The Zahorski theorem is valid in Gevrey classes

by

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Abstract. Let \( \{ \Omega, F, G \} \) be a partition of \( \mathbb{R}^n \) such that \( \Omega \) is open, \( F \) is \( F_\sigma \) and of the first category, and \( G \) is \( G_\delta \). We prove that, for every \( \gamma \in (1, \infty) \), there is an element of the Gevrey class \( \Gamma_\gamma \) which is analytic on \( \Omega \), has \( F \) as its set of defect points and has \( G \) as its set of divergence points.

1. Introduction. Let \( f \) be a real \( C_\infty \)-function on \( \mathbb{R}^n \). The set where \( f \) is analytic is of course an open subset of \( \mathbb{R}^n \); denote it by \( \Omega_f \). It is clear that \( x \) belongs to \( \Omega_f \) if and only if the radius of convergence \( \rho_f(x) \) of the Taylor series of \( f \) at \( x \) is strictly positive and this series represents \( f \) on some neighbourhood of \( x \).

As the set \( T_f \) of \( x \in \mathbb{R}^n \) such that \( \rho_f(x) > 0 \) is easily seen to be an \( F_\sigma \)-set, the set \( F_f = T_f \setminus \Omega_f \) is also \( F_\sigma \) and its elements \( x \) are characterized by the fact that \( \rho_f(x) > 0 \) and that the Taylor series of \( f \) at \( x \) represents \( f \) on no neighbourhood of \( x \). By use of a lemma of R. P. Boas ([1], p. 234), one easily sees that \( F_f \) is also a first category set (cf. [3] or [4]).

Finally, one may consider the set \( G_f = \mathbb{R}^n \setminus T_f \), a \( G_\delta \)-set given by \( \rho_f(x) = 0 \).

It is clear that \( \{ \Omega_f, G_f, F_f \} \) is a partition of \( \mathbb{R}^n \).

The Zahorski theorem (cf. [6]) asserts conversely that for every partition \( \{ \Omega, F, G \} \) of \( [0, 1] \), where \( \Omega \) is an open subset of \( [0, 1] \), \( F \) is a first category \( F_\sigma \)-subset of \( [0, 1] \) and \( G \) is a \( G_\delta \)-subset of \( [0, 1] \), there is a real \( C_\infty \)-function \( f \) on \( [0, 1] \) such that \( \Omega = \Omega_f \), \( F = F_f \) and \( G = G_f \). In [3], H. Salzmann and K. Zeller have provided a shorter proof of the Zahorski theorem and in [4], J. Siciak has extended this result to \( \mathbb{R}^n \).
The purpose of this paper is to prove that, for every \( \gamma \in ]1, \infty[ \), the Zahorski theorem has a solution \( f \) belonging to the Gevrey class \( \Gamma_\gamma \).

Let us recall that, for an open subset \( \Omega \) of \( \mathbb{R}^n \) and for \( \gamma \in ]1, \infty[ \), the Gevrey class \( \Gamma_\gamma(\Omega) \) is the set of \( f \in C_\infty(\Omega) \) for which there are constants \( a, b > 0 \) such that

\[
\|D^\alpha f\|_\Omega \leq ab^{(|\alpha|)\gamma}, \quad \forall \alpha \in \mathbb{N}_0^n.
\]

If \( \Omega = \mathbb{R}^n \), we simply write \( \Gamma_\gamma \) instead of \( \Gamma_\gamma(\mathbb{R}^n) \).

It is known that

(a) a function \( f \in C_\infty(\Omega) \) belongs to \( \Gamma_\gamma(\Omega) \) if and only if there are constants \( c, d > 0 \) such that

\[
\|D^\alpha f\|_\Omega \leq cd^{(|\alpha|)\gamma}, \quad \forall \alpha \in \mathbb{N}_0^n.
\]

(b) the Denjoy–Carleman–Mandelbrojt result (cf. [2]) states that, for every closed ball \( b \) of \( \mathbb{R}^n \) and every \( \gamma \in ]1, \infty[ \), there is a nonzero function \( f \in \Gamma_\gamma \) with support contained in \( b \).

In order to get an efficient way to state the results, for a real \( C_\infty \)-function \( f \) on \( \mathbb{R}^n \), let us call the elements of \( \Omega_f \) (resp. \( F_f \); \( G_f \)) the analytic points (resp. the defect points; the divergence points) of \( f \).

The purpose of this article is to prove the following result.

**Theorem 1.1.** For every partition \( \{\Omega, F, G\} \) of \( \mathbb{R}^n \), where \( \Omega \) (resp. \( F \); \( G \)) is an open set (resp. a first category \( F_\sigma \)-set; a \( G_\delta \)-set) and every \( \gamma \in ]1, \infty[ \), there is an element of \( \Gamma_\gamma \) having \( \Omega \) (resp. \( F \); \( G \)) as its set of analytic points (resp. defect points; divergence points).

**Remark.** It is a direct matter to check that the Zahorski theorem extends to the case when \( \mathbb{R}^n \) is replaced by an open or a closed subset of \( \mathbb{R}^n \).

**2. An auxiliary result.** We begin with the following easy result, where, as usual, \( D_r(\Omega) \) denotes the space of \( C_r \)-functions on the open subset \( \Omega \) of \( \mathbb{R}^n \) which have a compact support contained in \( \Omega \).

**Proposition 2.1.** Let \( \Omega \) be a nonvoid open subset of \( \mathbb{R}^n \). For every \( f \in C_\infty(\Omega) \), \( g \in D_\infty(\Omega) \) and \( \lambda > 0 \), it is well known that

\[
h(x) = \pi^{-n/2} \lambda^n \int_\Omega f(y)g(y)e^{-\lambda^2|x-y|^2} \, dy
\]

belongs to \( C_\infty(\mathbb{R}^n) \). If moreover \( a_1, a_2, b_1, b_2 > 0 \) and \( \zeta > 1 \) are such that

\[
\|D^\alpha f\|_\Omega \leq a_1 b_1^{(|\alpha|)\zeta} \quad \text{and} \quad \|D^\alpha g\|_\Omega \leq a_2 b_2^{(|\alpha|)\zeta}, \quad \forall \alpha \in \mathbb{N}_0^n,
\]

then

\[
\|D^\alpha h\|_{\mathbb{R}^n} \leq a_1 a_2 (b_1 + b_2)^{(|\alpha|)\zeta}, \quad \forall \alpha \in \mathbb{N}_0^n.
\]
Proof. As \( g \) has a compact support contained in \( \Omega \), up to extension by 0 on \( \mathbb{R}^n \setminus \Omega \), we may suppose that the product \( fg \) is a \( D_\infty \)-function on \( \mathbb{R}^n \) with compact support contained in \( \Omega \). So in the definition of \( h(x) \), we may consider that we integrate on \( \mathbb{R}^n \). Therefore integrating by parts \(|\alpha|\) times gives

\[
D^\alpha h(x) = \pi^{-n/2} \lambda^n \int_{\mathbb{R}^n} D^\alpha (fg)(y) e^{-\lambda^2 |x-y|^2} \, dy,
\]

hence

\[
|D^\alpha h(x)| \leq \pi^{-n/2} \lambda^n \sum_{\beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) |D^{\alpha - \beta} f(y) D^\beta g(y)| e^{-\lambda^2 |x-y|^2} \, dy
\]

\[
\leq \sum_{\beta \leq \alpha} \left( \frac{\alpha}{\beta} \right) a_1 a_2 b_1^{\beta} b_2^{\alpha - \beta} (|\beta|)! (|\alpha - \beta|)! \zeta
\]

\[
\leq a_1 a_2 (b_1 + b_2)^{|\alpha| (|\alpha|)!} \zeta. \]

3. Special compact covers of open subsets of \( \mathbb{R}^n \). In the following results, we are going to use systematically the following construction and notations. Let \( \Omega \) be a nonvoid open subset of \( \mathbb{R}^n \). Then we set

\[
\Omega_m = (\mathbb{R}^n \setminus \Omega) \cup \{ x \in \mathbb{R}^n : |x| \geq m \sqrt{n} \}, \quad \forall m \in \mathbb{N},
\]

and denote by \( \mu \) the first positive integer \( m \) for which there is at least one cube \( Q \) of the type

\[
\prod_{j=1}^n [2^{-m} a_j, 2^{-m} (a_j + 1)] \quad \text{with} \ a \in \mathbb{Z}^n
\]

contained in \( \Omega \) and such that \( d(Q, \Omega_{\mu}) > 2^{-m} \sqrt{n} \). Let \( Q_{1,1}, \ldots, Q_{1,p_1} \) be these cubes (of course, we have \( p_1 \in \mathbb{N} \)) and set

\[
H_1 = \bigcup_{h=1}^{p_1} Q_{1,h}.
\]

Now we proceed by recursion. If the sets \( H_1, \ldots, H_r \) are obtained, we let \( Q_{r+1,1}, \ldots, Q_{r+1,p_{r+1}} \) denote all the cubes of the type

\[
\prod_{j=1}^n [2^{-\mu-r} a_j, 2^{-\mu-r} (a_j + 1)] \quad \text{with} \ a \in \mathbb{Z}^n
\]

contained in \( \Omega \), disjoint from the interior of \( H_1 \cup \ldots \cup H_r \) and such that \( d(Q, \Omega_{\mu+r}) > 2^{-\mu-r} \sqrt{n} \). Then we set

\[
H_{r+1} = \bigcup_{h=1}^{p_{r+1}} Q_{r+1,h}.
\]
At this point let us remark that
\[ d(H_r, H_{r+2}) \geq 2^{-\mu - r}, \quad \forall r \in \mathbb{N}. \]

Finally, we set
\[ K_r = H_1 \cup \ldots \cup H_r, \quad \forall r \in \mathbb{N}. \]
It is clear that \( \{ K_r : r \in \mathbb{N} \} \) is a compact cover of \( \Omega \) such that \( K_r \subset K_{r+1}^o \) for every \( r \in \mathbb{N} \).

4. The auxiliary functions \( v_r \). With this construction in mind and the notations therein, we now prove the following result.

**Proposition 4.1.** Let \( \Omega \) be a nonvoid open subset of \( \mathbb{R}^n \) and \( \zeta \in [1, \infty[ \). Then there are integers \( c, d \in \mathbb{N} \) and functions \( v_{r-2} \in C^\infty(\mathbb{R}^n) \) for \( r \in \{ 3, 4, 5, \ldots \} \) such that

(a) \( \text{supp}(v_{r-2}) \subset K_{r+1} \setminus K_{r-2} \),
(b) \( v_{r-2}(\mathbb{R}^n) \subset [0, 1] \),
(c) \( v_{r-2}(H_r) = \{ 1 \} \),
(d) \( \| D^\alpha v_{r-2} \|_{\mathbb{R}^n} \leq c(2^{r-2}d|\alpha|(|\alpha|)!)^\zeta, \quad \forall \alpha \in \mathbb{N}^n_0 \),

for every integer \( r > 2 \).

**Proof.** Let \( \varphi \) be an element of \( C^\infty(\mathbb{R}) \) for which there are \( l, d > 0 \) such that

\[ \varphi(t) > 0 \quad \text{if } |t| < n^{-1/2}2^{-\mu - 4}, \]
\[ \varphi(t) = 0 \quad \text{if } |t| \geq n^{-1/2}2^{-\mu - 4}, \]
\[ \| \varphi(s) \|_{\mathbb{R}} \leq ld^n s!^\zeta, \quad \forall s \in \mathbb{N}_0, \quad \int_{\mathbb{R}} \varphi(t) \, dt = 1. \]

The existence of such a \( \varphi \) is provided by the Denjoy–Carleman–Mandelbrojt theorem (cf. [2]). Then we define
\[ \psi(x) = \varphi(x_1) \ldots \varphi(x_n), \quad \forall x \in \mathbb{R}^n. \]

Clearly \( \psi \) belongs to \( C^\infty(\mathbb{R}^n) \), has compact support equal to \( [-n^{-1/2}2^{-\mu - 4}, n^{-1/2}2^{-\mu - 4}]^n \) and satisfies
\[ \| D^\alpha \psi \|_{\mathbb{R}^n} \leq l^n d|\alpha|(|\alpha|)!^\zeta, \quad \forall \alpha \in \mathbb{N}^n_0, \quad \int_{\mathbb{R}^n} \psi(x) \, dx = 1. \]

Now for every integer \( r \in \mathbb{N} \), we set
\[ L_r = \{ x \in \mathbb{R}^n : d(x, H_r) \leq 2^{-\mu - r - 1} \}, \]
\[ \psi_r(x) = 2^{(r-2)n} \psi(2^{r-2}x), \quad \forall x \in \mathbb{R}^n. \]
and note that \( \psi_r \) belongs to \( C_\infty(\mathbb{R}^n) \) and has a compact support of diameter \( 2^{-\mu-r-1} \). Then we set
\[
v_{r-2}(x) = \psi_r \ast \chi_{L_r}(x) = \int_{\mathbb{R}^n} \psi_r(y) \chi_{L_r}(x-y) \, dy, \quad \forall x \in \mathbb{R}^n.
\]

It is well known or easy to check that the function \( v_{r-2} \) belongs to \( C_\infty(\mathbb{R}^n) \) and has the properties (a)–(c) as well as
\[
\| D^\alpha v_{r-2} \|_{\mathbb{R}^n} \leq \int_{\mathbb{R}^n} |D^\alpha \psi_r| \, dx \leq \ell(\supp(\psi_r)) \| D^\alpha \psi_r \|_{\mathbb{R}^n} \leq c(2^{r-2}d)^{\alpha(\| \alpha \|)}
\]
for every \( \alpha \in \mathbb{N}_0^n \) if we set \( c = l^n(n^{-1/2}2^{-n-3})n \), a constant which does not depend on \( r > 2 \) nor on \( \alpha \in \mathbb{N}_0^n \). \( \blacksquare \)

5. Approximation in Gevrey classes and consequences. In the proof of Theorem 5.2, we shall make use of the following property which results immediately from the proof of Lemma 5 of [5].

**Proposition 5.1** Let \( r \in \mathbb{N} \) and \( g \in D_r(\mathbb{R}^n) \). Then, for every \( \varepsilon > 0 \), there is \( \lambda_0 > 0 \) such that, for every \( \lambda \geq \lambda_0 \), the function
\[
h(x) = \pi^{-n/2} \lambda^n \int_{\mathbb{R}^n} g(y) e^{-\lambda^2 \| x-y \|^2} \, dy
\]
belongs to \( C_\infty(\mathbb{R}^n) \) (in fact, it is analytic on \( \mathbb{R}^n \)) and satisfies
\[
\| D^\alpha h - D^\alpha g \|_{\mathbb{R}^n} \leq \varepsilon \quad \text{if} \quad |\alpha| \leq r. \quad \blacksquare
\]

**Theorem 5.2.** Let \( \Omega \) be a nonvoid open subset of \( \mathbb{R}^n \) and let \( \zeta, \gamma \) be real numbers such that \( 1 < \zeta < \gamma \). Then, for every \( f \in \Gamma_\zeta(\Omega) \), there is \( g \in \Gamma_\gamma(\Omega) \) which is analytic on \( \Omega \) and such that
\[
\| D^\alpha f - D^\alpha g \|_{\Omega \setminus K_{s+1}} \leq \frac{1}{s} \quad \text{if} \quad |\alpha| \leq s \quad \text{and} \quad s \geq 2,
\]
with \( K_{s+1} \) defined as in the special compact cover of \( \Omega \).

**Proof.** Of course there are numbers \( a, b > 1 \) such that
\[
\| D^\alpha f \|_{\Omega} \leq ab^{\| \alpha \|}(|\alpha|!)^\zeta \quad \text{and} \quad \| D^\alpha v_{r-2} \|_{\Omega} \leq a(2^{r-2}b)^{\| \alpha \|}(|\alpha|!)^\zeta
\]
for every integer \( r \geq 3 \) and \( \alpha \in \mathbb{N}_0^n \).

Now we introduce by recursion a sequence \( (g_s)_{s \in \mathbb{N}} \) in \( C_\infty(\mathbb{R}^n) \) such that
\[
\| D^\alpha g_s \|_{\mathbb{R}^n} \leq 2^{s^2}a^{s+1}b^{(s+1)^2}(|\alpha|!)^\zeta, \quad \forall \alpha \in \mathbb{N}_0^n, \forall s \in \mathbb{N}
\]
At this point, to get the functions \( g_s \), we just need to consider a strictly increasing sequence \( (\lambda_s)_{s \in \mathbb{N}} \) of \( ]0, \infty[ \) but later on we shall make more stringent restrictions on these positive numbers.

We start with
\[
g_1(x) = \pi^{-n/2} \lambda_1^n \int_{\mathbb{R}^n} v_1(y)f(y)e^{-\lambda_1^2 \| x-y \|^2} \, dy, \quad \forall x \in \mathbb{R}^n,
\]
where of course \( v_1 f \) has been extended by 0 on \( \mathbb{R}^n \setminus \Omega \). By Proposition 2.1, \( g_1 \) belongs to \( C_\infty (\mathbb{R}^n) \) and satisfies
\[
\| D^\alpha g_1 \|_{\mathbb{R}^n} \leq a 2^j |\alpha| b |\alpha| (|\alpha|!)^\zeta \leq 2^0 a 2^2 |\alpha| b |\alpha| (|\alpha|!)^\zeta
\]
for every \( \alpha \in \mathbb{N}_0^n \). Now if \( g_1, \ldots, g_s \) are obtained, we first remark that we certainly have
\[
\| D^\alpha \left( f - \sum_{j=1}^s g_j \right) \|_{\Omega} \leq \| D^\alpha f \|_{\Omega} + \sum_{j=1}^s \| D^\alpha g_j \|_{\Omega}
\]
\[
\leq ab |\alpha|(|\alpha|!)^\zeta + \sum_{j=1}^s 2^{j-1} a^{j+1} 2^{(j+1)|\alpha|} b |\alpha| (|\alpha|!)^\zeta
\]
\[
\leq 2^s a^{s+1} 2^{(s+1)|\alpha|} b |\alpha| (|\alpha|!)^\zeta
\]
for every \( \alpha \in \mathbb{N}_0^n \) and then check by direct use of Proposition 2.1 that the function \( g_{s+1} \) defined by
\[
g_{s+1}(x) = \pi^{-n/2} \lambda_{s+1} \int_{\mathbb{R}^n} v_{s+1}(y) \left( f(y) - \sum_{j=1}^s g_j(y) \right) e^{-\lambda_{s+1}^2 |x-y|^2} \, dy
\]
suits our purpose.

Of course we have
\[
\left\| D^\alpha \sum_{j=1}^s g_j \right\|_{\mathbb{R}^n} \leq \sum_{j=1}^s 2^{j-1} a^{j+1} 2^{(j+1)|\alpha|} b |\alpha| (|\alpha|!)^\zeta
\]
\[
\leq 2^s a^{s+1} 2^{(s+1)|\alpha|} b |\alpha| (|\alpha|!)^\zeta
\]
for every \( s \in \mathbb{N} \) and \( \alpha \in \mathbb{N}_0^n \).

With this majorant at our disposal, we are in a position to make a more precise (but not yet final) choice of the numbers \( \lambda_s \) (we are free to take them larger but strictly increasing). As we have
\[
\lim_{|\alpha| \to \infty} 2^s a^{s+1} 2^{(s+1)|\alpha|} b |\alpha| (|\alpha|!)^\zeta = 0, \quad \forall s \in \mathbb{N},
\]
there is a strictly increasing sequence \( (A_s)_{s \in \mathbb{N}} \) in \( \mathbb{N} \) such that, for every \( s \in \mathbb{N} \),
\[
2^s a^{s+1} 2^{(s+1)|\alpha|} b |\alpha| (|\alpha|!)^\zeta \leq (|\alpha|!)^\gamma \quad \text{if } |\alpha| \geq A_s.
\]
Then we can also fix a strictly increasing sequence \( (B_s)_{s \in \mathbb{N}} \) in \( \mathbb{N} \) such that
\[
\sup_{|\alpha| \leq A_s} a 2^{|\alpha|} (|\alpha|!)^\zeta \leq B_s, \quad \forall s \in \mathbb{N}.
\]

Next we introduce the following elements of \( D_\infty (\mathbb{R}^n) \):
\[
h_1(x) = \begin{cases} v_1(x) f(x), & \forall x \in \Omega, \\ 0, & \forall x \in \mathbb{R}^n \setminus \Omega, \end{cases}
\]
and, for every integer $s \geq 2$,
\[
h_s(x) = \begin{cases} v_s(x)(f(x) - \sum_{j=1}^{s-1} g_j(x)), & \forall x \in \Omega, \\
0, & \forall x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\]
Finally, by recursive use of Proposition 5.1, we may require that the numbers $\lambda_s$ are such that, for every $s \in \mathbb{N}$,
\[
\|D^\alpha h_s - D^\alpha g_s\|_{\mathbb{R}^n} \leq (2^{s+2} + A_{s+1} B_{s+1})^{-1} \quad \text{if } |\alpha| \leq A_{s+1}.
\]
Now we consider the series
\[
g(x) = \sum_{s=1}^{\infty} g_s(x), \quad \forall x \in \Omega.
\]
We first prove that $g$ is defined and belongs to $C_\infty(\Omega)$. Indeed, for every $s \in \mathbb{N}$ and every $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq A_{s+1}$, we get
\[
\|D^\alpha h_{s+1}\|_{\mathcal{K}_{s+2}} \leq \sum_{\beta \leq \alpha} \left(\frac{\alpha}{\beta}\right) \|D^\beta v_{s+1}\|_{\mathcal{K}_{s+2}} \|D^{\alpha-\beta}(f - \sum_{j=1}^{s} g_j)\|_{\mathcal{K}_{s+2}}
\]
\[
\leq \sum_{*} \left(\frac{\alpha}{\beta}\right) a(2^{s+1}b)|\beta|!(2^{s+2} + A_{s+1} B_{s+1})^{-1}
\]
\[
\leq \sum_{\beta \leq \alpha} \left(\frac{\alpha}{\beta}\right) B_{s+1}(2^{s+2} + A_{s+1} B_{s+1})^{-1}
\]\n\[
\leq 2^{-s-2-A_{s+1}2|\alpha|} \leq 2^{-s-2}
\]
(at *), we have used the fact that $f - \sum_{j=1}^{s} g_j = h_s - g_s$ on $\mathcal{K}_{s+2}$). As $v_{s+1}(x) = 0$ for every $x \in K_{s+1}$, we get
\[
\|D^\alpha h_{s+1}\|_{\mathcal{K}_{s+2}} \leq 2^{-s-2} \quad \text{if } s \in \mathbb{N} \text{ and } |\alpha| \leq A_{s+1},
\]
hence
\[
\|D^\alpha g_{s+1}\|_{\mathcal{K}_{s+2}} \leq \|D^\alpha g_{s+1} - D^\alpha h_{s+1}\|_{\mathcal{K}_{s+2}} + \|D^\alpha h_{s+1}\|_{\mathcal{K}_{s+2}}
\]
\[
\leq (2^{s+3} + A_{s+2} B_{s+2})^{-1} + 2^{-s-2} \leq 2^{-s-1}
\]
for every $s \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq A_{s+1}$. Now it is clear that $g \in C_\infty(\Omega)$.

We establish next that $g$ satisfies the inequalities announced in the statement of the theorem—in fact, we are going to prove more. Consider an integer $s \geq 2$ and a multi-index $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq A_s$—this is certainly the case if $|\alpha| \leq s$. For every $x_0 \in \Omega \setminus K_{s+1}$, there is a first integer $p \geq 2$ such that $x_0 \in K_{s+p}$. According to the last established inequality, we of course have
\[
|D^\alpha g_{s+r}(x_0)| \leq 2^{-s-r}, \quad \forall r \in \{p-1, p, p+1, \ldots\}.
\]
As $v_{s+p-2}(H_{s+p}) = \{1\}$, we also get
\[
\left| D^\alpha f(x_0) - D^\alpha \sum_{j=1}^{s+p-2} g_j(x_0) \right| = \left| D^\alpha h_{s+p-2}(x_0) - D^\alpha g_{s+p-2}(x_0) \right| \leq 2^{-s-p}.
\]
Thus
\[
\left| D^\alpha f(x_0) - D^\alpha g(x_0) \right| \leq \left| D^\alpha f(x_0) - D^\alpha \sum_{j=1}^{s+p-2} g_j(x_0) \right| + \sum_{r=p-1}^{\infty} \left| D^\alpha g_{s+r}(x_0) \right|
\]
\[
\leq 2^{-s-p} + \sum_{r=p-1}^{\infty} 2^{-s-r} \leq 2^{-s-p} + 2^{-s-p+2} < \frac{1}{s},
\]
and hence
\[
\| D^\alpha f - D^\alpha g\|_{\Omega\setminus K_{s+1}} \leq \frac{1}{s} \quad \text{if } s \geq 2 \text{ and } |\alpha| \leq A_s.
\]

At this step, if we proceed as in the proof of Lemma 6 of [5], we see that it is possible to select successively the numbers $\lambda_s$ in such a way that $g$ is an analytic function on $\Omega$. (This is our last refinement on the choice of $\lambda_s$.)

We still have to prove that $g \in \Gamma_\gamma(\Omega)$.

Set
\[
\sup_{|\alpha| \leq A_2} \| D^\alpha g\|_{K_3/\langle |\alpha|! \rangle} = a_1
\]
and consider $\alpha \in \mathbb{N}_0^n$ and $x_0 \in \Omega$.

On the one hand, if $|\alpha| \leq A_2$ and
\begin{enumerate}
  \item if $x_0 \in K_3$, we trivially have
  \[
  |D^\alpha g(x_0)| \leq \| D^\alpha g\|_{K_3} \leq a_1 \langle |\alpha|! \rangle \gamma,
  \]
  \item if $x_0 \in \Omega \setminus K_3$, we get
  \[
  |D^\alpha g(x_0)| \leq |D^\alpha f(x_0)| + |D^\alpha f(x_0) - D^\alpha g(x_0)|
  \]
  \[
  \leq ab^{|\alpha|\langle |\alpha|! \rangle} + \frac{1}{2} \leq 2ab^{|\alpha|\langle |\alpha|! \rangle} \gamma.
  \]
\end{enumerate}

Hence there are constants $a_2, b_2 > 0$ such that
\[
\| D^\alpha g\|_{\Omega} \leq a_2b_2^{|\alpha|\langle |\alpha|! \rangle} \quad \text{if } |\alpha| \leq A_2.
\]

On the other hand, if $|\alpha| > A_2$, we first let $s$ be the integer such that $A_s < |\alpha| \leq A_{s+1}$ (of course $s \geq 2$) and then consider the following two possibilities:
\begin{enumerate}
  \item if $x_0 \in \Omega \setminus K_{s+2}$, then we have at once
  \[
  |D^\alpha g(x_0)| \leq |D^\alpha f(x_0)| + |D^\alpha f(x_0) - D^\alpha g(x_0)|
  \]
  \[
  \leq ab^{|\alpha|\langle |\alpha|! \rangle} + \frac{1}{s+1} \leq 2ab^{|\alpha|\langle |\alpha|! \rangle} \gamma,
  \]
\end{enumerate}
(ii) if \( x_0 \in K_{s+2} \), we have \( |D^\alpha g_{s+r}(x_0)| \leq 2^{-s-r} \) for every \( r \in \mathbb{N} \), hence

\[
\sum_{r=1}^\infty |D^\alpha g_{s+r}(x_0)| \leq \sum_{r=1}^\infty 2^{-s-r} = 2^{-s}.
\]

But we also have

\[
|D^\beta \sum_{j=1}^s g_j(x_0)| \leq 2^s a_3 b_3 (s+1)! |\beta| |\beta|! \zeta
\]

for every \( \beta \in \mathbb{N}_0^n \) and \( s \in \mathbb{N} \), hence

\[
\left| D^\alpha \sum_{j=1}^s g_j(x_0) \right| \leq \left( |\alpha|! \right)^\gamma
\]

since \( |\alpha| \geq A_s \). Therefore

\[
\|D^\alpha g\|_\Omega \leq a_3 b_3 (|\alpha|!)^\gamma \quad \text{if } |\alpha| > A_2. \]

\textbf{Corollary 5.3.} For every open and nonvoid subset \( \Omega \) of \( \mathbb{R}^n \) and \( \gamma \in ]1, 1[ \), there is a function \( g \in \Gamma_\gamma \) which is

(a) analytic on \( \Omega \),
(b) identically 0 on no connected component of \( \Omega \),
(c) flat on \( \mathbb{R}^n \setminus \Omega \) (i.e. identically 0 together with all its derivatives on \( \mathbb{R}^n \setminus \Omega \)).

\textbf{Proof.} If \( \Omega \) is connected, we choose \( \zeta \in ]1, \gamma[ \) and \( f \in \Gamma_\zeta(\Omega) \) with compact support contained in \( K_4 \setminus K_3 \) and such that \( \|f\|_\Omega > 1/2 \). Then Theorem 5.2 provides \( g \in \Gamma_\gamma(\Omega) \) which is analytic on \( \Omega \) and such that

\[
\|D^\alpha f - D^\alpha g\|_{\Omega \setminus K_{s+1}} \leq \frac{1}{2} \quad \text{if } |\alpha| \leq 2
\]

(which implies that \( g \) is not identically 0 on \( \Omega \)) as well as

\[
\|D^\alpha g\|_{\Omega \setminus K_{s+1}} = \|D^\alpha g - D^\alpha f\|_{\Omega \setminus K_{s+1}} \leq \frac{1}{s} \quad \text{if } |\alpha| \leq s \text{ and } s \geq 3.
\]

It is then well known that extending \( g \) by 0 on \( \mathbb{R}^n \setminus \Omega \) provides a solution.

If \( \Omega \) has a finite number of connected components—say \( \Omega_1, \ldots, \Omega_m \)—then, for every \( k \in \{1, \ldots, m\} \), there is \( g_k \in \Gamma_\gamma \) which is analytic on \( \Omega_k \), not identically 0 on \( \Omega_k \) and flat on \( \mathbb{R}^n \setminus \Omega_k \). It is then clear that \( g = \sum_{k=1}^m g_k \) is a solution.

As \( \Omega \) always has countably many connected components, to conclude, we just have to settle the case when \( \{\Omega_m : m \in \mathbb{N}\} \) is the set of connected
components of \(\Omega\). For this purpose, fix \(\zeta \in ]1, \gamma[\). By the first part of the proof, for every \(m \in \mathbb{N}\), there is \(g_m \in \Gamma_{\zeta}\) which is analytic on \(\Omega_m\), not identically 0 on \(\Omega_m\), flat on \(\mathbb{R}^n \setminus \Omega_m\) and such that
\[
\|D^\alpha g_m\|_{\Omega_m \setminus K_{m,r+1}} \leq \frac{1}{s} \quad \text{if } |\alpha| \leq s \text{ and } s \geq 2,
\]
where of course \(K_{m,r}\) is the \(r\)th element of the special cover of \(\Omega_m\). So, for every \(m \in \mathbb{N}\), there are constants \(a_m, b_m > 1\) such that
\[
\|D^\alpha g_m\|_{\mathbb{R}^n} \leq a_m b_m^{|\alpha|}(|\alpha|!)^\zeta, \quad \forall \alpha \in \mathbb{N}_0^n,
\]
therefore there is an integer \(k_m \geq m\) such that
\[
a_m b_m^{k_m}(|\alpha|!)^\zeta \leq (|\alpha|!)^\gamma \quad \text{if } |\alpha| \geq k_m.
\]
Then we set
\[
c_m = (\sup_{|\alpha| \leq k_m} a_m b_m^{|\alpha|}(|\alpha|!)^\zeta)^{-1} \quad \text{and} \quad g = \sum_{m=1}^{\infty} 2^{-m} c_m g_m.
\]
It is clear that \(g\) is a function defined on \(\mathbb{R}^n\) which is analytic on \(\Omega\), identically 0 on no connected component of \(\Omega\) and identically 0 on \(\mathbb{R}^n \setminus \Omega\). Now we prove that \(g\) belongs to \(C_\infty(\mathbb{R}^n)\) and is flat on \(\mathbb{R}^n \setminus \Omega\). Let \(x \in \mathbb{R}^n \setminus \Omega\). For every integer \(k \geq 3\), there is \(r > 0\) such that the ball \(b = \{y \in \mathbb{R}^n : |x - y| \leq r\}\) is disjoint from the compact sets \(K_{1,k+1}, \ldots, K_{k,k+1}\). For every \(\alpha \in \mathbb{N}_0^n\) such that \(|\alpha| \leq k\), this leads to
\[
\sup_{x \in b} \sum_{m=1}^{\infty} 2^{-m} c_m |D^\alpha g_m| \\
\leq \sup_{m \leq k} \left\{ \sup_{m \leq k} \frac{2^{-m} c_m}{k} \sup_{m > k} \|D^\alpha g_m\|_{\mathbb{R}^n} \right\} \leq \sup\{1/k, 2^{-k}\},
\]
hence \(g\) belongs to \(C_\infty(\mathbb{R}^n)\) and is flat on \(\mathbb{R}^n \setminus \Omega\). We still have to prove that \(g \in \Gamma_\gamma\). This is immediate: for every \(\alpha \in \mathbb{N}_0^n\), we have
\[
\|D^\alpha g\|_{\mathbb{R}^n} = \sup_{m \in \mathbb{N}} 2^{-m} c_m \|D^\alpha g_m\|_{\mathbb{R}^n} \leq \sup_{m \in \mathbb{N}} \{2^{-m}, 2^{-m}(|\alpha|!)^\gamma\} \leq (|\alpha|!)^\gamma
\]
by consideration of the cases \(|\alpha| \leq k_m\) and \(|\alpha| > k_m\). \(\blacksquare\)

**Corollary 5.4.** For every \(\gamma \in ]1, \infty[\) and nondegenerate compact intervals \(I, J\) of \(\mathbb{R}\) such that \(J \subset I^o\), there are \(f \in C_\infty(\mathbb{R})\) and \(c > 0\) such that
(a) \(f\) has no divergence point,
(b) \(f(\mathbb{R}) \subset [0, 1]\),
(c) \(f(\mathbb{R} \setminus I) = \{0\}, f(I^o) \subset ]0, 1[\) and \(f(J) = \{1\}\),
(d) \(\|D^k f\|_{\mathbb{R}} \leq c k^{\gamma k}, \forall k \in \mathbb{N}\).
Proof. Let $I = [a_1, b_1]$ and $J = [a_2, b_2]$. We choose $\zeta \in ]1, \gamma[$ and apply Corollary 5.3 with $\Omega = ]a_1, a_2[$: there is a nonzero $g \in \Gamma_\zeta$ which is analytic on $]a_1, a_2[$ and such that $\text{supp}(g) = [a_1, a_2]$. Now we choose $k > 0$ such that $g_1 = kg^2$ satisfies $\int_R g_1(x) \, dx = 1$ and define the function $f_1$ on $R$ by

$$f_1(x) = \int_{-\infty}^{x} g_1(t) \, dt, \quad \forall x \in R.$$  

It is clear that $f_1$ belongs to $\Gamma_\zeta$, is analytic on $]a_1, a_2[$ and satisfies

$$f_1([-\infty, a_1]) = \{0\}, \quad f_1([a_1, a_2[) \subset ]0, 1] \quad \text{and} \quad f_1([a_2, \infty[) = \{1\}.$$  

So it is clear that $f_1$ has no divergence point.

Similarly there is $f_2 \in \Gamma_\zeta$ which is analytic on $]b_2, b_1[$, has no divergence point and satisfies

$$f_2([-\infty, b_2]) = \{0\}, \quad f_2([b_2, b_1[) \subset ]0, 1] \quad \text{and} \quad f_2([b_1, \infty[) = \{1\}.$$  

Finally, we set

$$f(x) = f_1(x)f_2(b_1 + b_2 - x), \quad \forall x \in R.$$  

Of course $f$ belongs to $\Gamma_\zeta$ and satisfies (a)–(c). Let us establish that $f$ also satisfies (d). As $f \in \Gamma_\zeta$, there are $a, b > 0$ such that

$$\|f^k\|_R \leq ab^k \zeta^k, \quad \forall k \in \mathbb{N}_0.$$  

Since

$$\lim_{k \to \infty} ab^k \zeta^k \frac{k^c}{\gamma^k} = 0,$$  

there is $k_0 \in \mathbb{N}$ such that $ab^k \zeta^k \leq k^{c \gamma^k}$ for every $k \geq k_0$; therefore there is $c > 0$ such that

$$\|f^k\|_R \leq c \gamma^k, \quad \forall k \in \mathbb{N}. \quad \blacksquare$$  

Proposition 5.5. Let $\gamma \in ]1, \infty[$, let $p \in \mathbb{N}$ and let $I, J$ be nondegenerate compact intervals of $R$ such that $J \subset I^n$. Then there is $m_0 \in \mathbb{N}$ such that, for every integer $m \geq m_0$, there is a function $u \in C_\infty(R)$ satisfying the following conditions:

(a) $u$ has no divergence point,
(b) $\text{supp}(u) \subset I$,
(c) $\|u^k\|_R \leq 2^{-m}, \quad \forall k \in \{0, 1, \ldots, m\}$,
(d) $\|u^k\|_R \leq 2^k \gamma^k, \quad \forall k \in \mathbb{N}$,
(e) for every $x \in J$, one has either

$$|u^{(pm)}(x)| \geq 5^{-m} \gamma^{(p-1)m}$$  

or

$$|u^{(pm+1)}(x)| \geq 5^{-(m+1/p)} \left( \frac{pm + 1}{p + 1} \right)^{\gamma^{(p-1)(m+1)/p}}.$$  

Proof. Corollary 5.4 provides a function \( f \in C_\infty(\mathbb{R}) \) and a constant \( c > 0 \) such that \( f \) has no divergence point and satisfies
\[
 f(\mathbb{R}) \subset [0, c], \quad f(\mathbb{R} \setminus I) = \{0\}, \quad f(J) = \{1\}
\]
as well as
\[
 \|f^{(k)}\|_\mathbb{R} \leq c k^{\gamma k}, \quad \forall k \in \mathbb{N}.
\]
For any \( m \in \mathbb{N} \), we can introduce
\[
 a = m^\gamma, \quad b = (2^{2m} m^\gamma m^{-1}),
\]
\[
 u(x) = bf(x) \sin(ax), \quad \forall x \in \mathbb{R}.
\]
It is then clear that \( u \) is a \( C_\infty \)-function on \( \mathbb{R} \) satisfying the conditions (a) and (b) as well as
\[
 \|u\|_\mathbb{R} \leq bc = (2^{2m} m^\gamma m^{-1}) \leq 2^{-m}.
\]
Moreover, for every \( x \in \mathbb{R} \) and \( k \in \mathbb{N} \), we have
\[
 |u^{(k)}(x)| \leq b \sum_{h=0}^{k} \binom{k}{h} |f^{(h)}(x)| a^{k-h}
\]
\[
 \leq bc \sum_{h=0}^{k} \binom{k}{h} h^{\gamma h} a^{k-h} \leq (2^{2m} m^\gamma m^{-1}) (k^{\gamma} + a)^{k},
\]
hence
\[
 |u^{(k)}(x)| \leq (2^{2m} m^\gamma m^{-1}) (m^{\gamma} + m^{\gamma})^{m} = 2^{-m} \quad \text{if } 1 \leq k \leq m
\]
as well as
\[
 |u^{(k)}(x)| \leq (2^{2m} m^\gamma m^{-1}) (k^{\gamma} + k^{\gamma})^{k} \leq 2^{k} k^{\gamma k} \quad \text{if } k > m;
\]
i.e. \( u \) also satisfies the conditions (c) and (d).

Now we investigate (e). Let \( x \in J \). Of course we have
\[
 u^{(k)}(x) = a^{k} b \sin(k \pi/2 + ax), \quad \forall k \in \mathbb{N}.
\]
Now, for every \( m \in \mathbb{N} \), we certainly have
\[
 \sup \{|\sin(pm \pi/2 + ax)|, |\sin((pm + 1) \pi/2 + ax)|\} \geq 2^{-1/2}.
\]
So on the one hand, if \( |\sin(pm \pi/2 + ax)| \geq 2^{-1/2} \), we get
\[
 |u^{(pm)}(x)| \geq 2^{-1/2} a^{pm} b = 4^{-m} (\sqrt{2}c)^{-1} m^{\gamma(p-1)m},
\]
and on the other hand, if \( |\sin((pm + 1) \pi/2 + ax)| \geq 2^{-1/2} \), then
\[
 |u^{(pm+1)}(x)| \geq 2^{-1/2} a^{pm+1} b = 4^{-m} (\sqrt{2}c)^{-1} m^{\gamma(p-1)(m+1)}
\]
\[
 \geq 4^{-m+1/p} (\sqrt{2}c)^{-1} m^{\gamma(p-1)(m+1)/p} 
\]
\[
 \geq 4^{-m+1/p} (\sqrt{2}c)^{-1} \left( \frac{pm + 1}{p + 1} \right)^{\gamma(p-1)(m+1)/p}.
\]
To conclude, it is enough to take as \( m_0 \) any positive integer \( m_0 \) such that
\[
\sup\{4(\sqrt{2}c)^{1/m_0}, 4(\sqrt{2}c)^{1/(m_0+1/p)}\} \leq 5. \]

6. Characterizing the sets of divergence points. In this section, we establish the following result.

**Theorem 6.1.** For every \( \gamma \in [1, \infty] \) and every \( G_5 \)-subset \( G \) of \( \mathbb{R}^n \), there is an element of \( \Gamma_\gamma \) having \( G \) as its set of divergence points.

**Proof.** We proceed in several steps.

**Step 1:** the numbers \( \gamma_j \) and \( p_j \). We fix a strictly increasing sequence \((\gamma_j)_{j \in \mathbb{N}_0} \subset [1, \gamma]\) and, for every \( j \in \mathbb{N} \), denote by \( p_j \) a positive integer such that \( p_j(\gamma_j - \gamma_{j-1}) > \gamma_j \).

**Step 2:** some auxiliary inequalities and the numbers \( q_r \). For every \( r \in \mathbb{N} \), we certainly have
\[
\frac{p_r - 1}{p_r} \gamma_r > \gamma_{r-1} > \ldots > \gamma_0 > 1.
\]
Therefore, for every \( j \in \{0, \ldots, r - 1\} \), it is a straightforward matter to check the following limits:

1. \[
\lim_{m \to \infty} 5^m 2^{p_r m} (p_r m)^{\gamma_r p_r - m} m^{\gamma_r (p_r - 1)m} \frac{1}{m^{\gamma_r (p_r - 1)m}} = \lim_{m \to \infty} \left( \frac{4.5^{1/p_r} p_r^{\gamma_r}}{m^{\gamma_r (p_r - 1)/p_r - \gamma_r \gamma_j}} \right)^{p_r m} = 0,
\]
2. \[
\lim_{m \to \infty} 5^{m+1/p_r} 2^{(p_r m + 1)} (p_r m + 1)^{\gamma_r (p_r m + 1) + 1} \frac{p_r + 1}{p_r m + 1} \gamma_r (p_r - 1)(m + 1/p_r) = \lim_{m \to \infty} \left( \frac{4.5^{1/p_r} (p_r + 1)^{\gamma_r (p_r - 1)/p_r}}{(p_r m + 1)^{\gamma_r (p_r - 1)/p_r - \gamma_r \gamma_j}} \right)^{p_r m + 1} = 0,
\]
3. \[
\lim_{m \to \infty} 5^m (nr)^{p_r m} (p_r m + 1)^{p_r m} m^{\gamma_r (p_r - 1)m} \frac{1}{m^{\gamma_r (p_r - 1)m}} = \lim_{m \to \infty} \left( \frac{5^{1/p_r} nr p_r^{\gamma_r (p_r - 1)/p_r}}{(p_r m + 1)^{\gamma_r (p_r - 1)/p_r - \gamma_r \gamma_j}} \right)^{p_r m + 1} \left(\frac{p_r + 1}{p_r m + 1}\right)^{p_r m} = 0,
\]
4. \[
\lim_{m \to \infty} 5^{m+1/p_r} (nr)^{p_r m + 1}(p_r m + 2)^{p_r m + 1} \frac{p_r + 1}{p_r m + 1} \gamma_r (p_r - 1)(m + 1/p_r) = \lim_{m \to \infty} \left( \frac{5^{1/p_r} nr (p_r + 1)^{\gamma_r (p_r - 1)/p_r}}{(p_r m + 1)^{\gamma_r (p_r - 1)/p_r - \gamma_r \gamma_j}} \right)^{p_r m + 1} \left(\frac{p_r + 2}{p_r m + 1}\right)^{p_r m + 1} = 0.
\]

With these limits at our disposal, we find that, for every \( r \in \mathbb{N} \), there is \( q_r \in \mathbb{N} \) such that, for every integer \( m \geq q_r \) and \( j \in \{1, \ldots, r - 1\} \), we have the following auxiliary inequalities:
(I) \[ 4 \cdot 2^{2p^m} (p_r)^{\gamma_r} < 5^{-m} m^{\gamma_r (p_r - 1)} m, \]

(II) \[ 4 \cdot 2^{(p_r + 1)} (p_r m + 1)^{\gamma_r (p_r + 1)} < 5^{-(m + 1) / p_r \gamma_r (p_r - 1) m}, \]

(III) \[(nr)^{p_r} (p_r m + 1)^{p_r} < 5^{-m} m^{\gamma_r (p_r - 1)} m, \]

(IV) \[(nr)^{p_r + 1} (p_r m + 2)^{p_r + 1} < 5^{-(m + 1) / p_r \gamma_r (p_r - 1) m}. \]

**Step 3:** the sets \( G_l, Q_r, P_r, I_{r,j} \) and \( J_{r,j} \). Being a \( G_\delta \)-subset of \( \mathbb{R}^n \), \( G \) is equal to the intersection of a sequence \( (G_l)_{l \in \mathbb{N}} \) of open subsets of \( \mathbb{R}^n \) that we may suppose decreasing.

Proceeding as in the construction of the special compact cover of an open set, we find that each \( G_l \) is the union of countably many compact cubes \( Q_{l,m,h} \) that we may renumber as a sequence, say \( (Q_{l,k})_{k \in \mathbb{N}} \). Then for every \( l, k \in \mathbb{N} \), we denote by \( P_{l,k} \) the compact cube in \( \mathbb{R}^n \) having the same center as \( Q_{l,k} \) and \( \frac{1}{2} \text{diam}(Q_{l,k}) \) as diameter. Now we arrange \( \mathbb{N}^2 \) into a sequence \( ((l_r, k_r))_{r \in \mathbb{N}} \), set

\[ Q_r = Q_{l_r, k_r} \quad \text{and} \quad P_r = P_{l_r, k_r}, \]

and let \( I_{r,j} \) and \( J_{r,j} \) for \( j \in \{1, \ldots, n\} \) be the compact intervals in \( \mathbb{R} \) such that

\[ Q_r = \prod_{j=1}^{n} J_{r,j} \quad \text{and} \quad P_r = \prod_{j=1}^{n} I_{r,j}. \]

Of course this construction leads to \( J_{r,j} \subset I_{r,j}^0 \) for every \( r \in \mathbb{N} \) and \( j \in \{1, \ldots, n\} \).

**Step 4:** the functions \( u_{r,j} \) and the numbers \( m_r \). At this point, everything is set up to introduce the functions \( u_{r,j} \) for \( r \in \mathbb{N} \) and \( j \in \{1, \ldots, n\} \), as well as the sequence \( (m_r)_{r \in \mathbb{N}} \) of \( \mathbb{N} \) by the following recursion.

An application of Proposition 5.5 to \( \gamma = \gamma_1 \) and \( p = p_1 \) leads to an integer \( m_1 > q_1 \) and to functions \( u_{1,1}, \ldots, u_{1,n} \in C_\infty(\mathbb{R}) \) such that, for every \( j \in \{1, \ldots, n\} \),

(a) \( u_{1,j} \) has no divergence point,

(b) \( \text{supp}(u_{1,j}) \subset I_{1,j}, \)

(c) \( \|u_{1,j}^{(k)}\|_\infty \leq 2^{-m_1}, \forall k \in \{0, 1, \ldots, m_1\}, \)

(d) \( \|u_{1,j}^{(k)}\|_2 \leq 2^k k^{\gamma_1 k}, \forall k \in \mathbb{N}, \)

(e) for every \( t \in J_{1,j} \), one has either

\[ |u_{1,j}^{(p_{1}m_1)}(t)| \geq 5^{-m_1} m_1^{\gamma_1 (p_1 - 1) m_1} \]
or
\[ |u_{r,j}^{(p, m r)}(t)| \geq 5^{-(m r + 1/p r)} \left( \frac{p r m r + 1}{p r + 1} \right)^{r/(p r - 1)}(m r + 1/p r). \]

Now, for an integer \( r \geq 2 \), if the functions \( u_{r,j} \) for \( t \in \{1, \ldots, r - 1\} \) and \( j \in \{1, \ldots, n\} \) and the integers \( m_1, \ldots, m_{r-1} \) are obtained, we apply Proposition 5.5 to \( \gamma = \gamma_r \) and \( p = p_r \) and obtain an integer \( m_r \geq \sup\{p_{r-1} m_{r-1}, q_r\} \) and functions \( u_{r,1}, \ldots, u_{r,n} \in C_\infty(\mathbb{R}) \) such that, for every \( j \in \{1, \ldots, n\} \),

(a) \( u_{r,j} \) has no divergence point,
(b) \( \text{supp}(u_{r,j}) \subset J_{r,j} \),
(c) \( \|u_{r,j}^{(k)}\|_R \leq 2^{-m r}, \forall k \in \{0, 1, \ldots, m_r\} \),
(d) \( \|u_{r,j}^{(k)}\|_R \leq 2^k k\gamma_r^{-k}, \forall k \in \mathbb{N} \),
(e) for every \( t \in J_{r,j} \), one has either
\[ |u_{r,j}^{(p, m r)}(t)| \geq 5^{-m r} m_r^{\gamma_r}(p_r - 1)m r \]
or
\[ |u_{r,j}^{(p, m r + 1)}(t)| \geq 5^{-(m r + 1/p r)} \left( \frac{p r m r + 1}{p r + 1} \right)^{r/(p r - 1)}(m r + 1/p r). \]

Step 5: the functions \( u_r \) and \( u \). Finally, for every \( r \in \mathbb{N} \), we define,
\[ u_r(x) = u_{r,1}(x_1) \ldots u_{r,n}(x_n), \quad \forall x \in \mathbb{R}^n \]
and consider the series \( u = \sum_{r=1}^{\infty} u_r \). For every \( k \in \mathbb{N} \), we certainly have \( k \leq m_k \). Therefore, for every \( \alpha \in \mathbb{N}_0^n \),
\[ \sum_{r=\sup\{1,|\alpha|\}}^{\infty} \|D^\alpha u_r\|_{\mathbb{R}^n} \leq \sum_{r=\sup\{1,|\alpha|\}}^{\infty} 2^{-nm r} \leq 1; \]
this implies that \( u \) is a bounded \( C_\infty \)-function on \( \mathbb{R}^n \). Moreover, for every \( \alpha \in \mathbb{N}_0^n \) such that \( |\alpha| \geq 1 \), we have
\[ \|D^\alpha u\|_{\mathbb{R}^n} \leq \sum_{r=1}^{\infty} \|D^\alpha u_r\|_{\mathbb{R}^n} + \sum_{r=|\alpha|}^{\infty} \|D^\alpha u_r\|_{\mathbb{R}^n} \]
\[ \leq \sum_{r=1}^{\infty} 2^{|\alpha|}|\alpha|^{\gamma_r}|\alpha| + 1 \leq 3^{|\alpha|}|\alpha|^{\gamma_r}|\alpha|, \]
hence \( u \in \Gamma_\gamma \).

To conclude, we prove that \( G \) is the set of divergence points of \( u \).

On the one hand, if \( x \in \mathbb{R}^n \) does not belong to \( G \), then \( x \not\in G_{l_0} \) for some \( l_0 \), hence \( x \not\in G_l \) for all \( l \geq l_0 \). This implies that \( x \) belongs to an at most
finite number of the $P_r$'s. Therefore for $r$ large enough we have $D^\alpha u_r(x) = 0$ for every $\alpha \in \mathbb{N}_0^n$ and clearly $x$ is not a divergence point of $u$.

On the other hand, let us prove by contradiction that every element of $G$ is a divergence point of $u$. Suppose that $x \in G$ is not a divergence point of $u$. This implies the existence of $s \in \mathbb{N}$ such that

$$|D^\beta u(x)| \leq s|\beta|^{1/|\beta|}$$

if $|\beta| \geq 1$.

As $x$ belongs to each $G_l$, there is an integer $r > 3s$ such that $x \in Q_r$; in particular, $x_j \in J_{r,j}$ for every $j \in \{1, \ldots, n\}$.

Fix $j \in \{1, \ldots, n\}$. The consideration of the property (e) leads to the following two possibilities.

Case 1: We have

$$\left| u_{r,j}^{(p_r m_r)}(x_j) \right| \geq 5^{-m_r} m_r^{\gamma r (p_r - 1) m_r}.$$ 

Then we set $\alpha_j = p_r m_r$ and remark that

(1.i) the auxiliary inequality (III) leads to

$$\left| u_{r,j}^{(\alpha_j)}(x_j) \right| \geq (nr)^{\alpha_j} (\alpha_j + 1)^{\alpha_j},$$

(1.ii) the use of (I) in (\star) leads to

$$\sum_{t=1}^{r-1} |u_{t,j}^{(\alpha_j)}(x_j)| + \sum_{t=r+1}^{\infty} |u_{t,j}^{(\alpha_j)}(x_j)|$$

$$\leq \sum_{t=1}^{r-1} 2^{\alpha_j} \gamma^{\alpha_j} + \sum_{t=r+1}^{\infty} 2^{-m_t} \leq 2 \sum_{t=1}^{r-1} 2^{p_r m_r} m_r^{\gamma r (p_r - 1) m_r}$$

$$\leq 1 \sum_{t=1}^{r-1} 2^{-m_r} m_r^{\gamma r (p_r - 1) m_r} \leq \frac{1}{2} |u_{r,j}^{(\alpha_j)}(x_j)|.$$ 

Case 2: (\star) does not hold. Then we have

$$\left| u_{r,j}^{(p_r m_r + 1)}(x_j) \right| \geq 5^{-m_r + 1/p_r} \left( \frac{p_r m_r + 1}{p_r + 1} \right)^{\gamma r (p_r - 1) (m_r + 1/p_r)}.$$ 

we set $\alpha_j = p_r m_r + 1$ and remark that

(2.i) the auxiliary inequality (IV) leads to

$$\left| u_{r,j}^{(\alpha_j)}(x_j) \right| \geq (nr)^{\alpha_j} (\alpha_j + 1)^{\alpha_j},$$

(2.ii) the use of (II) in (\star) leads to
\[
\sum_{t=1}^{r-1} |u_{t,j}(x_j)| + \sum_{t=r+1}^{\infty} |u_{t,j}^{(\alpha_j)}(x_j)| \leq \sum_{t=1}^{r-1} 2^{\alpha_j} \alpha_j^{\gamma t} + \sum_{t=r+1}^{\infty} 2^{-m_t} \leq 2 \sum_{t=1}^{r-1} 2^{\alpha t_m r+1} (p_r m_r + 1)^{\alpha (p_r m_r + 1)}
\]

\[
\leq \frac{1}{2} \sum_{t=1}^{r-1} 2^{(p_r m_r + 1) \frac{s}{m_t} - (m_t + 1/p_r)} \left( \frac{p_r m_r + 1}{m_t + 1/p_r} \right)^{\gamma_t \alpha (p_r - 1)(m_t + 1/p_r)}
\]

\[
\leq \frac{1}{2} |u_{r,j}^{(\alpha_j)}(x_j)|.
\]

So setting \( \alpha = (\alpha_1, \ldots, \alpha_n) \) yields

\[
|D^\alpha u(x)| \geq |D^\alpha u_r(x) - \sum_{t=1}^{r-1} |D^\alpha u_t(x)| - \sum_{t=r+1}^{\infty} |D^\alpha u_t(x)|
\]

\[
\geq |D^\alpha u_r(x)| - \prod_{j=1}^{n} \left( \sum_{t=1}^{r-1} |u_{t,j}^{(\alpha_j)}(x_j)| + \sum_{t=r+1}^{\infty} |u_{t,j}^{(\alpha_j)}(x_j)| \right)
\]

\[
\geq |D^\alpha u_r(x)| - \prod_{j=1}^{n} \frac{1}{2} |u_{r,j}^{(\alpha_j)}(x_j)| \geq \frac{1}{2} |D^\alpha u_r(x)|.
\]

For every \( j \in \{1, \ldots, n\} \), as \( \alpha_j \) belongs to \( \{m_r p_r, m_r p_r + 1\} \), we certainly have \( \alpha_j + 1 \geq |\alpha|/n \). Therefore

\[
|D^\alpha u(x)| \geq \frac{1}{2} |D^\alpha u_r(x)| = \frac{1}{2} \prod_{j=1}^{n} |u_{r,j}^{(\alpha_j)}(x_j)|
\]

\[
\geq \frac{1}{2} \prod_{j=1}^{n} (nr)^{\alpha_j} (\alpha_j + 1)^{\alpha_j} \geq \frac{1}{2} (nr)^{|\alpha|} \left( \frac{|\alpha|}{n} \right)^{|\alpha|} = \frac{1}{2} r^{|\alpha|} |\alpha|^{||\alpha||}
\]

and finally, as we have chosen \( r > 3s \), we arrive at the following contradiction:

\[
|D^\alpha u(x)| \geq \frac{1}{2} (3s)^{|\alpha|} |\alpha|^{||\alpha||} > s^{|\alpha|} |\alpha|^{||\alpha||}.
\]

7. Proof of Theorem 1.1. We first fix some \( \zeta \in ]1, \gamma[ \). We next apply Theorem 6.1 to get \( u \in \Gamma_{\zeta} \) having \( G \) as its set of divergence points. We then apply Theorem 5.2 to get \( h \in \Gamma_{\gamma}(\mathbb{R}^n \setminus G^-) \) which is analytic on \( \mathbb{R}^n \setminus G^- \) and such that

\[
||D^\alpha u - D^\alpha h||_{(\mathbb{R}^n \setminus G^-), K_{\alpha+1}} \leq \frac{1}{s} \quad \text{if } |\alpha| \leq s \text{ and } s \geq 2
\]

(where of course \( K_s \) is the \( s \)th compact set corresponding to the special
compact cover of the open set $\mathbb{R}^n \setminus G^-$. So the function

$$f : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \begin{cases} u(x) & \text{if } x \in G^-, \\ h(x) & \text{if } x \in \mathbb{R}^n \setminus G^-, \end{cases}$$

belongs to $\Gamma_{\gamma}$, is analytic on $\mathbb{R}^n \setminus G^-$ and has $G$ as its set of divergence points.

We now apply Corollary 5.3 to get $g \in \Gamma_{\gamma}$ which is analytic on $\Omega$, identically 0 on no connected component of $\Omega$ and flat on $\mathbb{R}^n \setminus \Omega$; in particular, $g$ has no divergence point.

To conclude one just has to check that the function $f + g$ suits our purpose: $f + g$ certainly belongs to $\Gamma_{\gamma}$, is analytic on $\Omega$ (since $\Omega \subset \mathbb{R}^n \setminus G^-$) and has $G$ as its set of divergence points. Moreover, no point $x$ of $F$ can be a divergence point (since $F$ and $G$ are disjoint), nor an analytic point (this would imply that $f + g$ is analytic on some open ball $b$ centered at $x$; this in turn implies that $b$ and $G$ are disjoint, so $f$ must be analytic on $b$; finally, $g$ is analytic hence flat on $b$, contrary to the fact that $x$ must belong to the boundary of $\Omega$).

References


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