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Monotone σ-complete groups with unbounded refinement

by

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Abstract. The real line $\mathbb{R}$ may be characterized as the unique non-atomic directed partially ordered abelian group which is monotone σ-complete (countable increasing bounded sequences have suprema), has the countable refinement property (countable sums $\sum_m a_m = \sum_n b_n$ of positive (possibly infinite) elements have common refinements) and is linearly ordered. We prove here that the latter condition is not redundant, thus solving an old problem by A. Tarski, by proving that there are many spaces (in particular, of arbitrarily large cardinality) satisfying all the above listed axioms except linear ordering.

0. Introduction. The real line $\mathbb{R}$ may be characterized up to isomorphism as the unique partially ordered abelian group G with the following properties: G is *non-atomic* (i.e., there are no minimal elements of $G^+ \setminus \{0\}$), *directed* (i.e., every element is the difference of two positive elements), *monotone σ-complete* (i.e., every bounded increasing sequence of elements has a supremum), $G^+ \cup \{\infty\}$ has the *countable refinement property* (i.e., if $(a_m)_m$ and $(b_n)_n$ are sequences of elements of $G^+ \cup \{\infty\}$ such that $\sum_m a_m = \sum_n b_n$, then there exists a double sequence $(c_{mn})_{m,n}$ of elements of $G^+ \cup \{\infty\}$ such that for all m, $a_m = \sum_n c_{mn}$ and for all n, $b_n = \sum_m c_{mn}$)—call *cardinal groups* (Definition 2.1) those partially ordered abelian groups satisfying all these conditions—and, last but not least, G is *linearly ordered* (i.e., $G = G^+ \cup (-G^+)$).

The question whether the latter condition results from the others was posed in Tarski's 1949 book [9] (in the form "are there non-linearly ordered simple cardinal algebras?"), and, since then, has remained unsolved. The papers [3] and [4] indicate that if there exists a non-linearly ordered cardinal group, then it has to be a rather unusual space, while the statement of the classification theorem presented in [5] involves these hypothetical objects. The main advance made about these objects is probably Chuaqui's result

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[3, Corollary 3.3] that if a cardinal group is not linearly ordered, then it is *divisible* (thus a partially ordered vector space over the reals); the hard core of the proof of this result is Bradford's very difficult Decomposition Theorem [2]. Another property of non-linearly ordered cardinal groups is that they are *prime*, i.e., any two strictly positive elements lie above some strictly positive element. In [8, Theorem IV.18.4 + additional remark] two examples are shown of non-linearly ordered prime directed monotone σ-complete partially ordered abelian groups whose positive cone has the *finite* refinement property (one of them is divisible, the other is not), but unfortunately, they fail to be cardinal groups. Nevertheless, although no example of non-linearly ordered cardinal group has ever been constructed, the alleged answer to Tarski's question has been positive.

In this paper, we confirm this view and thus we solve Tarski's problem; in fact, we show that every directed Archimedean partially ordered abelian group embeds cofinally into a cardinal group, in a way preserving bounded countable suprema when they exist (*monotone σ-complete embeddings*, Definition 1.6). Thus not only are there non-linearly ordered cardinal groups, but they can be taken of arbitrarily large cardinality. The embedding methods that we use are elementary, and their context is the one of *partially ordered vector spaces*. The hard core of the proof is, when a and b_n (n a non-negative integer) are positive elements such that $\sum_n b_n = \infty$, to find elements x_n (for all n) in some extension such that $0 \leq x_n \leq a_n$ (for all n) and $a = \sum_{n\in\omega} x_n$ (Lemmas 2.3–2.6).

For all sets X and Y, X^Y denotes the set of all mappings from Y to X. We denote as usual by ω the first infinite ordinal, that is, $\omega = \{0, 1, 2, \ldots\}$, and we put $\mathbb{N} = \omega \setminus \{0\}$.

If X and Y are two subsets of a partially ordered abelian group $(G, +, \leq)$, write $X + Y = \{x + y : x \in X \text{ and } y \in Y\}$, and write $X \leq Y$ if and only if $x \leq y$ for all $x \in X$ and $y \in Y$; in the latter case, write $x \leq Y$ (resp. $X \leq y$) when $X = \{x\}$ (resp. $Y = \{y\}$). If m is a non-negative integer, write $mX = \{mx : x \in X\}$.

If G is a partially ordered abelian group, then we let ∞ be an object not in G and let $G^+ \cup \{\infty\}$ be the commutative monoid whose addition extends the one of G^+ in such a way that $x + \infty = \infty$ for all x; the ordering of G^+ is extended by stating that ∞ is maximum. If m is a positive integer, say as in [7] that G is *m-unperforated* when it satisfies $(\forall x)(mx \geq 0 \Rightarrow x \geq 0)$, *unperforated* when it is m-unperforated for all $m \in \mathbb{N}$, and *Archimedean* when for all $a, b \in G$, if $a \leq mb$ for every positive integer m, then $0 \leq b$.

The *finite refinement property* is the axiom (in the language $(+, =)$)

$$(\forall_{i<2} a_i, b_i)(a_0 + a_1 = b_0 + b_1 \\ \Rightarrow (\exists_{i,j<2} c_{ij})(\forall i < 2)(a_i = c_{i0} + c_{i1} \text{ and } b_i = c_{0i} + c_{1i})).$$

An *interpolation group* is a partially ordered abelian group G whose positive cone G^+ has the finite refinement property (for an explanation of this terminology, see [7, Proposition 2.1]). For example, every abelian lattice-ordered group is an interpolation group. This is in particular the case for $G = \ell^\infty$, the space of all bounded sequences of real numbers, with positive cone $(\ell^\infty)^+$, the subset of all bounded sequences (indexed by ω) of non-negative real numbers. We will denote by $\mathbf{0}$ (resp. $\mathbf{1}$) the constant sequence with value 0 (resp. 1), and for every $n \in \omega$, we will denote by $\boldsymbol{e}_n$ the element of ℓ^∞ defined by $\boldsymbol{e}_n(i) = 0$ if $i \neq n$, and $\boldsymbol{e}_n(n) = 1$. We will denote by $\vee$, $\bigvee$ the supremum operation, and by $\wedge$, $\bigwedge$ the infimum operation (in any partially ordered set). If x is a real number (resp. a sequence of real numbers), we will write $x^+ = x \vee 0$ (resp. $x \vee \mathbf{0}$). Unless specified otherwise, all vector spaces will be over the reals.

1. Preliminary embedding results. The techniques and results presented in this section are essentially standard, but they may not be of immediate access; thus, since the proofs are anyway easy, we give some of them here for convenience. We first define monotone σ-complete partially ordered abelian groups as in [7]:

1.1. Definition. A partially ordered abelian group G is *monotone σ-complete* when every bounded countable increasing sequence of elements of G admits a supremum.

Thus if G is a partially ordered abelian group, then it is monotone σ-complete if and only if every countable increasing sequence of elements of $G^+ \cup \{\infty\}$ admits a supremum. In general, if G is a partially ordered abelian group and if $a \in G^+ \cup \{\infty\}$ and $(a_i)_{i \in I}$ is a family of elements of $G^+ \cup \{\infty\}$, write $a = \sum_{i \in I} a_i$ when a is the supremum of all finite sums $\sum_{i \in J} a_i$, where J ranges over all finite subsets of I. We record the following classical (and easily checked) properties of suprema and infinite sums:

1.2. Lemma. *Let G be a partially ordered group, and let X and Y be two subsets of G. If both $\bigvee X$ and $\bigvee Y$ exist, then $\bigvee(X + Y)$ exists, and $\bigvee(X + Y) = \bigvee X + \bigvee Y$.* ■

1.3. Lemma. *Let G be a monotone σ-complete partially ordered abelian group. Then the following holds*:

(i) *If I is a countable set and I_k ($k \in \omega$) are mutually disjoint subsets such that $I = \bigcup_{k \in \omega} I_k$ and if $(a_i)_{i \in I}$ is a family of elements of $G^+ \cup \{\infty\}$, then*

$$\sum_{i \in I} a_i = \sum_{k \in \omega} \sum_{i \in I_k} a_i.$$

(ii) *If I is a countable set and $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ are families of elements of $G^+\cup\{\infty\}$, then*

$$\sum_{i\in I}(a_i+b_i)=\sum_{i\in I}a_i+\sum_{i\in I}b_i.$$

In addition, if G is a partially ordered vector space, then the following holds:

(iii) *If I is a countable set and $(a_i)_{i\in I}$ is a family of elements of $G^+\cup\{\infty\}$, then for every $\lambda\in\mathbb{R}^+$,*

$$\lambda\cdot\sum_{i\in I}a_i=\sum_{i\in I}\lambda\cdot a_i \quad (\textit{with the usual convention } 0\cdot\infty=0). \ \blacksquare$$

Now, if G is a monotone σ-complete partially ordered vector space and $\boldsymbol{a}=(a_n)_{n\in\omega}$ is a sequence of elements of $G^+\cup\{\infty\}$ while $\boldsymbol{s}=(s_n)_{n\in\omega}\in(\mathbb{R}^+)^\omega$, we shall write $\boldsymbol{s}(\boldsymbol{a})=\sum_{n\in\omega}s_na_n$. Thus, in particular, $\boldsymbol{s}(\boldsymbol{a})=\infty$ if and only if either there exists n such that $s_n>0$ while $a_n=\infty$, or all the a_n's are finite (i.e., they belong to G^+) and the set of all partial sums $\sum_{i<n}s_ia_i$ for $n\in\omega$ is unbounded in G.

1.4. Lemma. (i) *Let E be an Archimedean partially ordered vector space, let $(\lambda_n)_{n\in\omega}$ be a sequence of real numbers with supremum $\lambda\in\mathbb{R}$ and let $a\in E^+$. Then $\bigvee_{n\in\omega}(\lambda_na)$ exists in E and is equal to λa.*

(ii) *Let E be an Archimedean monotone σ-complete partially ordered vector space, and let $\boldsymbol{a}\in(E^+\setminus\{0\})^\omega$. Let $(\boldsymbol{s}_n)_{n\in\omega}$ be an increasing sequence of elements of $(\mathbb{R}^+)^\omega$ such that the set $\{\boldsymbol{s}_n(\boldsymbol{a}):n\in\omega\}$ is bounded above in E. Then the supremum $\boldsymbol{s}=\bigvee_{n\in\omega}\boldsymbol{s}_n$ belongs to $(\mathbb{R}^+)^\omega$ and $\boldsymbol{s}(\boldsymbol{a})=\bigvee_{n\in\omega}\boldsymbol{s}_n(\boldsymbol{a})$.*

Proof. (i) Without loss of generality, $\lambda\le\lambda_n+1/(n+1)$ for all n. Therefore, if b is an upper bound for $\{\lambda_na:n\in\omega\}$, then for all n, we have $\lambda a-b\le(1/(n+1))a$ for all n, thus, since E is Archimedean, $\lambda a\le b$.

(ii) Put $\boldsymbol{a}=(a_k)_{k\in\omega}$, $\boldsymbol{s}=(s^k)_{k\in\omega}$ and $\boldsymbol{s}_n=(s^k_n)_{k\in\omega}$ for all n. By assumption, for all k, the set $\{s^k_na_k:n\in\omega\}$ is bounded, thus, since $a_k>0$ and E is Archimedean, $\{s^k_n:n\in\omega\}$ is bounded, whence $s^k\in\mathbb{R}^+$. Thus to conclude, it suffices to show that every element b of G^+ which is an upper bound for all $\boldsymbol{s}_n(\boldsymbol{a})$ $(n\in\omega)$ is larger than or equal to $\boldsymbol{s}(\boldsymbol{a})$. For all $m,n\in\omega$, we have $\sum_{k<m}s^k_na_k\le\boldsymbol{s}_n(\boldsymbol{a})\le b$, whence, by taking the supremum over n and using Lemmas 1.2 and 1.4(i), we obtain $\sum_{k<m}s^ka_k\le b$. This holds for every m, whence $\boldsymbol{s}(\boldsymbol{a})\le b$. $\blacksquare$

In view of Lemma 1.4(i), if x and a are two elements of an Archimedean partially ordered vector space E such that $a>0$, we shall write

$$(x:a)=\begin{cases}\bigvee\{\lambda\in\mathbb{R}:\lambda a\le x\} & \text{if } (\exists\lambda\in\mathbb{R})(\lambda a\le x),\\ -\infty & \text{otherwise;}\end{cases}$$

therefore, $(x:a)\in\mathbb{R}$ if and only if $(\exists\lambda\in\mathbb{R})(\lambda a\le x)$, and then $(x:a)a\le x$.

1.5. Lemma. *Let m be a non-negative integer and let G be a partially ordered abelian group that is k-unperforated for all k such that $0 \leq k \leq m$. Then for every subset X of G, if $\bigvee X$ exists, then $\bigvee(mX)$ exists and $\bigvee(mX) = m \cdot \bigvee X$.*

Proof (by induction on m). This is trivial for $m = 0$, so suppose that $m > 0$. Put $a = \bigvee X$; it suffices to prove that if b is an upper bound for mX, then $ma \leq b$. For all elements x and y of X, we have $mx \leq b$ and $my \leq b$, whence $(m-1)my \leq (m-1)b$, thus, adding the first and third inequalities, we obtain $m(x + (m-1)y) \leq mb$, whence, by m-unperforation, $(m-1)y \leq b - x$. When x is fixed this holds for all y, whence, by induction hypothesis, $(m-1)a \leq b - x$. This holds for all x, whence, by definition of a, $a \leq b - (m-1)a$, whence $ma \leq b$. ∎

1.6. Definition. Let $f : G \to H$ be a homomorphism of partially ordered abelian groups. Then f is *complete* (resp. *monotone σ-complete*) when for every subset (resp. range of a bounded increasing sequence) X of G, if $\bigvee X$ exists in G, then $\bigvee f[X]$ exists in H and $\bigvee f[X] = f(\bigvee X)$.

Recall now that for every directed Archimedean partially ordered abelian group G, there exists a unique (up to isomorphism) embedding from G into a (Dedekind) complete lattice-ordered group $\widehat{G}$ such that every element of $\widehat{G}$ is a supremum of elements of G (see for example [1] for more information). Then denote by G^σ (the *Dedekind σ-completion* of G) the closure of G in $\widehat{G}$ under countable suprema and infima.

1.7. Lemma. *For every directed Archimedean partially ordered abelian group G, the natural embedding from G into $\widehat{G}$ is complete*; *thus so is the natural embedding from G into G^σ.*

Proof. Let X be a subset of G, with supremum $a \in G$. To prove the result about $\widehat{G}$, it suffices to prove that for every element y of $\widehat{G}$, if y is an upper bound of X, then $a \leq y$. Since $-y$ is a supremum of elements of G, there exists a subset Y of G such that $y = \bigwedge Y$, and so $X \leq y$ means that $X \leq Y$; but now, both X and Y are subsets of G, thus $a \leq Y$ by definition of a; whence $a \leq y$. Thus the natural embedding from G into $\widehat{G}$ is complete. Since $G \subseteq G^\sigma \subseteq \widehat{G}$, the result for G^σ follows immediately. ∎

The result of this lemma will be of importance in the following proposition:

1.8. Proposition. *Let G be a directed Archimedean partially ordered abelian group. Then G admits a complete cofinal embedding into a Dedekind σ-complete vector space E such that $|E| \leq |G|^{\aleph_0}$.*

Proof. First, let G' be the divisible, unperforated closure of G (it can for example be realized as the tensor product $G \otimes \mathbb{Q}$): thus G' is a partially ordered vector space over $\mathbb{Q}$ and every element of G' can be written as $(1/n)x$ for some $n \in \mathbb{N}$ and $x \in G$. It is easy to verify that G' is also directed and Archimedean. Using Lemma 1.5, it is easy to verify that the natural embedding from G into G' is complete. We conclude by taking for E the Dedekind σ-completion of G' (taking the σ-completion instead of the completion yields the bound on the cardinality). ∎

Thus from now on, we are going to focus attention on partially ordered vector spaces.

2. The main result; unbounded refinement property. For every interpolation group G, it results immediately from the definitions that G is monotone σ-complete if and only if $G^+ \cup \{\infty\}$ is a *weak cardinal algebra* (in the sense, for example, of [10, Definition 2.2]). Of course, in all well-known cases except for the closed subgroups of $\mathbb{R}$ (where G is isomorphic either to $\mathbb{R}$ or to $\mathbb{Z}$), $G^+ \cup \{\infty\}$ fails to be a cardinal algebra. The caveat for this lies in the following definition:

2.1. Definition. Let G be a monotone σ-complete partially ordered abelian group. Then G^+ has the *unbounded refinement property* when for all a, b_n $(n \in \omega)$ in G^+ such that $\sum_{n\in\omega} b_n = \infty$, there exists a sequence $(a_n)_{n\in\omega}$ of elements of G^+ such that $(\forall n \in \omega)(a_n \le b_n)$ and $a = \sum_{n\in\omega} a_n$. If G is a directed monotone σ-complete partially ordered abelian group and G^+ has both the finite refinement property and the unbounded refinement property, we will say that G is a *cardinal group*; if in addition G is a vector space, then we will say that G is a *cardinal space*.

Note that it is sufficient to verify the condition above for a and all the b_n's in $E^+ \setminus \{0\}$. It is also to be noted that, for example by [7, Theorem 16.10], *every directed monotone σ-complete interpolation group* (*thus every cardinal group*) *is Archimedean* (by contrast, there exist non-Archimedean directed monotone σ-complete partially ordered abelian groups, for example $G = \mathbb{Z} \times \mathbb{Z}$ endowed with the positive cone G^+ defined by $(x, y) \in G^+ \Leftrightarrow (x = y = 0 \text{ or } (x > 0 \text{ and } y \ge 0)))$.

Although the following proposition will not be used in the sequel, it is worth recording:

2.2. Proposition. *For every cardinal group G, the positive cone $G^+ \cup \{\infty\}$ has the* general (countable) refinement property, *i.e.*, *for all elements a_m, b_n $(m, n \in \omega)$ of $G^+ \cup \{\infty\}$ such that $\sum_{m\in\omega} a_m = \sum_{n\in\omega} b_n$, there exists a double sequence $(c_{mn})_{m,n\in\omega}$ such that $(\forall m \in \omega)(a_m = \sum_{n\in\omega} c_{mn})$ and $(\forall n \in \omega)(b_n = \sum_{m\in\omega} c_{mn})$.*

Proof. By [6, Theorem 1.6] (whose proof is far from being trivial!), $G^+ \cup \{\infty\}$ is a cardinal algebra. ∎

By Chuaqui's result [3, Corollary 3.3], itself resulting from Bradford's Decomposition Theorem [2], every non-linearly ordered cardinal group is a cardinal space.

Note also that uncountable versions of Proposition 2.2 do not hold, even for the very simple structure $\mathbb{R}^+ \cup \{\infty\}$: for example, in this structure, there is no refinement for the equality $\underbrace{1 + \ldots + 1}_{\omega \text{ times}} = \underbrace{1 + \ldots + 1}_{\omega_1 \text{ times}}$.

Now, from 2.3 to 2.6, we will fix a directed Archimedean monotone σ-complete partially ordered vector space E and elements a, b_n $(n \in \omega)$ of $E^+ \setminus \{0\}$ (write $\boldsymbol{b} = (b_n)_{n\in\omega}$) such that $\sum_{n\in\omega} b_n = \infty$, and we will construct a monotone σ-complete embedding of E into a directed Archimedean partially ordered abelian group $\widetilde{E}$ with elements x_n $(n \in \omega)$ such that for all n, $0 < x_n < b_n$ and the sum $\sum_{n\in\omega} x_n$ exists and is equal to a.

Let I be the set of all *bounded* sequences of non-negative real numbers $\boldsymbol{s}$ such that $\boldsymbol{s}(\boldsymbol{b}) < \infty$. For all $(x, \boldsymbol{t}) \in E \times \ell^\infty$, let $\Lambda(x, \boldsymbol{t})$ be the set of all real numbers λ such that $(-\boldsymbol{t} - \lambda\mathbf{1})^+ \in I$ and $x \geq \lambda a + (-\boldsymbol{t} - \lambda\mathbf{1})^+(\boldsymbol{b})$. Furthermore, let P be the set of all those $(x, \boldsymbol{t}) \in E \times \ell^\infty$ such that $\Lambda(x, \boldsymbol{t}) \neq \emptyset$ and let F be the subspace of $E \times \ell^\infty$ generated by the vector $(a, -\mathbf{1})$.

2.3. Lemma. *For all $(x, \boldsymbol{t}) \in E \times \ell^\infty$, $\Lambda(x, \boldsymbol{t})$ is a compact subset of the interval $[-\sup(\boldsymbol{t}), (x : a)]$ (the latter being by convention empty if $-\sup(\boldsymbol{t}) > (x : a)$).*

Proof. Let $\lambda \in \Lambda(x, \boldsymbol{t})$. If $\lambda < -\sup(\boldsymbol{t})$, then there exists $\varepsilon > 0$ such that $(\lambda + \varepsilon)\mathbf{1} \leq -\boldsymbol{t}$, whence $\varepsilon\mathbf{1} \leq (-\lambda\mathbf{1} - \boldsymbol{t})^+$; since $(-\lambda\mathbf{1} - \boldsymbol{t})^+ \in I$ and $0 \leq \varepsilon\mathbf{1}$, it follows that $\varepsilon\mathbf{1} \in I$, whence $\mathbf{1} \in I$, a contradiction. Hence $-\sup(\boldsymbol{t}) \leq \lambda$. Moreover, $\lambda a \leq \lambda a + (-\lambda\mathbf{1} - \boldsymbol{t})^+(\boldsymbol{b}) \leq x$, whence $\lambda \leq (x : a)$.

Now let us prove that $\Lambda(x, \boldsymbol{t})$ is compact. Thus let λ be a point in the closure of $\Lambda(x, \boldsymbol{t})$. Thus λ is the limit of a sequence $(\lambda_n)_{n\in\omega}$ of points of $\Lambda(x, \boldsymbol{t})$, and we may assume that $(\lambda_n)_{n\in\omega}$ is either increasing or decreasing. Thus we distinguish two cases:

Case 1: *$(\lambda_n)_{n\in\omega}$ is increasing.* Then for all $n \in \omega$, we have $x \geq \lambda_n a + (-\lambda_n\mathbf{1} - \boldsymbol{t})^+(\boldsymbol{b}) \geq \lambda_n a + (-\lambda\mathbf{1} - \boldsymbol{t})^+(\boldsymbol{b})$ (in particular, $(-\lambda\mathbf{1} - \boldsymbol{t})^+ \in I$), whence, by Lemma 1.4(i), $x \geq \lambda a + (-\lambda\mathbf{1} - \boldsymbol{t})^+(\boldsymbol{b})$. Hence, $\lambda \in \Lambda(x, \boldsymbol{t})$.

Case 2: *$(\lambda_n)_{n\in\omega}$ is decreasing.* For all $n \in \omega$, put $\boldsymbol{s}_n = (-\lambda_n\mathbf{1} - \boldsymbol{t})^+$. Then $\boldsymbol{s}_n \in I$ and $x \geq \lambda_n a + \boldsymbol{s}_n(\boldsymbol{b})$, whence $x \geq \lambda a + \boldsymbol{s}_n(\boldsymbol{b})$. Therefore, by Lemma 1.4(ii), $\boldsymbol{s} = (-\lambda\mathbf{1} - \boldsymbol{t})^+ = \bigvee_{n\in\omega} \boldsymbol{s}_n$ belongs to I, and $x \geq \lambda a + \boldsymbol{s}(\boldsymbol{b})$. Thus we can conclude again that $\lambda \in \Lambda(x, \boldsymbol{t})$.

In both cases, $\Lambda(x, \boldsymbol{t})$ is compact. ∎

2.4. Lemma. *The set P is the positive cone of a structure of partially preordered vector space on $E \times \ell^\infty$, and $P \cap (-P) = F$. Furthermore, the* *quotient space* $\widetilde{E} = (E \times \ell^\infty, +, (0, \mathbf{0}), P)/F$, *is a directed Archimedean partially ordered vector space*, *and the natural map* $j : E \to \widetilde{E}$, $x \mapsto (x, \mathbf{0}) + F$, *is a cofinal embedding of partially ordered vector spaces.*

Proof. It is easy to verify that in fact, for all elements $(x, \boldsymbol{t})$ and $(x', \boldsymbol{t}')$ of $E \times \ell^\infty$ and for all real $\lambda \geq 0$, we have $\Lambda(x, \boldsymbol{t}) + \Lambda(x', \boldsymbol{t}') \subseteq \Lambda(x + x', \boldsymbol{t} + \boldsymbol{t}')$ and $\lambda \cdot \Lambda(x, \boldsymbol{t}) \subseteq \Lambda(\lambda x, \lambda \boldsymbol{t})$; hence, $P + P \subseteq P$ and $\lambda P \subseteq P$.

Next, let $(x, \boldsymbol{t})$ be an element of $P \cap (-P)$. Let $\lambda \in \Lambda(x, \boldsymbol{t})$ and $\lambda' \in \Lambda(-x, -\boldsymbol{t})$; put $\boldsymbol{s} = (-\lambda \mathbf{1} - \boldsymbol{t})^+$ and $\boldsymbol{s}' = (-\lambda' \mathbf{1} + \boldsymbol{t})^+$, so that both $\boldsymbol{s}$ and $\boldsymbol{s}'$ belong to I and $x \geq \lambda a + \boldsymbol{s}(\boldsymbol{b})$ and $-x \geq \lambda' a + \boldsymbol{s}'(\boldsymbol{b})$. By adding both inequalities together we obtain $0 = x + (-x) \geq (\lambda + \lambda')a$, whence $\lambda + \lambda' \leq 0$. On the other hand, $-(\lambda + \lambda')\mathbf{1} \leq \boldsymbol{s} + \boldsymbol{s}' \in I$, whence one cannot have $\lambda + \lambda' < 0$ (because $\mathbf{1} \notin I$); thus, $\lambda + \lambda' = 0$. Thus $0 = x + (-x) \geq (\boldsymbol{s} + \boldsymbol{s}')(\boldsymbol{b})$ with $\boldsymbol{s}$ and $\boldsymbol{s}'$ in $(\ell^\infty)^+$ and all the b_n's (strictly) positive, whence $\boldsymbol{s} = \boldsymbol{s}' = \mathbf{0}$, so that $\boldsymbol{t} = -\lambda \mathbf{1}$ and $x = \lambda a$; therefore, $(x, \boldsymbol{t}) = (\lambda a, -\lambda \mathbf{1}) \in F$.

We now prove that $\widetilde{E}$ is Archimedean. It suffices to prove that if $(x, \boldsymbol{t})$ and $(x_0, \boldsymbol{t}_0)$ are elements of $E \times \ell^\infty$ such that for all $n \in \mathbb{N}$, $(x, \boldsymbol{t}) + (1/n)(x_0, \boldsymbol{t}_0) \in P$, then $(x, \boldsymbol{t}) \in P$. First, since $E^+ \times (\ell^\infty)^+ \subseteq P$ and both E and ℓ^∞ are directed, we may assume without loss of generality that $x_0 \geq 0$ and $\boldsymbol{t}_0 \geq \mathbf{0}$. Next, for all $n \in \mathbb{N}$, let λ_n be any element of $\Lambda(x + (1/n)x_0, \boldsymbol{t} + (1/n)\boldsymbol{t}_0)$. Then $-\sup(\boldsymbol{t} + \boldsymbol{t}_0) \leq \lambda_n \leq (x + x_0 : a)$, thus $(\lambda_n)_{n \in \mathbb{N}}$ has a convergent subsequence, say $(\lambda_n)_{n \in S}$ for some infinite subset S of $\mathbb{N}$. Let $\lambda = \lim_{n \in S, n \to \infty} \lambda_n$. Without loss of generality, $(\lambda_n)_{n \in S}$ is either increasing or decreasing.

Case 1: $(\lambda_n)_{n \in S}$ *is increasing.* Then for all $n \in S$, we have $x + (1/n)x_0 \geq \lambda_n a + (-\lambda_n \mathbf{1} - \boldsymbol{t} - (1/n)\boldsymbol{t}_0)^+(\boldsymbol{b}) \geq \lambda_n a + \boldsymbol{s}_n(\boldsymbol{b})$, where $\boldsymbol{s}_n = (-\lambda \mathbf{1} - \boldsymbol{t} - (1/n)\boldsymbol{t}_0)^+$. Thus for all $n \in S$, $\boldsymbol{s}_n(\boldsymbol{b}) \leq x + x_0 - \lambda_0 a$, whence $\boldsymbol{s} = (-\lambda \mathbf{1} - \boldsymbol{t})^+ = \bigvee_{n \in S} \boldsymbol{s}_n$ belongs to I and, by Lemmas 1.2 and 1.4 and the fact that E is Archimedean, $x \geq \lambda a + \boldsymbol{s}(\boldsymbol{b})$; thus $\lambda \in \Lambda(x, \boldsymbol{t})$, whence $(x, \boldsymbol{t}) \in P$.

Case 2: $(\lambda_n)_{n \in S}$ *is decreasing.* Then for all $n \in S$, we have $x + (1/n)x_0 \geq \lambda_n a + \boldsymbol{s}_n(\boldsymbol{b})$, where $\boldsymbol{s}_n = (-\lambda_n \mathbf{1} - \boldsymbol{t} - (1/n)\boldsymbol{t}_0)^+$; thus $x + (1/n)x_0 \geq \lambda a + \boldsymbol{s}_n(\boldsymbol{b})$; it follows that $\boldsymbol{s}_n(\boldsymbol{b}) \leq x + x_0 - \lambda a$, thus $\boldsymbol{s} = (-\lambda \mathbf{1} - \boldsymbol{t})^+ = \bigvee_{n \in S} \boldsymbol{s}_n$ belongs to I and, by Lemma 1.4(ii) and the fact that E is Archimedean, $x \geq \lambda a + \boldsymbol{s}(\boldsymbol{b})$; thus we obtain $\lambda \in \Lambda(x, \boldsymbol{t})$, whence $(x, \boldsymbol{t}) \in P$ again.

The fact that j is a homomorphism of partially ordered vector spaces is obvious. If $x \in E$ and $(x, \mathbf{0}) \in P$, then, for all $\lambda \in \Lambda(x, \mathbf{0})$, we have $(-\lambda \mathbf{1})^+ \in I$, whence $\lambda \geq 0$ (again because $\mathbf{1} \notin I$); thus $x \geq 0$, and it follows that j is an embedding of partially ordered vector spaces. For all $(x, \boldsymbol{t}) \in E \times \ell^\infty$, we have $(\lambda a, -\boldsymbol{t}) \in P$ where $\lambda = \sup(\boldsymbol{t})$, whence $(x, \boldsymbol{t}) + F \leq$

$(x+\lambda a,\mathbf{0})+F\in j[E]$; thus j is cofinal. Since E is directed, it follows that $\widetilde{E}$ is also directed. ■

For all $(x,\boldsymbol{t})\in E\times\ell^\infty$, denote by $[x,\boldsymbol{t}]$ its projection on $\widetilde{E}$ (that is, $[x,\boldsymbol{t}]=(x,\boldsymbol{t})+F$).

2.5. Lemma. *The embedding j is monotone σ-complete.*

Proof. Let $(c_n)_{n\in\omega}$ be a bounded increasing sequence of elements of E, with supremum c. We prove that for all $(x,\boldsymbol{t})\in E\times\ell^\infty$, if $(\forall n\in\omega)([c_n,\mathbf{0}]\le[x,\boldsymbol{t}])$, then $[c,\mathbf{0}]\le[x,\boldsymbol{t}]$. For all $n\in\omega$, let λ_n be the infimum of $\Lambda(x-c_n,\boldsymbol{t})$. Note that $(\lambda_n)_{n\in\omega}$ is increasing and that for all n, $\lambda_n\le(x-c_n:a)\le(x-c_0:a)$, thus $\lambda=\bigvee_{n\in\omega}\lambda_n$ is a real number. Moreover, for all n, $\boldsymbol{s}_n=(-\lambda_n\mathbf{1}-\boldsymbol{t})^+$ belongs to I and $x-c_n\ge\lambda_n a+\boldsymbol{s}_n(\boldsymbol{b})\ge\lambda_n a+\boldsymbol{s}(\boldsymbol{b})$, where $\boldsymbol{s}=\bigwedge_{n\in\omega}\boldsymbol{s}_n=(-\lambda\mathbf{1}-\boldsymbol{t})^+$. This holds for all n, thus, by Lemmas 1.2 and 1.4(i), $x\ge c+\lambda a+\boldsymbol{s}(\boldsymbol{b})$; whence $\lambda\in\Lambda(x-c,\boldsymbol{t})$, so that $[c,\mathbf{0}]\le[x,\boldsymbol{t}]$. ■

Now, for all $n\in\omega$, put $x_n=[0,\boldsymbol{e}_n]$.

2.6. Lemma. *The space $\widetilde{E}$ satisfies the following statements*:

(i) $(\forall n\in\omega)(0<x_n<j(b_n))$.
(ii) $j(a)=\sum_{n\in\omega}x_n$.

Proof. (i) It is easy to verify that $0\in\Lambda(0,\boldsymbol{e}_n)$, $0\in\Lambda(b_n,-\boldsymbol{e}_n)$ and that both $(0,\boldsymbol{e}_n)$ and $(b,-\boldsymbol{e}_n)$ do not belong to F.

(ii) For all $n\in\omega$, put $\boldsymbol{f}_n=\sum_{k<n}\boldsymbol{e}_k$. Since $1\in\Lambda(a,-\boldsymbol{f}_n)$, we have $\sum_{k<n}x_k\le j(a)$. Thus, to conclude, it suffices to show that for every upper bound $[x,\boldsymbol{t}]$ of $\{\sum_{k<n}x_k:n\in\omega\}$, we have $j(a)\le[x,\boldsymbol{t}]$. For all $n\in\omega$, $\Lambda(x,\boldsymbol{t}-\boldsymbol{f}_n)$ is non-empty and thus it contains as an element its supremum λ_n; note that $(\lambda_n)_{n\in\omega}$ is decreasing and $\lambda_n\ge-\sup(\boldsymbol{t}-\boldsymbol{f}_n)\ge-\sup(\boldsymbol{t})$ for all n, thus $\lambda=\bigwedge_{n\in\omega}\lambda_n$ is a real number. For all n, $\boldsymbol{s}_n=(-\lambda_n\mathbf{1}-\boldsymbol{t}+\boldsymbol{f}_n)^+$ belongs to I and $x\ge\lambda_n a+\boldsymbol{s}_n(\boldsymbol{b})\ge\lambda a+\boldsymbol{s}_n(\boldsymbol{b})$, whence, by Lemma 1.4(ii), $\boldsymbol{s}=(-\lambda\mathbf{1}-\boldsymbol{t}+\mathbf{1})^+=\bigvee_{n\in\omega}\boldsymbol{s}_n$ belongs to I and $\boldsymbol{s}(\boldsymbol{b})=\bigvee_{n\in\omega}\boldsymbol{s}_n(\boldsymbol{b})$. Thus, by Lemma 1.4(ii), $x\ge\lambda a+\boldsymbol{s}(\boldsymbol{b})$; since $\boldsymbol{s}=(-(\lambda-1)\mathbf{1}-\boldsymbol{t})^+$, it follows that $\lambda-1\in\Lambda(x-a,\boldsymbol{t})$. Hence $j(a)\le[x,\boldsymbol{t}]$ and we are done. ■

In the sequel, we shall identify E and $j[E]$, and write $\widetilde{E}=E[a;\sum_{n\in\omega}b_n=\infty]$. We can now state our main theorem:

2.7. Lemma. *Every directed Archimedean partially ordered abelian group G admits a monotone σ-complete cofinal embedding into a cardinal space E such that $|E|=|G|^{\aleph_0}$.*

Proof. By Proposition 1.8, it suffices to prove the theorem when G is a (non-trivial) monotone σ-complete (or even Dedekind σ-complete) vector space. Thus for every directed Archimedean monotone σ-complete partially ordered vector space E, we shall first construct a certain extension E' of E.

Start with $E_0 = E$. Enumerate all ordered pairs $(a, (b_n)_{n\in\omega})$ such that $a, b_n \in E^+ \setminus \{0\}$ (for all n) and $\sum_{n\in\omega} b_n = \infty$ in a list $(a_\xi, (b_{\xi n})_{n\in\omega})_{0<\xi<\theta}$, where $\theta = |E|^{\aleph_0}$. Define inductively E_ξ $(\xi < \theta)$ and F_ξ $(0 < \xi \le \theta)$ by the following rule:

$$F_\xi = \bigcup_{\eta<\xi} E_\eta, \qquad E_\xi = \Big(F_\xi\Big[a_\xi; \sum_{n\in\omega} b_{\xi n} = \infty\Big]\Big)^\sigma.$$

Clearly, $|E_\xi| \le |E|^{\aleph_0}$. Since $(E_\xi)_{\xi<\theta}$ is strictly increasing for inclusion, $E' = F_\theta$ has cardinality exactly $|E|^{\aleph_0}$. By Lemmas 1.7, 2.4 and 2.5, for $\xi \le \eta < \theta$, the transition map $E_\xi \to E_\eta$ is monotone σ-complete cofinal, thus so is the natural embedding from E into E'. Since, by König's Theorem, θ has uncountable cofinality, and by construction (in particular, we use again the fact that the transition maps are monotone σ-complete), E' is monotone σ-complete.

Moreover, all the E_ξ^+ have the finite refinement property (because E_ξ is Dedekind σ-complete), thus E'^+ has the finite refinement property. Finally, if a and b_n $(n \in \omega)$ are elements of $E^+ \setminus \{0\}$ such that $\sum_{n\in\omega} b_n = \infty$, then there exists $\xi < \theta$ such that $a = a_\xi$ and $(b_n)_{n\in\omega} = (b_{\xi n})_{n\in\omega}$, thus, since the natural embedding from E_ξ into E' is monotone σ-complete, and by Lemma 2.6, there are elements x_n $(n \in \omega)$ of E_ξ (thus of E') such that for all n, $0 < x_n < b_n$ while $a = \sum_{n\in\omega} x_n$.

Finally, put $E^{(0)} = E$, $E^{(\alpha+1)} = (E^{(\alpha)})'$ for all $\alpha < \omega_1$, and for every countable limit ordinal λ, $E^{(\lambda)} = (\bigcup_{\beta<\lambda} E^{(\beta)})^\sigma$. Then $E^* = \bigcup_{\alpha<\omega_1} E^{(\alpha)}$ satisfies the required conditions. ∎

2.8. Problem. By Theorem 2.7, there are non-linearly ordered cardinal spaces of cardinality $2^{\aleph_0}$, thus they can be encoded by subsets of $\mathbb{R}$. What is the complexity of these subsets? Can they for example be taken in the Borel hierarchy? Note that in order to make the construction of non-trivial cardinal spaces as *effective* as possible, one should at least be able to avoid the consideration of the enumeration $(a_\xi, (b_{\xi n})_{n\in\omega})_{\xi<\theta}$ of the proof of Theorem 2.7, thus to carry out the construction of $\widetilde{E}$ (from 2.3 to 2.6) for all those families *simultaneously* (i.e., to consider the amalgamated sum of all the $E[a; \sum_{n\in\omega} b_n = \infty]$'s over E). One may also try to modify the construction of [8, Theorem IV.18.4].

2.9. Problem. If G is a cardinal space, is $|G|$ equal to $|G|^{\aleph_0}$? How many cardinal spaces are there of a given cardinality?

2.10. Problem. In [12], we construct "non-measurable" directed partially ordered vector spaces (over the rationals) with interpolation and order-unit, of cardinality $\aleph_2$; in particular, they cannot be isomorphic to $K_0(R)$

for any (von Neumann) regular ring R. Study the analogue of this for the more restrictive class of cardinal groups.

2.11. PROBLEM. Generalize the results of this paper to *monotone κ-complete* partially ordered abelian groups (which means for example that suprema of bounded increasing families indexed by an ordinal $< \kappa$ exist). Note, as we have remarked above, that all the possible "reasonable" versions of infinite refinement are not equivalent.

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