) one side of the cut has cofinality $\kappa$.

**Proof.** Let $I = (\{0\} \times \mu) \cup (\{1\} \times \kappa)$, $I_i = (\{0\} \times \mu_i) \cup (\{1\} \times \kappa)$ ordered by $(i, \alpha) < (j, \beta)$ iff $i < j$ or $0 = i = j$ and $\alpha < \beta$, or $1 = i = j$ and $\alpha > \beta$.

**Lemma 5.2.** Suppose that $T$ is categorical in $\lambda > \text{cf}(\lambda)$ and $\kappa + |T| < \mu \leq \lambda$. If $M \in \mathcal{K}_\lambda$, then there exists a continuous increasing $\prec_T$-chain $\langle M_i : i < \text{cf}(\lambda) \rangle$ of models such that

1. $M \preceq_T \bigcup_{i < \text{cf}(\lambda)} M_i$;
2. $\| \bigcup_{j < \text{cf}(\lambda)} M_i \| = \lambda$;
3. $\kappa + |T| \leq \| M_i \| < \| M_{i+1} \| < \lambda$;
4. for each $i < \text{cf}(\lambda)$, $M_i \preceq_T \bigcup_{j < \text{cf}(\lambda)} M_j$.

**Proof.** If $\lambda$ is a limit cardinal, choose a continuous increasing sequence $\langle \mu_i : i < \text{cf}(\lambda) \rangle$ with $\lambda = \sup_{i < \text{cf}(\lambda)} \mu_i$ and $\kappa + |T| \leq \mu_0 < \lambda$. If $\lambda$ is a successor let $\mu_i = 1 + i$. Let $\langle I, I_i : i < \text{cf}(\lambda) \rangle$ be as in 5.1. By $\lambda$-categoricity of $T$, without loss of generality, $M = EM(\lambda)$. Let $M_i = EM(I_i)$ for $i < \text{cf}(\lambda)$. Clearly (1), (2) and (3) hold. To obtain (4), observe that by 2.6 and 3.5 it suffices to remark that by demand (3) from 5.1 on $\langle I_i : i < \text{cf}(\lambda) \rangle$ clause (N) or (2) in 2.6 holds for each $t \in I \setminus I_i$.

**Theorem 5.3.** For every $\mu \in [\kappa + |T|, \lambda]$ and $M \in K_\mu$, there exists $M' \in K_\mu$ with $M \preceq_T M'$ such that

$\ast_{M'}$ for every $A \subset |M'|$ with $|A| < \lambda$ and $|A| \leq \mu$, there is $N \in K_{\kappa + |T| + |A|}$ such that $A \subset N \preceq T M'$ and $N$ is nice.
Proof. The proof is by induction on \( \mu \).

Case 1: \( \mu = \kappa + |T| \). By 3.3 there is \( M' \in K_\mu \) such that \( M \preceq M' \) and \( M' \) is nice. Given \( A \subset |M'| \) let \( N = M' \) and note that \( N \) is as required in \((*)_{M'}\).

Case 2: \( \kappa + |T| < \mu \). Without loss of generality, one can replace \( M \) by any \( \prec_{M} \)-extension in \( K_\mu \). Choose a continuous increasing sequence \( \langle \mu_i : i < \text{cf}(\mu) \rangle \) such that if \( \mu \) is a limit cardinal it is a strictly increasing sequence with limit \( \mu \); if \( \mu \) is a successor, use \( \mu_i^+ = \mu \) and in both cases \( \kappa + |T| \leq \mu_i < \mu \).

Find \( \bar{M} = \langle M_i : i < \text{cf}(\mu) \rangle \) such that

(a) \( M \preceq M \bar{M} \bar{M} \bar{M} \preceq M \bar{M} \bar{M} \bar{M} \in K_\mu \);  
(b) \( \| M_i \| = \mu_i \);  
(c) \( \| M_i \| = \mu_i \);  
(d) \( M_i \preceq \bigcup_{j < \text{cf}(\mu)} M_j \).

Why does \( \bar{M} \) exist? If \( \mu = \lambda \) by 5.2, otherwise by 4.4 (\( \mu \) regular) and 4.11 (\( \mu \) singular).

Choose by induction on \( i < \text{cf}(\mu) \) models \( L_i^0, L_i^1, L_i^2 \) in that order such that

(8) \( M_i \preceq L_i^0 \preceq M_i \preceq L_i^1 \preceq L_i^2 \in K_\mu \);  
(9) \( j < i \Rightarrow L_j^2 \preceq L_i^2 \);  
(10) \( (*)_{L_i^1} \) holds, i.e. for each \( A \subset |L_i^1| \), there is \( N \in K_{\leq \kappa + |T| + |A|} \) such that \( A \subset N \preceq L_i^1 \) and \( N \) is nice (so in particular \( L_i^1 \) is nice, letting \( A = |L_i^1| \));  
(11) \( L_i^2 \) is nice and \( \mu_i \)-universal over \( L_i^1 \);  
(12) \( L_i^0 \) is increasing continuous;  
(13) \( L_i^1 \cap \bigcup_{j < \text{cf}(\mu)} M_j = M_i \) (or use a system of \( \preceq_{M} \)-embeddings).

For \( i = 0 \), let \( L_i^0 = M_0 \). For \( i = j+1 \), note that by 2.1 there is an amalgam \( L_i^0 \in K_\mu \) of \( M_i, L_j^0 \) over \( M_j \) since \( M_j \preceq M_i \) and \( M_j \preceq L_j^2 \) (use the last phrase of 2.1 for clause (13)); actually not really needed. For limit \( i \), continuity necessitates choosing \( L_i^0 = \bigcup_{j < i} L_j^0 \) (note that in this case \( L_i^0 = \bigcup_{j < i} L_j^2 \)).

To choose \( L_i^2 \) apply the inductive hypothesis with respect to \( \mu_i, L_i^0 \) to find \( L_i^1 \) so that \( L_i^0 \preceq L_i^1 \) and \( (*)_{L_i^1} \) holds. To choose \( L_i^2 \) apply 3.10 to \( L_i^1 \in K_\mu \), giving \( L_i^1 \preceq L_i^2 \), \( L_i^2 \) is nice and \( \mu_i \)-universal over \( L_i^1 \) (so (11) holds).

Let \( L = \bigcup_{i < \text{cf}(\mu)} L_i^0 = \bigcup_{i < \text{cf}(\mu)} L_i^1 = \bigcup_{i < \text{cf}(\mu)} L_i^2 \), and let \( L_i = L_i^0 \) if \( i \) is a limit, \( L_i^1 \) otherwise. Now show by induction \( L_i \) is nice. [Why? show by induction on \( i \) for \( i = 0 \) or \( i \) successor that \( L_i = L_i^1 \), hence use clause (9); if \( i \) is limit then \( L_i \) is \((\mathfrak{B}(i))\)-saturated, hence \( L_i \) is nice by 4.8, 4.10.] Now \( \langle L_i : i < \text{cf}(\mu) \rangle \) witnesses that if \( \mu \) is regular, then \( L \) is \((\mu, \mu)\)-saturated by 4.4 and if \( \mu \) is singular, then \( L \) is \( \mathfrak{p} \)-saturated; in all cases \( L \) is \( \mathfrak{p} \)-saturated.
of power $\mu$, hence by the results of Section 4 (i.e. 4.8, 4.10), if $\mu < \lambda$ then $L$ is nice.

CLAIM. $M' = L$ is as required.

Proof. $M \not\preceq \bigcup_{i < \text{cf}(\mu)} M_i \not\preceq \bigcup_{i < \text{cf}(\mu)} L^0_i = L \in K_\mu$. Suppose that $A \subset |L|$. If $|A| = \mu$, then necessarily $\mu < \lambda$ and we take $N = L$. So without loss of generality, $|A| < \mu$. If $\mu = \text{cf}(\mu)$ or $|A| < \text{cf}(\mu)$, then there is $i < \text{cf}(\mu)$ such that $A \subset L^0_i$ and, by ($\exists$), ($\ast$)$_{L^0_i}$ holds, so there is $N \in K_{\kappa + |T| + |A|}$ such that $A \subset N \not\preceq L^1_i$, $N$ is nice and $N \not\preceq L$ as required. So suppose that $\mu < |A| < \mu$. Choose by induction on $i < \text{cf}(\mu)$ models $N^0_i, N^1_i, N^2_i$ in that order such that

1. $N^0_i \not\preceq N^1_i \not\preceq N^2_i$;
2. $N^2_i \not\preceq N^0_i$;
3. $A \cap L^0_i \subseteq N^0_i \not\preceq L^0_i$;
4. $N^1_i \not\preceq L^1_i$ and $N^1_i$ is nice;
5. $N^2_i \not\preceq L^2_i$, $N^2_i$ is nice and universal over $N^1_i$;
6. $N^0_i, N^1_i, N^2_i$ have power at most $\text{min}(\{|T| + \kappa + |A|, \mu_i\})$.

For $i = 0$, apply 1.2 for $A \cap L^0_0, L^0_0$; for $i = j + 1$, apply 1.2 to find $N^0_i \in K_\mu$, such that $(A \cap L^0_i) \cup N^2_i \subset N^0_i \not\preceq L^0_i$ (in particular, $N^0_i \not\preceq N^0_i$); for limit $i$, $N^0_i = \bigcup_{j < i} N^0_j$. To choose $N^1_i$, use ($\ast$)$_{L^1_i}$ for the set $A_i = N^0_i$ to find a nice $N^1_i \in K_{\kappa + |T| + |A|}$ with $N^0_i \not\preceq N^1_i \not\preceq L^1_i$. Note that $\|N^1_i\| \leq \mu_i$. Finally, to choose $N^2_i$ note that by 3.9 the model $N^1_i$ has a nice extension $N^1_i^+$ (of power $\|N^1_i\|$) weakly universal over $N^1_i$. Now $N^1_i$ is nice, hence $N^2_i$ is universal over $N^1_i$ (by 3.6A(5)) and by 2.1 there is an amalgam $N_i$ of $N^1_i, L_i^1$ over $N^1_i$ such that $\|N_i\| \leq \mu_i$. Since $L^2_i$ is universal over $L^1_i$ one can find an $\mathcal{F}$-elementary submodel $N^2_i$ of $L^2_i$ isomorphic to $N_i$. Let $N_i$ be $N^0_i$ if $i$ is a limit, $N^1_i$ otherwise; prove by induction on $i$ that $N_i$ is nice (by 4.2).

Now $\bigcup_{i < \text{cf}(\mu)} N^0_i$ is an $\mathcal{F}$-elementary submodel of $L$ of power at most $\kappa + |T| + |A|$, including $A$ (by (4)) and $\bigcup_{i < \text{cf}(\mu)} N^0_i$ is $(\kappa + |T| + |A|, \text{cf}(\mu))$-saturated, hence (by 4.2) nice, as required.

COROLLARY 5.4. Suppose that $T$ is categorical in $\lambda$. Then every element of $K_{< \lambda}$ is nice.

Proof. Suppose otherwise and let $N_0 \in K_{< \lambda}$ be a model which is not nice. Choose a suitable Op such that $\|\text{Op}(N_0)\| \geq \lambda$ and by 1.2 find $M_0 \in K_\lambda$ with $N_0 \not\preceq M_0 \not\preceq \text{Op}(N_0)$, i.e. $N_0 \not\preceq M_0$. It follows that

1. $N_0 \not\preceq N \not\preceq M_0$ and $N \in K_{< \lambda}$ then $N$ is not nice.

[Why? By 4.3; alternatively suppose by contradiction that $N$ is nice. So there is $N_1 \in K_{< \lambda}$ such that $N_0 \not\preceq N_1, N_0 \not\preceq N_1 \cdot N_0 \not\preceq N$ since $N_0 \not\preceq M_0$ and

$\text{nice}$ $\text{nice}$ $\text{nice}$
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\[ N \preceq M_0, \text{ hence there is an amalgam } N' \in K_{<\lambda} \text{ of } N_1, N \text{ over } N_0. \text{ Since } N \text{ is nice, } N \preceq N'; N_0 \preceq N, \text{ so } N_0 \preceq N_1, \text{ a contradiction.} \]

On the other hand, applying 5.3 for \( \mu = \lambda \) there exists \( M' \in K_{\lambda} \) satisfying \((*)_{M'}\). By \( \lambda \)-categoricity of \( T \), without loss of generality, \((*)_{M'}\) and for \( A = |N_0| \) yields a nice model \( N \in K_{\kappa+|T|+\|N_0\|} \) such that \( N_0 \preceq N \preceq M_0 \), contradicting \((+).\)

**Corollary 5.5.** Suppose that \( T \) is categorical in \( \lambda \). Then \( K_{<\lambda} \) has the amalgamation property.

**Proof.** 2.1 and the previous corollary.

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