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Examples of sequential topological groups under the continuum hypothesis

by

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Abstract. Using **CH** we construct examples of sequential topological groups:

1. a pair of countable Fréchet topological groups whose product is sequential but is not Fréchet,
2. a countable Fréchet and α_1 topological group which contains no copy of the rationals.

1. Introduction. The classical methods of study of continuity involve consideration of convergent sequences and their images. Although the continuity as it is understood in modern topology cannot be treated only in terms of classical convergent sequences there is a field of topology and the corresponding subclass of topological spaces where classical convergence plays an important rôle. Like general topology itself the field has its origin in metric space theory.

The first natural generalization of metrizability is first-countability. Going further in generalization one can emphasize the following property of the closure operator in a first-countable space: $x \in \overline{A}$ implies existence of a sequence in A converging to x. Spaces having this property are called *Fréchet spaces*. The next step is to require only that convergent sequences determine the topology of the space. The corresponding definition is: a space X is *sequential* if for every $A \subseteq X$ such that $A \neq \overline{A}$ there exists a sequence in A converging to a point outside A.

Sequentiality and its behaviour in several situations were studied by a number of authors (see [F], [No1], [No2], [NT], [AF]). In the course of their investigation some new convergence properties were introduced and some problems were stressed. Among those problems is the study of products of Fréchet and, more generally, sequential spaces. It is relatively easy to de-

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stroy Fréchetness by product operation (see [vD], [GMT]) so one has to put some restrictions on the factors to see subtle phenomena. The most popular and useful restrictions are compactness and the property to be a topological group. Several techniques were developed to study products with some or all factors being compact (see [A1], [O], [M2]). It was shown that the product of two Fréchet compact spaces may be non-Fréchet (see [Si], [BR], [MS]). Properties α_i were introduced to obtain theorems about preservation of Fréchetness in products (see [A1], [A2]). Since then α_i-properties have found several applications in the theory of sequential spaces ([No1], [No2], [N], [NT]). In particular, for topological groups P. Nyikos proved in [N] that a sequential topological group is Fréchet if and only if it is α_4. D. Shakhmatov showed ([Sm]) that one can say no more about Fréchet topological groups: in a model of ZFC obtained by adding uncountably many Cohen reals there exists a Fréchet non-α_3 topological group. As α_i-properties play an important rôle in the study of the product operation and preservation of Fréchetness, the products of Fréchet topological groups are also of interest. S. Todorčević in [T] constructed (in ZFC) a pair of σ-compact Fréchet topological groups whose product is non-Fréchet (even of uncountable tightness).

In Section 2 we construct using **CH** a pair of countable Fréchet topological groups whose product is sequential but not Fréchet. The sequentiality of the product imposes some restrictions on the factors. For example, at least one group of the pair is a non-α_3-space. Indeed, if both groups were α_3-spaces then so would be their product by [No1, Theorem 2.2] but being sequential it would be Fréchet by the result of P. Nyikos cited above.

The technique by which that example was obtained is applied to the construction of a countable Fréchet and α_1 topological group containing no copy of the rationals in the conclusion of Section 2. Being necessarily a non-first-countable space, it cannot be obtained without extra set-theoretic assumptions (see [DS]). In fact, slight modification of the technique permits obtaining a topological field with such properties.

The topology of each group is constructed by induction. At each step a pair of topologies is considered and the finer topology is coarsened by adding a new convergent sequence from the usual topology of $\mathbb{Q}$ while the coarser one is refined so that the resulting new pair of topologies remains comparable and stays between the usual topology of $\mathbb{Q}$ and the discrete topology. The construction is arranged so that the processes of coarsening and refining come together in a single topology. The properties of the topology thus constructed are obtained by considering an appropriate pair of topologies involved in the inductive procedure which "approximate" the topology from above and below.

Let us recall the terminology used in the study of sequential spaces. A family $\mathcal{S} = \{S^i \mid i \in \omega\}$ of sequences converging to a common point $x \in X$

is called a *sheaf*, the point x is called the *vertex* of the sheaf (see [A1]). A space X is called an α_1-*space* (or $X \in \langle 1\rangle$ in the notation of [A1]) if for every sheaf there is a sequence S converging to its vertex such that $S^i \setminus S$ is finite for all $i \in \omega$. X is called an α_4-*space* if for every sheaf there is a sequence converging to its vertex which meets infinitely many sequences of the sheaf. A quotient image of a topological sum of countably many compact spaces is called a k_ω-*space*. A product of two k_ω-spaces is itself a k_ω-space (see [M2]).

Put $\omega(n) = \{k \mid k > n\} \subseteq \omega$. A set $\sigma \subseteq \omega^2$ will be called *thin* (resp. *small*) if for every $n \in \omega$ the set $\sigma \cap \{n\} \times \omega$ is finite (resp. $\sigma \cap \omega(n) \times \omega$ is thin for some $n \in \omega$). Let σ be a small set and $n = \min\{k \mid \sigma \cap \omega(k) \times \omega$ is thin$\}$. Then $\sigma \cap \omega(n) \times \omega = \mathrm{ess}(\sigma)$ will be called the *essential part* of σ.

Consider the set $S = \omega^2 \cup \omega \cup \{\omega\}$. Define a topology on S as follows. Every point of ω^2 is isolated, a typical neighborhood of n is $\{n\} \cup (\{n\} \times \omega \setminus$ finitely many points), $U \ni \{\omega\}$ is open if $(U \cap \{n\} \times \omega) \cup \{n\}$ is a neighborhood of n for every $n \in U$ and $\omega \setminus U$ is finite. The resulting space is called *Arens' space* S_2. Another canonical space S_ω is obtained from countably many convergent sequences by identifying their limit points.

Let $\mathbb{Q}$ be the set of rationals. Let $\mathcal{K} = \{K_\alpha\}_{\alpha\in A}$ be an arbitrary family of subsets of $\mathbb{Q}$. Suppose $\vec{a} \in \mathbb{Q}^n$ and $\vec{K} \in \mathcal{K}^n$ where $n \in \omega \setminus \{0\}$. Set $\langle \vec{a}, \vec{K}\rangle = \langle (a_1, \ldots, a_n), (K_{\alpha_1}, \ldots, K_{\alpha_n})\rangle = a_1 \cdot K_{\alpha_1} + \ldots + a_n \cdot K_{\alpha_n} \subseteq \mathbb{Q}$, where $a_i \in \mathbb{Q}$. Define $\mathbb{Q}^\infty = \bigcup_{n\in\omega} \mathbb{Q}^n$ and $\mathbb{Q}^0 = \{0\}$. If $K \subseteq \mathbb{Q}$ and $\vec{a} \in \mathbb{Q}^n$ we set $\vec{a}\langle K\rangle = a_1 \cdot K + \ldots + a_n \cdot K$. If $\vec{a} \in \mathbb{Q}^0$ then $\vec{a}\langle K\rangle = 0$. Let $\mathbb{Q} = \{b_i \mid i \in \omega\}$ with $b_i \neq b_j$ for $i \neq j$, $\mathbb{Q}(i) = \{b_j \mid j \leq i\}$ and $\mathbb{Q}_k = \bigcup_{i,j\leq k} (\mathbb{Q}(i))^j$. If $a \in \mathbb{Q} \setminus \{0\}$ let $n_{\mathbb{Q}}(a) = n$ provided $a = b_n$, and $n_{\mathbb{Q}}(0) = \infty > k$ for any $k \in \omega$. All spaces are assumed to be Hausdorff.

The following simple lemma was proved in [Sh, Lemma 1.1].

LEMMA 1.1. *A countable nondiscrete sequential topological group contains a closed copy of S_2 provided the group is a k_ω-space.*

2. Examples. Lemmas 2.1–2.6 were proved in [Sh, Lemmas 2.1–2.6].

LEMMA 2.1. *Let $\mathcal{K} = \{K_n\}_{n\in\omega}$ be an arbitrary family of subsets of $\mathbb{Q}$. Then there exists a countable family $C(\mathcal{K}) \supseteq \mathcal{K}$ such that*:

(1) $\{a\} \in C(\mathcal{K})$ *for all* $a \in \mathbb{Q}$,
(2) *if* $\vec{a} \in \mathbb{Q}^n$ *and* $\vec{K} \in C(\mathcal{K})^n$ *then* $\langle \vec{a}, \vec{K}\rangle \in C(\mathcal{K})$,
(3) *if* $K^1 \in C(\mathcal{K}), \ldots, K^n \in C(\mathcal{K})$ *then* $\bigcup_{i\leq n} K^i \in C(\mathcal{K})$,
(4) *if* $\mathcal{K} \subseteq \mathcal{K}'$ *and* $\mathcal{K}'$ *has properties* (1)–(3) *then* $C(\mathcal{K}) \subseteq \mathcal{K}'$.

LEMMA 2.2. *If $\mathcal{K} = \{K_n\}_{n\in\omega}$ is a family of compact subsets of $\mathbb{Q}$ then so is $C(\mathcal{K})$.*

LEMMA 2.3. *Let $\mathcal{K} = \{K_n\}_{n\in\omega}$ be an arbitrary family of compact subsets of $\mathbb{Q}$. Introduce a new topology on $\mathbb{Q}$ by declaring $U \subseteq \mathbb{Q}$ to be open if and* *only if $U \cap F$ is relatively open for every $F \in C(\mathcal{K})$. Denote $\mathbb{Q}$ with this topology as $G(\mathcal{K})$. Then*:

(5) *if $\vec{a} \in \mathbb{Q}^n$ then the mapping $p : G(\mathcal{K})^n \to G(\mathcal{K})$, $p(\vec{b}) = \langle \vec{a}, \vec{b} \rangle$, is continuous*,

(6) *$G(\mathcal{K})$ is a k_ω-space.*

LEMMA 2.4. *If $\mathcal{K} = \bigcup_{\beta<\alpha} \mathcal{K}_\beta$ and $\mathcal{K}_\beta \subseteq \mathcal{K}_{\beta'}$ for $\beta \le \beta'$ then $C(\mathcal{K}) = \bigcup_{\beta<\alpha} C(\mathcal{K}_\beta)$.*

LEMMA 2.5. *For every family $\mathcal{K} = \{K_i\}_{i\in\omega}$ of compact subsets of $\mathbb{Q}$ and every family $\mathcal{U} = \{U_i\}_{i\in\omega}$ of open subsets of $G(\mathcal{K})$ one can fix a topology $\tau(\mathcal{U}, \mathcal{K})$ on $\mathbb{Q}$ such that*:

(a) *the mapping $p : \mathbb{Q}^n \to \mathbb{Q}$ where $p(\vec{a}) = \langle \vec{b}, \vec{a} \rangle$, $\vec{b} \in \mathbb{Q}^n$, is continuous in $\tau(\mathcal{U}, \mathcal{K})$*,

(b) *$\tau(\mathcal{U}, \mathcal{K})$ is a Hausdorff group topology with a countable base*,

(c) *$U_i \in \tau(\mathcal{U}, \mathcal{K})$ for all $i \in \omega$*,

(d) *$\tau(\mathcal{U}, \mathcal{K})$ is stronger than the usual topology of $\mathbb{Q}$ and weaker than the topology of $G(\mathcal{K})$, and*

(e) *if $\mathcal{U} \supseteq \tau_0(\mathcal{U}', \mathcal{K}')$ then $\tau(\mathcal{U}, \mathcal{K})$ is stronger than $\tau(\mathcal{U}', \mathcal{K}')$ where $\mathcal{K}$ and $\mathcal{K}'$ are countable families of compact subsets of $\mathbb{Q}$ and $\tau_0(\mathcal{U}, \mathcal{K})$ is a fixed countable base at $0 \in \mathbb{Q}$ in $\tau(\mathcal{U}, \mathcal{K})$.*

LEMMA 2.6. *$C(\mathcal{K} \cup \{K\}) = \{\bigcup_{i\le k} (\vec{a}_i \langle K \rangle + K^i) \mid \vec{a}_i \in \mathbb{Q}^\infty,\ K^i \in C(\mathcal{K}),\ k \in \omega\}$.*

We need the following technical definition. Let $t : \omega^2 \to \mathbb{Q}$ be an injection. We shall call t a *correct table* in $G(\mathcal{K})$ if the following properties hold with $S_t = t(\omega^2)$ and $S_t^n = t(\{n\} \times \omega)$:

1(t) $t(n, k) \to s_t^n$ as $k \to \infty$ in $G(\mathcal{K})$,

2(t) $s_t^n \to 0$ as $n \to \infty$ in $G(\mathcal{K})$,

3(t) $S_t^n \cup \{s_t^n\} \subseteq K_n^t$ and $\{s_t^n \mid n \in \omega\} \cup \{0\} \subseteq B^t$ where $K_n^t \in C(\mathcal{K})$ and $B^t \in C(\mathcal{K})$.

LEMMA 2.7. *Let t and u be correct tables, $\mathcal{U} = \{U_n\}_{n\in\omega}$ be a family of open subsets of $G(\mathcal{K})$, and $\mathcal{K} = \{K_i\}_{i\in\omega}$ be a family of compact subsets of $\mathbb{Q}$. Suppose that for any $F \in C(\mathcal{K})$ the set $t^{-1}(F)$ is small. Then there exists an infinite thin subset $\sigma = \{\sigma_i \mid i \ge 1\} \subseteq \omega^2$ such that*:

(7) *$u(\sigma_i) \to 0$ as $i \to \infty$ in $\tau(\mathcal{U}, \mathcal{K})$* (see Lemma 2.5),

(8) *for each $\vec{a} \in \mathbb{Q}^\infty$ and $F \in C(\mathcal{K})$ the set $t^{-1}(\vec{a}\langle u(\sigma) \cup \{0\}\rangle + F)$ is small.*

Proof. Let $C(\mathcal{K}) = \{F_i\}_{i\in\omega}$. Suppose that for some $0 \neq a \in \mathbb{Q}$, $\{b_1,\ldots,b_n\} \subseteq \mathbb{Q}$, $k \in \omega$ and $\vec{a} \in \mathbb{Q}^\infty$ the set

$$u^{-1}\Big(\Big(\Big(\bigcup_{i\le k} K_i^t \cup B^t\Big) - \Big(\bigcup_{i\le k} F_i + \vec{a}\langle\{b_1,\ldots,b_n\}\rangle\Big)\Big)\cdot a^{-1}\Big) \tag{9}$$

is not small. Denote the part $(\ldots)\cdot a^{-1}$ as F'. Then $F' \in C(\mathcal{K})$. It easily follows from (5) that F' is a compact subspace of $G(\mathcal{K})$ and thus the topology on F' inherited from $G(\mathcal{K})$ coincides with that inherited from $\mathbb{Q}$. Thus F' is a metrizable compact subspace of $G(\mathcal{K})$. Suppose $u^{-1}(F')$ is not small. Then $1(u)$ and $2(u)$ imply that $0 \in \overline{F' \cap u(\omega^2)}$. So there exists $\sigma \subseteq \omega^2$ such that $\sigma = \{\sigma_i \mid i \ge 1\}$ is infinite and thin, $u(\sigma) \subseteq F'$ and $u(\sigma_i) \to 0$ as $i \to \infty$ in $G(\mathcal{K})$. Thus $u(\sigma_i) \to 0$ in $\tau(\mathcal{U},\mathcal{K})$ by (d). Now for any $\vec{b} \in \mathbb{Q}^\infty$ and any $F \in C(\mathcal{K})$ we have $t^{-1}(\vec{b}\langle u(\sigma)\cup\{0\}\rangle + F) \subseteq t^{-1}(\vec{b}\langle F'\rangle + F) = t^{-1}(G)$ where $G \in C(\mathcal{K})$. Thus $t^{-1}(\vec{b}\langle u(\sigma)\cup\{0\}\rangle + F)$ is small by the assumption of the lemma. So σ satisfies both (7) and (8). Thus we may assume without loss of generality that every set of the form (9) is small.

It follows easily from $1(u)$ and $2(u)$ that if $U \ni 0$ is open in $G(\mathcal{K})$ then $\omega^2 \setminus u^{-1}(U)$ is small. Now choose σ_k, $k \ge 1$, by induction so that:

$$u(\sigma_k) \notin \bigcup_{\substack{n_{\mathbb{Q}}(a)\le k\\ \vec{a}\in\mathbb{Q}_k}} \Big(\Big(\bigcup_{i\le k} K_i^t \cup B^t\Big) - \Big(\bigcup_{i\le k} F_i + \vec{a}\langle u(\{\sigma_i \mid i<k\})\cup\{0\}\rangle\Big)\Big)\cdot a^{-1}, \tag{10}$$

$$u(\sigma_k) \in \bigcap_{i\le k} U^i, \quad \{U^i\}_{i\in\omega} = \tau_0(\mathcal{U},\mathcal{K}) \quad \text{(see Lemma 2.5)}, \tag{11}$$

$$\sigma_k \in S_u^{n_k}, \qquad n_{k+1} > n_k. \tag{12}$$

The preimage under u of the union on the right hand side of (10) is small by the assumption so using the remark preceding (10) it is easy to choose σ_k satisfying (10)–(12). Now by (11), $u(\sigma_k) \to 0$ as $k \to \infty$ so (7) holds.

Consider now the set $R = \vec{a}\langle u(\sigma)\cup\{0\}\rangle + F_n$ where $\vec{a} \in \mathbb{Q}^\infty$ and $n \in \omega$. We have $\vec{a} = (a_1,\ldots,a_k)$ for some $k \in \omega$. So $\vec{a} \in \mathbb{Q}_{i(\vec{a})}$ for some $i(\vec{a}) \in \omega$. The set $A = \{\langle\vec{a},\vec{b}\rangle \mid \vec{b} \in \{0,1\}^k\} \setminus \{0\}$ is finite, so $r = \max\{n_{\mathbb{Q}}(a) \mid a \in A\} < \infty$. Put $M = \max\{i(\vec{a}), r, n\}$. Now

$$R = a_1\cdot(u(\sigma)\cup\{0\}) + \ldots + a_k\cdot(u(\sigma)\cup\{0\}) + F_n. \tag{13}$$

Define $u(\sigma_i) = p_i$ for $i \ge 1$ and $p_0 = 0$ and rewrite (13) as

$$R = \bigcup_{(i_1,\ldots,i_k)\in\omega^k} a_1\cdot p_{i_1} + \ldots + a_k\cdot p_{i_k} + F_n.$$

We write $i \in_e (i_1, \ldots, i_k)$ if and only if $\sum_{i_\nu = i} a_\nu \neq 0$ or $p_i = 0$. It is easy to see that if $a_1 \cdot p_{i_1} + \ldots + a_k \cdot p_{i_k} = b \in \mathbb{Q}$ then there are $\{p_{j_1}, \ldots, p_{j_k}\} \subseteq \{p_{i_1}, \ldots, p_{i_k}\} \cup \{p_0\}$ such that $a_1 \cdot p_{j_1} + \ldots + a_k \cdot p_{j_k} = b$ and $j_m \in_e (j_1, \ldots, j_k)$ for all $m \leq k$. A point $(i_1, \ldots, i_k) \in \omega^k$ is called *essential* if $i_m \in_e (i_1, \ldots, i_k)$ for all $m \leq k$. Let $\Omega \subseteq \omega^k$ be the set of all the essential points. It is easy to check now, using the properties of essential points discussed above, that

$$R = \bigcup_{(i_1, \ldots, i_k) \in \Omega} a_1 \cdot p_{i_1} + \ldots + a_k \cdot p_{i_k} + F_n.$$

Set

$$L = \bigcup_{(i_1, \ldots, i_k) \in \Omega \setminus \{i \mid i \leq M\}^k} a_1 \cdot p_{i_1} + \ldots + a_k \cdot p_{i_k} + F_n$$

Obviously

$$\vec{a}\langle u(\sigma) \cup \{0\}\rangle + F_n = R = (\vec{a}\langle u(\{\sigma_i \mid i \leq M\}) \cup \{0\}\rangle + F_n) \cup L.$$

Let us prove that $S_t^m \cap L$ is finite for $m > M$. Suppose there is $m > M$ such that $S_t^m \cap L$ is infinite. So we have

$$a_1 \cdot p_{i(1,l)} + \ldots + a_k \cdot p_{i(k,l)} + f^l = t(m, n_l) \tag{14}$$

where $n_{l+1} > n_l$, $f^l \in F_n$ and $(i(1,l), \ldots, i(k,l)) \in \Omega \setminus \{i \mid i \leq M\}^k$. Suppose that there are $s, l \in \omega$ such that $i(s,l) > m > M$. Without loss of generality assume that $i(s,l) = \max\{i(s',l) \mid s' \leq k\}$. Then substituting every occurrence of $p_{i(s,l)}$ in (14) by $p_0 = 0$, leaving the occurrences of other $p_{i(\nu,l)}$ untouched and thus obtaining $p_{j(\nu,l)}$ we have

$$\Big(\sum_{i(\nu,l) = i(s,l)} a_\nu\Big) \cdot p_{i(s,l)} = t(m, n_l) - (f^l + a_1 \cdot p_{j(1,l)} + \ldots + a_k \cdot p_{j(k,l)})$$

where $j(\nu,l) < i(s,l)$ if $\nu \leq k$ and $\sum_{i(\nu,l) = i(s,l)} a_\nu = a \neq 0$ because $(i(1,l), \ldots, i(k,l)) \in \Omega$; moreover, $a \in A$ and thus $n_{\mathbb{Q}}(a) \leq r \leq M < m < i(s,l)$. It follows that

$$p_{i(s,l)} \in \Big(\bigcup_{i \leq i(s,l)} K_i^t - \Big(\bigcup_{i \leq i(s,l)} F_i + \vec{a}\langle u(\{\sigma_i \mid i < i(s,l)\})\rangle\Big)\Big) \cdot a^{-1}$$

where $n_{\mathbb{Q}}(a) < i(s,l)$ and $\vec{a} \in \mathbb{Q}_M \subseteq \mathbb{Q}_{i(s,l)}$, which contradicts (10).

Therefore

$$a_1 \cdot p_{i(1,l)} + \ldots + a_k \cdot p_{i(k,l)} + f^l = t(m, n_l)$$

where $n_{l+1} > n_l$, $i(s,l) \leq m$ and $(i(1,l), \ldots, i(k,l)) \in \Omega \setminus \{i \mid i \leq M\}^k$. The set

$$\bigcup_{l \in \omega} a_1 \cdot p_{i(1,l)} + \ldots + a_k \cdot p_{i(k,l)} \subseteq \vec{a}\langle u(\{\sigma_i \mid i \leq m\}) \cup \{0\}\rangle$$

is finite and thus the set

$$F = \bigcup_{l\in\omega} \langle \vec{a}, (p_{i(1,l)}, \ldots, p_{i(k,l)})\rangle + F_n$$

is compact in $G(\mathcal{K})$. But $F \cap S_t^m$ is infinite and thus by $1(t)$ and $3(t)$ there is a point

$$a_1 \cdot p_{i(1,l)} + \ldots + a_k \cdot p_{i(k,l)} + f = s_t^m = b^t \in B^t, \qquad f \in F_n.$$

Let $j = \max\{i(j',l) \mid j' \le k\}$. Note that since $(i(1,l), \ldots, i(k,l)) \notin \{i \mid i \le M\}^k$, it follows that $j > M$. Analogously to the consideration of the previous case we have

$$\Big(\sum_{i(\nu,l)=j} a_\nu\Big) \cdot p_j = b^t - (f + a_1 \cdot p_{j_1} + \ldots + a_k \cdot p_{j_k})$$

and

$$p_j \in \Big(B_t - \Big(\bigcup_{i\le j} F_i + \vec{a}\langle u(\{\sigma_i \mid i < j\})\rangle\Big)\Big) \cdot a^{-1}$$

where $a = \sum_{i(\nu,l)=j} a_\nu$, $n_{\mathbb{Q}}(a) < j$ and $\vec{a} \in \mathbb{Q}_j$, which contradicts (10). Thus $S_t^m \cap L$ is finite for $m > M$, which implies that $t^{-1}(L)$ is small.

Now $N = \vec{a}\langle u(\{\sigma_i \mid i \le M\}) \cup \{0\}\rangle + F_n \in C(\mathcal{K})$ and thus $t^{-1}(N)$ is small. Then $R = N \cup L$ and $t^{-1}(N \cup L)$ is small. Thus (8) also holds. The lemma is proved. ∎

Let us consider an example of a group $G(\mathcal{S})$. Define $S_1 = \{1 \mid n \in \mathbb{N}\} \cup \{0\}$ and $\mathcal{S} = \{S_1\}$. Consider the topological group $G(\mathcal{S})$. It is obviously nondiscrete and is a k_ω-space by (6). Then it contains a closed copy of S_2 by Lemma 1.1. So we can fix an injection $t : \omega^2 \to \mathbb{Q}$ such that:

(f) $t(n,k) \to s_t^n$ as $k \to \infty$ in $G(\mathcal{S})$,

(g) $s_t^n \to 0$ as $n \to \infty$ in $G(\mathcal{S})$,

(h) $0 \notin S_t = t(\omega^2)$ and $0 \ne s_t^n \ne s_t^k \notin S_t$ if $n \ne k$,

(i) if $S_t^n = t(\{n\} \times \omega)$ then $S_t^n \cup \{s_t^n\} \subseteq K_n^t$ and $\{s_t^n \mid n \in \omega\} \cup \{0\} \subseteq B^t$ where $K_n^t, B^t \in C(\mathcal{S})$,

(j) $t(\omega^2) \cup \{s_t^i \mid i \in \omega\} \cup \{0\}$ is a closed subset of $G(\mathcal{S})$ homeomorphic to S_2.

Then properties (f)–(g) and (i) imply $1(t)$–$3(t)$ so t is a correct table in $G(\mathcal{S})$. Property (j) implies that $t^{-1}(F)$ is small for all $F \in C(\mathcal{S})$. In all further considerations t denotes the injection discussed above.

Assume **CH**. Let $\{O_\alpha\}_{\alpha<\omega_1}$ be all subsets of $\mathbb{Q}$, and $\{Z_\alpha\}_{\alpha<\omega_1}$ be all subsets of $\mathbb{Q}^2$. We assume for convenience that $O_0 = \emptyset$, $Z_0 = \emptyset$ and $Z_1 = \{(t(n,k), t(n,k)) \mid n,k \in \omega\} \cup \{(s_t^n, s_t^n) \mid n \in \omega\} \cup \{(0,0)\}$. Let $\omega_1 \backslash 0 = \Lambda_0 \cup \Lambda_1$ and $\Lambda_0 \cap \Lambda_1 = \emptyset$, with Λ_ν uncountable ($\nu \in \{0,1\}$ here and further on). Let

$\{u_\alpha\}_{\alpha<\omega_1}$ be the family of all the injections $u_\alpha : \omega^2 \to \mathbb{Q}$ such that every $u \in \{u_\alpha\}_{\alpha<\omega_1}$ repeats ω_1 times in $\{u_\alpha\}_{\alpha\in\Lambda_0}$ as well as in $\{u_\alpha\}_{\alpha\in\Lambda_1}$.

LEMMA 2.8 (**CH**). *For every $\alpha < \omega_1$ there exist*:

- *countable families $\mathcal{K}^\nu_\alpha$ of compact subsets of $\mathbb{Q}$*,
- *countable families $\mathcal{U}^\nu_\alpha$ of subsets of $\mathbb{Q}$*,
- *compact subsets K^ν_α of $\mathbb{Q}$*,

such that:

(15) $\mathcal{K}^\nu_\alpha = \bigcup_{\beta<\alpha}\mathcal{K}^\nu_\beta \cup \{K^\nu_\alpha\}$, $S_1 \in \mathcal{K}^\nu_\alpha$,
(16) *if $\alpha \in \Lambda_\nu$ and u_α is a correct table in $G(\mathcal{K}^\nu_\beta)$ for some $\beta < \alpha$ then $K^\nu_\alpha \subseteq S_{u_\alpha} \cup \{0\}$ and $u^{-1}_\alpha(K^\nu_\alpha)$ is infinite and thin*; *otherwise $K^\nu_\alpha = S_1$*,
(17) *K^ν_α is a nontrivial convergent sequence with limit point 0 in $G(\mathcal{K}^\nu_\alpha)$*,
(18) *if $U^\nu \in \mathcal{U}^\nu_\beta$ and $\beta \le \alpha$ then U^ν is open in $G(\mathcal{K}^\nu_\alpha)$*,
(19) $\mathcal{U}^\nu_\alpha \supseteq \bigcup_{\beta<\alpha}\tau_0(\mathcal{U}^\nu_\beta,\mathcal{K}^\nu_\beta)$,
(20) *for every $\beta \le \alpha$ the topology of $G(\mathcal{K}^\nu_\alpha)$ is stronger than $\tau(\mathcal{U}^\nu_\beta,\mathcal{K}^\nu_\beta)$*,
(21) *if O_α is open in $G(\mathcal{K}^\nu_\alpha)$ then $O_\alpha \in \mathcal{U}^\nu_\alpha$*,
(22) *Z_α is either not closed in $G(\mathcal{K}^0_\alpha) \times G(\mathcal{K}^1_\alpha)$ or closed in $\tau(\mathcal{U}^0_\alpha,\mathcal{K}^0_\alpha) \times \tau(\mathcal{U}^1_\alpha,\mathcal{K}^1_\alpha)$*,
(23) *for every $F^\nu \in C(\mathcal{K}^\nu_\beta)$ with $\beta \le \alpha$ the following hold*:

(a) *$t^{-1}(F^\nu)$ is small*,
(b) *$\mathrm{ess}(t^{-1}(F^\nu)) \cap \mathrm{ess}(t^{-1}(F^{1-\nu}))$ is finite*,
(c) *$t(\mathrm{ess}(t^{-1}(F^\nu)))$ is closed and discrete in $\tau(\mathcal{U}^{1-\nu}_\alpha, K^{1-\nu}_\alpha)$.*

Proof. Put $\mathcal{K}^0_0 = \mathcal{K}^1_0 = \{S_1\}$, $K^0_0 = K^1_0 = S_1$ and $\mathcal{U}^0_0 = \mathcal{U}^1_0 = \{\emptyset\}$. Then (15)–(23) are easy to check. Suppose the families $\mathcal{K}^\nu_\alpha$, $\mathcal{U}^\nu_\alpha$ and the sets K^ν_α are already constructed so that they satisfy the conditions (15)–(23) for all $\alpha < \kappa$. Put

$$\mathcal{U}^\nu_{(1)} = \bigcup_{\alpha<\kappa}\tau_0(\mathcal{U}^\nu_\alpha,\mathcal{K}^\nu_\alpha) \cup \bigcup_{\alpha<\kappa}\mathcal{U}^\nu_\alpha, \quad \mathcal{K}^\nu_{(1)} = \bigcup_{\alpha<\kappa}\mathcal{K}^\nu_\alpha. \tag{24}$$

Suppose that Z_α is closed in $G(\mathcal{K}^0_{(1)}) \times G(\mathcal{K}^1_{(1)})$. Since $\mathbb{Q}$ is countable there exist countable families $\{L^\nu_i\}_{i\in\omega}$ such that for every $i \in \omega$, L^ν_i is open in $G(\mathcal{K}^\nu_{(1)})$, and for any $(a,b) \in \mathbb{Q}^2 \setminus Z_\alpha$ there are $i,j \in \omega$ such that $(a,b) \in L^0_i \times L^1_j \subseteq \mathbb{Q}^2 \setminus Z_\alpha$. Put

$$\mathcal{U}^\nu_{(2)} = \mathcal{U}^\nu_{(1)} \cup \{L^\nu_i\}_{i\in\omega}. \tag{25}$$

If O_κ is open in $G(\mathcal{K}^\nu_{(1)})$ then put

$$\mathcal{U}^\nu_{(3)} = \mathcal{U}^\nu_{(2)} \cup \{O_\kappa\}. \tag{26}$$

Otherwise $\mathcal{U}^\nu_{(3)} = \mathcal{U}^\nu_{(2)}$. Consider the families $\{F^i_\nu\}_{i\in\omega} = C(\mathcal{K}^\nu_{(1)})$. By (24), (15) and Lemma 2.4 every F^i_ν is in $C(\mathcal{K}^\nu_\alpha)$ for some $\alpha < \kappa$. Now consider the families $\{\theta_i^\nu\}_{i\in\omega}$ where $\theta_i^\nu = \mathrm{ess}(t^{-1}(F_{1-\nu}^i))$. This definition is correct by induction, (23)(a) and the remark above. It now follows by induction and (23)(c) that for any $i \in \omega$ the set $t(\theta_i^\nu)$ is closed and discrete in $\tau(\mathcal{U}_\beta^\nu, \mathcal{K}_\beta^\nu)$ for some $\beta < \kappa$. Thus by (20), $t(\theta_i^\nu)$ is closed and discrete in any $G(\mathcal{K}_\alpha^\nu)$ where $\beta \le \alpha < \kappa$ and thus by (15) in any $G(\mathcal{K}_\alpha^\nu)$ with $\alpha < \kappa$ since (15) obviously implies that the topology of $G(\mathcal{K}_\gamma^\nu)$ is stronger than that of $G(\mathcal{K}_\alpha^\nu)$ for $\gamma \le \alpha$. Thus by (24), Lemma 2.4 and the definition of $G(\mathcal{K}_{(1)}^\nu)$ every $t(\theta_i^\nu)$ is closed and discrete in $G(\mathcal{K}_{(1)}^\nu)$. Put $W_{a,i}^\nu = (\mathbb{Q} \setminus t(\theta_i^\nu)) \cup \{a\}$ for $a \in \mathbb{Q}$. Now every $W_{a,i}^\nu$ is open in $G(\mathcal{K}_{(1)}^\nu)$. Put

$$\mathcal{U}_{(4)}^\nu = \mathcal{U}_{(3)}^\nu \cup \{W_{a,i}^\nu\}_{a\in\mathbb{Q}, i\in\omega}. \tag{27}$$

It follows by induction, (24)–(27)(d), (18)–(20) and the construction of $\mathcal{U}_{(4)}^\nu$ that every $U^\nu \in \mathcal{U}_{(4)}^\nu$ is open in $G(\mathcal{K}_{(1)}^\nu)$. So we can consider the topology $\tau(\mathcal{U}_{(4)}^\nu, \mathcal{K}_{(1)}^\nu)$.

Assume without loss of generality that $\kappa \in \Lambda_0$ and u_κ is a correct table in $G(\mathcal{K}_\alpha^0)$ for some $\alpha < \kappa$. Then obviously u_κ is a correct table in $G(\mathcal{K}_{(1)}^0)$. By induction, (23)(a), (15) and Lemma 2.4, $t^{-1}(F^0)$ is small for each $F^0 \in C(\mathcal{K}_{(1)}^\nu)$. Then by Lemma 2.7 choose an infinite and thin $\sigma = \{\sigma_i \mid i \ge 1\} \subseteq \omega^2$ such that

$$u_\kappa(\sigma_i) \to 0 \quad \text{as } i \to \infty \quad \text{in } \tau(\mathcal{U}_{(4)}^0, \mathcal{K}_{(1)}^0) \tag{28}$$

and

(29) for all $\vec{a} \in \mathbb{Q}^\infty$ and $F^0 \in C(\mathcal{K}_{(1)}^0)$ the set $t^{-1}(\vec{a}\langle u_\kappa(\sigma) \cup \{0\}\rangle + F^0)$ is small.

Put $K_\kappa^0 = u_\kappa(\sigma) \cup \{0\}$ and $K_\kappa^1 = S_1$. Then by (d) and (28), K_κ^ν is a compact subset of $\mathbb{Q}$. Now put $\mathcal{K}_\kappa^\nu = \bigcup_{\alpha<\kappa} \mathcal{K}_\alpha^\nu \cup \{K_\kappa^\nu\}$. Then (15) holds.

Let $U^\nu \in \mathcal{U}_{(4)}^\nu$. Let us show that U^ν is open in $G(\mathcal{K}_\kappa^\nu)$. It is enough to prove that $U^\nu \cap F^\nu$ is relatively open (in the usual topology of $\mathbb{Q}$) for every $F^\nu \in C(\mathcal{K}_\kappa^\nu)$. By Lemma 2.6, (24) and Lemma 2.4 every F^ν is of the form

$$F^\nu = \bigcup_{i\le k} \vec{a}_i\langle K_\kappa^\nu\rangle + F_i \tag{30}$$

where $\vec{a}_i \in \mathbb{Q}^\infty$, $F_i \in C(\mathcal{K}_\alpha^\nu)$ for some $\alpha < \kappa$, and $k \in \omega$. Now K_κ^ν is compact in $\tau(\mathcal{U}_{(4)}^\nu, \mathcal{K}_{(1)}^\nu)$ and by (a) every $\vec{a}_i\langle K_\kappa^\nu\rangle$ is compact in $\tau(\mathcal{U}_{(4)}^\nu, \mathcal{K}_{(1)}^\nu)$. Thus F^ν is compact in $\tau(\mathcal{U}_{(4)}^\nu, \mathcal{K}_{(1)}^\nu)$ and thus has the topology induced from $\mathbb{Q}$ by (d). But $U^\nu \in \tau(\mathcal{U}_{(4)}^\nu, \mathcal{K}_{(1)}^\nu)$ by (c) so $U^\nu \cap F^\nu$ is relatively open.

Now let us show that every set of the form

$$t(\mathrm{ess}(t^{-1}(F^0))), \qquad F^0 \in C(\mathcal{K}_\kappa^0), \tag{31}$$

is closed and discrete in $G(\mathcal{K}_\kappa^1)$. Note that $\mathrm{ess}(t^{-1}(F^0))$ exists due to (29), Lemma 2.6 and the construction of K_κ^0. First we have $C(\mathcal{K}_\kappa^1) = C(\mathcal{K}_{(1)}^1)$.

So if $F^1 \in C(\mathcal{K}^1_\kappa)$ then $F^1 \in C(\mathcal{K}^1_\alpha)$ for some $\alpha < \kappa$ and thus $F^1 = F^n_1$ and $\mathrm{ess}(t^{-1}(F^n_1)) = \theta^0_n$. Then for any point $a \in \mathbb{Q}$ there is a neighborhood $a \in (\mathbb{Q} \setminus t(\theta^0_n)) \cup \{a\} = W^0_{a,n} \in \mathcal{U}^0_{(4)}$ open in $G(\mathcal{K}^0_\kappa)$ by what we have proved above. Thus $t(\theta^0_n)$ is closed and discrete in $G(\mathcal{K}^0_\kappa)$. So $F^0 \cap t(\theta^0_n)$ is finite. Then $t(\mathrm{ess}(t^{-1}(F^0))) \cap t(\mathrm{ess}(t^{-1}(F^1)))$ is finite for all $F^1 \in C(\mathcal{K}^1_\kappa)$. So $t(\mathrm{ess}(t^{-1}(F^0))) \cap F^1$ is finite for all $F^1 \in C(\mathcal{K}^1_\kappa)$. Thus $t(\mathrm{ess}(t^{-1}(F^0)))$ is closed and discrete in $G(\mathcal{K}^1_\kappa)$.

Consider the family $\{V_{a,i}\}_{a\in\mathbb{Q},i\in\omega}$ where $V_{a,i} = (\mathbb{Q}\setminus t(\mathrm{ess}(t^{-1}(H_i)))) \cup \{a\}$ and $\{H_i\}_{i\in\omega} = C(\mathcal{K}^0_\kappa)$. By what we have proved above every $V_{a,i}$ is open in $G(\mathcal{K}^1_\kappa)$. Put

$$\mathcal{U}^0_\kappa = \mathcal{U}^0_{(4)}, \quad \mathcal{U}^1_\kappa = \mathcal{U}^1_{(4)} \cup \{V_{a,i}\}_{a\in\mathbb{Q},i\in\omega}. \tag{32}$$

Let $U^\nu \in \mathcal{U}^\nu_\alpha$ with $\alpha \le \kappa$. If $\alpha < \kappa$ we have already proved that U^ν is open in $G(\mathcal{K}^\nu_\kappa)$. If $\alpha = \kappa$ then if $U^\nu \in \mathcal{U}^\nu_{(4)}$ we have proved before that U^ν is open in $G(\mathcal{K}^\nu_\kappa)$. Now it follows from (32) that (18) holds. Then (20) is obvious because if $\beta < \kappa$ then by (24) and (e), $\tau(\mathcal{U}^\nu_\kappa, \mathcal{K}^\nu_\kappa)$ is stronger than $\tau(\mathcal{U}^\nu_\beta, \mathcal{K}^\nu_\beta)$ and the topology of $G(\mathcal{K}^\nu_\kappa)$ is stronger than $\tau(\mathcal{U}^\nu_\kappa, \mathcal{K}^\nu_\kappa)$ by (d). If O_κ is open in $G(\mathcal{K}^\nu_\kappa)$ then it is open in $G(\mathcal{K}^\nu_{(1)})$ and thus $O_\kappa \in \mathcal{U}^\nu_{(3)} \subseteq \mathcal{U}^\nu_\kappa$ by (26)–(27). So (21) holds.

If Z_κ is closed in $G(\mathcal{K}^0_\kappa) \times G(\mathcal{K}^1_\kappa)$ then it is closed in $G(\mathcal{K}^0_{(1)}) \times G(\mathcal{K}^1_{(1)})$. Then the construction of L^ν_i and (25) give that Z_κ is closed in $\tau(\mathcal{U}^0_\kappa, \mathcal{K}^0_\kappa) \times \tau(\mathcal{U}^1_\kappa, \mathcal{K}^1_\kappa)$. Thus (22) holds. Now (16), (17) and (19) are obvious. Let now $F^1 \in C(\mathcal{K}^1_\alpha)$ with $\alpha \le \kappa$. Then in fact $F^1 \in C(\mathcal{K}^1_\beta)$ for some $\beta < \kappa$. So by induction and (23)(a), $t^{-1}(F^1)$ is small. If $F^0 \in C(\mathcal{K}^0_\alpha)$ with $\alpha \le \kappa$ then by Lemma 2.6, Lemma 2.4 and (15),

$$F^0 = \bigcup_{i\le k} \vec{a}_i\langle K^0_\kappa\rangle + F_i, \quad \text{where } F_i \in C(\mathcal{K}^0_{(1)}).$$

Now by (29) each $t^{-1}(\vec{a}_i\langle K^0_\kappa\rangle + F_i) = t^{-1}(\vec{a}_i\langle u_\kappa(\sigma) \cup \{0\}\rangle + F_i)$ is small so (23)(a) holds. By the choice of $\{V_{a,i}\}_{a\in\mathbb{Q},i\in\omega}$ and the fact that every $V_{a,i}$ is open in $\tau(\mathcal{U}^1_\kappa, \mathcal{K}^1_\kappa)$ every set of the form $t(\mathrm{ess}(t^{-1}(F^0)))$ where $F^0 \in C(\mathcal{K}^0_\kappa)$ is closed and discrete in $\tau(\mathcal{U}^1_\kappa, \mathcal{K}^1_\kappa)$. It follows that $t(\mathrm{ess}(t^{-1}(F^0))) \cap F^1$ is finite for all $F^1 \in C(\mathcal{K}^1_\kappa)$. Thus $\mathrm{ess}(t^{-1}(F^0)) \cap \mathrm{ess}(t^{-1}(F^1))$ is finite for all $F^\nu \in C(\mathcal{K}^\nu_\kappa)$. So (23)(b) holds. To prove (23)(c) it remains to show that for every $F^1 \in C(\mathcal{K}^1_\kappa)$ the set $t(\mathrm{ess}(t^{-1}(F^1)))$ is closed and discrete in $\tau(\mathcal{U}^0_\kappa, \mathcal{K}^0_\kappa)$. This can be proved using the properties of $W^0_{a,i}$. ∎

Let us now construct a pair of countable Fréchet topological groups whose product is sequential but is not Fréchet.

Example 2.9 (**CH**). Let $\mathcal{K}_\nu = \bigcup_{\alpha<\omega_1} C(\mathcal{K}^\nu_\alpha)$ where the families $\mathcal{K}^\nu_\alpha$ are constructed in Lemma 2.8. Let τ^ν be the topology on $\mathbb{Q}$ defined as follows.

$U \subseteq \mathbb{Q}$ is open in τ^ν if and only if $U \cap F^\nu$ is relatively open for each $F^\nu \in \mathcal{K}_\nu$. The following fact follows easily from (20) and the definition of $G(\mathcal{K}^\nu_\alpha)$:

FACT. *For every $\alpha < \omega_1$ the topology τ^ν is stronger than $\tau(\mathcal{U}^\nu_\alpha, \mathcal{K}^\nu_\alpha)$.*

Consider now an arbitrary $O \in \tau^\nu$. Then $O = O_\alpha$ for some $\alpha < \omega_1$ and O_α is open in the topology of $G(\mathcal{K}^\nu_\alpha)$, which is stronger than τ^ν. Thus by (21) and (c), O_α is open in $\tau(\mathcal{U}^\nu_\alpha, \mathcal{K}^\nu_\alpha)$. It follows from the Fact and the above considerations that τ^ν is a common refinement for the family $\{\tau(\mathcal{U}^\nu_\alpha, \mathcal{K}^\nu_\alpha) \mid \alpha < \omega_1\}$. So τ^ν is a group topology.

Let $Z \subseteq (\mathbb{Q}, \tau^0) \times (\mathbb{Q}, \tau^1)$ be an arbitrary subset. Then $Z = Z_\alpha$ for some $\alpha < \omega_1$. Let Z be a nonclosed subset of $G(\mathcal{K}^0_\alpha) \times G(\mathcal{K}^1_\alpha)$. Then since $G(\mathcal{K}^\nu_\alpha)$ has a k_ω-topology it follows that $G(\mathcal{K}^0_\alpha) \times G(\mathcal{K}^1_\alpha)$ is sequential and thus there is a sequence in Z converging to a point outside Z in the topology of $G(\mathcal{K}^0_\alpha) \times G(\mathcal{K}^1_\alpha)$ and thus in the weaker topology $\tau^0 \times \tau^1$. If Z is closed in $G(\mathcal{K}^0_\alpha) \times G(\mathcal{K}^1_\alpha)$ then Z is closed in $\tau(\mathcal{U}^0_\alpha, \mathcal{K}^0_\alpha) \times \tau(\mathcal{U}^1_\alpha, \mathcal{K}^1_\alpha)$ by (22) and thus Z is closed in the stronger topology $\tau^0 \times \tau^1$. So $\tau^0 \times \tau^1$ is sequential.

Suppose τ^ν is not Fréchet. Then there exists an injection $u : \omega^2 \to \mathbb{Q}$ such that $u(n,k) \to s^n_u$ as $k \to \infty$ in $(\mathbb{Q}, \tau^\nu)$ and $s^n_u \to 0$ as $n \to \infty$ in $(\mathbb{Q}, \tau^\nu)$ and there is no sequence in $u(\omega^2)$ converging to 0 in $(\mathbb{Q}, \tau^\nu)$. Using the definition of τ^ν we may assume without loss of generality that $u(\{n\} \times \omega) \cup \{s^n_u\} \subseteq K^u_n$, $\{s^n_u \mid n \in \omega\} \cup \{0\} \subseteq B^u$ where $K^u_n \in C(\mathcal{K}^\nu_{\alpha_n})$ and $B^u \in C(\mathcal{K}^\nu_\alpha)$. Let $\gamma = \sup(\{\alpha_n \mid n \in \omega\} \cup \{\alpha\})$. Obviously u is a correct table in $G(\mathcal{K}^\nu_{\gamma+1})$. By the choice of u_α there exists $\beta \in \Lambda_\nu$ with $\beta > \gamma + 1$ such that $u = u_\beta$. Now by (16) and (17), $K^\nu_\beta \subseteq S_u \cup \{0\}$ and K^ν_β is homeomorphic to a nontrivial convergent sequence with limit point 0 in $G(\mathcal{K}^\nu_\beta)$. So K^ν_β is a convergent sequence in the weaker topology τ^ν. A contradiction. So τ^ν is Fréchet.

Obviously $(t(n,k), t(n,k)) \to (s^n_t, s^n_t)$ as $k \to \infty$ in $\tau^0 \times \tau^1$ and $(s^n_t, s^n_t) \to (0,0)$ as $n \to \infty$ in $\tau^0 \times \tau^1$. Suppose $(t(n_i,k_i), t(n_i,k_i)) \to (0,0)$ as $i \to \infty$ in $\tau^0 \times \tau^1$. Then we may assume without loss of generality that $\{t(n_i,k_i) \mid i \in \omega\} \subseteq F^0 \in \mathcal{K}_0$ and $\{t(n_i,k_i) \mid i \in \omega\} \subseteq F^1 \in \mathcal{K}_1$ for some F^0, F^1. Also, $F^0 \in G(\mathcal{K}^0_\alpha)$ and $F^1 \in G(\mathcal{K}^1_\alpha)$ for some $\alpha < \omega_1$. The set $\{(n_i,k_i) \mid i \in \omega\}$ is infinite and thin. Then $\operatorname{ess}(t^{-1}(F^0)) \cap \operatorname{ess}(t^{-1}(F^1)) \supseteq \{(n_i,k_i) \mid i \in \omega\}$, which contradicts (23)(b). So $\tau^0 \times \tau^1$ is not Fréchet. The argument above also shows that the set $Z_1 = \{(t(m,n), t(m,n)) \mid m,n \in \omega\} \cup \{(s^n_t, s^n_t) \mid n \in \omega\} \cup \{(0,0)\}$ is homeomorphic to S_2 in the topology induced by $(\mathbb{Q}^2, \tau^0 \times \tau^1)$ and the proof of Lemma 2.8 shows that Z_1 is closed in $(\mathbb{Q}^2, \tau^0 \times \tau^1)$. ∎

In the next example we construct two countable Fréchet topological groups whose product is not sequential.

EXAMPLE 2.10 (**CH**). Let τ^ν be the topologies constructed in the previous example. Put $G_0 = (\mathbb{Q}, \tau^0)$, $G_1 = (\mathbb{Q}, \tau^1)$ and $G^0 = G_0 \times \mathbb{Q}$. Then G^0

can be embedded into $G_0 \times [0,1]$ and since G_0 is an α_4-space by the result of [N], it follows from [A3, Corollary 5.26] that G^0 is Fréchet. The product $G_0 \times G_1$ contains a closed copy of S_2 as was shown in Example 2.9. Now $S_2 \times \mathbb{Q}$ is a closed subset of the product $G^0 \times G_1$. Since $S_2 \times \mathbb{Q}$ is not a k-space (see [M2]), neither is $G^0 \times G_1$. ∎

Let $\{v_\alpha\}_{\alpha<\omega_1}$ be the family of all mappings $v_\alpha : \omega^2 \to \mathbb{Q}$, and $\{P_\alpha\}_{\alpha<\omega_1}$ be the family of all compact subsets of $\mathbb{Q}$.

The following lemma may be proved by an argument similar to that of Lemma 2.8 (see [EKN] for a discussion of spaces containing a copy of the rationals).

Lemma 2.11 (**CH**). *For every $\alpha < \omega_1$ there is a convergent sequence $K_\alpha \subseteq \mathbb{Q}$, a countable family $\mathcal{K}_\alpha$ of compact subsets of $\mathbb{Q}$, a subset D_α of $\mathbb{Q}$ and a countable family $\mathcal{U}_\alpha$ of subsets of $\mathbb{Q}$ such that*:

(33) $\mathcal{K}_\alpha = \bigcup_{\beta<\alpha} \mathcal{K}_\beta \cup \{K_\alpha\}$ *and* $S_1 \in \mathcal{K}_\alpha$,

(34) *if for all* $i \in \omega$, $v_\alpha(i,j) \to 0$ *as* $j \to \infty$ *in some* $G(\mathcal{K}_\beta)$ *with* $\beta < \alpha$ *then* $K_\alpha \subseteq v_\alpha(\omega^2) \cup \{0\}$ *and* $\{v_\alpha(i,j) \mid j \in \omega\} \setminus K_\alpha$ *is finite for all* $i \in \omega$,

(35) *if* O_α *is open in* $G(\mathcal{K}_\alpha)$ *then* $O_\alpha \in \mathcal{U}_\alpha$,

(36) $\mathcal{U}_\alpha \supseteq \bigcup_{\beta<\alpha} \tau_0(\mathcal{U}_\beta, \mathcal{K}_\beta)$,

(37) *if* $U \in \mathcal{U}_\alpha$ *then* U *is open in* $G(\mathcal{K}_\alpha)$,

(38) *the topology of* $G(\mathcal{K}_\alpha)$ *is stronger than* $\tau(\mathcal{U}_\beta, \mathcal{K}_\beta)$ *for* $\beta \le \alpha$,

(39) *if there is no finite* $\kappa \subseteq \mathcal{K}_\alpha$ *such that* $P_\alpha \subseteq \bigcup \kappa$ *then* D_α *is an infinite closed and discrete subset of* P_α *in* $\tau(\mathcal{U}_\alpha, \mathcal{K}_\alpha)$.

Let us now indicate briefly how to construct an α_1 and Fréchet countable topological group which contains no copy of the rationals. Let us recall the definition of a well known topological invariant. For a topological space K let $K^0 = K \setminus$isolated points of K, $K^{\alpha+1} = K^\alpha \setminus$isolated points of K^α and $K^\alpha = \bigcap_{\beta<\alpha} K^\beta$ for limit α. Let $\mathrm{sc}(K) = \min\{\alpha \mid K^\alpha = \emptyset\}$. It is well known that $\mathrm{sc}(K)$ is well defined for every countable compact space and that if $\mathrm{sc}(K_1)$ and $\mathrm{sc}(K_2)$ are finite for $K_1, K_2 \subseteq \mathbb{Q}$ then $\mathrm{sc}(K_1 \cup K_2)$ and $\mathrm{sc}(K_1 + K_2)$ are finite. So in the construction of Lemma 2.2 it can be shown that $\mathrm{sc}(K)$ is finite for all $K \subseteq \mathcal{K}_\alpha$.

Example 2.12 (**CH**). Let $\mathcal{K} = \bigcup_{\alpha<\omega_1} C(\mathcal{K}_\alpha)$ where $\mathcal{K}_\alpha$ were constructed in Lemma 2.11. Define a topology on $\mathbb{Q}$ as in Example 2.9. We obtain a topological group G. The conditions (34), (35) and (38) easily give that G is α_1 and sequential and hence Fréchet. Now it follows from (39) that for each compact $P \subseteq G$ there is a finite $\kappa \subseteq \mathcal{K}_\alpha$ for some $\alpha < \omega_1$ such that $P \subseteq \bigcup \kappa$. This implies that $\mathrm{sc}(P)$ is finite. But if G contained a copy of the rationals it would contain a compact P such that $\mathrm{sc}(P) = \omega + 1$. ∎

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