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A complement to the theory of equivariant finiteness obstructions

by

Paweł Andrzejewski (Szczecin)

Abstract. It is known ([1], [2]) that a construction of equivariant finiteness obstructions leads to a family $w_\alpha^H(X)$ of elements of the groups $K_0(\mathbb{Z}[\pi_0(WH(X))_\alpha^*])$. We prove that every family $\{w_\alpha^H\}$ of elements of the groups $K_0(\mathbb{Z}[\pi_0(WH(X))_\alpha^*])$ can be realized as the family of equivariant finiteness obstructions $w_\alpha^H(X)$ of an appropriate finitely dominated G-complex X. As an application of this result we show the natural equivalence of the geometric construction of equivariant finiteness obstruction ([5], [6]) and equivariant generalization of Wall's obstruction ([1], [2]).

Introduction. The purpose of this paper is a clarification of the theory of equivariant finiteness obstructions. At present there are four different approaches to this subject. Two of them are equivariant generalizations of Wall's and Ferry's ideas (see [1]–[3] and [4] respectively). In 1985 W. Lück [5] suggested a purely geometric construction of the finiteness obstruction and then he developed the global algebraic approach to the equivariant finiteness obstruction [6] which covers all the constructions mentioned above.

In [7], Theorem F, C. T. C. Wall proved that if Y is a finite CW-complex then each element of the group $\widetilde{K}_0(\mathbb{Z}[\pi_1(Y)])$ can be realized as the finiteness obstruction of a finitely dominated CW-complex.

We shall establish among other things a similar theorem for equivariant finiteness obstructions proving in Section 2 that if Y is a finite G-complex then every family $\{w_\alpha^H\}$ of elements of the groups $\widetilde{K}_0(\mathbb{Z}[\pi_0(WH(Y))_\alpha^*])$ can be realized as the family of equivariant finiteness obstructions $w_\alpha^H(X)$ of an appropriate finitely dominated G-complex X. This result, in turn, will be used in Section 3 to show the existence of a natural equivalence between the geometric finiteness obstruction introduced by Lück [5] and the obstructions $w_\alpha^H(X)$.

Throughout the paper G denotes a compact Lie group.

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1. A short review of the equivariant finiteness obstruction. In this introductory section we recall a construction of the equivariant finiteness obstruction based on the ideas of C. T. C. Wall [7] and described by the author in [1] and [2]. As a result of this construction one gets a family of invariants which decide whether a finitely G-dominated G-complex is G-homotopy finite.

Roughly speaking, the family of obstructions we want to introduce is defined for each component X_α^H by means of the invariants $w_G(X,A)$ (see [1], §1, or [2], §2). Precisely, let H denote a closed subgroup of G and let X_α^H be a connected component of $X^H \neq \emptyset$. We define an equivalence relation $\approx$ in the set of such components X_α^H by setting $X_\alpha^H \approx X_\beta^H$ iff there exists an element $n \in G$ such that $nHn^{-1} = K$ and $n(X_\alpha^H) = X_\beta^H$. We denote the set of equivalence classes of this relation by $\underline{CI}(X)$. Note that this definition is functorial, i.e. a G-map $f : X \to Y$ induces a map $\underline{CI}(f) : \underline{CI}(X) \to \underline{CI}(Y)$.

If X is finitely G-dominated by a complex K and X_α^H denotes a component of $X^H \neq \emptyset$ which represents an element of the set $\underline{CI}(X)$ then the group $(WH)_\alpha$ acts on the pairs $(X_\alpha^H, X_\alpha^{>H})$ and $(K_\beta^H, K_\beta^{>H})$ in such a way that $(X_\alpha^H, X_\alpha^{>H})$ is relatively free and $(K_\beta^H, K_\beta^{>H})$ is relatively free and relatively finite. By the relative version of Proposition 1.3 in [1] we see that the pair $(K_\beta^H, K_\beta^{>H})$ $(WH)_\alpha$-dominates the pair $(X_\alpha^H, X_\alpha^{>H})$.

Definition ([1], [2]). We define a *Wall-type invariant* $w_\alpha^H(X)$ to be

$$\begin{aligned} w_\alpha^H(X) &= w_{(WH)_\alpha}(X_\alpha^H, X_\alpha^{>H}) \\ &= w(C_*(\widetilde{X_\alpha^H}, \widetilde{X_\alpha^{>H}})) \in \widetilde{K}_0(\mathbb{Z}[\pi_0(WH)_\alpha^*]). \end{aligned}$$

The elements $w_\alpha^H(X)$ are invariants of the equivariant homotopy type and they vanish for finite G-complexes. Moreover, the invariant $w_\alpha^H(X)$ does not depend (up to canonical isomorphism) on the choice of the representative X_α^H from the equivalence class $[X_\alpha^H]$ in $\underline{CI}(X)$ (see [1]). The fundamental property of the invariants $w_\alpha^H(X)$ is that they are actually obstructions to homotopy finiteness of X:

Theorem 1.1 ([1]–[3]). *Let a G-complex X be G-dominated by a finite G-complex K. Then there exist a finite G-complex Y and a G-homotopy equivalence $h : Y \to X$ iff all the invariants $w_\alpha^H(X)$ vanish. Moreover, if the complex X contains a finite G-subcomplex B and* $\dim K = n$ *then Y and h can be chosen in such a manner that $B \subset Y$,* $\dim Y = \max(3,n)$ *and* $h|_B = \mathrm{id}_B$.

2. The realization theorems for the equivariant finiteness obstruction. As in the proof of Theorem 1.1 (see [1] or [2]) we begin with the case of a relatively free action which will serve as an inductive step in the proof of the main result.

PROPOSITION 2.1. *Let (Y,A) be a relatively free, relatively finite G-CW-pair and $w_0 \in \widetilde{K}_0(\mathbb{Z}[\pi_0(G(Y)^*)])$ be an arbitrary element. Then there exist relatively free G-CW-pairs (X,A) and (K,A) and a G-retraction $r : X \to Y$ inducing the isomorphism of fundamental groups such that $Y \subset X$, $Y \subset K$, (K,A) is a relatively finite G-CW-pair and G-dominates (X,A) and the equality $r_*(w_G(X,A)) = w_0$ holds where r_* denotes the isomorphism induced by r on $\widetilde{K}_0$.*

Remark. Here $w_G(X,A)$ denotes the algebraic Wall finiteness obstruction of a finitely dominated chain complex $C_*(\widetilde{X},\widetilde{A})$ of free $\mathbb{Z}[\pi_0(G(Y)^*)]$-modules (see [1], p. 12, or [2], §2).

Proof. Let P and Q be finitely generated, projective $\mathbb{Z}[\pi_0(G(Y)^*)]$-modules with $P \oplus Q = B$ a free module. Let $w_0 = (-1)^n[P] = (-1)^{n+1}[Q]$ where $n > 2$. Let $p : B \to P$ and $q : B \to Q$ denote projections and C_* be the chain complex of the form

$$\ldots \to B \xrightarrow{q} B \xrightarrow{p} B \xrightarrow{q} B \to 0 \to 0 \to \ldots$$

with $C_k = 0$ for $k < n$.

We shall construct a relatively free G-CW-pair (X,Y) such that $C_* = C_*(\widetilde{X},\widetilde{Y})$.

Suppose $\operatorname{rank}(B) = m$ and let Y_1 be a G-complex obtained from Y by attaching m free G-n-cells via trivial G-maps

$$\phi_i : G \times S^{n-1} \to Y,$$

$\phi_i(g,x) = g \cdot y_0$, where $y_0 \in Y$ is fixed.

We shall show inductively that for each $k \geq 0$ there exists a relatively free G-CW-pair (X_k, Y) and a G-map $r_k : X_k \to Y_1$ such that $C_* = C_*(\widetilde{X}_k, \widetilde{Y})$ for $* \leq n+k-1$ and that P (respectively Q) is a direct summand in $\pi_{n+k}(r_k)$ for odd (resp. even) k. We start with the inclusion $r_0 : Y = X_0 \hookrightarrow Y_1$. Since the attaching maps of free G-n-cells in Y_1 are equivariantly trivial there exists an exact sequence

$$\ldots \to \pi_n(Y_1) \to \pi_n(r_0) \xrightarrow{\partial} \pi_{n-1}(Y) \to \pi_{n-1}(Y_1) \to \ldots$$

with $\pi_n(r_0) = B$ and $\partial = 0$. Let ξ_j $(j = 1,\ldots,m)$ denote free generators of the module B and $a_j = q(\xi_j) \in B = \pi_n(r_0)$. If $r_1 : X_1 \to Y_1$ is obtained from r_0 by attaching m free G-n-cells to $Y = X_0$ via $a_j \in \pi_n(r_0)$ then one has the split exact sequence

$$\ldots \to \pi_{n+1}(r_0) \to \pi_{n+1}(r_1) \rightleftarrows P \to 0$$

and P is a direct summand in $\pi_{n+1}(r_1)$.

Since $\partial = 0$, the attaching maps of G-n-cells in X_1 are equivariantly trivial. Hence there is a G-homotopy equivalence $k_1 : Y_1 \to X_1$.

Let further $b_j = p(\xi_j) \in P \subset \pi_{n+1}(r_1)$ and let $r_2 : X_2 \to Y_1$ be obtained from r_1 by attaching free G-$(n+1)$-cells via b_j. We have the split exact sequence

$$\ldots \to \pi_{n+2}(r_1) \to \pi_{n+2}(r_2) \rightleftarrows Q \to 0$$

and Q is a direct summand in $\pi_{n+2}(r_2)$.

It follows from the construction that $C_*(\widetilde{X}_1, \widetilde{Y}) = C_*$ for $* \le n$ and $C_*(\widetilde{X}_2, \widetilde{Y}) = C_*$ for $* \le n+1$.

The inductive step goes alternately.

Set $X = \bigcup_{k\ge 0} X_k$ and $r : X \to Y_1$ by $r|_{X_k} = r_k$. Then for $K = X_1$ we see that the pair (K, A) G-dominates the pair (X, A) with the section given by the composition

$$(X, A) \xrightarrow{r} (Y_1, A) \xrightarrow{k_1} (K, A).$$

Finally, we have by definition

$$\begin{aligned} r_*(w_G(X, A)) &= (-1)^{n+1}[C_{n+1}(\widetilde{X}, \widetilde{Y})/B_{n+1}(\widetilde{X}, \widetilde{Y})] \\ &= (-1)^{n+1}[C_{n+1}/\operatorname{im} \partial_{n+2}] \\ &= (-1)^{n+1}[B/P] = (-1)^{n+1}[Q] = w_0. \ \blacksquare \end{aligned}$$

We will also need the following technical result concerning the glueing equivariant domination maps.

Lemma 2.2. *Let $A \to X$ be a G-cofibration, Y a G-space and $r : Y \to A$ a G-domination map with a section $s : A \to Y$. Then in the commutative diagram*

$$\begin{array}{ccccc} X & \longleftarrow & A & \xrightarrow{\mathrm{id}} & A \\ \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \mathrm{id}} & & {\scriptstyle r}\big\uparrow\big\downarrow{\scriptstyle s} \\ X & \longleftarrow & A & \xrightarrow{s} & Y \end{array}$$

the map r extends to a G-domination map $R : X \cup_s Y \to X \cup_{\mathrm{id}} A \cong X$.

Now we can formulate the realization theorem.

Theorem 2.3. *Let Y be a finite G-complex and $\{w_\alpha^H\}$ be a family of elements indexed by the set $\underline{CI}(Y)$, with $w_\alpha^H \in \widetilde{K}_0(\mathbb{Z}[\pi_0(WH(Y)^*)])$. Then there exist a G-complex X and a G-retraction $r : X \to Y$ inducing bijections*

$$r_* : \pi_0(X^H) \to \pi_0(Y^H)$$

and isomorphisms

$$r_* : \pi_1(X_\alpha^H) \to \pi_1(Y_\alpha^H)$$

such that $Y \subset X$, X is finitely G-dominated and $r_*(w_\alpha^H(X)) = w_\alpha^H$.

Proof. Note that the set $\underline{CI}(Y)$ consists of one connected component from each WH-component $(WH)Y_\alpha^H$. One can assume, in view of Proposition 2.14 in [6], that H runs through a complete set of representatives for all the isotropy types (H) occurring in X.

We may suppose, in view of Proposition 2.12 in [6], that the set $\underline{CI}(Y)$ is finite. Let $Y_{\alpha_q}^{H_p}$, with $1 \le p \le r$, $1 \le q \le s_p$, denote the representatives of WH_p-components in the set $\underline{CI}(Y)$. Order the set of pairs $\{(p,q) : 1 \le p \le r,\ 1 \le q \le s_p\}$ lexicographically. For each pair (p,q) we shall construct inductively a G-complex $X_{p,q}$ with the following properties:

(1) $Y \subset X_{p,q}$ and there exists a G-retraction $r_{p,q} : X_{p,q} \to Y$ inducing bijections on the π_0-level and isomorphisms of fundamental groups of appropriate fixed point set components.

(2) If $(p,q) \le (m,n)$ then $X_{p,q} \subset X_{m,n}$.

(3) The complex $X_{p,q}$ is G-dominated by the finite G-complex $K_{p,q}$.

(4) $w_\alpha^H(X_{p,q}) = w_\alpha^H$ for $(H) = (H_i), 1 \le i < p$ and for any α.

(5) $w_{\alpha_j}^{H_p}(X_{p,q}) = w_{\alpha_j}^{H_p}$ for $1 \le j \le q$.

(6) $w_{\alpha_j}^{H_p}(X_{p,q}) = 0$ for $j > q$.

(7) $w_\alpha^H(X_{p,q}) = 0$ for $(H) = (H_i)$, $i > p$ and for any α.

Then the complex X_{r,s_r} obtained as a result of the final inductive step satisfies the assertion of the theorem.

Let $X_{0,0} = Y$ and suppose that $X_{p,q}$ has been constructed. There are two cases to consider.

Case I: $q < s_p$. Simplify the notation by setting $H = H_p$ and $\alpha = \alpha_{q+1}$. Then $((X_{p,q})_\alpha^H, (X_{p,q})_\alpha^{>H})$ is a relatively free and relatively finite $(WH)_\alpha$-CW-pair (by property (6) and Theorem 1.1). Since $\pi_1(Y_\alpha^H) \cong \pi_1((X_{p,q})_\alpha^H)$ we can assume that $\underline{CI}(Y) = \underline{CI}(X_{p,q})$ and $w_\alpha^H \in \widetilde{K}_0(\mathbb{Z}[\pi_0(WH(X_{p,q}))_\alpha^*])$. By Proposition 2.1 there exists a relatively free $(WH)_\alpha$-CW-pair $(Z, (X_{p,q})_\alpha^{>H})$ such that

(a) $(Z, (X_{p,q})_\alpha^{>H})$ is $(WH)_\alpha$-dominated by a relatively free, relatively finite $(WH)_\alpha$-CW-pair $(K, (X_{p,q})_\alpha^{>H})$,

(b) $(X_{p,q})_\alpha^H \subset Z$ and there exists a $(WH)_\alpha$-retraction $r : Z \to (X_{p,q})_\alpha^H$, and

(c) $r_*(w_{(WH)_\alpha}(Z, (X_{p,q})_\alpha^{>H})) = w_\alpha^H$.

Let

$$d : (K, (X_{p,q})_\alpha^{>H}) \to (Z, (X_{p,q})_\alpha^{>H})$$

denote a $(WH)_\alpha$-domination map with a section

$$s : (Z, (X_{p,q})_\alpha^{>H}) \to (K, (X_{p,q})_\alpha^{>H}).$$

One can treat the pair $(Z, (X_{p,q})_\alpha^{>H})$ as an $(NH)_\alpha$-pair and then the inclusion (b) extends to the inclusion of G-pairs

$$(G \times_{(NH)_\alpha} (X_{p,q})_\alpha^H, G \times_{(NH)_\alpha} (X_{p,q})_\alpha^{>H}) \subset (G \times_{(NH)_\alpha} Z, G \times_{(NH)_\alpha} (X_{p,q})_\alpha^{>H})$$

and the retraction $r : Z \to (X_{p,q})_\alpha^H$ to the G-retraction

$$r : G \times_{(NH)_\alpha} Z \to G \times_{(NH)_\alpha} (X_{p,q})_\alpha^H.$$

If

$$Z_1 = (G \times_{(NH)_\alpha} Z) \cup_q G(X_{p,q})_\alpha^{>H}$$

then by Lemma 2.2 we have the inclusion $(X_{p,q})_\alpha^{(H)} \subset Z_1$ and the G-retraction $r_1 : Z_1 \to (X_{p,q})_\alpha^{(H)}$. By the inductive assumption (conditions (6), (7) and Theorem 1.1) the pair $(X_{p,q}, (X_{p,q})_\alpha^{(H)})$ is relatively finite and taking

$$Z_2 = X_{p,q} \cup Z_1$$

one can extend the inclusion $(X_{p,q})_\alpha^{(H)} \subset Z_1$ to the inclusion $X_{p,q} \subset Z_2$ and the retraction $r_1 : Z_1 \to (X_{p,q})_\alpha^{(H)}$ to a G-retraction $r_2 : Z_2 \to X_{p,q}$ such that the G-pair (Z_2, Z_1) is relatively finite.

If $K_1 = (G \times_{(NH)_\alpha} K) \cup_q G(X_{p,q})_\alpha^{>H}$, then we can extend the domination d to the G-domination map

$$d_1 : (K_1, G(X_{p,q})_\alpha^{>H}) \to (Z_1, G(X_{p,q})_\alpha^{>H})$$

such that the pair $(K_1, G(X_{p,q})_\alpha^{>H})$ is relatively finite. By the inductive assumption (property (3)) $G(X_{p,q})_\alpha^{>H}$ is G-dominated by a finite G-complex $G(K_{p,q})_\alpha^{>H} = K'$. Let

$$\phi : K' \to G(X_{p,q})_\alpha^{>H}$$

denote this domination and

$$s_1 : G(X_{p,q})_\alpha^{>H} \to K'$$

its section. Applying Lemma 2.2 to the diagram

$$\begin{array}{ccccc}
K_1 & \longleftarrow & G(X_{p,q})_\alpha^{>H} & \xrightarrow{\text{id}} & G(X_{p,q})_\alpha^{>H} \\
\downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle s_1} \\
K_1 & \longleftarrow & G(X_{p,q})_\alpha^{>H} & \xrightarrow{s_1} & K'
\end{array}$$

we get the G-domination map

$$\phi_1 : K_1 \cup_{s_1} K' \to K_1$$

where $K_1' = K_1 \cup_{s_1} K'$ is a finite G-complex. Then the composition

$$K_1' \xrightarrow{\phi_1} K_1 \xrightarrow{d_1} Z_1$$

is a finite domination over Z_1. Invoking Lemma 2.2 again we get a G-domination map

$$d_2 : K_2 \to Z_2$$

with K_2 a finite G-complex. Hence $X_{p,q+1} = Z_2$ is G-dominated by a finite G-complex $K_{p,q+1} = K_2$ and the composition

$$X_{p,q+1} \xrightarrow{r_2} X_{p,q} \xrightarrow{r_{p,q}} Y$$

defines a G-retraction $r_{p,q+1} : X_{p,q+1} \to Y$.

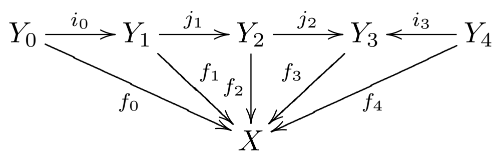
Finally, it follows from the construction that $X_{p,q+1}$ has the desired properties (4)–(7).

Case II: $q = s_p$. This is similar to Case I. ■

3. The geometric finiteness obstruction of W. Lück and the invariants $w^H_\alpha(X)$. Let X be a G-complex G-dominated by a finite one. W. Lück [5], [6] defined geometrically a group $Wa^G(X)$ and an element $w^G(X) \in Wa^G(X)$ that decides when the G-complex X has the G-homotopy type of a finite G-complex.

The aim of this section is to connect Lück's obstruction with the invariants $w^H_\alpha(X)$. This theorem along with results of [1], §4, completes the proof of the equivalence of three out of four definitions of the equivariant obstruction to finiteness.

We start by recalling the construction from [5], [6]. Let X be an arbitrary G-complex. Consider the set of G-maps $f : Y \to X$, where Y ranges through finitely G-dominated G-complexes. We define an equivalence relation as follows: $f_0 : Y_0 \to X$ and $f_4 : Y_4 \to X$ are equivalent iff there exists a commutative diagram



such that j_1 and j_2 are G-homotopy equivalences and i_0, i_3 are inclusions such that the G-CW-pairs (Y_1, Y_0) and (Y_3, Y_4) are relatively finite. Let $Wa^G(X)$ denote the set of equivalence classes. The disjoint union induces an addition on $Wa^G(X)$ and the inclusion of the empty space defines a neutral element. One can show that this addition gives $Wa^G(X)$ the structure of an abelian group ([5], p. 370, or [6], p. 51).

Definition. Let X be a finitely G-dominated G-complex. We define its *geometric obstruction to finiteness* as $w^G(X) = [\mathrm{id} : X \to X] \in Wa^G(X)$.

Then we have the following result.

THEOREM 3.1. ([5], Theorem 1.1, or [6], §3). *Let X be finitely G-dominated. Then*

(a) $Wa^G : G\text{-}CW \to \mathcal{A}b$ *is a covariant functor from the category of equivariant CW-complexes to the category of abelian groups.*

(b) $w^G(X)$ *is an invariant of the G-homotopy type.*

(c) *A G-complex X is G-homotopy equivalent to a finite G-complex iff* $w^G(X) = 0$.

Let X be a G-complex. We define a homomorphism

$$F : Wa^G(X) \to \bigoplus_{\underline{CI}(X)} \widetilde{K}_0(\mathbb{Z}[\pi_0(WH(X))^*_\alpha])$$

by the formula $F([f : Y \to X]) = \sum f_*(w^H_\alpha(Y))$ where

$$f_* : \widetilde{K}_0(\mathbb{Z}[\pi_0(WH(Y))^*_\alpha]) \to \widetilde{K}_0(\mathbb{Z}[\pi_0(WH(X))^*_\alpha])$$

denotes the homomorphism induced on $\widetilde{K}_0$ by f. The following result gives the precise relation between Lück's obstruction $w^G(X)$ and Wall-type invariants $w^H_\alpha(X)$.

THEOREM 3.2. *Suppose X is a G-complex such that*

(1) *X has finitely many orbit types*,

(2) $\pi_0(X^H)$ *is finite for any subgroup H of G occurring on X as an isotropy subgroup*,

(3) $\pi_1(X^H_\alpha, x)$ *is finitely presented for any representative X^H_α from the class $[X^H_\alpha] \in \underline{CI}(X)$ and for any $x \in X^H_\alpha$.*

Then the natural homomorphism

$$F : Wa^G(X) \to \bigoplus_{\underline{CI}(X)} \widetilde{K}_0(\mathbb{Z}[\pi_0(WH(X))^*_\alpha])$$

is an isomorphism. If the G-complex X is finitely G-dominated then $F(w^G(X)) = \sum w^H_\alpha(X)$.

Remark. Observe that any finitely G-dominated G-complex satisfies conditions (1)–(3) of Theorem 3.2.

Before presenting a proof of the theorem let us recall one technical lemma from [6] which will be used in the proof.

LEMMA 3.3 ([6], Lemma 14.7). *Let $f : Y \to X$ be a G-map between G-complexes. Suppose the sets* $\mathrm{Iso}(X)$ *and* $\mathrm{Iso}(Y)$ *of orbit types on X and Y, respectively, are finite. Suppose that for any $H \in \mathrm{Iso}(X) \cup \mathrm{Iso}(Y)$ the sets $\pi_0(X^H)$ and $\pi_0(Y^H)$ are finite and the fundamental groups $\pi_1(Y^H_\alpha, y)$ and $\pi_1(X^H_\beta, x)$ are finitely presented for any $y \in Y^H_\alpha$, $x \in X^H_\beta$. Then one* *can extend the map* f *to a* G*-map* $g : Z \to X$ *such that for any subgroup* H *of* G,

$$g_* : \pi_0(Z^H) \to \pi_0(X^H)$$

is bijective and

$$g_* : \pi_1(Z^H_\alpha, z) \to \pi_1(X^H_\alpha, g(z))$$

is an isomorphism for any component Z^H_α *and any point* $z \in Z^H_\alpha$.

Proof of Theorem 3.2. Suppose an element $[f : Y \to X]$ belongs to the kernel of F. The assumptions on X and Lemma 3.3 imply that there exists a G-complex Z obtained from Y by attaching finitely many G-cells and an extension $g : Z \to X$ of the map f such that

$$g_* : \pi_0(Z^H) \to \pi_0(X^H) \tag{1}$$

is a bijection and

$$g_* : \pi_1(Z^H_\alpha) \to \pi_1(X^H_\alpha) \tag{2}$$

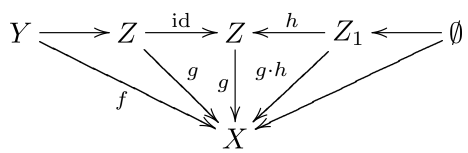
is an isomorphism. Note that $[f] = [g]$ in the group $Wa^G(X)$. Hence

$$F([f]) = F([g]) = \sum g_*(w^H_\alpha(Z)) = 0.$$

Since (1), (2) and

$$g_* : \widetilde{K}_0(\mathbb{Z}[\pi_0(WH(Z))^*_\alpha]) \to \widetilde{K}_0(\mathbb{Z}[\pi_0(WH(X))^*_\alpha])$$

are bijections we have $\underline{CI}(Z) = \underline{CI}(X)$ and $w^H_\alpha(Z) = 0$ for any component Z^H_α which represents an element of the set $\underline{CI}(Z)$. It follows from Theorem 1.1 that there exists a finite G-complex Z_1 and a G-homotopy equivalence $h : Z_1 \to Z$. Then the diagram



shows that $[f] = 0$ in the group $Wa^G(X)$. Thus F is a monomorphism.

Similarly, the assumptions on the space X and Lemma 3.3 applied to the map $\emptyset \to X$ imply that we can find a finite G-complex K and a G-map $g : K \to X$ such that

$$g_* : \pi_0(K^H) \to \pi_0(X^H)$$

is a bijection and

$$g_* : \pi_1(K^H_\alpha) \to \pi_1(X^H_\alpha)$$

is an isomorphism. Then $\underline{CI}(K) = \underline{CI}(X)$ and g induces an isomorphism

$$g_* : \widetilde{K}_0(\mathbb{Z}[\pi_0(WH(K))^*_\alpha]) \to \widetilde{K}_0(\mathbb{Z}[\pi_0(WH(X))^*_\alpha]).$$

By the commutativity of the diagram

$$\begin{array}{ccc} Wa^G(K) & \xrightarrow{F_1} & \bigoplus_{\underline{CI}(K)} \widetilde{K}_0(\mathbb{Z}[\pi_0(WH(K))^*_\alpha]) \\ \downarrow g_* & & \downarrow \cong \\ Wa^G(X) & \xrightarrow{F} & \bigoplus_{\underline{CI}(X)} \widetilde{K}_0(\mathbb{Z}[\pi_0(WH(X))^*_\alpha]) \end{array}$$

it suffices to show that

$$F_1 : Wa^G(K) \to \bigoplus_{\underline{CI}(K)} \widetilde{K}_0(\mathbb{Z}[\pi_0(WH(K))^*_\alpha])$$

is an epimorphism. Let $w^H_\alpha \in \widetilde{K}_0(\mathbb{Z}[\pi_0(WH(K))^*_\alpha])$ be an arbitrary element. By Theorem 2.3 there exists a G-complex L, G-dominated by a finite G-complex and a G-retraction $r : L \to K$ such that $r_*(w^H_\alpha(L)) = w^H_\alpha$. Then $[r : L \to K] \in Wa^G(K)$ and

$$F_1([r]) = \sum r_*(w^H_\alpha(L)) = \sum w^H_\alpha. \ \blacksquare$$

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Department of Mathematics
University of Szczecin
Wielkopolska 15
70-451 Szczecin 3, Poland
E-mail: pawelan@uoo.univ.szczecin.pl
pawelan@euler.mat.univ.szczecin.pl

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