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The geometry of laminations

by

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Abstract. A lamination is a continuum which locally is the product of a Cantor set and an arc. We investigate the topological structure and embedding properties of laminations. We prove that a nondegenerate lamination cannot be tree-like and that a planar lamination has at least four complementary domains. Furthermore, a lamination in the plane can be obtained by a lakes of Wada construction.

1. Introduction. In this paper we study the structure and embedding properties of spaces which locally are a product of a Cantor set and an arc. This study can be motivated by the following problems and results. During a lecture at the 1984 Spring Topology Conference held in Birmingham, Alabama, R. D. Edwards discussed the notion of a lamination, i.e., spaces which locally are a product of a Cantor set and an arc. He raised the question whether a planar lamination must separate the plane (i.e. is not tree-like). He solved this question in the affirmative but, as far as we know, his solution has not appeared in print. Another result is due to Konstantinov [K], and states that the closure of a smooth curve in the plane has at least four complementary domains, under the conditions that the curve is self entwined and of bounded curvature. Related is a result of Plykin [P1] that the complement of every expanding attractor in the plane has at least four complementary domains. There is an open problem of Kato [Ka] as to how many complementary domains a plane continuum which admits an expansive homeomorphism must have. In this paper we combine Edwards' problem and the results of Konstantinov and Plykin. We show that a continuum with

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local product structure of a Cantor set and an arc is not tree-like. Embedded in the plane, such a continuum has at least four complementary domains. In this paper a *continuum* is a compact and connected metrizable space.

(1.1) Definition. Let X be a continuum which does not contain a topological copy of the circle. We call X a *lamination* if its topology contains a base of open sets homeomorphic to $\mathcal{C} \times (-1,1)$.

Many topological structures in dynamics are related to laminations. For instance, the geodesic laminations as introduced by Thurston [T], [C-B]. A *geodesic lamination* is a collection of pairwise disjoint geodesics on a hyperbolic surface. Non-degenerate geodesic laminations are locally $\mathcal{C}\times(-1,1)$.

Another example are inverse limit spaces over $f : M \to M$, where M is a branched manifold and f is expanding. These spaces are the topological models of attractors, studied by Williams [W1]. Well known attractors like Plykin's attractor and solenoids are examples of laminations.

Finally, matchbox manifolds, the topological models of one-dimensional flows [A-M], are related to laminations.

We introduce some terminology. Following Aarts and Martens we call a neighborhood homeomorphic to $\mathcal{C}\times(-1,1)$ an (*open*) *matchbox* if its closure is homeomorphic to $\mathcal{C} \times [-1,1]$. The closure is called a (*closed*) *matchbox*. The copies of $\{x\} \times (-1,1)$ are called the *matches*. An arc component of a lamination is called a *leaf*. Since, by definition, a lamination does not contain a simple closed curve, all leaves of a lamination are one-to-one continuous images of the reals. A lamination is *minimal* if all of its leaves are dense. A minimal lamination is indecomposable, i.e., it cannot be written as the union of two proper subcontinua.

A continuum X is called *tree-like* if it admits open tree covers of arbitrarily small mesh. In other words, for every $\varepsilon > 0$ there exists a finite cover $\mathcal{U} = \{U_i \mid i = 1, \ldots, N\}$ of open sets, such that diam $U_i < \epsilon$ and the nerve $\mathcal{N}(\mathcal{U})$ is a tree.

2. Examples. In this section we discuss some motivating examples.

2.1. *Self-entwined curves.* A curve, i.e., a one-to-one continuous image of the reals, is called *self-entwined* if no open set intersects it in an arc. If a self-entwined planar curve has bounded curvature, it divides the plane into four components [K]. Self-entwined planar curves of bounded curvature have local product structure $\mathbb{Q} \times (-1,1)$. A priori, the closure of such a curve may not have local product structure $\mathcal{C} \times (-1,1)$. Possibly the closure is a lamination with some pieces of the boundary leaves glued together. If we open up the closure in some places, which can only decrease the number of complementary domains, we get a lamination. Hence, Konstantinov's theorem follows from Corollary (3.4).

Konstantinov has constructed an example of a self-entwined curve of bounded curvature which divides the plane into precisely four components [K2]. A leaf of the Plykin attractor also has this property.

2.2. *The Plykin attractor.* Let $f : X \to X$ be a map on a topological space. An *attractor* A of f is an invariant subset which has a neighborhood $A \subset U$ such that $A = \bigcap_{n\in\mathbb{N}} f^n(U)$. Many attractors have a local product structure. For instance, it is an open problem whether an Axiom A attractor is always locally $\mathcal{C}\times(-1,1)^n$ [S]. In the one-dimensional case, the expanding attractors studied by Williams are locally $\mathcal{C}\times(-1,1)$. There exists a theory classifying most of the one-dimensional expanding attractors [P2].

The Plykin attractor is a well known example of an expanding attractor in the plane. We sketch how it can be constructed. Represent the torus as a product $\mathbb{R}/\mathbb{Z}\times\mathbb{R}/\mathbb{Z}$. If points are identified under the involution $(x,y)\to(-x,-y)$, the resulting quotient space is a sphere. The torus is a branched cover of the sphere and the branch points are the fixed points of the involution: $(0,0),(\frac{1}{2},0),(0,\frac{1}{2}),(\frac{1}{2},\frac{1}{2})$. Since the set of branch points is invariant under the linear map $L:(x,y)\to(2x+y,x+y)$, this map induces a homeomorphism on the sphere.

The linear map L has two eigenvalues $(3-\sqrt{5})/2,(3+\sqrt{5})/2$. Let v be the eigenvector with eigenvalue $(3+\sqrt{5})/2$. The family of lines $\{w+\lambda v \mid w\in\mathbb{R}\times\mathbb{R},\ \lambda\in\mathbb{R}\}$ foliates the torus and is invariant under L. By carefully blowing up the branch points, the map L can be perturbed into a diffeomorphism f and the foliation can be transformed into an attractor A of f. This construction is known as *Smale's DA* construction [P-M]. The diffeomorphism f can be projected onto the sphere, where it has an attractor P covered by A. This is the *Plykin attractor*, which has four complementary domains in the sphere, corresponding to the four branch points. Both P and A are minimal laminations.

2.3. *The bucket handle.* The *bucket handle* $\mathcal{K}$ is a tree-like continuum which is locally $\mathcal{C}\times(-1,1)$ except at one point p [Kn]. It is the attractor of Smale's horseshoe map [S]. The point p is the attractive fixed point of the horseshoe map; it is an accumulation point of folds in $\mathcal{K}$. The same phenomenon is suggested by pictures of more complicated planar attractors. At most points the attractors are locally $\mathcal{C}\times(-1,1)$ but there are accumulation points of folds.

2.4. *Lakes of Wada.* The *lakes of Wada construction* starts out with two lakes on an island. Three canals are dug in the island, two starting out from the lakes and one starting out from the sea. The construction gives a continuum K such that each point in K is a boundary point of all its complementary domains. The Plykin attractor has the same property. It is a corollary of our results that all planar laminations can be obtained from a lakes of Wada construction, starting out from at least four lakes (counting the sea as a lake).

3. The number of complementary domains. A lamination is *minimal* if it does not contain any lamination other than itself. By Zorn's lemma, every lamination contains a minimal lamination. We show that a minimal lamination in the plane can be extended to a foliation of the sphere. The complementary domains of the lamination correspond to the singularities of the foliation.

(3.1) LEMMA. *Let $U \approx \mathcal{C} \times [-1,1]$ be a matchbox in the plane. There exists a curve K which intersects each match in a point.*

A proof is given in the Appendix.

(3.2) THEOREM. *Let X be a minimal lamination in the sphere S^2. The lamination X can be extended to a foliation with singularities of S^2.*

Proof. Let K be an arc which intersects the lamination X transversely. Without loss of generality we may assume that the end points of K are not contained in X. Since X is minimal, K intersects each leaf of X and the intersection $\mathcal{K} = K \cap X$ is a Cantor set. The complement $X \setminus K$ is a disjoint union of arcs. Each arc has two end points in K and, conversely, each point in $K \cap X$ is an end point of two arcs in $X \setminus K$.

Denote the end points of K by a, b. Consider two copies of the arc K denoted by $K \times \{-,+\}$. Identify $(a,-)$ with $(a,+)$ and $(b,-)$ with $(b,+)$. In this way we get a circle which is called the *double* of K. Consider the sphere $S^2 \setminus K$ which has been opened up along the arc K. The opened up sphere $S^2 \setminus K$ can be compactified with the double of K. This compactification is denoted by S.

The arcs in $X \setminus K$ have end points on the double of K. Two arcs L_1, L_2 in $X \setminus K$ are called *homotopic* if there exists a homotopy on S which moves L_1 to L_2 and keeps the double of K invariant. There are finitely many homotopy types and if two arcs are sufficiently close, they are homotopic. Therefore, the double of K can be divided into finitely many intervals $I_1, \ldots, I_n$ such that arcs in X are homotopic if they have an end point in the same interval. The intervals are chosen such that their interiors are mutually disjoint.

Let e_1, e_2 be the end points of I_1. Consider the family $\mathcal{A}$ of all arcs in $X \setminus K$ which have an end point in the interval I_1. Let V be the set of opposite end points, i.e., V is the set of end points outside I_1 of arcs in $\mathcal{A}$. Let J be the smallest interval in the double of K, which contains V and has empty intersection with I_1. Let f_1, f_2 be the end points of J. Since all arcs in $\mathcal{A}$ are homotopic, there exist two arcs A_1, A_2 which connect e_1, e_2 to f_1, f_2, such that $A_1 \cup A_2 \cup I_1 \cup J$ is a simple closed curve which contains the entire family

$\mathcal{A}$ in a complementary domain. This domain T is called a *strip*. Thus the lamination X is contained in a union $\bigcup_i T_i$ of strips. The strips T_i can be chosen such that the complementary domains of $\bigcup_i T_i$ in S are bounded by simple closed curves. By Theorem (6.2), the lamination X can be extended to a foliation of $\bigcup_i T_i$. Collapse the complementary domains to obtain a foliation with singularities of S. Then collapse the double of K to obtain a foliation of the sphere S^2.

It follows from the construction that the foliation has a finite number of singularities. The singularities correspond to the complementary domains of X.

(3.3) THEOREM. *A minimal lamination in the plane has at least four complementary domains.*

Proof. The complementary domains of the lamination correspond to the singularities of the foliation. By the Index Theorem [C-N], the index sum of the singularities is equal to 2. A singularity has index 1 if it is a source or a sink. Otherwise, it has index $\leq 1/2$. The foliation constructed in the proof of Theorem (3.2) has no sources or sinks. Therefore, it has at least four singularities.

(3.4) COROLLARY. *A lamination in the plane has at least four complementary domains.*

4. Planar laminations are lakes of Wada continua. We demonstrate that a minimal planar lamination can be obtained from a lakes of Wada construction. We use some topological techniques from hyperbolic geometry.

The Poincaré disc, as a subset of the complex plane, is denoted by $\mathbb{H}^2 = \{z \in \mathbb{C} \mid |z| < 1\}$ and it is endowed with the Poincaré metric [C-B]. The circle at infinity is the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. Its union with $\mathbb{H}^2$ is called the *extended Poincaré disc*. Geodesics in $\mathbb{H}^2$ are the semi-circles orthogonal to the circle at infinity.

(4.1) LEMMA. *Let M be a surface with Euler characteristic $\chi(M) < 0$. The universal cover of M is homeomorphic to the Poincaré disc.*

A proof can be found in [C-B]. Moreover, the surface M can be endowed with a hyperbolic metric, which is complete, such that the covering projection is a local isometry.

(4.2) LEMMA. *Let X be a minimal lamination in the plane. The lamination can be lifted to a subspace $\overline{X}$ of the Poincaré disc.*

Proof. By Theorem (3.2) the lamination X can be extended to a foliation with singularities of the sphere. Delete the singularities, so we get a foliation $\mathcal{F}$ without singularities of a punctured sphere. According to Theorem (3.2), the Euler characteristic of the punctured sphere is at most -2, so it is a hyperbolic surface. It follows from Lemma (4.1) that X is covered by a subspace of the Poincaré disc.

(4.3) Lemma. *Let $\mathcal{F}$ be a foliation without singularities. Then $\mathcal{F}$ admits a flow without rest points if and only if it is orientable.*

A proof can be found in [C-N]. Moreover, every foliation has an orientable cover.

(4.4) Lemma. *Let X be a minimal lamination in the plane and let $\overline{X}$ be its lift to the Poincaré disc. In the extended Poincaré disc, $\overline{X}$ is a union of mutually disjoint arcs, with end points on the circle at infinity.*

Proof. The minimal lamination can be extended to a foliation $\mathcal{F}$ of a punctured sphere. The orientable cover of $\mathcal{F}$ admits a flow φ without rest points. The flow φ can be lifted to a flow $\overline{\varphi}$ on the Poincaré disc, which has no rest points and leaves $\overline{X}$ invariant. To prove the lemma, it suffices to show that for every trajectory in $\overline{X}$, the limit set $\omega(x)$ is a single point on the circle at infinity.

The orientable cover of $\mathcal{F}$ foliates a surface M. There exists a simple closed curve C in M which is transversal to φ, i.e., C is a section. The curve C cannot be nullhomotopic, so, by eventually perturbing the flow, we may assume that C is geodesic. Let p be a point on C and let q be its first return point to C under the flow φ. Hence, the part of the trajectory between p and q connects points of C. Lifted to $\mathbb{H}^2$, it is an arc which connects two disjoint geodesics. So, the positive trajectory with starting point p is lifted to an arc in $\mathbb{H}^2$ which intersects infinitely many disjoint geodesics. In the euclidean metric, the diameter of these geodesics goes to zero. It follows that the limit set of the lift of the positive trajectory contains one point on the circle at infinity.

Lemma (4.4) associates a geodesic lamination to a minimal lamination as follows. Each arc component of $\overline{X}$ is a curve with end points in the unit circle and these end points determine a geodesic in $\mathbb{H}^2$. The union of all such geodesics covers a geodesic lamination $\mathcal{L}$ in the punctured sphere. We want to show that X is a lakes of Wada continuum. So we are interested in the complementary domains of X. The following result is proved in [C-B].

(4.5) Lemma. *The complementary domains of a geodesic lamination are bounded by finitely many geodesics.*

Consider a complementary domain of $\mathcal{L}$ bounded by the geodesics $\phi_1, \phi_2, \ldots, \phi_n$. Lifted to the Poincaré disc, the geodesics are *asymptotic*, i.e., for consecutive indices, the lifts of ϕ_i and ϕ_{i+1} have one end point in common.

It is important to note that different arcs in $\overline{X}$ may have the same pair of end points on the circle at infinity. To one geodesic in $\mathcal{L}$ may correspond two leaves of $\overline{X}$. There cannot be more than two, for suppose that more than two curves in $\overline{X}$ have both the end points in common. Remove all curves but the two outer curves from $\overline{X}$. Do this for every pair of end points. The result is a closed space $\overline{X}'$ which covers a sublamination $X' \subset X$. This contradicts the minimality of X and hence only two different arcs in $\overline{X}$ may have the same end points. This leads us to the following result.

(4.6) Lemma. *A complementary domain of X is bounded by a finite number of leaves.*

Proof. Consider a complementary domain in $\overline{X}$. Either it is bounded by two leaves, which are collapsed in $\overline{\mathcal{L}}$, or it corresponds to a complementary domain of $\overline{\mathcal{L}}$, in which case it is bounded by a finite number of asymptotic leaves. In particular, the number of boundary leaves in $\overline{X}$ is finite. The same holds for X.

Now it is obvious that in a lakes of Wada construction, a canal must be bounded by a pair of asymptotic boundary leaves. To prove that X can be obtained by digging canals, we still have to verify that the canals grow narrower as they are dug deeper.

(4.7) Theorem. *A minimal lamination in the plane can be obtained by a lakes of Wada construction.*

Proof. Let $\mathcal{F}$ be a foliation of the punctured sphere which extends X. Its orientable cover is denoted $\widetilde{\mathcal{F}}$ and its subset covering X is denoted $\widetilde{X}$. Let $\overline{L}_1, \overline{L}_2$ be arcs in $\overline{X}$ which have a common end point e on the circle at infinity. They cover leaves L_1, L_2 in $\widetilde{\mathcal{F}}$. Consider a simple closed curve C transversal to φ. The curve C intersects $\widetilde{X}$ in a Cantor set. We show that the arcs $\widetilde{L}_i$ eventually intersect C in consecutive end points of this Cantor set.

Let $\overline{C}$ be a lift of C to the Poincaré disc which intersects $\overline{L}_i$ in two points p_i $(i = 1, 2)$. The arc $[p_1, p_2]$ in $\overline{C}$ connects these two points. Consider the region in $\mathbb{H}^2$ bounded by $[p_1, p_2]$ and the part of $\overline{L}_i$ from p_i to e. If an arc in $\overline{X}$ intersects $[p_1, p_2]$, then it is trapped in this region. Hence, it has end point e, which contradicts the fact that at most two arcs in $\overline{X}$ can have one end point in common. We conclude that L_1, L_2 eventually intersects C in consecutive boundary points. It follows that the trajectories $\widetilde{L}_i$ bound an ever narrower channel. Projecting this on the sphere we see that the underlying channel is ever narrower as well. This implies that X is a lakes of Wada continuum.

The construction in this section has the following corollary.

(4.8) Corollary. *Let X be a minimal planar lamination with finitely many boundary leaves. A continuous map from X onto itself is either homotopic to a homeomorphism or to a constant map.*

Let $f : X \to X$ be a continuous function on a minimal planar lamination. If the image $f(X)$ contains elements of two separate leaves, then by the minimality of X, $f(X)$ is equal to X. Hence, if f is not homotopic to a constant map, it is a surjection.

Suppose that f is a surjection. We have shown above that the boundary leaves of X are asymptotic. In fact, the boundary leaves are the only asymptotic leaves in X and therefore f permutes the boundary leaves of X. Lift f to a map $\overline{f}$ on $\overline{X}$ on the Poincaré disc. Since $\overline{f}$ permutes boundary leaves, there exists an extension $\overline{f} : \mathbb{H}^2 \to \mathbb{H}^2$ which leaves the flow $\overline{\varphi}$ invariant. Consider all arcs in $\overline{X}$ which cover boundary leaves of $\widetilde{X}$. They have a dense set of end points in the circle at infinity and $\overline{f}$ induces a bijection on these end points. So, $\overline{f}$ induces a bijection on the circle at infinity. It follows that $\overline{f}$ is homotopic to a homeomorphism $\overline{h}$ on $\mathbb{H}^2$. So, f permutes the leaves of X, only it may not be 1-1. In [F], it is shown that such a map f is homotopic to a homeomorphism of X.

5. General laminations. In this section we show that a general lamination cannot be tree-like. We start with some preliminary lemmas. For a closed matchbox $U \approx \mathcal{C} \times [0,1]$ the *ends* of U are the points in the set $\mathcal{C} \times \{0,1\}$. The set of ends of U is denoted $E(U)$. The ends at the top $\mathcal{C} \times \{1\}$ are denoted $T(U)$. The bottom ends are denoted $B(U)$. The first lemma follows directly from these definitions.

(5.1) Lemma. *Let X be a tree-like lamination. Then there exists a finite cover $\mathcal{U}$ of X such that each element of $\mathcal{U}$ is a closed matchbox, the nerve $\mathcal{N}(\mathcal{U})$ is a tree and for any two distinct elements $U, V \in \mathcal{U}$, $E(U) \cap E(V) = \emptyset$.*

The second lemma is taken from [A-M].

(5.2) Lemma (lemma of the long box). *Let J be an arc in a matchbox manifold X with initial point x_1 and end point x_2. Suppose that V_1 and V_2 are disjoint matchbox neighborhoods of x_1 and x_2 respectively. For $i = 1, 2$ let $h_i : \mathcal{C} \times (0,1) \to V_i$ be a homeomorphism between V_i and the standard matchbox. We write $Z_i = h_i(\mathcal{C} \times \{0\})$. There exists a homeomorphism $h : \mathcal{C} \times [0,1] \to V$ such that*

(1) $x_1 \in h(\mathcal{C} \times \{0\}) \subset Z_1$,
(2) $x_2 \in h(\mathcal{C} \times \{1\}) \subset Z_2$,
(3) $h(\mathcal{C} \times \{0\})$ *is clopen in* Z_1,
(4) $h(\mathcal{C} \times \{1\})$ *is clopen in* Z_2.

Stated less accurately, there is a long box V along the arc J with bottom in Z_1 and top in Z_2.

(5.3) LEMMA. *Let X be a minimal lamination and $\mathcal{U}$ a finite cover by closed matchboxes. Let $U \in \mathcal{U}$ and let $\delta(U)$ be the minimum distance between two distinct matches of U which both meet the closure of a single component of $X \setminus U$. Then $\delta(U) > 0$.*

P r o o f. Suppose the lemma fails. Then there exist sequences of matches M_i^1 and M_i^2 meeting components C_i of $X \setminus U$ such that $d(M_i^1, M_i^2) \to 0$. Note that every component of $X \setminus U$ is an arc. Without loss of generality, $\lim M_i^1 = \lim M_i^2 = M^0$ and $\lim C_i = C_0$, where M_0 is a match of U and C_0 is a continuum in $\overline{X \setminus U}$. Let a_i and b_i be the end points of the arc C_i. Since M_i^1 and M_i^2 are distinct matches of U, C_i must meet U in two distinct end points.

Let K be the component of $X \setminus U$ whose closure contains C_0. By Lemma (5.2), K has a matchbox neighborhood in $X \setminus U$. In this neighborhood, the components C_i are matches converging to K. It follows that C_0 is equal to K and therefore it meets U in two distinct end points a and b. In this case $C_0 \cup M_0$ is a simple closed curve, contradicting the minimality of the lamination.

Given a finite cover $\mathcal{U}$ of a minimal lamination X by closed matchboxes we denote

$$\delta(\mathcal{U}) = \min\{\delta(U) \mid U \in \mathcal{U}\}.$$

We discuss the notion of orientability of laminations, which is studied in more detail in [A-M]. The notion is almost analogous to the notion of orientability of foliations [C-N, p. 37]. There is one important difference: a matchbox can be oriented in infinitely many ways. This complicates matters slightly.

Let $V \approx \mathcal{C} \times [0,1]$ be a closed matchbox. Each match M carries two possible orientations. An *orientation* $\mathcal{O}(V)$ of V is a continuous choice of orientations of the matches of V. If $W \subset V$ are matchboxes, then the orientation $\mathcal{O}(V)$ induces an orientation $\mathcal{O}(W)$. Two matchboxes U and V are *coherently oriented* provided they induce the same orientation at points in the intersection $U \cap V$. A subset $A \subset X$ of a lamination X is coherently oriented provided for each point $x \in A$ there exist a matchbox U containing x in its interior and an orientation $\mathcal{O}(U)$ such that the orientation at each point of $U \cap A$ induced by $\mathcal{O}(U)$ agrees with the given orientation of A. Every non-orientable lamination has an orientable double cover, which is defined in the same way as the orientable cover of a foliation.

An open cover $\mathcal{V}$ is *taut* provided for each pair $U, V \in \mathcal{V}$,

$$U \cap V \neq \emptyset \quad \text{if and only if} \quad \overline{U} \cap \overline{V} \neq \emptyset.$$

It is known that each cover $\mathcal{U}$ has a taut refinement $\mathcal{V}$ such that $\mathcal{N}(\mathcal{U})$ is homeomorphic to $\mathcal{N}(\mathcal{V})$. By a *chain* of elements of a cover $\mathcal{V}$ we mean a finite subcollection $\{V_1, \ldots, V_n\}$ with the property that $V_i \cap V_j \neq \emptyset$ if and only if $|i-j| \leq 1$. In other words, the nerve of $\{V_i\}$ is an arc.

(5.4) Theorem. *Let $\mathcal{U}$ be a finite cover satisfying the conclusions of Lemma* (5.1) *and let $\mathcal{V}$ be a finite taut open tree cover refining $\mathcal{U}$ such that the closure of each element of $\mathcal{V}$ is a closed matchbox and*

$$\operatorname{mesh}(\mathcal{V}) < \tfrac{1}{3}\min\{\delta(\mathcal{U}), \min\{d(T(U), B(U)) \mid U \in \mathcal{U}\}\}.$$

Let $\{V_1, \ldots, V_n\}$ be a chain of elements of $\mathcal{V}$. Then there exists a coherent orientation of $\bigcup\{V_i\}$.

Proof. The proof is by induction on the length n of the chain. If $n = 1$, the lemma follows immediately since any orientation of the matchbox V_1 induces the required orientation. Hence, suppose the theorem is proved for all chains of length $\leq n$ and let $\{V_1, \ldots, V_{n+1}\}$ be a chain of length $n+1$. By using the inductive assumption, we obtain an orientation of the chain $\{V_1, \ldots, V_n\}$. By assumption $\overline{V}_{n+1}$ meets only $\overline{V}_n$. Since mesh $\mathcal{V} < \delta\mathcal{U}$, each match μ of $\overline{V}_{n+1}$ meets at most one match of $\overline{V}_n$. Hence the orientation of the matches of V_n induces an orientation of the matches of $\overline{V}_{n+1}$ which meet $\overline{V}_n$. Since the matches of $\overline{V}_{n+1}$ which meet $\overline{V}_n$ form a clopen subset, we can orient the remaining matches of V_{n+1} coherently. It remains to be shown that this choice of orientation is continuous. Suppose matches μ_k of V_{n+1} converge to a match μ which meets $\overline{V}_n$. Since $E(V_n) \cap E(V_{n+1}) = \emptyset$, $\mu_k \cap \overline{V}_n$ converges to $\mu \cap \overline{V}_n$. Since the choice of orientation was continuous on V_n, it follows that the choice of orientation is also continuous on V_{n+1}. This completes the proof.

Notice that the above proof guarantees that if $\{V_i \mid i = 1, \ldots, n\}$ and $\{V_i' \mid i = 1, \ldots, n'\}$ are two chains in $\mathcal{V}$, then we can orient their union coherently.

(5.5) Corollary. *Let $\mathcal{U}$ and $\mathcal{V}$ be as in Theorem* (5.3). *Then $\mathcal{V}$ admits a coherent orientation.*

(5.6) Theorem. *Let X be a lamination. Then X is not tree-like.*

Proof. Since every subcontinuum of a tree-like continuum is tree-like, we may assume that X does not contain a simple closed curve. Hence, X contains a minimal lamination. By Corollary (5.5), X admits a coherent orientation. It follows from [K-S] that X admits a flow and is homeomorphic to the suspension over a Cantor set. Since X is minimal, the suspension admits an essential map to a circle. This contradicts the fact that X is tree-like [C-C] and completes the proof of the theorem.

6. Appendix. We show that a closed matchbox is tamely embedded in the plane. The proof depends on the Riemann mapping theorem. A simple closed curve in the plane can be mapped onto the unit circle by a transformation of the plane. Moreover, if two curves are close, then the transformations are close.

A *quadrilateral* $Q = \{K, x_1, x_2, x_3, x_4\}$ in the plane is a simple closed curve K with four distinct points x_1, x_2, x_3, x_4. We say that $f : Q \to Q'$ is a map between quadrilaterals if it maps K onto K' and x_i onto x'_i.

The following theorem is a version of the Riemann mapping theorem for quadrilaterals.

(6.1) Theorem. *Let Q be a quadrilateral. There exists a rectangle R such that Q can be mapped onto R by a homeomorphism of the plane. The homeomorphism is holomorphic and uniquely determined on the interior of Q.*

If Q and Q' are close in the Hausdorff metric, then the holomorphic maps on the interior are close in the compact open topology.

(6.2) Theorem. *A matchbox U is tamely embedded in the plane. In other words, there exists a transformation of the plane which maps U onto the standard matchbox $C \times [0,1]$, where C denotes a Cantor set.*

Proof. Without loss of generality, we may suppose that U is contained in the unit square $[0,1] \times [0,1]$. Since U is a matchbox there exists a homeomorphism $\gamma : C \times [0,1] \to U$. As before, the top $T(U)$ is the set of end points $\gamma(C \times \{1\})$ and the bottom $B(U)$ is the set of end points $\gamma(C \times \{0\})$. We may assume that $\{0\} \times [0,1]$ and $\{1\} \times [0,1]$ are matches of U, and that $B(U) \subset I \times \{0\}$ and $T(U) \subset I \times \{1\}$. Note that this requires that the end points of matches of U are accessible from the complement. (For a complete proof of this see [A-O].)

The complement of U in $[0,1] \times [0,1]$ is a union of infinitely many domains Q_j, for $j \in \mathbb{N}$, bounded by matches L_j and R_j. Each match $\Gamma \subset U$ is parameterized by a homeomorphism $\gamma_c : \{c\} \times I \to U$. The two boundary matches L_j and R_j of Q_j are parameterized by $l_j(t) = \gamma_{L_j}(t)$ and $r_j(t) = \gamma_{R_j}(t)$, respectively. Note that each point $l_j(t)$ can be joined to $r_j(t)$ by a curve $\gamma_{j,t}$ embedded in Q_j, such that $\operatorname{diam}(\gamma_{j,t}) \to 0$ as $j \to \infty$ uniformly in t. For arbitrary j, s, t, the four arcs $\gamma_{j,s}$, $\gamma_{j,t}$, $l_j(s,t), r_j(s,t)$ bound a quadrilateral $T(j,s,t)$, provided $\gamma_{j,s}$ and $\gamma_{j,t}$ are disjoint. For any $\varepsilon > 0$ and for large enough j, there exists a subdivision $\{-1 = t_0 < t_1 < \ldots < t_{n(j)} = 1\}$ such that the curves γ_{j,t_i} are pairwise disjoint, and the quadrilaterals $T(j, t_i, t_{i+1})$ have diameter less than ε. We may choose subdivisions for every j such that the diameter of the $T(j, t_i, t_{i+1})$ goes to zero uniformly as $j \to \infty$. By Theorem (6.1), each quadrilateral can be identified with a rectangle.

We may assume that the arcs $l_j(s,t)$ and $r_j(s,t)$ are identified with the vertical sides of the rectangle, whereas $\gamma_{j,s}$ and $\gamma_{j,t}$ are identified with the top and the bottom of the rectangle. Hence the quadrilateral $T(j,t_i,t_{i+1})$ is foliated by arcs corresponding to the vertical lines of the rectangle. This can be done for all the quadrilaterals. The foliation of $T(j,t_{i-1},t_i)$ can be glued to the foliation of $T(j,t_i,t_{i+1})$, so that the matchbox U is extended to a foliation $\mathcal{F}$ of the unit rectangle. For each point $x \in [0,1]$ there exists exactly one parameterized leaf F_x in $\mathcal{F}$ which contains $(x,0)$. Since diam $T(j,t_i,t_{i+1}) \to 0$ uniformly as $j \to \infty$, the parameterization of the leaves F_x defines a homeomorphism on the unit square which maps the leaves of $\mathcal{F}$ onto the vertical lines $\{x\} \times I$.

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