FUNDAMENTA
MATHEMATICAE
151 (1996)

A note on strange nonchaotic attractors

by

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Abstract. For a class of quasiperiodically forced time-discrete dynamical systems of two variables $(\theta, x) \in \mathbb{T}^1 \times \mathbb{R}_+$ with nonpositive Lyapunov exponents we prove the existence of an attractor $\overline{\Gamma}$ with the following properties:

1. $\overline{\Gamma}$ is the closure of the graph of a function $x = \phi(\theta)$. It attracts Lebesgue-a.e. starting point in $\mathbb{T}^1 \times \mathbb{R}_+$. The set $\{\theta : \phi(\theta) \neq 0\}$ is meager but has full 1-dimensional Lebesgue measure.
2. The omega-limit of Lebesgue-a.e. point in $\mathbb{T}^1 \times \mathbb{R}_+$ is $\overline{\Gamma}$, but for a residual set of points in $\mathbb{T}^1 \times \mathbb{R}_+$ the omega limit is the circle $\{(\theta, x) : x = 0\}$ contained in $\overline{\Gamma}$.
3. $\overline{\Gamma}$ is the topological support of a BRS measure. The corresponding measure theoretical dynamical system is isomorphic to the forcing rotation.

Let $X = \mathbb{T}^1 \times [0, \infty)$. We study the dynamical system $T : X \to X$,

$$T(\theta, x) = (\theta + \omega, f(x) \cdot g(\theta))$$

where $\omega \in \mathbb{R} \setminus \mathbb{Q}$, $f : [0, \infty) \to [0, \infty)$ is bounded C^1 and $g : \mathbb{T}^1 \to [0, \infty)$ is continuous. We assume furthermore that $f(0) = 0$ and that f is increasing and strictly concave (i.e. $0 < f'(x) \searrow$). Define

$$\sigma := f'(0) \cdot \exp\Big(\int \log g(\theta)\, d\theta\Big).$$

As g is bounded, the integral in this definition is always well defined, although it may be equal to $-\infty$ in which case it is natural to set $\sigma := 0$. (This happens in particular, if $g(\theta) = 0$ for a set of θ's of positive Lebesgue measure.) Finally, if no ambiguity can arise, we use the notation $(\theta_n, x_n) = T^n(\theta, x)$. With this notation we define the *vertical Lyapunov exponent* at (θ, x) as $\lambda(\theta, x) = \lim_{n\to\infty} (1/n) \log \partial x_n/\partial x$ if this limit exists. By $\overline{\lambda}(\theta, x)$ we denote the corresponding limit superior. In order to make the dependence

1991 *Mathematics Subject Classification*: 58F11–13.
Work partially supported by the DFG.

of $\partial x_n/\partial x$ on θ more explicit we also use the notation

$$L_n(\theta,x) := \frac{\partial x_n}{\partial x} = \prod_{k=0}^{n-1} g(\theta+k\omega)\cdot f'(x_k).$$

Note also that $x_n \le y_n$ for all n if x and y are on the same θ-fiber and if $x<y$.

For a measurable function $\psi:\mathbb{T}^1\to[0,\infty)$ define

$$\lambda_\psi := \int \log g(\theta)\,d\theta + \int \log f'(\psi(\theta))\,d\theta.$$

(Here and henceforth all integrals with $d\theta$ are taken over $\mathbb{T}^1$.) λ_ψ is well defined because $\log f'$ and $\log g$ are both bounded from above. The graph of ψ is called *invariant* if

$$f(\psi(\theta))\cdot g(\theta) = \psi(\theta+\omega) \qquad \text{for a.e. } \theta\in\mathbb{T}^1.$$

An easy induction yields that in this case for a.e. $\theta\in\mathbb{T}^1$,

$$T^k(\theta,\psi(\theta)) = (\theta+k\omega,\psi(\theta+k\omega)) \quad \text{for all } k\in\mathbb{N}$$

and hence

$$\begin{aligned}(1)\qquad \lambda(\theta,\psi(\theta)) &= \lim_{n\to\infty}\frac{1}{n}\log L_n(\theta,\psi(\theta))\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}[\log g(\theta+k\omega)+\log f'(\psi(\theta+k\omega))]\\ &= \int \log g(\theta)\,d\theta + \int \log f'(\psi(\theta))\,d\theta = \lambda_\psi\end{aligned}$$

for a.e. θ by Birkhoff's ergodic theorem. (Observe that $\log g(\theta)$ is bounded from above.)

THEOREM 1. *Under the above assumptions there is an upper semicontinuous function $\phi:\mathbb{T}^1\to[0,\infty)$ with an invariant graph such that*:

1) $\lim_{n\to\infty}(1/n)\sum_{k=0}^{n-1}|x_k-\phi(\theta_k)| = 0$ *for a.e. $\theta\in\mathbb{T}^1$ and all $x>0$. In particular, the Lebesgue measure on $\mathbb{T}^1$ "lifted" to the graph of ϕ is a BRS (Bowen–Ruelle–Sinai) measure for T, i.e.*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} v(T^k(\theta,x)) = \int_{\mathbb{T}^1} v(\theta,\phi(\theta))\,d\theta$$

for all $v\in C(X)$ and a.e. $(\theta,x)\in X$.

2) *If $\sigma\le 1$, then $\phi\equiv 0$ and $\lambda(\theta,x)=\lambda_\phi=\log\sigma$ for a.e. $\theta\in\mathbb{T}^1$ and each $x\ge 0$.*

3) *If $\sigma>1$, then $\lambda(\theta,x)=\lambda_\phi<0$ for a.e. $\theta\in\mathbb{T}^1$ and all $x>0$. The set $\{\theta:\phi(\theta)>0\}$ has full Lebesgue measure. Furthermore*,

(a) *if* $g(\hat{\theta}) = 0$ *for at least one* $\hat{\theta} \in \mathbb{T}^1$, *then the set* $\{\theta : \phi(\theta) > 0\}$ *is at the same time meager and* ϕ *is Lebesgue-a.e. discontinuous*,

(b) *if* $g(\theta) > 0$ *for all* $\theta \in \mathbb{T}^1$, *then* $\phi(\theta) > 0$ *for all* $\theta \in \mathbb{T}^1$. *In this case* ϕ *is continuous*, *and if* g *is* C^1, *then so is* ϕ.

4) *If* $\sigma \neq 1$, *then* $|x_n - \phi(\theta_n)| \to 0$ *exponentially fast for Lebesgue-a.e.* θ *and each* $x > 0$.

Remark 1. 1) This type of models was previously investigated in [2, 6]. I thank A. Pikovsky for pointing out to me the problem addressed here. Indeed, the map S on $\mathbb{T}^1 \times \mathbb{R}$, $S(\theta, x) = (\theta + \omega, 2\sigma \tanh(x) \cos(2\pi\theta))$, which is studied in [6], has the map T on $\mathbb{T}^1 \times [0, \infty)$, $T(\theta, x) = (\theta + \omega, f(x)g(\theta))$ with $f(x) = 2\sigma \tanh(x)$ and $g(\theta) = |\cos(2\pi\theta)|$ as an obvious $2:1$-factor ([1]).

2) Case 3(a) of the theorem is the most interesting one. Let Γ be the graph of the function ϕ (which is Lebesgue-a.e. discontinuous). Then $\overline{\Gamma}$ contains the circle $\{(\theta, x) : x = 0\}$, and it is the ω-limit set of Lebesgue-a.e. (θ, x). As the Lyapunov exponents of T in θ- and x-direction are 0 and $\lambda_\phi < 0$ respectively, $\overline{\Gamma}$ is called a *strange nonchaotic attractor*.

3) Recently Bellack [1] proved a similar result where the base is a diffeomorphic map with a solenoidal attractor. He can show additionally that the graph of ϕ is dense in the set $\{(\theta, x) : 0 \leq x \leq \phi(\theta)\}$. For the proof he uses essentially the presence of periodic points in the solenoid. In the case considered here I am not able to prove or disprove this property.

4) Related models were also investigated in [4].

Acknowledgements. I thank W. Jansen and an anonymous referee for many helpful remarks and corrections to previous versions of this note.

The proof of the theorem is based on the following lemma on functions with an invariant graph.

Lemma 1. *Suppose* $\psi : \mathbb{T}^1 \to [0, \infty)$ *has an invariant graph. Then*

1) ψ *is bounded and either* $\psi(\theta) = 0$ *for a.e.* θ *or* $\psi(\theta) > 0$ *for a.e.* θ.

2) *If* $\psi(\theta) > 0$ *for a.e.* θ, *then* $\lambda_\psi < 0$.

3) *If* $\psi(\theta) = 0$ *for a.e.* θ *and if there is a decreasing sequence of bounded measurable functions* $\psi_n : \mathbb{T}^1 \to [0, \infty)$ *such that* $\lim_{n\to\infty} \psi_n(\theta) = \psi(\theta)$ *for all* $\theta \in \mathbb{T}^1$, $\psi_n(\theta) > 0$ *for a.e.* θ *and such that* $f(\psi_n(\theta)) \cdot g(\theta) = \psi_{n+1}(\theta + \omega)$, *then* $\lambda_\psi = \log \sigma \leq 0$.

4) *If* $\lambda_\psi < 0$, *then* $|x_n - \psi(\theta_n)| \to 0$ *exponentially fast for a.e.* $\theta \in \mathbb{T}^1$ *and all* $x > 0$.

5) *If* $\lambda_\psi = 0$, *then* $\psi(\theta) = 0$ *for a.e.* $\theta \in \mathbb{T}^1$ *and* $\lim_{n\to\infty} (1/n) \sum_{k=0}^{n-1} x_k = 0$ *for a.e.* $\theta \in \mathbb{T}^1$ *and all* $x \geq 0$.

([1]) Added in proof: This model was also investigated in a recent preprint by Bezhaeva and Oseledets (Report Nr. 356, Institut für Dynamische Systeme, Universität Bremen).

6) *If* $\lambda_\psi \le 0$, *then* $\lambda(\theta,x)=\lambda_\psi$ *for a.e.* $\theta\in\mathbb{T}^1$ *and all* $x>0$.

7) *If* $\lambda_\psi \le 0$ *and if* $\widetilde\psi$ *is another measurable function with invariant graph*, *then* $\widetilde\psi=0$ *or* $\widetilde\psi=\psi$ *a.e.*

Proof. 1) As $\psi(\theta+\omega)=f(\psi(\theta))\cdot g(\theta)$ and as f and g are bounded, also ψ is bounded. Since $f(0)=0$, the set $\{\theta:\psi(\theta)=0\}$ is invariant under rotation by ω. Hence this set has either Lebesgue measure 0 or 1.

3) If $\sigma>0$ we have the following estimate: As $f(0)=0$ and $f(\psi_n(\theta))=\psi_{n+1}(\theta+\omega)/g(\theta)$ and f is strictly concave,

$$f'(\psi_n(\theta))<\frac{f(\psi_n(\theta))}{\psi_n(\theta)}=\frac{\psi_{n+1}(\theta+\omega)}{\psi_n(\theta)g(\theta)}\le\frac{\psi_n(\theta+\omega)}{\psi_n(\theta)g(\theta)}$$

for a.e. θ. In particular, $\theta\mapsto\log(\psi_n(\theta+\omega)/\psi_n(\theta))$ has the integrable minorant $\theta\mapsto\log f'(\psi_n(\theta))+\log g(\theta)$ (observe that $\int\log g(\theta)\,d\theta=\log\sigma-\log f'(0)>-\infty$). Invoking the measure theoretic Lemma 2 that we provide at the end of the paper, it follows that $\log(\psi_n(\theta+\omega)/\psi_n(\theta))$ is integrable and that $\int\log(\psi_n(\theta+\omega)/\psi_n(\theta))\,d\theta=0$. Hence

$$\begin{aligned}\int\log f'(\psi_n(\theta))\,d\theta&<\int\log\frac{\psi_n(\theta+\omega)}{\psi_n(\theta)}\,d\theta-\int\log g(\theta)\,d\theta\\&=-\int\log g(\theta)\,d\theta\end{aligned}$$

such that $\lambda_{\psi_n}<0$.

If $\sigma=0$, we have $\int\log g(\theta)\,d\theta=-\infty$ and hence also $\lambda_{\psi_n}=-\infty<0$.

In both cases the monotone convergence theorem implies that $\lambda_\psi=\lim_{n\to\infty}\lambda_{\psi_n}\le 0$.

2) In the special case $\psi_n=\psi$ for all n the above reasoning yields $\lambda_\psi<0$.

4) For $x\ge\psi(\theta)$ this is an immediate consequence of the facts that $x\mapsto L_n(\theta,x)$ decreases, that $\lim_{n\to\infty}(1/n)\log L_n(\theta,\psi(\theta))=\lambda_\psi<0$, and of the mean value theorem. If $\psi=0$ a.e. we are thus done. Otherwise $\psi>0$ a.e. and we proceed as follows for $0<x<\psi(\theta)$: Let

$$q(x):=\frac{xf'(x)}{f(x)}\quad\text{if } x>0\quad\text{and}\quad q(0)=1.$$

Then $q:[0,\infty)\to\mathbb{R}$ is continuous, $0<q\le 1$, and as f is strictly concave, $q(x)=1$ if and only if $x=0$. Using the concavity of f once more it follows that

$$\begin{aligned}\frac{\psi(\theta_n)-x_n}{\psi(\theta_{n-1})-x_{n-1}}&=\frac{f(\psi(\theta_{n-1}))-f(x_{n-1})}{\psi(\theta_{n-1})-x_{n-1}}\cdot g(\theta_{n-1})\\&\le f'(x_{n-1})g(\theta_{n-1})\\&=q(x_{n-1})\cdot\frac{f(x_{n-1})}{x_{n-1}}g(\theta_{n-1})=q(x_{n-1})\cdot\frac{x_n}{x_{n-1}}.\end{aligned}$$

Hence

$$\frac{\psi(\theta_n)-x_n}{x_n}=\frac{\psi(\theta_{n-1})-x_{n-1}}{x_{n-1}}\cdot q(x_{n-1}),$$

and by induction

$$|\psi(\theta_n)-x_n|=\underbrace{x_n}_{\le M}\cdot\underbrace{\prod_{i=0}^{n-1}q(x_i)}_{\le 1}\cdot\left|\frac{\psi(\theta_0)-x_0}{x_0}\right|.$$

If $x_n\to 0$, then $|\psi(\theta_n)-x_n|\to 0$. Otherwise $x_n\not\to 0$, and it follows that $\prod_{i=0}^{n-1}q(x_i)\to 0$ so that also in this case $|\psi(\theta_n)-x_n|\to 0$. In particular,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log f'(x_k)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log f'(\psi(\theta_k))$$

because $x\mapsto\log f'(x)$ is continuous, and it follows from (1) that

$$\lim_{n\to\infty}\frac{1}{n}\log L_n(\theta,x)=\lambda_\psi<0.$$

Now the exponential convergence $|\psi(\theta_n)-x_n|\to 0$ follows as for $x\ge\psi(\theta)$ above.

5) If $\lambda_\psi=0$, then $\psi=0$ a.e. by 1) and 2) of the lemma. As f and g are bounded, also the sequence (x_k) is bounded, and it suffices to show that for any $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{Z_n}{n}:=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}1_{\{x_k>\varepsilon\}}=0. \tag{2}$$

As f is strictly concave, the function $\kappa(x)=f(x)/(xf'(0))$ $(x>0)$ is decreasing with $\lim_{x\to 0}\kappa(x)=1$ and $\kappa(x)<1$ for $x>0$.

Given (θ_0,x_0), fix some $\delta>0$ and let $A_\delta=\{n\in\mathbb{N}:Z_n/n>\delta\}$. We observe that

$$x_n=g(\theta_{n-1})f(x_{n-1})=\kappa(x_{n-1})g(\theta_{n-1})f'(0)x_{n-1}.$$

By induction we obtain, for $n\in A_\delta$,

$$x_n=\prod_{i=0}^{n-1}\kappa(x_i)\cdot\prod_{k=0}^{n-1}(g(\theta_k)f'(0))\cdot x_0\le\kappa(\varepsilon)^{n\delta}\cdot L_n(\theta,0)\cdot x_0.$$

As $\lambda(\theta,0)=\lambda_\psi=0$ for a.e. θ by assumption and as $\kappa(\varepsilon)^\delta<1$, this proves that $\lim_{n\in A_\delta,n\to\infty}x_n=0$. As f and g are continuous, it follows that for each $N>0$,

$$\lim_{n\in A_\delta,n\to\infty}\max_{0\le j\le N}x_{n+j}=0.$$

Applying this assertion to $N=[\delta^{-1}]$ we obtain some $n_0=n_0(\theta_0,x_0,\delta)$ such that for $n\ge n_0$ we have: If $n\in A_\delta$ but $(n-1)\notin A_\delta$, then

$$Z_{n+j}=Z_n=Z_{n-1}+1\le(n-1)\delta+1\le\begin{cases}2(n+j)\delta & \text{for } 0\le j<N,\\ (n+j)\delta & \text{for } j=N.\end{cases}$$

In particular, $(n+N)\notin A_\delta$, and it follows that $Z_n\le 2n\delta$ for all $n\ge n_0$. As $\delta>0$ was arbitrary, this implies (2).

6) Because of the continuity of f' this is an immediate consequence of 4) and 5).

7) If $\widetilde{\psi}(\theta)$ is not equal to 0 for a.e. θ, then $\widetilde{\psi}(\theta)>0$ for a.e. θ by 1). Applying 4) or 5) to ψ yields in view of the ergodic theorem

$$\begin{aligned}\int|\widetilde{\psi}(\theta)-\psi(\theta)|\,d\theta&=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|\widetilde{\psi}(\theta_k)-\psi(\theta_k)|\\&=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}|(\widetilde{\psi}(\theta_0))_k-\psi(\theta_k)|=0\end{aligned}$$

for a.e. $\theta_0\in\mathbb{T}^1$, i.e. $\widetilde{\psi}=\psi$ a.e. ■

Proof of Theorem 1.

1. *The definition of* ϕ. Denote by π_1 and π_2 the projections from X onto $\mathbb{T}^1$ and $[0,\infty)$ respectively. Define for $n\in\mathbb{N}$

$$\phi_n:\mathbb{T}^1\to[0,\infty),\quad \phi_n(\theta)=\pi_2\circ T^n(\theta-n\omega,M)$$

where $M:=\sup_{(\theta,x)}f(x)g(\theta)$. Then

$$\begin{aligned}\phi_{n+1}(\theta)&=\pi_2\circ T^n(T(\theta-(n+1)\omega,M))\\&=\pi_2\circ T^n(\theta-n\omega,f(M)g(\theta-(n+1)\omega))\end{aligned}$$

where the second argument is bounded by M. As an easy induction argument shows, $\pi_2\circ T^n$ is isotonic as a function of its second argument, and we conclude that

$$\phi_{n+1}(\theta)\le\pi_2\circ T^n(\theta-n\omega,M)=\phi_n(\theta). \tag{3}$$

Hence

$$\phi(\theta):=\lim_{n\to\infty}\phi_n(\theta)=\inf_n\phi_n(\theta)$$

is well defined. As the infimum of a decreasing sequence of continuous functions ϕ is upper semicontinuous, all sets $\{\theta:\phi(\theta)<\varepsilon\}$ with $\varepsilon>0$ are open. Hence $\{\theta:\phi(\theta)=0\}$ is a decreasing intersection of open sets. If $g(\hat{\theta})=0$ and if we set $\hat{\theta}_n:=\hat{\theta}-n\omega$, then $\phi_n(\hat{\theta}_k)=0$ for $k=0,\dots,n-1$ so that $\hat{\phi}(\hat{\theta}_k)=0$ for all k. So, in this case, the sets $\{\theta:\phi(\theta)<\varepsilon\}$ are also dense in $\mathbb{T}^1$ and $\{\theta:\phi(\theta)=0\}$ is residual, i.e. $\{\theta:\phi(\theta)>0\}$ is meager.

Observe also that ϕ has an invariant graph:

$$(4)\quad \begin{aligned} f(\phi(\theta))\cdot g(\theta) &= \lim_{n\to\infty} f(\phi_n(\theta))\cdot g(\theta) = \lim_{n\to\infty} \pi_2\circ T(\theta,\phi_n(\theta)) \\ &= \lim_{n\to\infty} \pi_2\circ T(T^n(\theta-n\omega,M)) = \lim_{n\to\infty} \phi_{n+1}(\theta+\omega) \\ &= \phi(\theta+\omega). \end{aligned}$$

2. *Consequences of the lemma.* If $\sigma\le 1$ we apply the lemma to $\psi\equiv 0$. As in this case $\lambda_\psi=\log\sigma\le 0$ by definition of λ_ψ, we conclude from 7) of the lemma applied to $\widetilde{\psi}=\phi$ that $\phi(\theta)=0$ for a.e. θ. The rest of assertions 1) and 2) of the theorem follow from 4), 5) and 6) of the lemma. Finally, as the statements of the theorem are only about a.e. $\theta\in\mathbb{T}^1$, we may assume that $\phi\equiv 0$.

If $\sigma>1$, we apply 1) and 3) of the lemma to $\psi=\phi$ to conclude that $\phi(\theta)>0$ for a.e. θ. Now 2) of the lemma implies $\lambda_\phi<0$, assertion 1) follows from 4) of the lemma, and $\lambda(\theta,x)=\lambda_\phi$ for a.e. $\theta\in\mathbb{T}^1$ and all $x>0$ follows from 6) of the lemma. Statement 3(a), i.e. the meagerness of the set $\{\theta:\phi(\theta)>0\}$ in case $g(\hat\theta)=0$, was already proved above, and the proof of 3(b) is deferred to item 3.

Finally, if $\sigma\ne 1$, then $\lambda_\phi<0$ by 2) and 3), and assertion 4) follows from 4) of the lemma.

3. *The "non-strange" case $g>0$, $\sigma>1$.* If $g(\theta)>0$ for all θ, the function $\theta\mapsto\log g(\theta)$ is continuous on $\mathbb{T}^1$. In this case $(1/n)\sum_{k=0}^{n-1}\log(f'(0)\cdot g(\theta_k))$ converges *uniformly in* θ to $\log\sigma>0$ by the Kronecker-Weyl equidistribution theorem. Hence there is $n_0>0$ such that $L_{n_0}(\theta,0)>\sigma^{n_0/2}>1$ for all $\theta\in\mathbb{T}^1$, and by continuity there is $\delta>0$ such that the same estimate holds for $L_{n_0}(\theta,x)$ with $0\le x\le\delta$. Hence, by the mean value theorem, the x-component of $T^{n_0}(\theta,\delta)$ is greater than δ.

Define functions

$$\psi_n:\mathbb{T}^1\to[0,\infty),\quad \psi_n(\theta)=\pi_2\circ T^n(\theta-n\omega,\delta)\quad (n\ge 0)$$

in analogy with the definition of the functions ϕ_n. Then $\psi_{n_0}>\delta=\psi_0$, and we obtain an increasing sequence $(\psi_{jn_0})_{j\ge 0}$ of continuous functions bounded above by M. Its pointwise limit has an invariant graph (cf. the proof of (4)) and thus coincides with ϕ a.e. by Lemma 1.7. Consider the sequence $(\psi_{k+jn_0})_{j\ge 0}$ for fixed k. As $\psi_{k+jn_0}(\omega)=\pi_2\circ T^k(\theta-k\omega,\psi_{jn_0}(\theta-k\omega))$, as T^k is continuous and as the graph of ϕ is invariant, the sequence $(\psi_{k+jn_0})_{j\ge 0}$ converges a.e. to ϕ, too, and it follows that $\lim_{n\to\infty}\psi_n(\theta)=\phi(\theta)$ for a.e. θ. In particular there is some $N>n_0$ such that $\lambda_{\psi_N}<(1/2)\lambda_\phi<0$. Invoking the equidistribution theorem once more it follows that there is $n_1>N$ such that

$$(5)\qquad \frac{1}{n}\log L_n(\theta,x) \le \frac{1}{n}\log L_n(\theta,\psi_N(\theta)) = \frac{1}{n}\sum_{k=0}^{n-1}\log(f'(\psi_N(\theta_k))\cdot g(\theta_k)) < \frac{1}{2}\lambda_{\psi_N} < 0$$

for *all* $\theta \in \mathbb{T}^1$, $x \ge \psi_N(\theta)$ and $n \ge n_1$. Hence the sequence $(\psi_n)_{n\ge n_1}$ of continuous functions converges uniformly (and exponentially fast!) to ϕ so that ϕ is continuous, too.

If g is even continuously differentiable, then

$$\begin{aligned}
&DT^n(\theta,x)\\
&= \prod_{k=1}^{n} DT(\theta_{n-k},x_{n-k}) = \prod_{k=1}^{n}\begin{pmatrix} 1 & 0\\ f(x_{n-k})g'(\theta_{n-k}) & f'(x_{n-k})g(\theta_{n-k})\end{pmatrix}\\
&= \begin{pmatrix} 1 & 0\\ \sum_{j=1}^{n} f(x_{n-j})g'(\theta_{n-j})\prod_{k=1}^{j-1} f'(x_{n-k})g(\theta_{n-k}) & \prod_{k=1}^{n} f'(x_{n-k})g(\theta_{n-k})\end{pmatrix}\\
&= \begin{pmatrix} 1 & 0\\ \sum_{j=1}^{n} f(x_{n-j})g'(\theta_{n-j})L_{j-1}(\theta_{n-j+1},x_{n-j+1}) & L_n(\theta,x)\end{pmatrix}
\end{aligned}$$

and it follows from (5) that

$$\begin{aligned}
\psi_n'(\theta) &= \frac{\partial}{\partial\theta}\pi_2\circ T^n(\theta-n\omega,\delta)\\
&= \sum_{j=1}^{n} f(\psi_{n-j}(\theta_{-j}))g'(\theta_{-j})\cdot L_{j-1}(\theta_{-(j-1)},\psi_{n-(j-1)}(\theta_{-(j-1)}))\\
&\to \sum_{j=1}^{\infty} f(\phi(\theta_{-j}))g'(\theta_{-j})\cdot L_{j-1}(\theta_{-(j-1)},\phi(\theta_{-(j-1)}))
\end{aligned}$$

uniformly as $n\to\infty$. Hence ϕ is differentiable, and $\phi' = \lim_{n\to\infty}\psi_n'$. ∎

The next theorem gives some insight into the dependence of ϕ and λ_ϕ on the parameter σ for σ close to its critical value 1:

THEOREM 2. *Fix a map f as above which is normalized to $f'(0) = 1$, and fix a constant $K > 0$. Consider the function g from above as a parameter that can be varied subject to the constraint $\sup_\theta |g(\theta)| \le K$. ($g$ thus determines σ.)*

1) *If $a(x) := \log f'(x)/\log(f(x)/x)$ $(0 < x \le M)$ extends continuously to $x = 0$ with $a(0) > 1$, then*

$$\lambda_\phi = (1-a(0))\cdot\log\sigma + o(\log\sigma)\quad \textit{if}\ \ \sigma\searrow 1.$$

2) *If $b(x) := -\log(f(x)/x)$ $(0 < x \le M)$ extends differentiably to $x = 0$*

with $b(0) = 0$ *and* $b'(0) > 0$, *then*

$$\int \phi(\theta)\, d\theta = \frac{\log \sigma}{b'(0)} + o(\log \sigma) \quad \text{if } \sigma \searrow 1.$$

Remark 2. 1) If $f(x) = x/(1 + cx^{a-1})^{1/(a-1)}$, $a > 1$, then $a(x) = a$ for all x and $\lambda_\phi = (1 - a) \cdot \log \sigma$ exactly.

2) If $f(x) = x \cdot e^{-bx}$ and if $b < M^{-1}$, then $b(x) = bx$, f is monotone and concave on $[0, M]$, and $\int \phi(\theta)\, d\theta = (\log \sigma)/b$ exactly.

Proof of Theorem 2. Without loss of generality we may assume that $f'(0) = 1$. As

$$\phi(\theta_{n+1}) = f(\phi(\theta_n)) \cdot g(\theta_n) = \phi(\theta_n) \cdot \frac{f(\phi(\theta_n)) \cdot g(\theta_n)}{\phi(\theta_n)}$$

we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log \phi(\theta_n) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(\log \frac{f(\phi(\theta_k))}{\phi(\theta_k)} + \log g(\theta_k) \right) \\ &= \int \log \frac{f(\phi(\theta))}{\phi(\theta)}\, d\theta + \int \log g(\theta)\, d\theta\end{aligned}$$

for a.e. θ, where $\theta_n = \theta + n\omega$ as before. On the other hand, as $\phi > 0$ a.e. and $\phi \le M < \infty$, we have $\limsup_{n\to\infty} (1/n) \log \phi(\theta_n) = 0$ for a.e. θ. Therefore

$$\int \log \frac{f(\phi(\theta))}{\phi(\theta)}\, d\theta = -\int \log g(\theta)\, d\theta = -\log \sigma. \tag{6}$$

Observe that $f(x) < x$ for all $x > 0$ as $f'(0) = 1$ and f is strictly concave. So (6) implies that $\phi = \phi_g \to 0$ in measure if $\log \sigma \searrow 0$. (Here we made use of the uniform bound M for $|fg|$.) Hence

$$\begin{aligned}\lambda_\phi &= \log \sigma + \int \log f'(\phi(\theta))\, d\theta \\ &= \log \sigma + \int a(\phi(\theta)) \cdot \log \frac{f(\phi(\theta))}{\phi(\theta)}\, d\theta \\ &= \log \sigma + a(0) \cdot \int \log \frac{f(\phi(\theta))}{\phi(\theta)}\, d\theta + \int (a(\phi(\theta)) - a(0)) \cdot \log \frac{f(\phi(\theta))}{\phi(\theta)}\, d\theta \\ &= (1 - a(0)) \cdot \log \sigma + o(\log \sigma) \quad \text{if } \log \sigma \searrow 0,\end{aligned}$$

because $\phi \to 0$ in measure if $\log \sigma \searrow 0$.

Similarly,

$$\log \sigma = \int b(\phi(\theta))\, d\theta = b'(0) \cdot \int \phi(\theta)\, d\theta + O\left(\int \phi(\theta)^2\, d\theta \right),$$

whence

$$\int \phi(\theta)\, d\theta = \frac{\log \sigma}{b'(0)} + o(\log \sigma) \quad \text{if } \log \sigma \searrow 0. \ \blacksquare$$

We close with a general measure theoretic result used in the proof of Lemma 1. It was first stated in [3, Lemma 14], but the proof given there was not quite correct. The present proof is taken from [5] (unpublished).

LEMMA 2. *Let* $(Y,\mathcal{F},\mu)$ *be a probability space*, $T:Y\to Y$ *a measurable transformation leaving the measure* μ *invariant*, *and* $f:Y\to\mathbb{R}$ *a measurable function. If the function* $f\circ T-f$ *has a minorant* $g\in L^1_\mu$, *then* $f\circ T-f\in L^1_\mu$ *and*

$$\int (f\circ T-f)\,d\mu=0.$$

Proof. Let $f_n:=\max(\min(f,n),-n)$. Then

$$0\le f_n\circ T-f_n\le f\circ T-f \text{ on the set } \{f\circ T-f\ge 0\} \text{ and}$$
$$0\ge f_n\circ T-f_n\ge f\circ T-f \text{ on the set } \{f\circ T-f\le 0\}.$$

Therefore $(f_n\circ T-f_n)_{n>0}$ is a sequence of bounded functions with common integrable minorant $\min(g,0)$ and converging to $f\circ T-f$. By the T-invariance of μ it thus follows from Fatou's lemma that

$$\int (f\circ T-f)\,d\mu\le\liminf_{n\to\infty}\int (f_n\circ T-f_n)\,d\mu=0.$$

Hence $f\circ T-f\in L^1_\mu$. Because of $|f_n\circ T-f_n|\le|f\circ T-f|$, the dominated convergence theorem finally yields $\int(f\circ T-f)\,d\mu=0$. ∎

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Received 10 October 1995;
in revised form 6 March 1996 and 24 June 1996