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The Zahorski theorem is valid in Gevrey classes

by

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Abstract. Let $\{\Omega, F, G\}$ be a partition of $\mathbb{R}^n$ such that Ω is open, F is F_σ and of the first category, and G is G_δ. We prove that, for every $\gamma \in\,]1,\infty[$, there is an element of the Gevrey class Γ_γ which is analytic on Ω, has F as its set of defect points and has G as its set of divergence points.

1. Introduction. Let f be a real C_∞-function on $\mathbb{R}^n$. The set where f is analytic is of course an open subset of $\mathbb{R}^n$; denote it by Ω_f. It is clear that x belongs to Ω_f if and only if the radius of convergence $\varrho_f(x)$ of the Taylor series of f at x is strictly positive and this series represents f on some neighbourhood of x.

As the set T_f of $x \in \mathbb{R}^n$ such that $\varrho_f(x) > 0$ is easily seen to be an F_σ-set, the set $F_f = T_f \setminus \Omega_f$ is also F_σ and its elements x are characterized by the fact that $\varrho_f(x) > 0$ and that the Taylor series of f at x represents f on no neighbourhood of x. By use of a lemma of R. P. Boas ([1], p. 234), one easily sees that F_f is also a first category set (cf. [3] or [4]).

Finally, one may consider the set $G_f = \mathbb{R}^n \setminus T_f$, a G_δ-set given by $\varrho_f(x) = 0$.

It is clear that $\{\Omega_f, G_f, F_f\}$ is a partition of $\mathbb{R}^n$.

The Zahorski theorem (cf. [6]) asserts conversely that for every partition $\{\Omega, F, G\}$ of $[0,1]$, where Ω is an open subset of $[0,1]$, F is a first category F_σ-subset of $[0,1]$ and G is a G_δ-subset of $[0,1]$, there is a real C_∞-function f on $[0,1]$ such that $\Omega = \Omega_f$, $F = F_f$ and $G = G_f$. In [3], H. Salzmann and K. Zeller have provided a shorter proof of the Zahorski theorem and in [4], J. Siciak has extended this result to $\mathbb{R}^n$.

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The purpose of this paper is to prove that, for every $\gamma \in]1,\infty[$, the Zahorski theorem has a solution f belonging to the Gevrey class Γ_γ.

Let us recall that, for an open subset Ω of $\mathbb{R}^n$ and for $\gamma \in]1,\infty[$, the *Gevrey class* $\Gamma_\gamma(\Omega)$ is the set of $f \in \mathrm{C}_\infty(\Omega)$ for which there are constants $a, b > 0$ such that

$$\|\mathrm{D}^\alpha f\|_\Omega \le ab^{|\alpha|}(|\alpha|!)^\gamma, \quad \forall \alpha \in \mathbb{N}_0^n.$$

If $\Omega = \mathbb{R}^n$, we simply write Γ_γ instead of $\Gamma_\gamma(\mathbb{R}^n)$.

It is known that

(a) a function $f \in \mathrm{C}_\infty(\Omega)$ belongs to $\Gamma_\gamma(\Omega)$ if and only if there are constants $c, d > 0$ such that

$$\|\mathrm{D}^\alpha f\|_\Omega \le cd^{|\alpha|}|\alpha|^{\gamma|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n.$$

(b) the Denjoy–Carleman–Mandelbrojt result (cf. [2]) states that, for every closed ball b of $\mathbb{R}^n$ and every $\gamma \in]1,\infty[$, there is a nonzero function $f \in \Gamma_\gamma$ with support contained in b.

In order to get an efficient way to state the results, for a real C_∞-function f on $\mathbb{R}^n$, let us call the elements of Ω_f (resp. F_f; G_f) the *analytic points* (resp. the *defect points*; the *divergence points*) of f.

The purpose of this article is to prove the following result.

Theorem 1.1. *For every partition $\{\Omega, F, G\}$ of $\mathbb{R}^n$, where Ω (resp. F; G) is an open set (resp. a first category F_σ-set; a G_δ-set) and every $\gamma \in]1,\infty[$, there is an element of Γ_γ having Ω (resp. F; G) as its set of analytic points (resp. defect points; divergence points).*

Remark. It is a direct matter to check that the Zahorski theorem extends to the case when $\mathbb{R}^n$ is replaced by an open or a closed subset of $\mathbb{R}^n$.

2. An auxiliary result. We begin with the following easy result, where, as usual, $\mathrm{D}_r(\Omega)$ denotes the space of C_r-functions on the open subset Ω of $\mathbb{R}^n$ which have a compact support contained in Ω.

Proposition 2.1. *Let Ω be a nonvoid open subset of $\mathbb{R}^n$. For every $f \in \mathrm{C}_\infty(\Omega)$, $g \in \mathrm{D}_\infty(\Omega)$ and $\lambda > 0$, it is well known that*

$$h(x) = \pi^{-n/2}\lambda^n \int_\Omega f(y)g(y)e^{-\lambda^2|x-y|^2}\,dy$$

belongs to $\mathrm{C}_\infty(\mathbb{R}^n)$. If moreover $a_1, a_2, b_1, b_2 > 0$ and $\zeta > 1$ are such that

$$\|\mathrm{D}^\alpha f\|_\Omega \le a_1 b_1^{|\alpha|}(|\alpha|!)^\zeta \quad \text{and} \quad \|\mathrm{D}^\alpha g\|_\Omega \le a_2 b_2^{|\alpha|}(|\alpha|!)^\zeta, \quad \forall \alpha \in \mathbb{N}_0^n,$$

then

$$\|\mathrm{D}^\alpha h\|_{\mathbb{R}^n} \le a_1 a_2 (b_1 + b_2)^{|\alpha|}(|\alpha|!)^\zeta, \quad \forall \alpha \in \mathbb{N}_0^n.$$

P r o o f. As g has a compact support contained in Ω, up to extension by 0 on $\mathbb{R}^n \setminus \Omega$, we may suppose that the product fg is a D_∞-function on $\mathbb{R}^n$ with compact support contained in Ω. So in the definition of $h(x)$, we may consider that we integrate on $\mathbb{R}^n$. Therefore integrating by parts $|\alpha|$ times gives

$$\mathrm{D}^\alpha h(x) = \pi^{-n/2}\lambda^n \int_{\mathbb{R}^n} \mathrm{D}^\alpha (fg)(y) e^{-\lambda^2|x-y|^2}\, dy,$$

hence

$$\begin{aligned}|\mathrm{D}^\alpha h(x)| &\le \pi^{-n/2}\lambda^n \int_{\mathbb{R}^n} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} |\mathrm{D}^\beta f(y) \mathrm{D}^{\alpha-\beta} g(y)| e^{-\lambda^2|x-y|^2}\, dy \\ &\le \sum_{\beta\le\alpha} \binom{\alpha}{\beta} a_1 a_2 b_1^{|\beta|} b_2^{|\alpha-\beta|} (|\beta|!)^\zeta (|\alpha-\beta|!)^\zeta \\ &\le a_1 a_2 (b_1+b_2)^{|\alpha|} (|\alpha|!)^\zeta. \ \blacksquare\end{aligned}$$

3. Special compact covers of open subsets of $\mathbb{R}^n$. In the following results, we are going to use systematically the following construction and notations. Let Ω be a nonvoid open subset of $\mathbb{R}^n$. Then we set

$$\Omega_m = (\mathbb{R}^n \setminus \Omega) \cup \{x \in \mathbb{R}^n : |x| \ge m\sqrt{n}\}, \qquad \forall m \in \mathbb{N},$$

and denote by μ the first positive integer m for which there is at least one cube Q of the type

$$\prod_{j=1}^n [2^{-m}a_j, 2^{-m}(a_j+1)] \quad \text{with } a \in \mathbb{Z}^n$$

contained in Ω and such that $\mathrm{d}(Q, \Omega_m) > 2^{-m}\sqrt{n}$. Let $Q_{1,1}, \ldots, Q_{1,p_1}$ be these cubes (of course, we have $p_1 \in \mathbb{N}$) and set

$$H_1 = \bigcup_{h=1}^{p_1} Q_{1,h}.$$

Now we proceed by recursion. If the sets $H_1, \ldots, H_r$ are obtained, we let $Q_{r+1,1}, \ldots, Q_{r+1,p_{r+1}}$ denote all the cubes of the type

$$\prod_{j=1}^n [2^{-\mu-r}a_j, 2^{-\mu-r}(a_j+1)] \quad \text{with } a \in \mathbb{Z}^n$$

contained in Ω, disjoint from the interior of $H_1 \cup \ldots \cup H_r$ and such that $\mathrm{d}(Q, \Omega_{\mu+r}) > 2^{-\mu-r}\sqrt{n}$. Then we set

$$H_{r+1} = \bigcup_{h=1}^{p_{r+1}} Q_{r+1,h}.$$

At this point let us remark that

$$d(H_r, H_{r+2}) \geq 2^{-\mu-r}, \quad \forall r \in \mathbb{N}.$$

Finally, we set

$$K_r = H_1 \cup \ldots \cup H_r, \quad \forall r \in \mathbb{N}.$$

It is clear that $\{K_r : r \in \mathbb{N}\}$ is a compact cover of Ω such that $K_r \subset K_{r+1}^{\circ}$ for every $r \in \mathbb{N}$.

4. The auxiliary functions v_r. With this construction in mind and the notations therein, we now prove the following result.

PROPOSITION 4.1. *Let Ω be a nonvoid open subset of $\mathbb{R}^n$ and $\zeta \in]1, \infty[$. Then there are integers $c, d \in \mathbb{N}$ and functions $v_{r-2} \in C_\infty(\mathbb{R}^n)$ for $r \in \{3, 4, 5, \ldots\}$ such that*

(a) $\operatorname{supp}(v_{r-2}) \subset K_{r+1} \setminus K_{r-2}^{\circ}$,
(b) $v_{r-2}(\mathbb{R}^n) \subset [0, 1]$,
(c) $v_{r-2}(H_r) = \{1\}$,
(d) $\|D^\alpha v_{r-2}\|_{\mathbb{R}^n} \leq c(2^{r-2}d)^{|\alpha|}(|\alpha|!)^\zeta$, $\forall \alpha \in \mathbb{N}_0^n$,

for every integer $r > 2$.

Proof. Let φ be an element of $C_\infty(\mathbb{R})$ for which there are $l, d > 0$ such that

$$\varphi(t) > 0 \quad \text{if } |t| < n^{-1/2}2^{-\mu-4},$$
$$\varphi(t) = 0 \quad \text{if } |t| \geq n^{-1/2}2^{-\mu-4},$$
$$\|\varphi^{(s)}\|_{\mathbb{R}} \leq l d^s (s!)^\zeta, \quad \forall s \in \mathbb{N}_0, \quad \int_{\mathbb{R}} \varphi(t)\, dt = 1.$$

The existence of such a φ is provided by the Denjoy–Carleman–Mandelbrojt theorem (cf. [2]). Then we define

$$\psi(x) = \varphi(x_1) \ldots \varphi(x_n), \quad \forall x \in \mathbb{R}^n.$$

Clearly ψ belongs to $C_\infty(\mathbb{R}^n)$, has compact support equal to $[-n^{-1/2}2^{-\mu-4}, n^{-1/2}2^{-\mu-4}]^n$ and satisfies

$$\|D^\alpha \psi\|_{\mathbb{R}^n} \leq l^n d^{|\alpha|}(|\alpha|!)^\zeta, \quad \forall \alpha \in \mathbb{N}_0^n, \quad \int_{\mathbb{R}^n} \psi(x)\, dx = 1.$$

Now for every integer $r \in \mathbb{N}$, we set

$$L_r = \{x \in \mathbb{R}^n : d(x, H_r) \leq 2^{-\mu-r-1}\},$$
$$\psi_r(x) = 2^{(r-2)n}\psi(2^{r-2}x), \quad \forall x \in \mathbb{R}^n,$$

and note that ψ_r belongs to $\mathrm{C}_\infty(\mathbb{R}^n)$ and has a compact support of diameter $2^{-\mu-r-1}$. Then we set

$$v_{r-2}(x) = \psi_r * \chi_{L_r}(x) = \int_{\mathbb{R}^n} \psi_r(y)\chi_{L_r}(x-y)\,dy, \qquad \forall x \in \mathbb{R}^n.$$

It is well known or easy to check that the function v_{r-2} belongs to $\mathrm{C}_\infty(\mathbb{R}^n)$ and has the properties (a)–(c) as well as

$$\|\mathrm{D}^\alpha v_{r-2}\|_{\mathbb{R}^n} \le \int_{\mathbb{R}^n} |\mathrm{D}^\alpha \psi_r|\,dx \le \ell(\mathrm{supp}(\psi_r))\|\mathrm{D}^\alpha \psi_r\|_{\mathbb{R}^n} \le c(2^{r-2}d)^{|\alpha|}(|\alpha|!)^\zeta$$

for every $\alpha \in \mathbb{N}_0^n$ if we set $c = l^n(n^{-1/2}2^{-\mu-3})^n$, a constant which does not depend on $r > 2$ nor on $\alpha \in \mathbb{N}_0^n$. ■

5. Approximation in Gevrey classes and consequences. In the proof of Theorem 5.2, we shall make use of the following property which results immediately from the proof of Lemma 5 of [5].

PROPOSITION 5.1 *Let $r \in \mathbb{N}$ and $g \in \mathrm{D}_r(\mathbb{R}^n)$. Then, for every $\varepsilon > 0$, there is $\lambda_0 > 0$ such that, for every $\lambda \ge \lambda_0$, the function*

$$h(x) = \pi^{-n/2}\lambda^n \int_{\mathbb{R}^n} g(y)e^{-\lambda^2|x-y|^2}\,dy$$

belongs to $\mathrm{C}_\infty(\mathbb{R}^n)$ (in fact, it is analytic on $\mathbb{R}^n$) and satisfies

$$\|\mathrm{D}^\alpha h - \mathrm{D}^\alpha g\|_{\mathbb{R}^n} \le \varepsilon \quad \text{if } |\alpha| \le r. \ ■$$

THEOREM 5.2. *Let Ω be a nonvoid open subset of $\mathbb{R}^n$ and let ζ, γ be real numbers such that $1 < \zeta < \gamma$. Then, for every $f \in \Gamma_\zeta(\Omega)$, there is $g \in \Gamma_\gamma(\Omega)$ which is analytic on Ω and such that*

$$\|\mathrm{D}^\alpha f - \mathrm{D}^\alpha g\|_{\Omega\setminus K_{s+1}} \le \frac{1}{s} \quad \text{if } |\alpha| \le s \text{ and } s \ge 2,$$

with K_{s+1} defined as in the special compact cover of Ω.

Proof. Of course there are numbers $a, b > 1$ such that

$$\|\mathrm{D}^\alpha f\|_\Omega \le ab^{|\alpha|}(|\alpha|!)^\zeta \quad \text{and} \quad \|\mathrm{D}^\alpha v_{r-2}\|_\Omega \le a(2^{r-2}b)^{|\alpha|}(|\alpha|!)^\zeta$$

for every integer $r \ge 3$ and $\alpha \in \mathbb{N}_0^n$.

Now we introduce by recursion a sequence $(g_s)_{s\in\mathbb{N}}$ in $\mathrm{C}_\infty(\mathbb{R}^n)$ such that

$$\|\mathrm{D}^\alpha g_s\|_{\mathbb{R}^n} \le 2^{s-1}a^{s+1}2^{(s+1)|\alpha|}b^{|\alpha|}(|\alpha|!)^\zeta, \qquad \forall \alpha \in \mathbb{N}_0^n,\ \forall s \in \mathbb{N}.$$

At this point, to get the functions g_s, we just need to consider a strictly increasing sequence $(\lambda_s)_{s\in\mathbb{N}}$ of $]0,\infty[$ but later on we shall make more stringent restrictions on these positive numbers.

We start with

$$g_1(x) = \pi^{-n/2}\lambda_1^n \int_{\mathbb{R}^n} v_1(y)f(y)e^{-\lambda_1^2|x-y|^2}\,dy, \qquad \forall x \in \mathbb{R}^n,$$

where of course $v_1 f$ has been extended by 0 on $\mathbb{R}^n \setminus \Omega$. By Proposition 2.1, g_1 belongs to $\mathrm{C}_\infty(\mathbb{R}^n)$ and satisfies

$$\|\mathrm{D}^\alpha g_1\|_{\mathbb{R}^n} \le a^2 3^{|\alpha|} b^{|\alpha|} (|\alpha|!)^\zeta \le 2^0 a^2 2^{2|\alpha|} b^{|\alpha|} (|\alpha|!)^\zeta$$

for every $\alpha \in \mathbb{N}_0^n$. Now if $g_1, \ldots, g_s$ are obtained, we first remark that we certainly have

$$\begin{aligned}\Big\|\mathrm{D}^\alpha\Big(f - \sum_{j=1}^s g_j\Big)\Big\|_\Omega &\le \|\mathrm{D}^\alpha f\|_\Omega + \sum_{j=1}^s \|\mathrm{D}^\alpha g_j\|_\Omega \\ &\le a b^{|\alpha|} (|\alpha|!)^\zeta + \sum_{j=1}^s 2^{j-1} a^{j+1} 2^{(j+1)|\alpha|} b^{|\alpha|} (|\alpha|!)^\zeta \\ &\le 2^s a^{s+1} 2^{(s+1)|\alpha|} b^{|\alpha|} (|\alpha|!)^\zeta\end{aligned}$$

for every $\alpha \in \mathbb{N}_0^n$ and then check by direct use of Proposition 2.1 that the function g_{s+1} defined by

$$g_{s+1}(x) = \pi^{-n/2} \lambda_{s+1}^n \int_{\mathbb{R}^n} v_{s+1}(y) \Big(f(y) - \sum_{j=1}^s g_j(y)\Big) e^{-\lambda_{s+1}^2 |x-y|^2}\, dy$$

suits our purpose.

Of course we have

$$\begin{aligned}\Big\|\mathrm{D}^\alpha \sum_{j=1}^s g_j\Big\|_{\mathbb{R}^n} &\le \sum_{j=1}^s 2^{j-1} a^{j+1} 2^{(j+1)|\alpha|} b^{|\alpha|} (|\alpha|!)^\zeta \\ &\le 2^s a^{s+1} 2^{(s+1)|\alpha|} b^{|\alpha|} (|\alpha|!)^\zeta\end{aligned}$$

for every $s \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$.

With this majorant at our disposal, we are in a position to make a more precise (but not yet final) choice of the numbers λ_s (we are free to take them larger but strictly increasing). As we have

$$\lim_{|\alpha|\to\infty} \frac{2^s a^{s+1} 2^{(s+1)|\alpha|} b^{|\alpha|} (|\alpha|!)^\zeta}{(|\alpha|!)^\gamma} = 0, \qquad \forall s \in \mathbb{N},$$

there is a strictly increasing sequence $(A_s)_{s\in\mathbb{N}}$ in $\mathbb{N}$ such that, for every $s \in \mathbb{N}$,

$$2^s a^{s+1} 2^{(s+1)|\alpha|} b^{|\alpha|} (|\alpha|!)^\zeta \le (|\alpha|!)^\gamma \quad \text{ if } |\alpha| \ge A_s.$$

Then we can also fix a strictly increasing sequence $(B_s)_{s\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\sup_{|\alpha|\le A_s} a(2^s b)^{|\alpha|} (|\alpha|!)^\zeta \le B_s, \qquad \forall s \in \mathbb{N}.$$

Next we introduce the following elements of $\mathrm{D}_\infty(\mathbb{R}^n)$:

$$h_1(x) = \begin{cases} v_1(x) f(x), & \forall x \in \Omega, \\ 0, & \forall x \in \mathbb{R}^n \setminus \Omega, \end{cases}$$

and, for every integer $s \geq 2$,

$$h_s(x) = \begin{cases} v_s(x)(f(x) - \sum_{j=1}^{s-1} g_j(x)), & \forall x \in \Omega, \\ 0, & \forall x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Finally, by recursive use of Proposition 5.1, we may require that the numbers λ_s are such that, for every $s \in \mathbb{N}$,

$$\|\mathrm{D}^\alpha h_s - \mathrm{D}^\alpha g_s\|_{\mathbb{R}^n} \leq (2^{s+2+A_{s+1}} B_{s+1})^{-1} \quad \text{if } |\alpha| \leq A_{s+1}.$$

Now we consider the series

$$g(x) = \sum_{s=1}^{\infty} g_s(x), \quad \forall x \in \Omega.$$

We first prove that g is defined and belongs to $\mathrm{C}_\infty(\Omega)$. Indeed, for every $s \in \mathbb{N}$ and every $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq A_{s+1}$, we get

$$\begin{aligned} \|\mathrm{D}^\alpha h_{s+1}\|_{H_{s+2}} &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \|\mathrm{D}^\beta v_{s+1}\|_{H_{s+2}} \Big\|\mathrm{D}^{\alpha-\beta}\Big(f - \sum_{j=1}^{s} g_j\Big)\Big\|_{H_{s+2}} \\ &\underset{(*)}{\leq} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} a(2^{s+1} b)^{|\beta|} (|\beta|!)^\zeta (2^{s+2+A_{s+1}} B_{s+1})^{-1} \\ &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} B_{s+1} (2^{s+2+A_{s+1}} B_{s+1})^{-1} \\ &\leq 2^{-s-2-A_{s+1}} 2^{|\alpha|} \leq 2^{-s-2} \end{aligned}$$

(at $(*)$, we have used the fact that $f - \sum_{j=1}^{s} g_j = h_s - g_s$ on H_{s+2}). As $v_{s+1}(x) = 0$ for every $x \in K_{s+1}$, we get

$$\|\mathrm{D}^\alpha h_{s+1}\|_{K_{s+2}} \leq 2^{-s-2} \quad \text{if } s \in \mathbb{N} \text{ and } |\alpha| \leq A_{s+1},$$

hence

$$\begin{aligned} \|\mathrm{D}^\alpha g_{s+1}\|_{K_{s+2}} &\leq \|\mathrm{D}^\alpha g_{s+1} - \mathrm{D}^\alpha h_{s+1}\|_\Omega + \|\mathrm{D}^\alpha h_{s+1}\|_{K_{s+2}} \\ &\leq (2^{s+3+A_{s+2}} B_{s+2})^{-1} + 2^{-s-2} \leq 2^{-s-1} \end{aligned}$$

for every $s \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq A_{s+1}$. Now it is clear that $g \in \mathrm{C}_\infty(\Omega)$.

We establish next that g satisfies the inequalities announced in the statement of the theorem—in fact, we are going to prove more. Consider an integer $s \geq 2$ and a multi-index $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq A_s$—this is certainly the case if $|\alpha| \leq s$. For every $x_0 \in \Omega \setminus K_{s+1}$, there is a first integer $p \geq 2$ such that $x_0 \in K_{s+p}$. According to the last established inequality, we of course have

$$|\mathrm{D}^\alpha g_{s+r}(x_0)| \leq 2^{-s-r}, \quad \forall r \in \{p-1, p, p+1, \ldots\}.$$

As $v_{s+p-2}(H_{s+p}) = \{1\}$, we also get

$$\Big|\mathrm{D}^\alpha f(x_0) - \mathrm{D}^\alpha \sum_{j=1}^{s+p-2} g_j(x_0)\Big| = |\mathrm{D}^\alpha h_{s+p-2}(x_0) - \mathrm{D}^\alpha g_{s+p-2}(x_0)| \le 2^{-s-p}.$$

Thus

$$\begin{aligned}|\mathrm{D}^\alpha f(x_0) - \mathrm{D}^\alpha g(x_0)| &\le \Big|\mathrm{D}^\alpha f(x_0) - \mathrm{D}^\alpha \sum_{j=1}^{s+p-2} g_j(x_0)\Big| + \sum_{r=p-1}^{\infty} |\mathrm{D}^\alpha g_{s+r}(x_0)| \\ &\le 2^{-s-p} + \sum_{r=p-1}^{\infty} 2^{-s-r} \le 2^{-s-p} + 2^{-s-p+2} < \frac{1}{s},\end{aligned}$$

and hence

$$\|\mathrm{D}^\alpha f - \mathrm{D}^\alpha g\|_{\Omega\setminus K_{s+1}} \le \frac{1}{s} \quad \text{if } s \ge 2 \text{ and } |\alpha| \le A_s.$$

At this step, if we proceed as in the proof of Lemma 6 of [5], we see that it is possible to select successively the numbers λ_s in such a way that g is an analytic function on Ω. (This is our last refinement on the choice of λ_s.)

We still have to prove that $g \in \Gamma_\gamma(\Omega)$.

Set

$$\sup_{|\alpha|\le A_2} \|\mathrm{D}^\alpha g\|_{K_3}/(|\alpha|!)^\gamma = a_1$$

and consider $\alpha \in \mathbb{N}_0^n$ and $x_0 \in \Omega$.

On the one hand, if $|\alpha| \le A_2$ and

(i) if $x_0 \in K_3$, we trivially have

$$|\mathrm{D}^\alpha g(x_0)| \le \|\mathrm{D}^\alpha g\|_{K_3} \le a_1(|\alpha|!)^\gamma,$$

(ii) if $x_0 \in \Omega\setminus K_3$, we get

$$\begin{aligned}|\mathrm{D}^\alpha g(x_0)| &\le |\mathrm{D}^\alpha f(x_0)| + |\mathrm{D}^\alpha f(x_0) - \mathrm{D}^\alpha g(x_0)| \\ &\le ab^{|\alpha|}(|\alpha|!)^\zeta + \frac{1}{2} \le 2ab^{|\alpha|}(|\alpha|!)^\gamma.\end{aligned}$$

Hence there are constants $a_2, b_2 > 0$ such that

$$\|\mathrm{D}^\alpha g\|_\Omega \le a_2 b_2^{|\alpha|}(|\alpha|!)^\gamma \quad \text{if } |\alpha| \le A_2.$$

On the other hand, if $|\alpha| > A_2$, we first let s be the integer such that $A_s < |\alpha| \le A_{s+1}$ (of course $s \ge 2$) and then consider the following two possibilities:

(i) if $x_0 \in \Omega\setminus K_{s+2}$, then we have at once

$$\begin{aligned}|\mathrm{D}^\alpha g(x_0)| &\le |\mathrm{D}^\alpha f(x_0)| + |\mathrm{D}^\alpha f(x_0) - \mathrm{D}^\alpha g(x_0)| \\ &\le ab^{|\alpha|}(|\alpha|!)^\zeta + \frac{1}{s+1} \le 2ab^{|\alpha|}(|\alpha|!)^\gamma,\end{aligned}$$

(ii) if $x_0 \in K_{s+2}$, we have $|\mathrm{D}^\alpha g_{s+r}(x_0)| \le 2^{-s-r}$ for every $r \in \mathbb{N}$, hence

$$\sum_{r=1}^{\infty} |\mathrm{D}^\alpha g_{s+r}(x_0)| \le \sum_{r=1}^{\infty} 2^{-s-r} = 2^{-s}.$$

But we also have

$$\Big|\mathrm{D}^\beta \sum_{j=1}^{s} g_j(x_0)\Big| \le 2^s a^{s+1} 2^{(s+1)|\beta|} b^{|\beta|} (|\beta|!)^\zeta$$

for every $\beta \in \mathbb{N}_0^n$ and $s \in \mathbb{N}$, hence

$$\Big|\mathrm{D}^\alpha \sum_{j=1}^{s} g_j(x_0)\Big| \le (|\alpha|!)^\gamma$$

since $|\alpha| \ge A_s$. Therefore

$$|\mathrm{D}^\alpha g(x_0)| \le \Big|\mathrm{D}^\alpha \sum_{j=1}^{s} g_j(x_0)\Big| + \sum_{r=1}^{\infty} |\mathrm{D}^\alpha g_{s+r}(x_0)| \le 2^{-s} + (|\alpha|!)^\gamma \le 2(|\alpha|!)^\gamma.$$

Consequently, there are constants a_3, $b_3 > 0$ such that

$$\|\mathrm{D}^\alpha g\|_\Omega \le a_3 b_3^{|\alpha|} (|\alpha|!)^\gamma \quad \text{if } |\alpha| > A_2. \ \blacksquare$$

COROLLARY 5.3. *For every open and nonvoid subset Ω of $\mathbb{R}^n$ and $\gamma \in]1,\infty[$, there is a function $g \in \Gamma_\gamma$ which is*

(a) *analytic on Ω*,

(b) *identically* 0 *on no connected component of Ω*,

(c) *flat on $\mathbb{R}^n \setminus \Omega$* (i.e. *identically* 0 *together with all its derivatives on $\mathbb{R}^n \setminus \Omega$*).

Proof. If Ω is connected, we choose $\zeta \in]1,\gamma[$ and $f \in \Gamma_\zeta(\Omega)$ with compact support contained in $K_4^\circ \setminus K_3$ and such that $\|f\|_\Omega > 1/2$. Then Theorem 5.2 provides $g \in \Gamma_\gamma(\Omega)$ which is analytic on Ω and such that

$$\|\mathrm{D}^\alpha f - \mathrm{D}^\alpha g\|_{\Omega\setminus K_{2+1}} \le \frac{1}{2} \quad \text{if } |\alpha| \le 2$$

(which implies that g is not identically 0 on Ω) as well as

$$\|\mathrm{D}^\alpha g\|_{\Omega\setminus K_{s+1}} = \|\mathrm{D}^\alpha g - \mathrm{D}^\alpha f\|_{\Omega\setminus K_{s+1}} \le \frac{1}{s} \quad \text{if } |\alpha| \le s \text{ and } s \ge 3.$$

It is then well known that extending g by 0 on $\mathbb{R}^n \setminus \Omega$ provides a solution.

If Ω has a finite number of connected components—say $\Omega_1, \ldots, \Omega_m$—then, for every $k \in \{1, \ldots, m\}$, there is $g_k \in \Gamma_\gamma$ which is analytic on Ω_k, not identically 0 on Ω_k and flat on $\mathbb{R}^n \setminus \Omega_k$. It is then clear that $g = \sum_{k=1}^m g_k$ is a solution.

As Ω always has countably many connected components, to conclude, we just have to settle the case when $\{\Omega_m : m \in \mathbb{N}\}$ is the set of connected components of Ω. For this purpose, fix $\zeta \in \,]1,\gamma[$. By the first part of the proof, for every $m \in \mathbb{N}$, there is $g_m \in \Gamma_\zeta$ which is analytic on Ω_m, not identically 0 on Ω_m, flat on $\mathbb{R}^n \setminus \Omega_m$ and such that

$$\|\mathrm{D}^\alpha g_m\|_{\Omega_m \setminus K_{m,s+1}} \le \frac{1}{s} \quad \text{if } |\alpha| \le s \text{ and } s \ge 2,$$

where of course $K_{m,r}$ is the rth element of the special cover of Ω_m. So, for every $m \in \mathbb{N}$, there are constants $a_m, b_m > 1$ such that

$$\|\mathrm{D}^\alpha g_m\|_{\mathbb{R}^n} \le a_m b_m^{|\alpha|} (|\alpha|!)^\zeta, \quad \forall \alpha \in \mathbb{N}_0^n,$$

hence there is an integer $k_m \ge m$ such that

$$a_m b_m^{|\alpha|} (|\alpha|!)^\zeta \le (|\alpha|!)^\gamma \quad \text{if } |\alpha| \ge k_m.$$

Then we set

$$c_m = (\sup_{|\alpha| \le k_m} a_m b_m^{|\alpha|} (|\alpha|!)^\zeta)^{-1} \quad \text{and} \quad g = \sum_{m=1}^\infty 2^{-m} c_m g_m.$$

It is clear that g is a function defined on $\mathbb{R}^n$ which is analytic on Ω, identically 0 on no connected component of Ω and identically 0 on $\mathbb{R}^n \setminus \Omega$. Now we prove that g belongs to $\mathrm{C}_\infty(\mathbb{R}^n)$ and is flat on $\mathbb{R}^n \setminus \Omega$. Let $x \in \mathbb{R}^n \setminus \Omega$. For every integer $k \ge 3$, there is $r > 0$ such that the ball $b = \{y \in \mathbb{R}^n : |x - y| \le r\}$ is disjoint from the compact sets $K_{1,k+1}, \ldots, K_{k,k+1}$. For every $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \le k$, this leads to

$$\sup_{x \in b} \sum_{m=1}^\infty 2^{-m} c_m |\mathrm{D}^\alpha g_m| \le \sup\Big\{\sup_{m \le k} \frac{2^{-m} c_m}{k}, \sup_{m > k} 2^{-m} c_m \|\mathrm{D}^\alpha g_m\|_{\mathbb{R}^n}\Big\} \le \sup\{1/k, 2^{-k}\},$$

hence g belongs to $\mathrm{C}_\infty(\mathbb{R}^n)$ and is flat on $\mathbb{R}^n \setminus \Omega$. We still have to prove that $g \in \Gamma_\gamma$. This is immediate: for every $\alpha \in \mathbb{N}_0^n$, we have

$$\|\mathrm{D}^\alpha g\|_{\mathbb{R}^n} = \sup_{m \in \mathbb{N}} 2^{-m} c_m \|\mathrm{D}^\alpha g_m\|_{\mathbb{R}^n} \le \sup_{m \in \mathbb{N}} \sup\{2^{-m}, 2^{-m}(|\alpha|!)^\gamma\} \le (|\alpha|!)^\gamma$$

by consideration of the cases $|\alpha| \le k_m$ and $|\alpha| > k_m$. ■

Corollary 5.4. *For every* $\gamma \in \,]1,\infty[$ *and nondegenerate compact intervals* I, J *of* $\mathbb{R}$ *such that* $J \subset I^{\mathrm{o}}$, *there are* $f \in \mathrm{C}_\infty(\mathbb{R})$ *and* $c > 0$ *such that*

(a) f *has no divergence point*,
(b) $f(\mathbb{R}) \subset [0,1]$,
(c) $f(\mathbb{R} \setminus I) = \{0\}$, $f(I^{\mathrm{o}}) \subset \,]0,1]$ *and* $f(J) = \{1\}$,
(d) $\|f^{(k)}\|_{\mathbb{R}} \le c k^{\gamma k}$, $\forall k \in \mathbb{N}$.

Proof. Let $I = [a_1, b_1]$ and $J = [a_2, b_2]$. We choose $\zeta \in]1, \gamma[$ and apply Corollary 5.3 with $\Omega =]a_1, a_2[$: there is a nonzero $g \in \Gamma_\zeta$ which is analytic on $]a_1, a_2[$ and such that $\mathrm{supp}(g) = [a_1, a_2]$. Now we choose $k > 0$ such that $g_1 = kg^2$ satisfies $\int_{\mathbb{R}} g_1(x)\,dx = 1$ and define the function f_1 on $\mathbb{R}$ by

$$f_1(x) = \int_{-\infty}^{x} g_1(t)\,dt, \qquad \forall x \in \mathbb{R}.$$

It is clear that f_1 belongs to Γ_ζ, is analytic on $]a_1, a_2[$ and satisfies

$$f_1(]-\infty, a_1]) = \{0\}, \quad f_1(]a_1, a_2[) \subset]0, 1] \quad \text{and} \quad f_1([a_2, \infty[) = \{1\}.$$

So it is clear that f_1 has no divergence point.

Similarly there is $f_2 \in \Gamma_\zeta$ which is analytic on $]b_2, b_1[$, has no divergence point and satisfies

$$f_2(]-\infty, b_2]) = \{0\}, \quad f_2(]b_2, b_1[) \subset]0, 1] \quad \text{and} \quad f_2([b_1, \infty[) = \{1\}.$$

Finally, we set

$$f(x) = f_1(x) f_2(b_1 + b_2 - x), \qquad \forall x \in \mathbb{R}.$$

Of course f belongs to Γ_ζ and satisfies (a)–(c). Let us establish that f also satisfies (d). As $f \in \Gamma_\zeta$, there are $a, b > 0$ such that

$$\|f^{(k)}\|_{\mathbb{R}} \le ab^k k^{\zeta k}, \qquad \forall k \in \mathbb{N}_0.$$

Since

$$\lim_{k \to \infty} \frac{ab^k k^{\zeta k}}{k^{\gamma k}} = 0,$$

there is $k_0 \in \mathbb{N}$ such that $ab^k k^{\zeta k} \le k^{\gamma k}$ for every $k \ge k_0$; therefore there is $c > 0$ such that

$$\|f^{(k)}\|_{\mathbb{R}} \le c k^{\gamma k}, \qquad \forall k \in \mathbb{N}. \ \blacksquare$$

PROPOSITION 5.5. *Let $\gamma \in]1, \infty[$, let $p \in \mathbb{N}$ and let I, J be nondegenerate compact intervals of $\mathbb{R}$ such that $J \subset I^{\circ}$. Then there is $m_0 \in \mathbb{N}$ such that, for every integer $m \ge m_0$, there is a function $u \in \mathrm{C}_\infty(\mathbb{R})$ satisfying the following conditions*:

(a) *u has no divergence point*,
(b) $\mathrm{supp}(u) \subset I$,
(c) $\|u^{(k)}\|_{\mathbb{R}} \le 2^{-m}$, $\forall k \in \{0, 1, \ldots, m\}$,
(d) $\|u^{(k)}\|_{\mathbb{R}} \le 2^k k^{\gamma k}$, $\forall k \in \mathbb{N}$,
(e) *for every $x \in J$, one has either*

$$|u^{(pm)}(x)| \ge 5^{-m} m^{\gamma(p-1)m}$$

or

$$|u^{(pm+1)}(x)| \ge 5^{-(m+1/p)} \left(\frac{pm+1}{p+1}\right)^{\gamma(p-1)(m+1/p)}.$$

Proof. Corollary 5.4 provides a function $f \in C_\infty(\mathbb{R})$ and a constant $c > 0$ such that f has no divergence point and satisfies

$$f(\mathbb{R}) \subset [0, c], \quad f(\mathbb{R} \setminus I) = \{0\}, \quad f(J) = \{1\}$$

as well as

$$\|f^{(k)}\|_{\mathbb{R}} \le ck^{\gamma k}, \quad \forall k \in \mathbb{N}.$$

For any $m \in \mathbb{N}$, we can introduce

$$a = m^\gamma, \quad b = (2^{2m} c m^{\gamma m})^{-1},$$
$$u(x) = bf(x)\sin(ax), \quad \forall x \in \mathbb{R}.$$

It is then clear that u is a C_∞-function on $\mathbb{R}$ satisfying the conditions (a) and (b) as well as

$$\|u\|_{\mathbb{R}} \le bc = (2^{2m} m^{\gamma m})^{-1} \le 2^{-m}.$$

Moreover, for every $x \in \mathbb{R}$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} |u^{(k)}(x)| &\le b \sum_{h=0}^{k} \binom{k}{h} |f^{(h)}(x)| a^{k-h} \\ &\le bc \sum_{h=0}^{k} \binom{k}{h} h^{\gamma h} a^{k-h} \le (2^{2m} m^{\gamma m})^{-1} (k^\gamma + a)^k, \end{aligned}$$

hence

$$|u^{(k)}(x)| \le (2^{2m} m^{\gamma m})^{-1} (m^\gamma + m^\gamma)^m = 2^{-m} \quad \text{if } 1 \le k \le m$$

as well as

$$|u^{(k)}(x)| \le (2^{2m} m^{\gamma m})^{-1} (k^\gamma + k^\gamma)^k \le 2^k k^{\gamma k} \quad \text{if } k > m;$$

i.e. u also satisfies the conditions (c) and (d).

Now we investigate (e). Let $x \in J$. Of course we have

$$u^{(k)}(x) = a^k b \sin(k\pi/2 + ax), \quad \forall k \in \mathbb{N}.$$

Now, for every $m \in \mathbb{N}$, we certainly have

$$\sup\{|\sin(pm\pi/2 + ax)|, |\sin((pm+1)\pi/2 + ax)|\} \ge 2^{-1/2}.$$

So on the one hand, if $|\sin(pm\pi/2 + ax)| \ge 2^{-1/2}$, we get

$$|u^{(pm)}(x)| \ge 2^{-1/2} a^{pm} b = 4^{-m} (\sqrt{2}c)^{-1} m^{\gamma(p-1)m},$$

and on the other hand, if $|\sin((pm+1)\pi/2 + ax)| \ge 2^{-1/2}$, then

$$\begin{aligned} |u^{(pm+1)}(x)| &\ge 2^{-1/2} a^{pm+1} b = 4^{-m} (\sqrt{2}c)^{-1} m^{\gamma((p-1)m+1)} \\ &\ge 4^{-(m+1/p)} (\sqrt{2}c)^{-1} m^{\gamma(p-1)(m+1/p)} \\ &\ge 4^{-(m+1/p)} (\sqrt{2}c)^{-1} \left(\frac{pm+1}{p+1}\right)^{\gamma(p-1)(m+1/p)}. \end{aligned}$$

To conclude, it is enough to take as m_0 any positive integer m_0 such that

$$\sup\{4(\sqrt{2}c)^{1/m_0}, 4(\sqrt{2}c)^{1/(m_0+1/p)}\} \le 5. \ \blacksquare$$

6. Characterizing the sets of divergence points. In this section, we establish the following result.

THEOREM 6.1. *For every $\gamma \in]1,\infty[$ and every G_δ-subset G of $\mathbb{R}^n$, there is an element of Γ_γ having G as its set of divergence points.*

Proof. We proceed in several steps.

Step 1: *the numbers γ_j and p_j.* We fix a strictly increasing sequence $(\gamma_j)_{j\in\mathbb{N}_0} \subset]1,\gamma[$ and, for every $j \in \mathbb{N}$, denote by p_j a positive integer such that $p_j(\gamma_j - \gamma_{j-1}) > \gamma_j$.

Step 2: *some auxiliary inequalities and the numbers q_r.* For every $r \in \mathbb{N}$, we certainly have

$$\frac{p_r - 1}{p_r}\gamma_r > \gamma_{r-1} > \ldots > \gamma_0 > 1.$$

Therefore, for every $j \in \{0,\ldots,r-1\}$, it is a straightforward matter to check the following limits:

$$(1)\quad \lim_{m\to\infty} 5^m 2^{2p_r m}(p_r m)^{\gamma_j p_r m}\frac{1}{m^{\gamma_r(p_r-1)m}} = \lim_{m\to\infty}\left(\frac{4.5^{1/p_r}p_r^{\gamma_j}}{m^{\gamma_r(p_r-1)/p_r-\gamma_j}}\right)^{p_r m} = 0,$$

$$(2)\quad \lim_{m\to\infty} 5^{m+1/p_r}2^{2(p_r m+1)}(p_r m+1)^{\gamma_j(p_r m+1)}\left(\frac{p_r+1}{p_r m+1}\right)^{\gamma_r(p_r-1)(m+1/p_r)} = \lim_{m\to\infty}\left(\frac{4.5^{1/p_r}(p_r+1)^{\gamma_r(p_r-1)/p_r}}{(p_r m+1)^{\gamma_r(p_r-1)/p_r-\gamma_j}}\right)^{p_r m+1} = 0,$$

$$(3)\quad \lim_{m\to\infty} 5^m(nr)^{p_r m}(p_r m+1)^{p_r m}\frac{1}{m^{\gamma_r(p_r-1)m}} = \lim_{m\to\infty}\left(\frac{5^{1/p_r}nrp_r^{\gamma_r(p_r-1)/p_r}}{(p_r m)^{\gamma_r(p_r-1)/p_r-1}}\right)^{p_r m}\left(\frac{p_r m+1}{p_r m}\right)^{p_r m} = 0,$$

$$(4)\quad \lim_{m\to\infty} 5^{m+1/p_r}(nr)^{p_r m+1}(p_r m+2)^{p_r m+1}\left(\frac{p_r+1}{p_r m+1}\right)^{\gamma_r(p_r-1)(m+1/p_r)} = \lim_{m\to\infty}\left(\frac{5^{1/p_r}nr(p_r+1)^{\gamma_r(p_r-1)/p_r}}{(p_r m+1)^{\gamma_r(p_r-1)/p_r-1}}\right)^{p_r m+1}\left(\frac{p_r m+2}{p_r m+1}\right)^{p_r m+1} = 0.$$

With these limits at our disposal, we find that, for every $r \in \mathbb{N}$, there is $q_r \in \mathbb{N}$ such that, for every integer $m \ge q_r$ and $j \in \{1,\ldots,r-1\}$, we have the following auxiliary inequalities:

(I) $\quad 4 \cdot 2^{2p_r m}(p_r m)^{\gamma_j p_r m} < 5^{-m} m^{\gamma_r (p_r - 1)m}$,

(II) $\quad 4 \cdot 2^{2(p_r m+1)}(p_r m + 1)^{\gamma_j (p_r m+1)}$

$$< 5^{-(m+1/p_r)} \left(\frac{p_r m + 1}{p_r + 1} \right)^{\gamma_r (p_r - 1)(m+1/p_r)},$$

(III) $\quad (nr)^{p_r m}(p_r m + 1)^{p_r m} < 5^{-m} m^{\gamma_r (p_r - 1)m}$,

(IV) $\quad (nr)^{p_r m+1}(p_r m+2)^{p_r m+1} < 5^{-(m+1/p_r)} \left(\frac{p_r m + 1}{p_r + 1} \right)^{\gamma_r (p_r - 1)(m+1/p_r)}$.

Step 3: *the sets G_l, Q_r, P_r, $I_{r,j}$ and $J_{r,j}$.* Being a G_δ-subset of $\mathbb{R}^n$, G is equal to the intersection of a sequence $(G_l)_{l\in\mathbb{N}}$ of open subsets of $\mathbb{R}^n$ that we may suppose decreasing.

Proceeding as in the construction of the special compact cover of an open set, we find that each G_l is the union of countably many compact cubes $Q_{l,m,h}$ that we may renumber as a sequence, say $(Q_{l,k})_{k\in\mathbb{N}}$. Then for every $l, k \in \mathbb{N}$, we denote by $P_{l,k}$ the compact cube in $\mathbb{R}^n$ having the same center as $Q_{l,k}$ and $\frac{3}{2}\mathrm{diam}(Q_{l,k})$ as diameter. Now we arrange $\mathbb{N}^2$ into a sequence $((l_r, k_r))_{r\in\mathbb{N}}$, set

$$Q_r = Q_{l_r,k_r} \quad \text{and} \quad P_r = P_{l_r,k_r},$$

and let $I_{r,j}$ and $J_{r,j}$ for $j \in \{1, \ldots, n\}$ be the compact intervals in $\mathbb{R}$ such that

$$Q_r = \prod_{j=1}^{n} J_{r,j} \quad \text{and} \quad P_r = \prod_{j=1}^{n} I_{r,j}.$$

Of course this construction leads to $J_{r,j} \subset I^{\circ}_{r,j}$ for every $r \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$.

Step 4: *the functions $u_{r,j}$ and the numbers m_r.* At this point, everything is set up to introduce the functions $u_{r,j}$ for $r \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$, as well as the sequence $(m_r)_{r\in\mathbb{N}}$ of $\mathbb{N}$ by the following recursion.

An application of Proposition 5.5 to $\gamma = \gamma_1$ and $p = p_1$ leads to an integer $m_1 > q_1$ and to functions $u_{1,1}, \ldots, u_{1,n} \in \mathrm{C}_\infty(\mathbb{R})$ such that, for every $j \in \{1, \ldots, n\}$,

(a) $u_{1,j}$ has no divergence point,

(b) $\mathrm{supp}(u_{1,j}) \subset I_{1,j}$,

(c) $\|u_{1,j}^{(k)}\|_{\mathbb{R}} \le 2^{-m_1}$, $\forall k \in \{0, 1, \ldots, m_1\}$,

(d) $\|u_{1,j}^{(k)}\|_{\mathbb{R}} \le 2^k k^{\gamma_1 k}, \forall k \in \mathbb{N}$,

(e) for every $t \in J_{1,j}$, one has either

$$|u_{1,j}^{(p_1 m_1)}(t)| \ge 5^{-m_1} m_1^{\gamma_1 (p_1 - 1)m_1}$$

or

$$|u_{1,j}^{(p_1 m_1+1)}(t)| \ge 5^{-(m_1+1/p_1)} \left(\frac{p_1 m_1+1}{p_1+1}\right)^{\gamma_1(p_1-1)(m_1+1/p_1)}.$$

Now, for an integer $r \ge 2$, if the functions $u_{t,j}$ for $t \in \{1,\dots,r-1\}$ and $j \in \{1,\dots,n\}$ and the integers $m_1,\dots,m_{r-1}$ are obtained, we apply Proposition 5.5 to $\gamma = \gamma_r$ and $p = p_r$ and obtain an integer $m_r > \sup\{p_{r-1}m_{r-1}, q_r\}$ and functions $u_{r,1},\dots,u_{r,n} \in \mathrm{C}_\infty(\mathbb{R})$ such that, for every $j \in \{1,\dots,n\}$,

(a) $u_{r,j}$ has no divergence point,
(b) $\mathrm{supp}(u_{r,j}) \subset I_{r,j}$,
(c) $\|u_{r,j}^{(k)}\|_{\mathbb{R}} \le 2^{-m_r}$, $\forall k \in \{0,1,\dots,m_r\}$,
(d) $\|u_{r,j}^{(k)}\|_{\mathbb{R}} \le 2^k k^{\gamma_r k}$, $\forall k \in \mathbb{N}$,
(e) for every $t \in J_{r,j}$, one has either

$$|u_{r,j}^{(p_r m_r)}(t)| \ge 5^{-m_r} m_r^{\gamma_r(p_r-1)m_r}$$

or

$$|u_{r,j}^{(p_r m_r+1)}(t)| \ge 5^{-(m_r+1/p_r)} \left(\frac{p_r m_r+1}{p_r+1}\right)^{\gamma_r(p_r-1)(m_r+1/p_r)}.$$

Step 5: *the functions u_r and u.* Finally, for every $r \in \mathbb{N}$, we define,

$$u_r(x) = u_{r,1}(x_1)\dots u_{r,n}(x_n), \qquad \forall x \in \mathbb{R}^n,$$

and consider the series $u = \sum_{r=1}^{\infty} u_r$. For every $k \in \mathbb{N}$, we certainly have $k \le m_k$. Therefore, for every $\alpha \in \mathbb{N}_0^n$,

$$\sum_{r=\sup\{1,|\alpha|\}}^{\infty} \|\mathrm{D}^\alpha u_r\|_{\mathbb{R}^n} \le \sum_{r=\sup\{1,|\alpha|\}}^{\infty} 2^{-nm_r} \le 1;$$

this implies that u is a bounded C_∞-function on $\mathbb{R}^n$. Moreover, for every $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \ge 1$, we have

$$\begin{aligned}\|\mathrm{D}^\alpha u\|_{\mathbb{R}^n} &\le \sum_{r=1}^{|\alpha|-1} \|\mathrm{D}^\alpha u_r\|_{\mathbb{R}^n} + \sum_{r=|\alpha|}^{\infty} \|\mathrm{D}^\alpha u_r\|_{\mathbb{R}^n} \\ &\le \sum_{r=1}^{|\alpha|-1} 2^{|\alpha|}|\alpha|^{\gamma|\alpha|} + 1 \le 3^{|\alpha|}|\alpha|^{\gamma|\alpha|},\end{aligned}$$

hence $u \in \Gamma_\gamma$.

To conclude, we prove that G is the set of divergence points of u.

On the one hand, if $x \in \mathbb{R}^n$ does not belong to G, then $x \notin G_{l_0}$ for some l_0, hence $x \notin G_l$ for all $l \ge l_0$. This implies that x belongs to an at most finite number of the P_r's. Therefore for r large enough we have $\mathrm{D}^\alpha u_r(x) = 0$ for every $\alpha \in \mathbb{N}_0^n$ and clearly x is not a divergence point of u.

On the other hand, let us prove by contradiction that every element of G is a divergence point of u. Suppose that $x \in G$ is not a divergence point of u. This implies the existence of $s \in \mathbb{N}$ such that

$$|\mathrm{D}^\beta u(x)| \le s^{|\beta|}|\beta|^{|\beta|} \quad \text{if } |\beta| \ge 1.$$

As x belongs to each G_l, there is an integer $r > 3s$ such that $x \in Q_r$; in particular, $x_j \in J_{r,j}$ for every $j \in \{1, \ldots, n\}$.

Fix $j \in \{1, \ldots, n\}$. The consideration of the property (e) leads to the following two possibilities.

Case 1: *We have*

$$(*) \qquad |u_{r,j}^{(p_r m_r)}(x_j)| \ge 5^{-m_r} m_r^{\gamma_r(p_r-1)m_r}.$$

Then we set $\alpha_j = p_r m_r$ and remark that

(1.i) the auxiliary inequality (III) leads to

$$|u_{r,j}^{(\alpha_j)}(x_j)| \ge (nr)^{\alpha_j}(\alpha_j+1)^{\alpha_j},$$

(1.ii) the use of (I) in $(*)$ leads to

$$\begin{aligned}
&\sum_{t=1}^{r-1}|u_{t,j}^{(\alpha_j)}(x_j)| + \sum_{t=r+1}^{\infty}|u_{t,j}^{(\alpha_j)}(x_j)| \\
&\qquad \le \sum_{t=1}^{r-1} 2^{\alpha_j}\alpha_j^{\gamma_t\alpha_j} + \sum_{t=r+1}^{\infty} 2^{-m_t} \le 2\sum_{t=1}^{r-1} 2^{p_r m_r}(p_r m_r)^{\gamma_t p_r m_r} \\
&\qquad \underset{(*)}{\le} \frac{1}{2}\sum_{t=1}^{r-1} 2^{-p_r m_r} 5^{-m_r} m_r^{\gamma_r(p_r-1)m_r} \le \frac{1}{2}|u_{r,j}^{(\alpha_j)}(x_j)|.
\end{aligned}$$

Case 2: $(*)$ *does not hold.* Then we have

$$|u_{r,j}^{(p_r m_r+1)}(x_j)| \ge 5^{-(m_r+1/p_r)}\left(\frac{p_r m_r+1}{p_r+1}\right)^{\gamma_r(p_r-1)(m_r+1/p_r)},$$

we set $\alpha_j = p_r m_r + 1$ and remark that

(2.i) the auxiliary inequality (IV) leads to

$$|u_{r,j}^{(\alpha_j)}(x_j)| \ge (nr)^{\alpha_j}(\alpha_j+1)^{\alpha_j},$$

(2.ii) the use of (II) in $(*)$ leads to

$$\sum_{t=1}^{r-1}|u_{t,j}^{(\alpha_j)}(x_j)|+\sum_{t=r+1}^{\infty}|u_{t,j}^{(\alpha_j)}(x_j)|$$
$$\le \sum_{t=1}^{r-1}2^{\alpha_j}\alpha_j^{\gamma_t\alpha_j}+\sum_{t=r+1}^{\infty}2^{-m_t}\le 2\sum_{t=1}^{r-1}2^{p_rm_r+1}(p_rm_r+1)^{\gamma_t(p_rm_r+1)}$$
$$\underset{(*)}{\le}\frac{1}{2}\sum_{t=1}^{r-1}2^{-(p_rm_r+1)}5^{-(m_r+1/p_r)}\left(\frac{p_rm_r+1}{p_r+1}\right)^{\gamma_r(p_r-1)(m_r+1/p_r)}$$
$$\le \frac{1}{2}|u_{r,j}^{(\alpha_j)}(x_j)|.$$

So setting $\alpha=(\alpha_1,\dots,\alpha_n)$ yields

$$|\mathrm{D}^\alpha u(x)|\ge|\mathrm{D}^\alpha u_r(x)|-\sum_{t=1}^{r-1}|\mathrm{D}^\alpha u_t(x)|-\sum_{t=r+1}^{\infty}|\mathrm{D}^\alpha u_t(x)|$$
$$\ge|\mathrm{D}^\alpha u_r(x)|-\prod_{j=1}^{n}\Big(\sum_{t=1}^{r-1}|u_{t,j}^{(\alpha_j)}(x_j)|+\sum_{t=r+1}^{\infty}|u_{t,j}^{(\alpha_j)}(x_j)|\Big)$$
$$\ge|\mathrm{D}^\alpha u_r(x)|-\prod_{j=1}^{n}\frac{1}{2}|u_{r,j}^{(\alpha_j)}(x_j)|\ge\frac{1}{2}|\mathrm{D}^\alpha u_r(x)|.$$

For every $j\in\{1,\dots,n\}$, as α_j belongs to $\{m_rp_r, m_rp_r+1\}$, we certainly have $\alpha_j+1\ge|\alpha|/n$. Therefore

$$|\mathrm{D}^\alpha u(x)|\ge\frac{1}{2}|\mathrm{D}^\alpha u_r(x)|=\frac{1}{2}\prod_{j=1}^{n}|u_{r,j}^{(\alpha_j)}(x_j)|$$
$$\ge\frac{1}{2}\prod_{j=1}^{n}(nr)^{\alpha_j}(\alpha_j+1)^{\alpha_j}\ge\frac{1}{2}(nr)^{|\alpha|}\left(\frac{|\alpha|}{n}\right)^{|\alpha|}=\frac{1}{2}r^{|\alpha|}|\alpha|^{|\alpha|}$$

and finally, as we have chosen $r>3s$, we arrive at the following contradiction:

$$|\mathrm{D}^\alpha u(x)|\ge\frac{1}{2}(3s)^{|\alpha|}|\alpha|^{|\alpha|}>s^{|\alpha|}|\alpha|^{|\alpha|}.\ \blacksquare$$

7. Proof of Theorem 1.1. We first fix some $\zeta\in\,]1,\gamma[$. We next apply Theorem 6.1 to get $u\in\Gamma_\zeta$ having G as its set of divergence points. We then apply Theorem 5.2 to get $h\in\Gamma_\gamma(\mathbb{R}^n\setminus G^-)$ which is analytic on $\mathbb{R}^n\setminus G^-$ and such that

$$\|\mathrm{D}^\alpha u-\mathrm{D}^\alpha h\|_{(\mathbb{R}^n\setminus G^-)\setminus K_{s+1}}\le\frac{1}{s}\quad\text{ if } |\alpha|\le s \text{ and } s\ge 2$$

(where of course K_s is the sth compact set corresponding to the special compact cover of the open set $\mathbb{R}^n \setminus G^-$). So the function

$$f : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \begin{cases} u(x) & \text{if } x \in G^-, \\ h(x) & \text{if } x \in \mathbb{R}^n \setminus G^-, \end{cases}$$

belongs to Γ_γ, is analytic on $\mathbb{R}^n \setminus G^-$ and has G as its set of divergence points.

We now apply Corollary 5.3 to get $g \in \Gamma_\gamma$ which is analytic on Ω, identically 0 on no connected component of Ω and flat on $\mathbb{R}^n \setminus \Omega$; in particular, g has no divergence point.

To conclude one just has to check that the function $f + g$ suits our purpose: $f+g$ certainly belongs to Γ_γ, is analytic on Ω (since $\Omega \subset \mathbb{R}^n \setminus G^-$) and has G as its set of divergence points. Moreover, no point x of F can be a divergence point (since F and G are disjoint), nor an analytic point (this would imply that $f + g$ is analytic on some open ball b centered at x; this in turn implies that b and G are disjoint, so f must be analytic on b; finally, g is analytic hence flat on b, contrary to the fact that x must belong to the boundary of Ω). ■

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