

On a discrete version of the antipodal theorem

by

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Abstract. The classical theorem of Borsuk and Ulam [2] says that for any continuous mapping $f : S^k \rightarrow \mathbb{R}^k$ there exists a point $x \in S^k$ such that $f(-x) = f(x)$. In this note a discrete version of the antipodal theorem is proved in which S^k is replaced by the set of vertices of a high-dimensional cube equipped with Hamming's metric. In place of equality we obtain some optimal estimates of $\inf_x \|f(x) - f(-x)\|$ which were previously known (as far as the author knows) only for f linear (cf. [1]).

We introduce standard notation: k and n will denote positive integers, $C_n^k = \{x \in [-1, 1]^n : \#\{i : |x_i| = 1\} \geq n - k\}$ will stand for the k -dimensional skeleton of the cube $C = [-1, 1]^n$ equipped with the standard CW-structure. Let $C_n = C_n^0 = \{-1, 1\}^n$. We consider Hamming's metric d on C_n defined as $d(x, y) = \#\{i : x_i \neq y_i\}$. $(X_k, \|\cdot\|)$ will stand for a normed k -dimensional linear space. By S^k (resp. B^{k+1}) we denote the unit Euclidean sphere (resp. ball) with centre at zero in \mathbb{R}^{k+1} .

This is the main result of the paper:

THEOREM 1. *Let $f : C_n \rightarrow X_k$ satisfy the following two conditions: $f(-x) = -f(x)$ and $\|f(x) - f(y)\| \leq d(x, y)$ for any $x, y \in C_n$. Then*

- (i) *there exists $x \in C_n$ such that $\|f(x)\| \leq \frac{1}{2} \min(k, n)$,*
- (ii) *if the norm is Euclidean then there exists $x \in C_n$ such that*

$$\|f(x)\| \leq \frac{1}{2} \sqrt{\min(k, n)}.$$

The examples of $X_k = l_1^k$ and $X_k = l_2^k$ with

$$f(x_1, \dots, x_n) = \frac{1}{2}(x_1, \dots, x_{\min(k, n)}, 0, 0, \dots, 0)$$

indicate that the constants cannot be improved.

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COROLLARY 1. Let $g : (C_n, d) \rightarrow (X_k, \|\cdot\|)$ be such that

$$\|g(x) - g(y)\| \leq d(x, y)$$

for any $x, y \in C_n$. Then there exists $z \in C_n$ such that $\|g(z) - g(-z)\| \leq \min(k, n)$.

This antipodal version follows from Theorem 1 when we set $f(x) = (g(x) - g(-x))/2$. In fact, one can easily see that Theorem 1(i) and Corollary 1 are equivalent. Therefore we will be interested only in the “anti-symmetric” case.

Theorem 1(i) is trivial if $n \leq k$. If $n < k$ then (i) easily follows from (ii), because for any k -dimensional linear normed space $(X_k, \|\cdot\|)$ there exists a Euclidean norm $|\cdot|$ such that $|v| \leq \|v\| \leq \sqrt{k}|v|$ for any $v \in X_k$. Therefore the proof will be devoted to the Euclidean case.

We will need two lemmas.

LEMMA 1. If $k < n$ then there exists a continuous mapping $h_k : S^k \rightarrow C_n^k$ such that $h_k(-x) = -h_k(x)$ for any $x \in S^k$.

PROOF. As the homotopy group π_i depends on the $(i+1)$ -dimensional skeleton of the CW-complex only, we know that $\pi_i(C_n^k) = \pi_i(C) = 0$ for any $i < k$. We inductively construct continuous mappings $h_i : S^i \rightarrow C_n^k$ for $i = 0, 1, \dots, k$ such that $h_i(-x) = -h_i(x)$ for any $x \in S^i$. We choose the function h_0 arbitrarily, just to keep anti-symmetry ($S^0 = \{-1, 1\}$). Assume h_{i-1} is well defined, continuous and anti-symmetric. Since $\pi_{i-1}(C_n^k) = 0$, there exists a continuous function $H_i : B^i \rightarrow C_n^k$ such that $H_i(x) = h_{i-1}(x)$ for any $x \in S^{i-1} = \partial B^i$. Let $G_i : B^i \rightarrow C_n^k$ be defined as $G_i(x) = -H_i(-x)$. For $x \in S^i$ we put $h_i(x_1, \dots, x_{i+1}) = H_i(x_1, \dots, x_i)$ if $x_{i+1} \geq 0$ and $h_i(x_1, \dots, x_{i+1}) = G_i(x_1, \dots, x_i)$ if $x_{i+1} \leq 0$. This completes our induction as h_k satisfies the desired conditions.

LEMMA 2. If $k < n$ and $F : C_n^k \rightarrow X_k$ is continuous and such that $F(-x) = -F(x)$ for any $x \in C_n^k$ then there exists $z \in C_n^k$ such that $F(z) = 0$.

PROOF. Let h_k be defined as in Lemma 1. The function $F \circ h_k : S^k \rightarrow X_k$ is continuous. Recall that $\dim X_k = k$. Therefore by Borsuk–Ulam’s Theorem

$$F(h_k(x)) = F(h_k(-x)) = F(-h_k(x)) = -F(h_k(x))$$

for some $x \in S^k$. Hence $z = h_k(x)$ satisfies the desired conditions.

The proofs above were given for the sake of completeness and because of their simplicity, but it should be noticed that they are only special cases of well known, far more general topological theorems (cf. [3]).

Proof of Theorem 1(ii). First consider the case $k < n$. Let $\widehat{f}: C \rightarrow X_k$ be defined by the formula

$$\widehat{f}(x_1, \dots, x_n) = \sum_{y \in C_n} \left(\prod_{j=1}^n \frac{1 + y_j x_j}{2} \right) f(y).$$

One can easily see that $\widehat{f}|_{C_n} = f$ and $\partial^2 \widehat{f} / \partial x_i^2 = 0$ for $i = 1, \dots, n$ (i.e. the function \widehat{f} is affine with respect to each variable). The function $F = \widehat{f}|_{C_n^k}$ satisfies the conditions of Lemma 2, hence $F(z) = 0$ for some $z \in C_n^k$. This is the crucial point of the proof. Without loss of generality we can assume that $|z_{k+1}|, \dots, |z_n| = 1$ and define $G: [-1, 1]^k \rightarrow X_k$ by $G(x_1, \dots, x_k) = F(x_1, \dots, x_k, z_{k+1}, \dots, z_n)$. For any $i \leq n$ and $x \in C$ by the triangle inequality we have

$$\begin{aligned} & \|\widehat{f}(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) - \widehat{f}(x_1, \dots, x_{i-1}, -1, x_{i+1}, \dots, x_n)\| \\ & \leq \sum_{y \in C_n, y_i=1} \left(\prod_{j \neq i} \frac{1 + y_j x_j}{2} \right) \|f(y_1, \dots, y_{i-1}, 1, y_{i+1}, \dots, y_n) \\ & \qquad \qquad \qquad - f(y_1, \dots, y_{i-1}, -1, y_{i+1}, \dots, y_n)\| \leq 1, \end{aligned}$$

since

$$\sum_{y \in C_n, y_i=1} \left(\prod_{j \neq i} \frac{1 + y_j x_j}{2} \right) = 1$$

for any $x \in C$ and $d((y_1, \dots, y_{i-1}, 1, y_{i+1}, \dots, y_n), (y_1, \dots, y_{i-1}, -1, y_{i+1}, \dots, y_n)) = 1$ for any $y \in C_n$.

Therefore for any $i \leq k$ and $x \in [-1, 1]^k$ we have

$$\|G(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_k) - G(x_1, \dots, x_{i-1}, -1, x_{i+1}, \dots, x_k)\| \leq 1.$$

Consider independent real random variables Y_1, \dots, Y_k such that $P(Y_i = 1) = (1 + z_i)/2$, $P(Y_i = -1) = (1 - z_i)/2$ for $i \leq k$. As $EY_i = z_i$ and $\partial^2 G / \partial x_i^2 = 0$ for $i \leq k$ (G is affine with respect to each variable), we have

$$\begin{aligned} & E\|G(Y_1, \dots, Y_k)\|^2 \\ & = \frac{1 + z_k}{2} E\|G(Y_1, \dots, Y_{k-1}, 1)\|^2 + \frac{1 - z_k}{2} E\|G(Y_1, \dots, Y_{k-1}, -1)\|^2 \\ & = \frac{1 + z_k}{2} E\|G(Y_1, \dots, Y_{k-1}, z_k) \\ & \quad + (G(Y_1, \dots, Y_{k-1}, 1) - G(Y_1, \dots, Y_{k-1}, z_k))\|^2 \\ & \quad + \frac{1 - z_k}{2} E\|G(Y_1, \dots, Y_{k-1}, z_k) \\ & \quad + (G(Y_1, \dots, Y_{k-1}, -1) - G(Y_1, \dots, Y_{k-1}, z_k))\|^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{1+z_k}{2} E \left\| G(Y_1, \dots, Y_{k-1}, z_k) \right. \\
&\quad \left. + \frac{1-z_k}{2} (G(Y_1, \dots, Y_{k-1}, 1) - G(Y_1, \dots, Y_{k-1}, -1)) \right\|^2 \\
&\quad + \frac{1-z_k}{2} E \left\| G(Y_1, \dots, Y_{k-1}, z_k) \right. \\
&\quad \left. - \frac{1+z_k}{2} (G(Y_1, \dots, Y_{k-1}, 1) - G(Y_1, \dots, Y_{k-1}, -1)) \right\|^2 \\
&= E \|G(Y_1, \dots, Y_{k-1}, z_k)\|^2 \\
&\quad + \frac{1-z_k^2}{4} E \|G(Y_1, \dots, Y_{k-1}, 1) - G(Y_1, \dots, Y_{k-1}, -1)\|^2 \\
&\leq E \|G(Y_1, \dots, Y_{k-1}, z_k)\|^2 + \frac{1}{4}
\end{aligned}$$

(by easy induction)

$$\leq \|G(z_1, \dots, z_k)\|^2 + \frac{k}{4} = \frac{k}{4}.$$

Hence there exists $y \in \{-1, 1\}^k$ such that $\|f(y_1, \dots, y_k, z_{k+1}, \dots, z_n)\| = \|G(y_1, \dots, y_k)\| \leq \sqrt{k}/2$, which proves (ii) if $k < n$. In the case $n \leq k$ we deal with $\widehat{f}(x_1, \dots, x_n)$ instead of $G(x_1, \dots, x_k)$ (then Lemmas 1 and 2 are unnecessary since $\widehat{f}(0) = 0$ follows just from the anti-symmetry of f) and the rest of the proof is essentially the same, so that we omit it. This completes the proof.

COROLLARY 2. *If $(X_k, \|\cdot\|) = l_p^k$ for $p \in [1, 2]$ then under the assumptions of Theorem 1 there exists $x \in C_n$ such that $\|f(x)\|_p \leq k^{1/p}/2$.*

To see this notice that $\|v\|_2 \leq \|v\|_p \leq k^{1/p} \|v\|_2$ for any $v \in \mathbb{R}^k$.

COROLLARY 3. *Let P be a convex, centrally symmetric polytope in \mathbb{R}^n and let P_0 be the set of its vertices. Let $(X_k, \|\cdot\|)$ be a normed linear space of finite dimension $k < n$. We will say that $x \sim y$ for $x, y \in P_0$ if there exists a k -dimensional face of P containing x and y . Then for any function $g : P_0 \rightarrow X_k$ there exists $z \in P_0$ such that*

$$\|g(z) - g(-z)\| \leq 2 \max_{x \sim y; x, y \in P_0} \|g(x) - g(y)\|.$$

We sketch a proof which is just a modification of the proof of Theorem 1. Instead of C_n^k we consider the k -dimensional skeleton of P . After obvious changes Lemmas 1 and 2 remain valid. Then we define a continuous function \widehat{f} by induction: $\widehat{f}|_{P_0} = g$ and \widehat{f} is harmonic (i.e. each of its coordinates is) inside any i -dimensional cell of P for $i = 1, 2, \dots$. Then Corollary 3 follows from the maximum property of harmonic functions.

THEOREM 2. *Let $p \in [1, 2]$. If $(X_k, \|\cdot\|)$ is p -smooth, i.e. there exists $K \geq 1$ such that*

$$\frac{\|x + y\|^p + \|x - y\|^p}{2} \leq \|x\|^p + K\|y\|^p$$

for any $x, y \in X_k$, then under the assumptions of Theorem 1 there exists $z \in C_n$ such that $\|f(z)\| \leq (K \min(k, n))^{1/p}$.

PROOF. Assume that $k < n$. Precisely as in the proof of Theorem 1 we construct a function $G : [-1, 1]^k \rightarrow X_k$ and deduce that $G(z_1, \dots, z_k) = 0$ for some $z \in [-1, 1]^k$. Our problem is reduced to the following lemma.

LEMMA 3. *Let a function $G : [-1, 1]^k \rightarrow X_k$ satisfy the conditions*

$$G(x) = \sum_{y \in \{-1, 1\}^k} \left(\prod_{j=1}^k \frac{1 + y_j x_j}{2} \right) G(y)$$

for any $x \in \{-1, 1\}^k$ and

$$\|G(x) - G(y)\| \leq \#\{i : x_i \neq y_i\}$$

for any $x, y \in [-1, 1]^k$. Let $(X_k, \|\cdot\|)$ satisfy the conditions of Theorem 2. Then for any $z \in [-1, 1]^k$ there exists $x \in \{-1, 1\}^k$ such that $\|G(x) - G(z)\| \leq (Kk)^{1/p}$.

PROOF. According to the properties of the function G shown in the proof of Theorem 1 we have $\|G(x) - G(z)\| \leq k$ for each $x \in \{-1, 1\}^k$. We will prove that if Lemma 3 is valid with a constant $L = L(k)$ in place of $(Kk)^{1/p}$ then it is true also with the constant $\frac{1}{2}L + \frac{1}{2}(Kk)^{1/p}$. Thus Lemma 3 can be proved by a limit argument.

Without loss of generality we can assume that $z_1, \dots, z_k \geq 0$. Applying Lemma 3 to the cube $[0, 1]^k$ we see that there exists $y \in \{0, 1\}^k$ such that $\|G(y) - G(z)\| \leq \frac{1}{2}L$. Now we only need to show that there exists $x \in \{-1, 1\}^n$ such that $\|G(x) - G(y)\|^p \leq (Kk)/(2^p)$. Without loss of generality we can assume that $y_1 = \dots = y_i = 0, y_{i+1} = \dots = y_k = 1$. Let r_1, \dots, r_i be independent symmetric random Bernoulli variables, i.e. $P(r_j = \pm 1) = \frac{1}{2}$ for $j \leq i$. From p -smoothness of the norm we easily deduce that

$$\begin{aligned} & E\|G(r_1, \dots, r_i, 1, \dots, 1) - G(0, 0, \dots, 1, \dots, 1)\|^p \\ &= E\left\| \sum_{j=1}^i (G(r_1, \dots, r_{j-1}, r_j, 0, 0, \dots, 1, \dots, 1) \right. \\ &\quad \left. - G(r_1, \dots, r_{j-1}, 0, 0, \dots, 1, \dots, 1)) \right\|^p \\ &\leq K \sum_{j=1}^i E\|G(r_1, \dots, r_{j-1}, r_j, 0, 0, \dots, 1, \dots, 1) \\ &\quad - G(r_1, \dots, r_{j-1}, 0, 0, \dots, 1, \dots, 1)\|^p \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^p} K \sum_{j=1}^i E \|G(r_1, \dots, r_{j-1}, 1, 0, 0, \dots, 1, \dots, 1) \\
&\quad - G(r_1, \dots, r_{j-1}, -1, 0, 0, \dots, 1, \dots, 1)\|^p \\
&\leq \frac{Ki}{2^p} \leq \frac{Kk}{2^p}.
\end{aligned}$$

We used the same properties of the function G which were verified during the proof of Theorem 1. This completes the proof of Lemma 3. The proof of Theorem 2 in the case $n \leq k$ follows by the argument ending the proof of Theorem 1.

Remark. Let us consider $X_k = \mathbb{R}^k$ with l_p and l_q norms for $p, q \in [1, 2]$. One can easily deduce from Corollary 2 and elementary inequalities between norms that for $f : C_n \rightarrow X_k$ satisfying the conditions

$$f(-x) = f(x) \quad \text{and} \quad \|f(x) - f(y)\|_p \leq d(x, y)$$

for any $x, y \in C_n$, there exists $z \in C_n$ such that $\|f(z)\|_q \leq k^{1/q}/2$. The constant $k^{1/q}/2$ is optimal with respect to k . It is of interest to know what happens when $p = 2$ and $q = \infty$. The well known and still unproved Komlós conjecture states that in this case the above estimate is universal (does not depend on k or n) if f is linear (as we consider C_n embedded in the natural way in \mathbb{R}^n).

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