FUNDAMENTA
MATHEMATICAE
151 (1996)

Almost-Bieberbach groups with prime order holonomy

by

Karel Dekimpe* and **Wim Malfait** (Kortrijk)

Abstract. The main issue of this paper is an attempt to find a decomposition theorem for infra-nilmanifolds in the same spirit as a result of A. Vasquez for flat Riemannian manifolds. That is: we look for infra-nilmanifolds with prime order holonomy which can be obtained as a fiber space with a non-trivial nilmanifold as fiber and an infra-nilmanifold as its base.

In this perspective, we prove the following algebraic result: if E is an almost-Bieberbach group with prime order holonomy, then there is a normal subgroup Π of E contained in the Fitting subgroup of E such that E/Π is an almost-Bieberbach group either having a Fitting subgroup with center isomorphic to the infinite cyclic group, or having an underlying crystallographic group with torsion and a center coinciding with that of its Fitting subgroup.

1. Introduction. Let us start with some notational remarks used throughout this paper. We also recall the (algebraic and geometric) fundamentals of the theory of almost-Bieberbach groups and infra-nilmanifolds and review the decomposition theorem of A. Vasquez for flat Riemannian manifolds which we want to generalize.

1.1. *Notational remarks*. For a group G, $Z(G)$ denotes the center of G. For a subset X in G, $C_G X$ is the centralizer of X in G. The *isolator* (or *root set*) of a subgroup H of G is defined by

$$\sqrt[G]{H} = \{g \in G : g^k \in H \text{ for some } k \geq 1\}.$$

Recall that the lower central series of G is defined inductively by $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$ ($n \in \mathbb{N}_0$, the set of positive integers). It is well known that, for all $k \geq 1$, $G/\sqrt[G]{\gamma_k(G)}$ is torsion-free ([9, p. 473]). G is said to be *c-step nilpotent* (or *nilpotent of class c*) if and only if $\gamma_c(G) \neq \{1\}$ and $\gamma_{c+1}(G) = \{1\}$. It then follows that $\gamma_c(G) \subseteq Z(G)$. If, moreover, $Z(G)$ is

1991 *Mathematics Subject Classification*: 20F17, 20F34.

* Postdoctoral Fellow of the Belgian National Fund for Scientific Research (N.F.W.O.).

torsion-free, then the inner automorphism group $\mathrm{Inn}(G) \cong G/Z(G)$ of G is also torsion-free ([9, p. 470]).

1.2. *Almost-crystallographic groups*: *an overview*. If L is a connected and simply connected, nilpotent Lie group and C a maximal compact subgroup of $\mathrm{Aut}(L)$, then an *almost-crystallographic group* is defined as a discrete and uniform subgroup E of the semi-direct product $L \rtimes C \subset L \rtimes \mathrm{Aut}(L) = \mathrm{Aff}(L)$. The translation subgroup $N = E \cap L$ is a lattice of L and it is the unique normal subgroup of E which is maximal nilpotent. The quotient $F = E/N$, called the *holonomy group*, is a finite group acting faithfully on L. If E is torsion-free, then E is an *almost-Bieberbach group* and the corresponding compact orbit space $M = E\backslash L$ is an *infra-nilmanifold*. Almost-Bieberbach groups are exactly the fundamental groups of infra-nilmanifolds. Clearly, this set-up is a straightforward generalization of the classical theory of Bieberbach groups and flat Riemannian manifolds (i.e. $L = \mathbb{R}^n$).

As an abstract group, a group E is almost-crystallographic if and only if E contains a finitely generated, torsion-free, nilpotent normal subgroup N of finite index, which is maximal nilpotent in E ([7]). A short exact sequence of groups $1 \to N \to E \to F \to 1$, with E finitely generated, N torsion-free nilpotent and F finite, such that N is maximal nilpotent in E, is called *essential* (see [6]). It follows then that N is the Fitting subgroup $\mathrm{Fitt}(E)$ of the almost-crystallographic group E ([10]). Its Hirsch length $h(N)$ is referred to as the *dimension* of E. In this case, the Lie group L, as introduced above, is the Mal'tsev completion of N. It is also well known that each finitely generated, torsion-free, virtually nilpotent group is an almost-Bieberbach group.

The abstract kernel $F \to \mathrm{Out}(N)$ induced by an essential extension $1 \to N \to E \to F \to 1$ is faithful ([4]), or equivalently $C_E(N) = Z(N)$, and hence, $Z(E)$ coincides with $Z(N)^F$, the subgroup of all elements of $Z(N)$ fixed under the action of F on $Z(N)$. Another basic fact concerning essential extensions is the following: if we factor out the characteristic subgroup $\sqrt[N]{\gamma_2(N)}$, then the resulting extension (of quotients) again is essential ([4]). We write $\mathrm{Cr}(E)$ for the quotient $E/\sqrt[N]{\gamma_2(N)}$ and call it the *underlying crystallographic group* of E.

1.3. *Flat toral extensions*. In [12], A. Vasquez introduces the notion of flat toral extensions. Recall that a *flat toral extension* of a k-dimensional flat Riemannian manifold $M = E\backslash\mathbb{R}^k$ is any Riemannian manifold isometric to $E\backslash(\mathbb{R}^k \times T)$, where T is a flat torus on which E acts via isometries. He proves that, under a condition stated in terms of the holonomy group of the manifold, each flat Riemannian manifold arises as a flat toral extension, i.e. as a fiber bundle over another flat Riemannian manifold with a flat torus as fiber:

THEOREM 1.1 [12, Theorem 2.3]. *Associated with each finite group F there is a positive integer $n(F)$ such that: if M is a flat Riemannian manifold with holonomy group F, then M is a flat toral extension of some flat Riemannian manifold of dimension $\leq n(F)$.*

Vasquez had no information about this function $n(F)$, except when F is of prime order, then $n(F)$ can be taken to be 1. At several places in the literature, attempts to estimate $n(F)$ are presented (see [11], [13], ...).

Here, we investigate to what extent the particular result for prime order holonomy can be generalized towards infra-nilmanifolds.

2. Almost-Bieberbach groups with prime order holonomy. Before we can formulate our main theorem, we need some group theoretic results.

LEMMA 2.1. *Let F be a finite group and $\mathbb{Z}^k$ an F-module. If $(\mathbb{Z}^k)^F$ is a proper subgroup of $\mathbb{Z}^k$, then there is a non-trivial F-submodule M of $\mathbb{Z}^k$ such that $(\mathbb{Z}^k)^F \cap M = \{1\}$ and $\mathbb{Z}^k/M$ is torsion-free.*

Proof. Assume that the F-module structure of $\mathbb{Z}^k$ is given by $\varphi : F \to \mathrm{Aut}(\mathbb{Z}^k)$. Consider $\mathbb{Z}^k$ as an F-submodule of $\mathbb{Q}^k$, where the module structure is given by $F \xrightarrow{\varphi} \mathrm{Aut}(\mathbb{Z}^k) \hookrightarrow \mathrm{Aut}(\mathbb{Q}^k)$. Consider the associated semi-direct product $\mathbb{Q}^k \rtimes F$. This group fits into a short exact sequence

$$1 \to (\mathbb{Q}^k)^F \to \mathbb{Q}^k \rtimes F \to (\mathbb{Q}^k \rtimes F)/(\mathbb{Q}^k)^F \to 1 \tag{1}$$

and $(\mathbb{Q}^k \rtimes F)/(\mathbb{Q}^k)^F \cong (\mathbb{Q}^k/(\mathbb{Q}^k)^F) \rtimes F \cong \mathbb{Q}^l \rtimes F$, for some $l \in \mathbb{N}_0$. Note that the restricted short exact sequence $1 \to (\mathbb{Q}^k)^F \to \mathbb{Q}^k \to \mathbb{Q}^l \to 1$ splits. So, if $\langle f \rangle \in H^2(\mathbb{Q}^l \rtimes F, (\mathbb{Q}^k)^F)$ denotes the cohomology class corresponding to the extension (1), then res : $H^2(\mathbb{Q}^l \rtimes F, (\mathbb{Q}^k)^F) \to H^2(\mathbb{Q}^l, (\mathbb{Q}^k)^F)$ sends $\langle f \rangle \mapsto \langle 0 \rangle$. But since $(\mathbb{Q}^k)^F$ is divisible and $\mathbb{Q}^l$ is of finite index in $\mathbb{Q}^l \rtimes F$, res is injective (use [1, p. 118]) or $\langle f \rangle = \langle 0 \rangle$. Hence, there exists a splitting section (i.e. a homomorphism) $s : \mathbb{Q}^l \rtimes F \to \mathbb{Q}^k \rtimes F$ of (1). Write $\widetilde{M}$ for $s(\mathbb{Q}^l)$, which is an F-submodule of $\mathbb{Q}^k$ (i.e. $\widetilde{M}$ is normal in $\mathbb{Q}^k \rtimes F$). Then

$$M = \sqrt[\mathbb{Z}^k]{\widetilde{M} \cap \mathbb{Z}^k} = \widetilde{M} \cap \mathbb{Z}^k$$

is a non-trivial F-submodule of $\mathbb{Z}^k$ such that $(\mathbb{Z}^k)^F \cap M$ is trivial and $\mathbb{Z}^k/M$ is torsion-free. ∎

The following well known lemma is the start of an induction process leading to our main theorem.

LEMMA 2.2. *Let $1 \to N \cong \mathbb{Z}^k \to E \cong \mathbb{Z}^k \to \mathbb{Z}_p \to 1$ be a short exact sequence of groups where $k \in \mathbb{N}_0$ and p is prime. Then there is a subgroup Π of E contained in N such that $E/\Pi \cong \mathbb{Z}$.*

This elementary result can now be generalized towards the nilpotent case.

PROPOSITION 2.3. *Let* $1 \to N \to E \to \mathbb{Z}_p \to 1$ *be a torsion-free, finitely generated, nilpotent extension of a group* N *by a finite cyclic group of prime order* p. *Then there is a normal subgroup* Π *of* E *contained in* N *such that* E/Π *is a torsion-free group with center isomorphic to* $\mathbb{Z}$.

Proof. We proceed by induction on the Hirsch length $h(E)$ of E. If $h(E) = 1$, then $E \cong \mathbb{Z}$ and the proposition is trivially true. Assume $h(E) > 1$ and $Z(E) \cong Z(N) \cong \mathbb{Z}^k$, where $k > 1$ (note that $Z(N)$ is a subgroup of $Z(E)$ of index $\le p$). Divide out $Z(N)$ to obtain the following extension of quotients:

$$1 \to N/Z(N) \to E/Z(N) \to \mathbb{Z}_p \to 1. \tag{2}$$

We distinguish two possibilities:

• $Z(N) = Z(E)$: Since $E/Z(E)$ is torsion-free and because the Hirsch length of $E/Z(E)$ is smaller than $h(E)$, by induction on (2), there is a normal subgroup $\Pi/Z(E)$ of $E/Z(E)$ (or Π normal in E) such that $\Pi \subseteq N$, $(E/Z(E))/(\Pi/Z(E)) \cong E/\Pi$ is torsion-free and $Z(E/\Pi) \cong \mathbb{Z}$.

• $Z(N) \ne Z(E)$: Since p is prime, $1 \to Z(N) \to Z(E) \to \mathbb{Z}_p \to 1$ is exact. Hence, there is a subgroup Π of $Z(E)$ contained in $Z(N)$ such that $Z(E)/\Pi \cong \mathbb{Z}$ (Lemma 2.2). Observe that Π cannot be trivial. Then

$$1 \to Z(E)/\Pi \to E/\Pi \to E/Z(E) \to 1$$

is exact and hence, because $E/Z(E)$ is torsion-free, E/Π is a finitely generated, torsion-free, nilpotent group with smaller Hirsch length than E. Now, the desired result follows immediately by induction on $1 \to N/\Pi \to E/\Pi \to \mathbb{Z}_p \to 1$. ∎

Let us now return to our main point of interest: the almost-Bieberbach groups. We recall that, in the context of Bieberbach groups, A. Vasquez proved

PROPOSITION 2.4 [12, Theorem 3.6, Remark 2.5]. *If* E *is a Bieberbach group with prime order holonomy, then there is a normal subgroup* Π *of* E *contained in* $\mathrm{Fitt}(E)$ *such that* $E/\Pi \cong \mathbb{Z}$.

As mentioned before, it is our purpose to generalize this result towards almost-Bieberbach groups with prime order holonomy.

For a Bieberbach group E with translation subgroup N and holonomy group $F \ne 1$, $Z(E) = N^F$ is always a proper subgroup of N since the F-module structure on N is injective. Unfortunately, in general, if E is an almost-Bieberbach group, the injectivity of the induced abstract kernel does not imply that $Z(E)$ is a proper subgroup of $Z(\mathrm{Fitt}(E))$, as the following elementary example shows:

EXAMPLE 2.5. The group E presented as

$$E : \langle a, b, c, \alpha : [b, a] = c^2,\ [c, a] = 1,\ [c, b] = 1,\\ \alpha a = a^{-1}\alpha,\ \alpha b = b^{-1}\alpha,\ \alpha c = c\alpha,\ \alpha^2 = c\rangle$$

is an almost-Bieberbach group fitting into an essential extension $1 \to N \to E \to \mathbb{Z}_2 \to 1$ where

$$N : \langle a, b, c : [b, a] = c^2,\ [c, a] = 1,\ [c, b] = 1\rangle.$$

Obviously, $Z(E) = Z(N)$, the subgroup generated by c.

As the reader will notice, this causes some difficulties to formulate a result similar to Proposition 2.4 for almost-Bieberbach groups. Anyhow, we can still prove the following generalization:

THEOREM 2.6. *Let E be an almost-Bieberbach group with prime order holonomy. Then there is a normal subgroup Π (which might be trivial) of E contained in* $\mathrm{Fitt}(E)$ *such that E/Π is again an almost-Bieberbach group, but now with the additional property that*

- $Z(\mathrm{Fitt}(E/\Pi)) \cong \mathbb{Z}$, *or*
- $\mathrm{Cr}(E/\Pi)$ *has torsion and* $Z(\mathrm{Fitt}(E/\Pi)) = Z(E/\Pi)$.

Proof. The almost-Bieberbach group E fits into an essential extension

$$1 \to N = \mathrm{Fitt}(E) \to E \xrightarrow{j} F \cong \mathbb{Z}_p \to 1 \qquad (p \text{ prime}).$$

We proceed by induction on the dimension of E (= the Hirsch number $h(N)$ of N). If $h(N) = 1$, then $N \cong \mathbb{Z}$ and the theorem holds trivially. Assume $h(N) > 1$, or even stronger $h(Z(N)) > 1$ (otherwise we can take $\Pi = \{1\}$).

If the underlying crystallographic group $\mathrm{Cr}(E) = E/\sqrt[N]{\gamma_2(N)}$ of E is torsion-free (or $\mathrm{Cr}(E)$ is a Bieberbach group), then Proposition 2.4 proves the existence of such a normal subgroup Π. On the other hand, if $\mathrm{Cr}(E)$ has torsion, then we assume that $Z(E)(= Z(N)^F)$ is a proper subgroup of $Z(N)$ in order to deal with a non-trivial situation.

Then there is a non-trivial F-submodule M of $Z(N)$ (or M is normal in E) such that $Z(E) \cap M = \{1\}$ and $Z(N)/M$ is torsion-free (Lemma 2.1). We factor out by M and obtain a short exact sequence

$$1 \to N/M \to E/M \to F \to 1. \tag{3}$$

Since $N/Z(N)$ is torsion-free and $1 \to Z(N)/M \to N/M \to N/Z(N) \to 1$ is exact, it follows that N/M has no torsion. Now, we claim that E/M is also torsion-free (and hence an almost-Bieberbach group of lower dimension than E).

Suppose that E/M is not torsion-free and let $e \neq 1$ be an element of E such that $e^k \in M$, for some $k > 1$. Since N/M is torsion-free, e is not an element of N. Then its image $x = j(e)$ can be seen as a generator of $F \cong \mathbb{Z}_p$. Fix a section $s : F \to E$ ($j \circ s = 1$) such that $s(x) = e$. Then e^k is of course fixed under the (induced) action of F on $Z(N)$. It follows that e^k (which already belongs to M) is also an element of the center of E and hence $e^k = 1$. Then it follows immediately that e should be equal to 1 since E is torsion-free, which contradicts our assumption.

Since p is prime, there are only two possibilities for the Fitting subgroup of the almost-Bieberbach group E/M. If $\mathrm{Fitt}(E/M) = N/M$, then the desired result follows immediately by induction on (3). In case $\mathrm{Fitt}(E/M) = E/M$, E/M is a finitely generated, torsion-free, nilpotent group. We use Proposition 2.3 (applied to extension (3)) to finish the proof. ■

This theorem indicates a class of almost-Bieberbach groups E with prime order holonomy having a normal subgroup Π contained in $N = \mathrm{Fitt}(E)$ such that E/Π is an almost-Bieberbach group of lower dimension; namely those E where the Hirsch length of the center of N is greater than 1 and

- the underlying crystallographic group of E is torsion-free (Example 3.1), or
- the center of E is a proper subgroup of the center of N (Examples 3.2, 3.3).

3. Some examples. We illustrate our theorem for some 4-dimensional almost-Bieberbach groups with 2-step nilpotent Fitting subgroup. A good algebraic source of examples of almost-crystallographic groups is found in [3] (all isomorphism types in dimension 3) and in [2] (dimension ≤ 4).

Example 3.1. We use almost-Bieberbach groups of type **4** in [2]. Consider the group E presented as:

$$\begin{aligned} E : \langle a, b, c, d, \alpha : {} & [b,a] = 1,\ [c,a] = d^k,\ [c,b] = 1, \\ & [d,a] = 1,\ [d,b] = 1,\ [d,c] = 1, \\ & \alpha a = a^{-1}\alpha,\ \alpha b = b\alpha,\ \alpha c = c^{-1}\alpha, \\ & \alpha d = d\alpha,\ \alpha^2 = b\rangle, \qquad k \in \mathbb{Z}_0. \end{aligned}$$

E fits into an essential extension $1 \to N \to E \to F \cong \mathbb{Z}_2 \to 1$ where

$$\begin{aligned} N : \langle a, b, c, d : {} & [b,a] = 1,\ [c,a] = d^k,\ [c,b] = 1, \\ & [d,a] = 1,\ [d,b] = 1,\ [d,c] = 1\rangle. \end{aligned}$$

Remark that $Z(E) = Z(N)$, the subgroup generated by b and d.

It is clear that $\sqrt[N]{\gamma_2(N)}$ is the subgroup generated by d and that the underlying crystallographic group of E is

$$\begin{aligned}\mathrm{Cr}(E) : \langle a, b, c, \alpha : {}& [b, a] = 1,\ [c, a] = 1,\ [c, b] = 1,\\ & \alpha a = a^{-1}\alpha,\ \alpha b = b\alpha,\ \alpha c = c^{-1}\alpha,\ \alpha^2 = b\rangle,\end{aligned}$$

a 3-dimensional Bieberbach group. As stated in Proposition 2.4, there is a normal subgroup Π of $\mathrm{Cr}(E)$ contained in the translation subgroup of $\mathrm{Cr}(E)$ such that $\mathrm{Cr}(E)/\Pi \cong \mathbb{Z}$; namely take Π to be the subgroup generated by a and c.

Example 3.2. We work with the almost-Bieberbach groups E of type **2** in [2]:

$$\begin{aligned}E : \langle a, b, c, d, \alpha : {}& [b, a] = d^{2k},\ [c, a] = 1,\ [c, b] = 1,\\ & [d, a] = 1,\ [d, b] = 1,\ [d, c] = 1,\\ & \alpha a = a^{-1}\alpha,\ \alpha b = b^{-1}\alpha,\ \alpha c = c^{-1}\alpha,\\ & \alpha d = d\alpha,\ \alpha^2 = d\rangle, \qquad k \in \mathbb{Z}_0.\end{aligned}$$

The group E fits into an essential extension $1 \to N \to E \to F \cong \mathbb{Z}_2 \to 1$ where

$$\begin{aligned}N : \langle a, b, c, d : {}& [b, a] = d^{2k},\ [c, a] = 1,\ [c, b] = 1,\\ & [d, a] = 1,\ [d, b] = 1,\ [d, c] = 1\rangle.\end{aligned}$$

Remark that $\sqrt[N]{\gamma_2(N)}$ is the subgroup generated by d and that

$$\begin{aligned}\mathrm{Cr}(E) : \langle a, b, c, \alpha : {}& [b, a] = 1,\ [c, a] = 1,\ [c, b] = 1,\\ & \alpha a = a^{-1}\alpha,\ \alpha b = b^{-1}\alpha,\ \alpha c = c^{-1}\alpha,\ \alpha^2 = 1\rangle,\end{aligned}$$

a 3-dimensional crystallographic group with torsion.

Clearly, $Z(E)$ is the subgroup generated by d while $Z(N)$ is generated by c and d. Note that the subgroup generated by c is a suitable F-module M. If we factor out, we obtain

$$\begin{aligned}E/M : \langle a, b, d, \alpha : {}& [b, a] = d^{2k},\ [d, a] = 1,\ [d, b] = 1,\\ & \alpha a = a^{-1}\alpha,\ \alpha b = b^{-1}\alpha,\ \alpha d = d\alpha,\ \alpha^2 = d\rangle,\end{aligned}$$

a torsion-free 3-dimensional almost-Bieberbach group of type **2** in [3]. Its Fitting subgroup is N/M, the underlying crystallographic group of E/Π has torsion and $Z(E/M) = Z(N/M) \cong \mathbb{Z}$.

Example 3.3. Now, take the following almost-Bieberbach groups (members of type **6** in [2]):

$$\begin{aligned}E : \langle a, b, c, d, \alpha : {}& [b, a] = 1,\ [c, a] = d^k,\ [c, b] = 1,\ [d, a] = 1,\\ & [d, b] = 1,\ [d, c] = 1,\ \alpha a = a\alpha,\ \alpha b = b^{-1}\alpha,\\ & \alpha c = c\alpha,\ \alpha d = d\alpha,\ \alpha^2 = d\rangle, \qquad k \in \mathbb{Z}_0.\end{aligned}$$

The group E fits into an essential extension $1 \to N \to E \to F \cong \mathbb{Z}_2 \to 1$ where

$$N : \langle a, b, c, d : [b, a] = 1,\ [c, a] = d^k,\ [c, b] = 1,\\ [d, a] = 1,\ [d, b] = 1,\ [d, c] = 1\rangle.$$

Clearly, $\sqrt[N]{\gamma_2(N)}$ is the subgroup generated by d and hence, the underlying crystallographic group of E has torsion.

Note that $Z(E)$ is the subgroup generated by d, $Z(N)$ is generated by b and d and the submodule generated by b is a suitable F-module M. If we factor out, we obtain

$$E/M : \langle a, c, \alpha : [c, a] = \alpha^{2k},\ [\alpha, a] = 1,\ [\alpha, c] = 1\rangle,$$

a torsion-free, 2-step nilpotent group with center isomorphic to $\mathbb{Z}$.

4. Infra-nilmanifolds as fiber spaces with a nilmanifold as fiber. A short exact sequence $1 \to \Pi \to N \to N/\Pi \to 1$ of finitely generated torsion-free nilpotent groups induces a short exact sequence of the corresponding Mal'tsev completions

$$1 \to L(\Pi) \to L(N) \to L(N)/L(\Pi) = L(N/\Pi) \to 1.$$

After choosing a smooth section $s : L(N/\Pi) \to L(N)$, we may identify $L(N)$ with $L(\Pi) \times L(N/\Pi)$ (as smooth manifolds, not as groups!). Indeed, any element $l \in L(N)$ can be written uniquely as a product $l = x\,s(y)$ with $x \in L(N)$ and $y \in L(N/\Pi)$, so that we can identify $l \in L(N)$ with $(x, y) \in L(\Pi) \times L(N/\Pi)$.

Now, let us return to the situation of the previous section and consider a short exact sequence

$$1 \to \Pi \to E \to \overline{E} \to 1$$

of almost-Bieberbach groups with $\Pi \subseteq N = \mathrm{Fitt}(E)$. As N/Π is a subgroup of finite index in $\overline{N} = \mathrm{Fitt}(\overline{E})$, we have $L(\overline{N}) = L(N/\Pi)$.

The affine action of E on $L(N) = L(\Pi) \times L(\overline{N})$ has the following properties:

1. $\forall e \in E$, $\forall (x, y) \in L(\Pi) \times L(\overline{N}) : {}^e(x, y) = (f_e(x, y), h_e(y))$, where f_e is a smooth map from $L(\Pi) \times L(\overline{N})$ to $L(\Pi)$ and h_e is a diffeomorphism of $L(\overline{N})$.

2. $\forall e \in \Pi$, $\forall (x, y) \in L(\Pi) \times L(\overline{N}) : {}^e(x, y) = (ex, y)$, where ex is just left translation in $L(\Pi)$.

3. $\forall e \in E : h_e \in \mathrm{Aff}(L(\overline{N}))$.

From these observations it follows that there is a commutative diagram

$$
\begin{array}{ccccc}
L(N) = L(\Pi) \times L(\overline{N}) & & \xrightarrow{\quad q_1 \quad} & & L(\overline{N}) \\
\Big\downarrow p_1 & \searrow^{p_2} & & & \Big\downarrow p_4 \\
& & \Pi\backslash L(\Pi) \times L(\overline{N}) & & \\
& \swarrow^{p_3} & & & \\
E\backslash L(N) = \overline{E}\backslash(\Pi\backslash L(\Pi) \times L(\overline{N})) & & \xrightarrow[\quad q \quad]{} & & \overline{E}\backslash L(\overline{N})
\end{array}
$$

where the p_i $(i = 1, 2, 3$ or $4)$ are covering projections, q_1 is projection onto the second component and q is the map making the diagram commute (see [5] and [7]). In fact, q is a fibering projection, so we have

CONCLUSION 4.1. *Every infra-nilmanifold of prime order holonomy has a fibration structure whose fiber is a nilmanifold and base is an infra-nilmanifold with fundamental group $\overline{E}$ such that $Z(\mathrm{Fitt}(\overline{E})) \cong \mathbb{Z}$ or such that $\mathrm{Cr}(\overline{E})$ has torsion and $Z(\mathrm{Fitt}(\overline{E})) = Z(\overline{E})$.*

Indeed, the above observations and Theorem 2.6 show that an infra-nilmanifold $E\backslash L(N)$, with prime order holonomy, fibers over an infra-nilmanifold $\overline{E}\backslash L(\overline{N})$ with fiber the nilmanifold $\Pi\backslash L(\Pi)$. Remark that in some cases the fiber might be trivial.

In order to illustrate the possible situations, we reconsider the examples of the previous section. First of all, we remark that all groups E considered in these examples are fundamental groups of 4-dimensional infra-nilmanifolds, each obtained as a quotient of the Lie group $L = H \times \mathbb{R}$, where H denotes the 3-dimensional Heisenberg Lie group.

1. In Example 3.1, we factored out (in two steps) by the group generated by a, c and d. This shows that the infra-nilmanifold with fundamental group E can be considered as a fiber space with fiber a 3-dimensional nilmanifold (covered by H) and base space the circle. The fact that we used two steps to obtain this factorization illustrates that the fiber (nilmanifold) itself can be seen as a fiber space with fiber a circle and base T^2, the two-dimensional torus.

2. In Example 3.2, we split the infra-nilmanifold as a fiber space with fiber a circle and base space a 3-dimensional infra-nilmanifold with holonomy $\mathbb{Z}_2$.

3. Finally, in Example 3.3, we again obtain a circle as fiber but now the base space is a 3-dimensional nilmanifold (covered by H).

References

[1] A. Babakhanian, *Cohomological Methods in Group Theory*, Pure and Appl. Math. 11, Marcel Dekker, New York, 1972.
[2] K. Dekimpe, *Almost Bieberbach groups*: *cohomology, construction and classification*, Doctoral Thesis, K.U. Leuven, 1993.
[3] K. Dekimpe, P. Igodt, S. Kim and K. B. Lee, *Affine structures for closed 3-dimensional manifolds with NIL-geometry*, Quart. J. Math. Oxford (2) 46 (1995), 141–167.
[4] K. Dekimpe, P. Igodt and W. Malfait, *On the Fitting subgroup of almost crystallographic groups*, Tijdschrift van het Belgisch Wiskundig Genootschap, 1993, B 1, 35–47.
[5] Y. Kamishima, K. B. Lee and F. Raymond, *The Seifert construction and its applications to infra-nilmanifolds*, Quart. J. Math. Oxford (2) 34 (1983), 433–452.
[6] K. B. Lee, *There are only finitely many infra-nilmanifolds under each nilmanifold*, ibid. (2) 39 (1988), 61–66.
[7] K. B. Lee and F. Raymond, *Geometric realization of group extensions by the Seifert construction*, in: Contemp. Math. 33, Amer. Math. Soc., 1984, 353–411.
[8] W. Malfait, *Symmetry of infra-nilmanifolds*: *an algebraic approach*, Doctoral Thesis, K.U. Leuven, 1994.
[9] D. S. Passman, *The Algebraic Structure of Group Rings*, Pure and Appl. Math., Wiley, New York, 1977.
[10] D. Segal, *Polycyclic Groups*, Cambridge University Press, 1983.
[11] A. Szczepański, *Decomposition of flat manifolds*, preprint, 1995.
[12] A. T. Vasquez, *Flat Riemannian manifolds*, J. Differential Geom. 4 (1970), 367–382.
[13] S. T. Yau, *Compact flat Riemannian manifolds*, ibid. 6 (1972), 395–402.

Katholieke Universiteit Leuven
Campus Kortrijk
Universitaire Campus
B-8500 Kortrijk, Belgium
E-mail: Karel.Dekimpe@kulak.ac.be
Wim.Malfait@kulak.ac.be

Received 21 November 1995;
in revised form 26 June 1996