

Embedding partially ordered sets into ${}^\omega\omega$

by

Ilijas Farah (Toronto, Ont.)

Abstract. We investigate some natural questions about the class of posets which can be embedded into $\langle {}^\omega\omega, \leq^* \rangle$. Our main tool is a simple ccc forcing notion \mathcal{H}_E which generically embeds a given poset E into $\langle {}^\omega\omega, \leq^* \rangle$ and does this in a “minimal” way (see Theorems 9.1, 10.1, 6.1 and 9.2).

We describe a simple ccc forcing notion \mathcal{H}_E which embeds a given poset E into $\langle {}^\omega\omega, \leq^* \rangle$ (see Definition 0.1 and Definition 4.1). It has the property that \mathcal{H}_{E_0} is a regular subordering of \mathcal{H}_E whenever E_0 is a subordering of E . If \mathcal{P} is a forcing notion, then “ \mathcal{P} adds a κ -chain to ${}^\omega\omega$ ” means “In a forcing extension by \mathcal{P} there is a κ -chain in ${}^\omega\omega$ ”, so in particular this phrase applies even if there is already a κ -chain in ${}^\omega\omega$. We prove the following results about \mathcal{H}_E (the symbol ${}^\omega\omega$ stands for the poset $\langle {}^\omega\omega, \leq^* \rangle$, while \mathcal{C} stands for a poset for adding a single Cohen real):

THEOREM 9.1 (Main Theorem). *If $\kappa > \omega_1$ is a regular cardinal then \mathcal{H}_E adds a κ -chain to ${}^\omega\omega$ iff one of the following happens:*

- (†1) E has a κ - or a κ^* -chain, or
- (†2) \mathcal{C} adds a κ -chain to ${}^\omega\omega$.

In the case when E is an antichain of size κ the poset \mathcal{H}_E reduces to a poset for adding κ many Cohen reals, so Theorem 9.1 implies Kunen’s theorem ([16]) that after adding any number of Cohen reals in ${}^\omega\omega$ there are no well-ordered chains of size larger than the ground-model continuum. In the following two theorems κ and λ are uncountable regular cardinals; for undefined notions see Definition 6.1 (\dot{g} is a name for the generic embedding of \mathcal{H}_E into ${}^\omega\omega$).

- THEOREM 6.1.** (a) *A \dot{g} -image of a limit $\langle a_\xi, b \rangle_{\xi < \kappa}$ in E is a limit in ${}^\omega\omega$.*
(b) *A \dot{g} -image of a $\langle \kappa, \omega \rangle$ -gap $\langle a_\xi, b_i \rangle_{\xi < \kappa, i < \omega}$ in E is a gap in ${}^\omega\omega$.*

1991 *Mathematics Subject Classification*: 03E35, 04A20, 06A07, 90D44.
Research supported by the Science Fund of Serbia grant number 0401A.

- (c) A \dot{g} -image of a gap $\langle a_\xi, b_\eta \rangle_{\xi < \kappa, \eta < \lambda}$ in E is a gap in ${}^\omega\omega$.
- (d) A \dot{g} -image of an unbounded chain $\langle a_\xi \rangle_{\xi < \kappa}$ is an unbounded chain in ${}^\omega\omega$.

A partial converse of the previous theorem is given in

THEOREM 9.2. *If $\kappa > \mathfrak{c}$ and \mathcal{H}_E adds a $\langle \kappa, \lambda \rangle$ -gap to ${}^\omega\omega$, then there is such a gap in E or in E^* .*

The following theorem is an attempt at describing which dense linearly ordered sets embed into ${}^\omega\omega$ after forcing with \mathcal{H}_E . Note that looking at those linearly ordered sets which are suborderings of $\langle 2^\kappa, <_{\text{Lex}} \rangle$ (the symbol $<_{\text{Lex}}$ stands for the lexicographical ordering) for some κ is not a loss of generality. Moreover, the theorem below has interesting applications (see Propositions 1.4 and 1.5).

THEOREM 10.1. *If \mathcal{H}_E forces that $\langle 2^{\omega_1}, <_{\text{Lex}} \rangle^V$ embeds into $\langle {}^\omega\omega, \leq^* \rangle$, then either a Cohen real forces this or $\langle 2^{\omega_1}, <_{\text{Lex}} \rangle^V$ embeds into E .*

The ‘‘Cohen real’’ alternative in Theorems 9.1 and 10.1 can be avoided if we start from a model of CH (which is the situation where these theorems are most often used), but in general the following question is open:

QUESTION. Does forcing with \mathcal{C} add an ω_2 -chain to ${}^\omega\omega$ iff there is an ω_2 -chain in ${}^\omega\omega$?

We start by presenting applications of Theorems 9.1 and 10.1 in §§1–3. In §1 we answer some questions of Dordal and Scheepers and prove some other related statements. In §2 we use a poset obtained by Todorćević to answer a question of Galvin, proving that the poset ${}^\omega({}^\omega\omega)$ is not necessarily embeddable into ${}^\omega\omega$. In §3 we use a poset obtained by Galvin to describe a forcing extension of the universe in which an ultrapower ${}^\omega\omega/\mathcal{U}$ is not embeddable into ${}^\omega\omega$ for every nonprincipal ultrafilter \mathcal{U} on ω . In §4 we define \mathcal{H}_E , describe the quotient $\mathcal{H}_E/\mathcal{H}_{E_0}$ for $E_0 \subseteq E$, and prove various properties of these posets. In §5 we prepare for the proofs of the above theorems. §6 is an investigation of gaps and limits in ${}^\omega\omega$ in an extension by \mathcal{H}_E . Chapters 7 and 8 include some prerequisites for the proofs of Theorems 9.1 and 10.1 which are independent of the rest of the paper and interesting in their own right: in the former we give a strengthening of an old result of Kurepa that every uncountable well-founded poset with finite levels has an uncountable chain, while in the latter we investigate the Banach–Mazur game of length ω_1 . In §9 we state and prove a ZFC, ‘‘local’’, version of the Δ -system lemma for countable sets (Lemma 9.1) which is, in the absence of CH, used in the proofs of Theorems 9.1 and 10.1. In §11 we show that ${}^\omega\omega$ in an extension by a single Cohen real is reflected in a certain ground-model ordering ${}^c\omega/\mathcal{N}$. In particular, by Theorem 11.1, the question above can be

reformulated as (\mathcal{N} is the ideal of nowhere dense subsets of a Cohen poset \mathcal{C} ; see Definition 11.1):

QUESTION. Does the existence of an ω_2 -chain in the poset ${}^c\omega/\mathcal{N}$ imply the existence of an ω_2 -chain in ${}^\omega\omega$?

Our notation is standard, and undefined notions can be found in [18]. If ϕ is a statement of a forcing language, then the phrase “ ϕ is true with probability one” is an abbreviation for “every condition of the poset forces ϕ ”.

0. Introduction. Let $\langle E, <_E \rangle$ be a partially ordered set. For $a \in E$ and $X \subseteq E$ let

$$X(\leq_E a) = \{x \in X : x \leq_E a\};$$

$X(\geq_E a)$, $X(<_E a)$ etc. have similar definitions. A subset X of E is *countably bounded* iff there is a countable $A \subseteq E$ such that $X = \bigcup_{a \in A} X(\leq_E a)$. A subset X of E is *countably bounded from below* iff there is a countable $A \subseteq E$ such that $X = \bigcup_{a \in A} X(\geq_E a)$. If every $a \in A$ is $<_E$ -incomparable with every $b \in B$ then we say that A and B are $<_E$ -incomparable. If a, b are elements of a poset E then the *interval of E with endpoints a and b* is the set

$$(a, b)_E = \{c \in E : a <_E c <_E b \text{ or } b <_E c <_E a\}.$$

In particular, $(a, b)_E = (b, a)_E$ always, and $(a, b)_E$ is empty if a and b are incomparable. A mapping $f : \langle E_0, <_0 \rangle \rightarrow \langle E_1, <_1 \rangle$ is an *embedding* iff we have $a <_0 b$ iff $f(a) <_1 f(b)$ for all $a, b \in E_0$, as opposed to a *strictly increasing* mapping which is one such that $a <_0 b$ implies $f(a) <_1 f(b)$ for all $a, b \in E_0$. Of course, in the case when E_0 is linearly ordered these two notions coincide.

DEFINITION 0.1. For $f, g \in {}^\omega\omega$ we define:

- (1) $f \leq^* g$ iff $\{n : f(n) \leq g(n)\}$ is cofinite.
- (2) $f \not\leq^* g$ iff $f \leq^* g$ and not $f \geq^* g$.
- (3) $f <^* g$ iff $\{n : f(n) < g(n)\}$ is cofinite.
- (4) $f \prec g$ iff $\lim_{n \rightarrow \infty} (g(n) - f(n)) = \infty$.
- (5) $f \leq^n g$ iff $f(m) \leq g(m)$ for all $m \geq n$.

Our forcing notion \mathcal{H}_E generically embeds E into $\langle {}^\omega\omega, \leq^* \rangle$. Similar forcings were used in [13], [27], [20], [4], but the poset E was usually embedded into the structure $\langle {}^\omega\omega, \prec \rangle$. We choose the ordering \leq^* because it is not clear how we can get the desirable property from Theorem 4.1 with other partial orderings on ${}^\omega\omega$. We first prove that the ordering we have chosen is good enough for our purposes.

PROPOSITION 0.1. *There is an embedding from $\langle {}^\omega\omega, \leq^* \rangle$ into $\langle {}^\omega\omega, \prec^* \rangle$.*

PROOF. Without loss of generality we consider only the subordering consisting of strictly increasing functions. Fix an infinite matrix of positive integers a_{mn} ($m, n \in \omega$) such that

$$a_{mn} > \sum_{i=0}^m \sum_{j=0}^{n-1} a_{ij}.$$

for all m, n . For each $=^*$ -equivalence class pick a representative, and for $f \in {}^\omega\omega$ let $\Psi(f) =^* f$ be the chosen representative. Let $f \mapsto \hat{f}$ be defined by

$$\hat{f}(n) = \sum_{i=0}^n \sum_{j=0}^{\Psi(f)(i)} a_{ij}$$

for $n \in \omega$. Then it is easy to check that for strictly increasing f and g we have $f \leq^* g$ iff $\hat{f} <^* \hat{g}$. ■

COROLLARY. *A linearly ordered set is embeddable into $\langle {}^\omega\omega, <^* \rangle$ iff it is embeddable into $\langle {}^\omega\omega, \leq^* \rangle$ iff it is embeddable into $\langle {}^\omega\omega, < \rangle$.*

PROOF. If a linearly ordered set L is embeddable into E_0 and there is a strictly increasing mapping from E_0 into E_1 , then L is embeddable into E_1 ; so it suffices to define some increasing mappings. Obviously the identity is a strictly increasing mapping from $\langle {}^\omega\omega, < \rangle$ into $\langle {}^\omega\omega, <^* \rangle$, as well as from $\langle {}^\omega\omega, <^* \rangle$ into $\langle {}^\omega\omega, \leq^* \rangle$. Finally, the mapping $f \mapsto \hat{f}$ defined by $\hat{f}(n) = n + f(n)$ is strictly increasing from $\langle {}^\omega\omega, <^* \rangle$ into $\langle {}^\omega\omega, < \rangle$. ■

So our saying that e.g. “there is an ω_2 -chain in ${}^\omega\omega$ ” without specifying an ordering is justified (as long as it is assumed that the ordering is one of the “mod finite” orderings).

1. Applications of the main theorem

PROPOSITION 1.1. *There is a forcing extension of V in which there are no ω_2 -chains in ${}^\omega\omega$, but there is a poset which adds such a chain without adding new ω_1 -sequences of ordinals.*

PROOF. Start from a model of CH in which there is an ω_2 -Suslin tree T . Go to a forcing extension by \mathcal{H}_T : by Theorem 9.1 there are no ω_2 -chains in ${}^\omega\omega$, T remains a Suslin tree (because \mathcal{H}_E has precaliber \aleph_2 ; see Lemma 4.1) and therefore further forcing with T does not add ω_1 -sequences of ordinals while it adds an ω_2 -branch to itself, and the image of this branch is an ω_2 -chain in ${}^\omega\omega$. ■

The following answers a question of Dordal [7, Remark 9.5, p. 269] and Scheepers [26, #81]. It is solved independently by Cummings, Scheepers and Shelah in [5].

PROPOSITION 1.2. *The existence of an ω_ω -chain in ${}^\omega\omega$ does not imply the existence of an $\omega_{\omega+1}$ -chain in ${}^\omega\omega$.*

PROOF. We can either start from a model of CH and force with $\mathcal{H}_{\omega_\omega}$, or we can start from a model with an $\omega_{\omega+1}$ -Suslin tree T and force with \mathcal{H}_T . The latter model has the property that there are no $\omega_{\omega+1}$ -chains but a forcing notion (namely T) adds one without adding new sequences of ordinals of length ω_ω . ■

Scheepers noticed that $\omega_{\omega+1}$ embeds into $\langle {}^\omega\omega_\omega, \leq^* \rangle$, and therefore in both models constructed in Proposition 1.2 the poset ${}^\omega({}^\omega\omega)$ (see §2) is not embeddable into ${}^\omega\omega$. So this answers an unpublished question of Galvin which was also asked in [26, #81]. In the above models the continuum is rather large, and in §2 we will prove that this can happen even when the continuum is equal to \aleph_2 (obviously this is the best possible because CH implies that all posets of size \mathfrak{c} embed into $\langle {}^\omega\omega, \leq^* \rangle$).

Consider a cardinal invariant of the continuum equal to the supremum of all cardinals κ such that there is a κ -chain in ${}^\omega\omega$. A natural question arises—is this supremum always attained, i.e. can “supremum” be replaced by “maximum” in the above definition? It is easy to show (e.g. by using lemmas from [25]) that if κ is singular and there are λ -chains in ${}^\omega\omega$ for all $\lambda < \kappa$, then there is a κ -chain in ${}^\omega\omega$ as well. Therefore in a model in which the answer to our question is negative this supremum must be a weakly inaccessible cardinal, so the use of an inaccessible cardinal to get a model where $\text{sup} \neq \text{max}$ in our next proposition is justified.

PROPOSITION 1.3. *If κ is an inaccessible cardinal then there is a cardinal-preserving forcing extension of V in which there is a λ -chain in ${}^\omega\omega$ for all $\lambda < \kappa$ but there is no κ -chain in ${}^\omega\omega$.*

PROOF. Let E be any poset with no κ -chains and with λ -chains for all $\lambda < \kappa$ and force with \mathcal{H}_E . ■

PROPOSITION 1.4. *There is a forcing extension of the universe in which there is a linearly ordered set $\langle L, <_L \rangle$ and a partition $L = L_0 \dot{\cup} L_1$ such that L is not embeddable into ${}^\omega\omega$, while both L_0 and L_1 are.*

PROOF. Start from a model of CH, let L be $\langle 2^{\omega_1}, <_{\text{Lex}} \rangle$ and let $L_0 \dot{\cup} L_1$ be its Bernstein decomposition (i.e. a decomposition such that L does not embed into L_0 or into L_1). Let E be $\langle L_0, <_{\text{Lex}} \rangle + \langle L_1, <_{\text{Lex}} \rangle$. Then by Theorem 10.1 the forcing extension by \mathcal{H}_E is as required. ■

In some models of Set Theory (e.g. when CH holds; also see [20]) linearly ordered sets which embed into ${}^\omega\omega$ are exactly those of size at most \mathfrak{c} , so the statement of Proposition 1.4 fails in such models.

Scheepers observed that under Martin's Axiom, (a) every ordinal of cardinality at most \mathfrak{c} embeds into ${}^\omega\omega$, and (b) every linearly ordered set of size strictly less than \mathfrak{c} embeds into ${}^\omega\omega$. He asked whether one of these statements implies the other. In the next proposition we show that (a) does not imply (b).

PROPOSITION 1.5. *It is not provable in ZFC that if ω_3 embeds into ${}^\omega\omega$ then all linearly ordered sets of size \aleph_2 embed into ${}^\omega\omega$.*

PROOF. Start from a model of GCH and let $\mathcal{E} = \langle 2^{\omega_1}, <_{\text{Lex}} \rangle^V$, so \mathcal{E} is of size \aleph_2 . After adding one Cohen real CH remains true, so \mathcal{E} is not embeddable into ${}^\omega\omega$ in this extension. Therefore after we force with \mathcal{H}_{ω_3} (or any other \mathcal{H}_κ), by Theorem 10.1 the poset \mathcal{E} is not embeddable into ${}^\omega\omega$. ■

REMARK. Our first proof of Proposition 1.5 was to add \aleph_2 many Cohen subsets of ω_1 , say $\langle c_\xi : \xi < \omega_2 \rangle$, and then to force with \mathcal{H}_{ω_3} ; in this model the set $\langle c_\xi : \xi < \omega_2 \rangle$ with the lexicographical ordering is not isomorphic to any $\langle X, \prec \rangle$, where X is a set of reals and \prec is a Borel ordering.

The following extends a result of Brendle–LaBerge, who in [3, Theorems 2.7 and 2.8] proved a special case when \mathcal{I} as below is taken to be the family of all subsets of κ of size smaller than κ . The forcing extensions given in [3] are similar to ones obtained by \mathcal{H}_E .

PROPOSITION 1.6. *If \mathcal{I} is a proper σ -ideal on the cardinal κ which includes all countable subsets of κ , then there is a forcing extension of V in which there are no $(\mathfrak{c}^+)^V$ chains in ${}^\omega\omega$ and there is a set $\{x_\xi : \xi < \kappa\}$ in ${}^\omega\omega$ such that $\{x_\xi : \xi \in A\}$ is bounded in ${}^\omega\omega$ iff $A \in \mathcal{I}$.*

PROOF. Let $E = \kappa \cup \mathcal{I}$ with the ordering $\xi < A$ iff $\xi \in \kappa$, $A \in \mathcal{I}$ and $\xi \in A$. A forcing extension by \mathcal{H}_E satisfies the requirements by Theorem 9.1 and Lemma 6.1. ■

2. A problem of Galvin. On the set ${}^\omega({}^\omega\omega)$ of all sequences $\vec{f} = \langle f_n \rangle$ of elements of ${}^\omega\omega$ we define the ordering of *eventual dominance*, \leq^* , by:

$$\vec{f} \leq^* \vec{g} \quad \text{iff} \quad f_n \leq^* g_n \text{ for all large enough } n.$$

[Observe that the symbol “ \leq^* ” in the above line denotes two different orderings on two different sets. The second \leq^* can be replaced by either $<^*$ or \prec (see Definition 1.1), but by Proposition 1.1 a linearly ordered set is embeddable into ${}^\omega({}^\omega\omega)$ with the ordering that we defined iff it is embeddable into ${}^\omega({}^\omega\omega)$ with any of these orderings.] We will denote the poset $\langle {}^\omega({}^\omega\omega), \leq^* \rangle$ by ${}^\omega({}^\omega\omega)$.

THEOREM 2.1. *There is a forcing extension of the universe such that*

- (1) *There is an ω_2 -chain in ${}^\omega({}^\omega\omega)$.*
- (2) *There are no ω_2 -chains in ${}^\omega\omega$.*
- (3) *${}^\omega({}^\omega\omega)$ is not embeddable into ${}^\omega\omega$.*
- (4) *Adding a dominating real adds an ω_2 -chain to ${}^\omega\omega$.*

Our model will be a forcing extension by \mathcal{H}_E , where E is supplied by the following result of Todorćević.

THEOREM 2.2 ([28]). (\square_{ω_1}) *There is a sequence $\langle \cdot \rangle^n$ ($n < \omega$) of tree orderings on ω_2 such that for all n ,*

- (T1) $\langle \cdot \rangle^n \subseteq \langle \cdot \rangle^{n+1} \subseteq \in$,
- (T2) $\in = \bigcup_{n < \omega} \langle \cdot \rangle^n$, and
- (T3) *no $T^n = \langle \omega_2, \langle \cdot \rangle^n \rangle$ has an ω_2 -branch. ■*

Let T denote the disjoint sum of T^n , i.e. $T = \langle \omega_2 \times \omega, \langle \cdot \rangle_T \rangle$ and $\langle \cdot \rangle_T$ is defined by

$$\langle \xi, m \rangle \langle \cdot \rangle_T \langle \eta, n \rangle \quad \text{iff} \quad n = m \text{ and } \xi \langle \cdot \rangle^n \eta.$$

Proof of Theorem 2.1. The model is obtained by forcing with \mathcal{H}_T over a model of CH and \square_{ω_1} .

(1) It is enough to provide a sequence $D_\xi = \{x_{\xi i} : i < \omega\}$ ($\xi < \omega_2$) of subsets of T such that for all $\xi < \eta$ and some n we have $x_{\xi i} \langle \cdot \rangle_T x_{\eta i}$ for all $i \geq n$. Let $D_\xi = \{\langle \xi, n \rangle : n < \omega\}$; obviously this family satisfies the requirements.

To prove (2), just notice that T does not have ω_2 -chains and apply Theorem 9.1.

(3) follows immediately from (1) and (2).

CLAIM. *If d is a dominating real, then in $V[d]$ there is a strictly increasing mapping from $({}^\omega({}^\omega\omega))^V$ into ${}^\omega\omega$.*

Proof. Map $\vec{f} = \langle f_n : n < \omega \rangle$ to g defined by $g(n) = f_n(d(n))$. To see that this mapping is strictly increasing, note that if \vec{f} and \vec{g} are in the ground model, then the function $\Delta_{fg} : \omega \rightarrow \omega$ defined by letting $\Delta_{fg}(n)$ be the least i such that $f_n(j) \geq g_n(j)$ for all $j \geq i$, is dominated by d . This shows that our embedding is increasing, and it is strictly increasing by genericity. ■

(4) follows immediately from the above claim. ■

COROLLARY. *It is not provable in ZFC that there is a strictly increasing mapping from ${}^\omega({}^\omega\omega)$ into ${}^\omega\omega$. ■*

As an application of the above, we mention an unpublished work of Galvin ([11]). Until the end of this section we will adopt Galvin's original terminology and say that " E_0 is embeddable into E_1 " iff there is a mapping

$f : E_0 \rightarrow E_1$ such that $a <_{E_0} b$ implies $f(a) <_{E_1} f(b)$, i.e. if there is a *strictly increasing map* from E_0 into E_1 in our terminology. For an indecomposable ordinal α let $\mathcal{P}(\alpha)$ be the poset of all $f : \alpha \rightarrow \omega$, ordered by (otp denotes the order type of a set)

$$f \prec g \quad \text{iff} \quad \text{otp}(\{\xi < \alpha : f(\xi) \geq g(\xi)\}) < \alpha.$$

Galvin observed that $\mathcal{P}(\alpha)$ is embeddable into $\mathcal{P}(\beta)$ whenever there is a function $g : \alpha \rightarrow \beta$ such that $\text{otp}(A) = \beta$ implies $\text{otp}(g^{-1}(A)) = \alpha$, for all $A \subseteq \beta$. So in particular (note that $\mathcal{P}(\omega)$ here denotes our $\langle \omega^\omega, <^* \rangle$ and $\mathcal{P}(\omega^2)$ is $\langle \omega^{\omega^\omega}, <^* \rangle$):

- (1) $\mathcal{P}(\omega)$ is embeddable into $\mathcal{P}(\alpha)$ for all α .
- (2) $\mathcal{P}(\omega^2)$ is embeddable into $\mathcal{P}(\alpha)$ for all $\alpha \geq \omega^2$.

Galvin asked a general question when $\mathcal{P}(\alpha)$ is embeddable into $\mathcal{P}(\beta)$, in particular:

- (Q1) Is it provable that $\mathcal{P}(\omega^2)$ is embeddable into $\mathcal{P}(\omega)$?
- (Q2) Is it provable that $\mathcal{P}(\omega^3)$ is embeddable into $\mathcal{P}(\omega^\omega)$?

[“Provable” means “provable in ZFC”; observe that both questions have a positive answer if CH is assumed.] We can reformulate our above Corollary to answer (Q1), namely

COROLLARY. *It is not provable in ZFC that $\mathcal{P}(\omega^2)$ is embeddable into $\mathcal{P}(\omega)$.* ■

Remark. The tree orderings \leq^n obtained in [28] have another interesting property:

- (T4) the set of \leq^n -predecessors of α is a closed subset of $\alpha + 1$ for all $\alpha < \omega_2$.

(Note that this implies that T^n is not Aronszajn.) This easily implies that the natural σ -closed poset \mathcal{P}_n which specializes T^n has \aleph_2 -cc. So Theorem 2.2 has another curious consequence: under the assumptions of CH and \square_{ω_1} there is a sequence \mathcal{P}_n ($n < \omega_2$) of σ -closed, \aleph_2 -cc posets such that every finite product of \mathcal{P}_n is \aleph_2 -cc, but $\prod_{n < \omega} \mathcal{P}_n$ is not. The fact that $\prod_{n < \omega} \mathcal{P}_n$ is not \aleph_2 -cc follows from another fact proved in [28]: if the orderings \leq^n satisfy (T1)–(T4), then one of the trees T^n is nonspecial.

3. Ultrapowers of ω^ω . Now we construct a model of ZFC in which there are no ω_2 -chains in $\langle \omega^\omega, \prec \rangle$, but for every nonprincipal ultrafilter \mathcal{U} on ω there is an ω_2 -chain in $\langle \omega^\omega / \mathcal{U}, <_{\mathcal{U}} \rangle$. This scenario is originally used by Solovay in the context of automatic continuity in Banach algebras (see [27]). In fact, in the model of Theorem 3.1 all homomorphisms of Banach algebras are continuous. This is so because the existence of a discontinuous homomorphism implies that there is a strictly increasing mapping from

$\langle {}^\omega\omega, \leq_{\mathcal{U}} \rangle$ into $\langle {}^\omega\omega, <^* \rangle$ for some nonprincipal ultrafilter \mathcal{U} on ω (see [6]). If \mathcal{U} is a nonprincipal ultrafilter on ω then a poset $\mathcal{P}_{\mathcal{U}}$ is defined as follows: A typical condition in $\mathcal{P}_{\mathcal{U}}$ is $\langle s, A \rangle$, where s is a finite subset of ω , $A \in \mathcal{U}$, and $\max s < \min A$. The ordering is defined by letting $\langle s, A \rangle \leq \langle t, B \rangle$ iff t is an initial segment of s , $A \subseteq B$, and $t \setminus s \subseteq B$. This poset is σ -centered and it generically adds a subset of ω (called a *Prikry real*) which is almost included in all elements of \mathcal{U} (see [22]).

THEOREM 3.1. (*CH*) *Let κ be a regular cardinal larger than \aleph_1 . Then there is a poset E such that in a forcing extension of the universe by \mathcal{H}_E , for every nonprincipal ultrafilter \mathcal{U} on ω :*

- (1) *There are κ -chains in $\langle {}^\omega\omega/\mathcal{U}, <_{\mathcal{U}} \rangle$.*
- (2) *There are no ω_2 -chains in $\langle {}^\omega\omega, \leq^* \rangle$.*
- (3) *$\langle {}^\omega\omega/\mathcal{U}, <_{\mathcal{U}} \rangle$ is not embeddable into $\langle {}^\omega\omega, \leq^* \rangle$.*
- (4) *Adding a \mathcal{U} -Prikry real adds a κ -chain to $\langle {}^\omega\omega, \leq^* \rangle$.*

The poset E is provided by the following special case of an old unpublished result of Galvin, which is included here with his kind permission.

THEOREM 3.2 ([10]). *If κ is a regular cardinal, then there is a poset $\langle G, < \rangle$ of size κ with no infinite chains but if \mathcal{E} is a linear ordering such that there is a strictly increasing $\Phi : G \rightarrow \mathcal{E}$, then \mathcal{E} has a κ - or a κ^* -chain.*

Proof. Let G be $\kappa \times \kappa^*$ with the *strict Cartesian ordering* $<_{sc}$, i.e.

$$\langle \alpha, \beta \rangle <_{sc} \langle \gamma, \delta \rangle \quad \text{iff} \quad \alpha < \gamma \text{ and } \beta > \delta.$$

Obviously, every chain in $\kappa \times \kappa^*$ is finite. Suppose that $\langle \mathcal{E}, < \rangle$ is a linearly ordered set with no κ - or κ^* -chains and that $\Phi : \kappa \times \kappa^* \rightarrow \mathcal{E}$ is strictly increasing.

Case 1: *There is a $\beta < \kappa$ such that for all $\alpha < \kappa$ the set $\{\gamma < \kappa : \Phi(\gamma, \beta) \leq \Phi(\alpha, \beta)\}$ is of size strictly less than κ .* Then we can pick α_ξ ($\xi < \kappa$) such that $\Phi(\alpha_\xi, \beta)$ is an increasing κ -chain.

Case 2: *For all $\beta < \kappa$ there is $\alpha_\beta < \kappa$ such that $\{\gamma < \kappa : \Phi(\gamma, \beta) \leq \Phi(\alpha_\beta, \beta)\}$ is of size κ .* We claim that the chain $\Phi(\alpha_\beta, \beta)$ ($\beta < \kappa$) is strictly decreasing. Suppose the contrary, that $\Phi(\alpha_\beta, \beta) \geq \Phi(\alpha_\gamma, \gamma)$ and $\beta < \gamma$. By the choice of α_γ we can pick $\xi > \alpha_\beta$ such that $\Phi(\alpha_\beta, \beta) \geq \Phi(\xi, \gamma)$, but $\langle \alpha_\beta, \beta \rangle <_{sc} \langle \xi, \gamma \rangle$ —a contradiction.

So there is a κ - or a κ^* -chain in \mathcal{E} . ■

Proof of Theorem 3.1. E is $\kappa \times \kappa^*$ ordered by $<_{sc}$.

(1) By Proposition 0.1, $\kappa \times \kappa^*$ is embeddable into $\langle {}^\omega\omega, <^* \rangle$, and $f \mapsto f/\mathcal{U}$ is a strictly increasing mapping from $\langle {}^\omega\omega, <^* \rangle$ into $\langle {}^\omega\omega/\mathcal{U}, <_{\mathcal{U}} \rangle$. So there are κ -chains in $\langle {}^\omega\omega/\mathcal{U}, <_{\mathcal{U}} \rangle$ by Theorem 3.2.

(2) follows immediately from Theorem 3.2 and Theorem 4.1.

(3) is a consequence of (1) and (2).

CLAIM. *If x is a \mathcal{U} -Prikry generic real, then in $V[x]$ there is a Borel strictly increasing mapping from $\langle {}^\omega\omega/\mathcal{U}, <_{\mathcal{U}} \rangle$ into $\langle {}^\omega\omega, \leq^* \rangle$.*

PROOF. [The ultrafilter \mathcal{U} restricted to the set x coincides with the Fréchet filter on x .] Working in the extension, it is enough to define a Borel mapping $\Phi : {}^\omega\omega \rightarrow {}^\omega\omega$ such that the Φ -image of $[f]_{\mathcal{U}}$ is included in $[\Phi(f)]_{=^*}$ for all f and $f <_{\mathcal{U}} g$ implies that $\Phi(f) \leq^* \Phi(g)$. Let e_x be the enumeration function of x (i.e. $e_x(n)$ is the n th element of the set x), and let $\Phi(f)(n) = f(e_x(n))$. This mapping obviously works. ■

(4) follows immediately from the above claim. ■

Stress in Theorem 3.1 is on the fact that (1), (3) and (4) are true for *all* nonprincipal ultrafilters on ω ; namely, it is easy to construct an ultrafilter \mathcal{U} such that there is a \mathfrak{c} -chain (or a copy of any given linearly ordered set of size at most \mathfrak{c}) in $\langle {}^\omega\omega/\mathcal{U}, <_{\mathcal{U}} \rangle$. [Let $<_0$ be the ordering on \mathfrak{c} which we want to embed into ${}^\omega\omega/\mathcal{U}$. Start from a family f_ξ ($\xi < \mathfrak{c}$) in ${}^\omega\omega$ which is *independent*, i.e. $A_{\xi\eta} = \{n : f_\xi(n) < f_\eta(n)\}$ is an independent family of subsets of ω and $f_\xi(n) = f_\eta(n)$ for at most finitely many n , for all $\xi \neq \eta$. Then every ultrafilter \mathcal{U} extending the filter base $\mathcal{F} = \{A_{\xi\eta} : \xi <_0 \eta\} \cup \{\omega \setminus A_{\xi\eta} : \eta <_0 \xi\}$ works.] Our next example shows that there can be nonprincipal ultrafilters \mathcal{U} such that in $\langle {}^\omega\omega/\mathcal{U}, <_{\mathcal{U}} \rangle$ there are no ω_2 -chains and the continuum is large.

PROPOSITION 3.1. *If we start from a model of CH and add any number of side-by-side Sacks reals with countable supports, then for many ultrafilters \mathcal{U} there are no ω_2 -chains in $\langle {}^\omega\omega/\mathcal{U}, <_{\mathcal{U}} \rangle$.*

PROOF. For the undefined notions see [2] or [29, §6.C]. Let \mathcal{S}_κ denote the poset for adding κ many side-by-side Sacks reals. It is well known that after forcing with \mathcal{S}_κ every ground-model selective ultrafilter still generates a selective ultrafilter (see e.g. [29, Theorem 6.8]). Since CH implies that there exists a selective ultrafilter, it will suffice to prove the claim for the case when \mathcal{U} is a ground-model selective ultrafilter. Let $\mathcal{B} = \{B_\alpha : \alpha < \omega_1\}$ be a base for \mathcal{U} . Let $\langle \dot{r}_\xi : \xi < \kappa \rangle$ be a name for a sequence of generic Sacks reals. Suppose that \dot{f}_ξ ($\xi < \omega_2$) is a name for a strictly increasing chain in ${}^\omega\omega/\mathcal{U}$. By [2] for every $\xi < \kappa$ there is a countable $A_\xi \subseteq \kappa$, a perfect set $P_\xi \subseteq \mathbb{R}^{A_\xi}$, and a continuous function $g_\xi : P_\xi \rightarrow {}^\omega\omega$ such that P_ξ forces $g_\xi(\langle \dot{r}_\alpha : \alpha \in A_\xi \rangle) = \dot{f}_\xi$. We can assume that $\kappa = \aleph_2$. By CH, we can assume that A_ξ 's form a Δ -system, and that there is a partial function $\bar{g} : \mathbb{R}^\omega \rightarrow {}^\omega\omega$ such that every g_ξ is isomorphic to \bar{g} . Fix $\xi < \eta < \omega_2$, and let $p_{\xi\eta} \in \mathcal{S}_\kappa$ and $A \in \mathcal{B}$ be such that

$$p_{\xi\eta} \leq P_\xi, P_\eta \quad \text{and} \quad p_{\xi\eta} \Vdash (\forall n \in \check{A}) \dot{g}_\xi(n) < \dot{g}_\eta(n).$$

Let $\Phi : \mathcal{S}_\kappa \rightarrow \mathcal{S}_\kappa$ be an automorphism of \mathcal{S}_κ (compare with paragraph before Definition 4.2) whose extension to \mathcal{S}_κ -names swaps \dot{g}_ξ and \dot{g}_η . Then $\Phi(p_{\xi\eta})$ forces $\dot{g}_\eta <_{\mathcal{U}} \dot{g}_\xi$, a contradiction. ■

4. \mathcal{H}_E and its basic properties

DEFINITION 4.1. If $\langle E, <_E \rangle$ is a partially ordered set, then we define the poset \mathcal{H}_E as follows: A typical condition p is $\langle F_p, n_p, f_p \rangle$, where

($\mathcal{H}1$) F_p is a finite subset of E , $n_p < \omega$, $f_p : F_p \times n_p \rightarrow \omega$.

We say that p extends q iff (as the notation of ($\mathcal{H}3$) suggests, we will sometimes consider f_p as a mapping from F_p into ${}^{n_p}\omega$):

($\mathcal{H}2$) $F_p \supseteq F_q$, $n_p \geq n_q$, $f_p \supseteq f_q$,

($\mathcal{H}3$) $f_p(a)(i) \leq f_p(b)(i)$ for all $a <_E b$ in F_q and all $i \in [n_q, n_p)$.

So if \dot{g} is a name for the mapping of E into ${}^\omega\omega$ defined by $a \mapsto \bigcup_{p \in \dot{G}} f_p(a)$ (\dot{G} is a name for the generic filter), then every condition p in \mathcal{H}_E forces that $\dot{g}(a) \leq^{n_p} \dot{g}(b)$ for all $a <_E b \in F_p$. By genericity $\dot{g}(a) \neq^* \dot{g}(b)$ for all distinct a and b in ${}^\omega\omega$. Note that the generic filter \dot{G} is *not* equal to the set $\{p : f_p(a) \subset \dot{g}(a) \text{ for all } a \in F_p\}$. Instead, we have (let n_{ab} be the least positive integer n such that $g(a) \leq^n g(b)$ if $g(a) \leq^* g(b)$ and 0 otherwise)

$$\dot{G} = \{p : f_p(a) \subset \dot{g}(a) \text{ and } n_{ab} \leq n_p \text{ for all } a, b \in F_p\}.$$

The following useful fact is an immediate consequence of Definition 4.1 (see also Lemma 4.4).

PROPOSITION 4.1. *If $p, q \in \mathcal{H}_E$ are such that $n_p = n_q$ and f_p, f_q agree on $F_p \cap F_q$, then p and q are compatible, with $\langle F_p \cup F_q, n_p, f_p \cup f_q \rangle$ extending both. ■*

The assumption $n_p = n_q$ is not necessary if e.g. F_p and F_q are disjoint, but in general it is (see Proposition 4.2). We will often write \Vdash_E instead of $\Vdash_{\mathcal{H}_E}$ when this does not lead to confusion. By the above (plus a standard Δ -system argument) we have:

LEMMA 4.1. *\mathcal{H}_E is ccc (moreover, it has precaliber κ for every uncountable regular κ) and \dot{g} is forced to be an embedding of $\langle E, <_E \rangle$ into $\langle {}^\omega\omega, \leq^* \rangle$. ■*

If E_0 is a subordering of E and p is in \mathcal{H}_E , then let $p \upharpoonright E_0$ be the condition p' such that $F_{p'} = F_p \cap E_0$, $n_{p'} = n_p$, and $f_{p'} = f_p \upharpoonright F_{p'} \times n_p$; so in particular $p \upharpoonright \emptyset$ is the maximal condition in \mathcal{H}_E . Recall that \mathcal{P} is a *regular subordering* of \mathcal{Q} (denoted $\mathcal{P} \triangleleft \mathcal{Q}$) iff for every condition q of \mathcal{Q} there is a $q_{\mathcal{P}} \in \mathcal{P}$ (a *projection* of q to \mathcal{P}) such that p is compatible with $q_{\mathcal{P}}$ iff q is, for all $p \in \mathcal{P}$. [In the terminology of [18], \mathcal{P} is *completely embedded* into \mathcal{Q} .]

THEOREM 4.1. *If E_0 is any subordering of E , then $\mathcal{H}_{E_0} \prec \mathcal{H}_E$. In particular, the projection mapping is $q \mapsto q \upharpoonright E_0$.*

Proof. We fix $q \in \mathcal{H}_E$ and $p \in \mathcal{H}_{E_0}$ which extends $q \upharpoonright \mathcal{H}_{E_0}$ and prove that q and p are compatible by finding $r \leq q, p$ such that $F_r = F_q \cup F_p$. It is enough to consider the case when $F_q \setminus E_0$ is a singleton, because the general case follows from this special one by obvious induction. So let $F_q \setminus E_0 = \{c\}$. Let $F_0 = F_q \cap E_0$; if F_0 is empty then p and q are by default comparable, so we can assume that F_0 is nonempty, and therefore that $n_p \geq n_q$. So by Proposition 4.1 we have to do some work only when $n_p > n_q$, and this work is in defining $f_r(c) \upharpoonright [n_q, n_p)$. If $F_0(< c)$ is nonempty, pick a_i in this last set such that $f_p(a_i)(i)$ is maximal for all $i \in [n_q, n_p)$. If $F_0(< c)$ is empty but $F_0(> c)$ is not, then pick a_i in this last set so that $f_p(a_i)(i)$ is minimal. If no element of F_0 is comparable with c then pick a_i 's arbitrarily. Let $f_r(c)(i) = f_p(a_i)(i)$ for $i \in [n_q, n_p)$. We then claim that

$$r = \langle F_p \cup F_q, n_p, f_p \cup f_r \rangle$$

extends both p and q . To see this, we only have to check if condition $(\mathcal{H}3)$ is valid between q and r . Suppose first that $F_0(< c) \neq \emptyset$. Pick $i \in [n_q, n_p)$ and $d \in F_q \cap E_0$.

If $d <_E c$, then $f_r(d)(i) = f_p(d)(i) \leq f_p(a_i)(i) = f_r(c)(i)$, by the choice of a_i .

If $d >_E c$, then $d >_E a_i$, so $f_p(d)(i) \geq f_p(a_i)(i) = f_r(c)(i)$ (because p extends $q \upharpoonright E_0$). The case when $F_0(< c) = \emptyset$ and $F_0(> c) \neq \emptyset$ is handled similarly, and if both sets are empty then the claim is by default true. So p and q are compatible and $q \upharpoonright E_0$ is the projection of q to \mathcal{H}_{E_0} . ■

The following gives us an internal characterization of the comparability relation in \mathcal{H}_E .

PROPOSITION 4.2. (a) *Conditions p and q in \mathcal{H}_E such that $n_p \geq n_q$ are incompatible iff one of the following happens:*

- ($\perp 1$) $f_p(a)(i) \neq f_q(a)(i)$ for some $a \in F_p \cap F_q$ and some $i < n_p, n_q$,
- ($\perp 2$) for $\varrho \in \{<, >\}$: $f_p(a)(i) \varrho f_p(b)(i)$ for some $b \varrho_E a \in F_q$ and $i \in [n_q, n_p)$.

(b) *Let $F = F_p \cap F_q$. Then p and q are incompatible iff $p \upharpoonright F$ and $q \upharpoonright F$ are.*

Proof. (a) We will prove only the nonobvious direction, so assume that $p \perp q$ and that $f_p \cup f_q$ is a function (i.e. ($\perp 1$) does not apply). If $n_p = n_q$ then p and q are comparable by Proposition 4.1, so we can assume that $n_p > n_q$. But if ($\perp 2$) does not apply, p and $q \upharpoonright F_p$ are comparable, so p and q are comparable by Theorem 4.1.

(b) This follows immediately from Theorem 4.1 applied with $E_0 = F$. ■

By [18, VII.5.12] we can assume that every \mathcal{H}_E -name τ for a real (that is, a subset of ω) is in a canonical form, called “nice name” in [18]. Namely, we assume that for a sequence $\{A_n^\tau\}$ of antichains we have

$$\tau = \{\{n\} \times A_n^\tau : n \in \omega\}.$$

[So $p \Vdash \check{n} \in \tau$ if $p \in A_n^\tau$, and $p \Vdash \check{n} \notin \tau$ iff p is incompatible with all elements of A_n^τ .] In particular, τ is countable. So we can define a *support* of a name τ by

$$\text{supp } \tau = \bigcup_{n \in \omega} A_n^\tau.$$

In particular, $\text{supp } \tau$ is a countable subset of E .

COROLLARY. (a) *For every real \dot{x} in an extension by \mathcal{H}_E there is a countable (i.e. Cohen) subordering of \mathcal{H}_E which adds \dot{x} .*

(b) *The real $\dot{g}(\check{a})$ is Cohen over V for every $a \in E$.*

PROOF. (a) By the above, $\mathcal{H}_{\text{supp } \dot{x}}$ is a regular subordering of \mathcal{H}_E .

(b) $\mathcal{H}_{\{a\}}$ is a regular subordering of \mathcal{H}_E , and the assertion follows by the definition of $\mathcal{H}_{\{a\}}$. ■

Observe that if $\langle E_0, <_0 \rangle$ and $\langle E_1, <_1 \rangle$ are isomorphic, then every isomorphism naturally extends to an isomorphism between \mathcal{H}_{E_0} and \mathcal{H}_{E_1} and to an isomorphism between the classes of \mathcal{H}_{E_0} - and \mathcal{H}_{E_1} -names. An \mathcal{H}_{E_0} -name \dot{f}_0 and an \mathcal{H}_{E_1} -name \dot{f}_1 are *isomorphic* iff there are $\text{supp } \dot{f}_i \subseteq A_i \subseteq E_i$ ($i = 0, 1$) such that the posets $\langle A_0, <_0 \rangle$ and $\langle A_1, <_1 \rangle$ are isomorphic and the extension of the isomorphism sends \dot{f}_0 to \dot{f}_1 .

We will describe the quotient $\mathcal{H}_E/\mathcal{H}_{E_0}$, after a definition which is slightly more general than we need.

DEFINITION 4.2. Let $E = E_0 \dot{\cup} E_1$ and g_0 be an embedding of E_0 into $\langle {}^\omega\omega, \leq^* \rangle$. For $a, b \in E_0$ let n_{ab} be the least positive integer n such that $g_0(a) \leq^n g_0(b)$ if such an n exists; otherwise let $n_{ab} = 0$. For $p \in \mathcal{H}_E$ let $F_p^0 = F_p \cap E_0$ and $F_p^1 = F_p \cap E_1$. We define the poset $\mathcal{H}_E(E_0, g_0)$ as the subordering of \mathcal{H}_E consisting of all p such that:

(H4) $f_p \upharpoonright F_p^0 \times n_p \subset g_0$, and

(H5) if $a <_E b$ are in F_p^0 , then $n_{ab} \leq n_p$.

The ordering is inherited from \mathcal{H}_E .

So $\mathcal{H}_E(E_0, g_0)$ adds a generic $\dot{g}_1 : (E \setminus E_0) \rightarrow {}^\omega\omega$ such that $g_0 \cup \dot{g}_1$ is an embedding. For p in this poset $p \upharpoonright F_p^0$ is a side-condition and $p \upharpoonright F_p^1$ is a working part. Note that without requiring (H5) the set of all conditions p such that $n_p \geq n$ would not be dense in $\mathcal{H}_E(E_0, \dot{g}_0)$ for every integer n , and that $\mathcal{H}_E(E_0, g_0)$ need not be separative. [E.g. if E_0 and E_1 are incomparable

then $\mathcal{H}_E(E_0, g_0)$ is equivalent to \mathcal{H}_{E_1} , because if p, q in this poset are such that $p \upharpoonright E_1 = q \upharpoonright E_1$ then for all r we have $r \perp p$ iff $r \perp q$.]

EXAMPLE 4.1. An analogous result to Theorem 4.1 fails in the case of $\mathcal{H}_E(E_0, g_0)$, namely there can be $X \subseteq E$ such that $\mathcal{H}_X(E_0 \cap X, g_0 \upharpoonright X)$ is not a regular subordering of $\mathcal{H}_E(E_0, g_0)$. E.g. if $a <_E b$ are such that $a \in E_0 \setminus X$ and $b \in E_1 \cap X$, then a condition q such that $a, b \in F_q$ does not have a projection to $\mathcal{H}_X(E_0 \cap X, g_0 \upharpoonright X)$. This is because q forces that n_{ab} is at most n_q , while $\mathcal{H}_X(E_0 \cap X, g_0 \upharpoonright X)$ by genericity forces that $\dot{g}(b)$ and $g_0(a)$ are \leq^* -incomparable.

The fact that the ordering on $\mathcal{H}_E(E_0, g_0)$ is inherited from \mathcal{H}_E does not imply that the compatibility relation is inherited from \mathcal{H}_E as well; compare the following proposition with Proposition 4.2(a).

PROPOSITION 4.3. *Assume that $E = E_0 \dot{\cup} E_1$ and that p, q are conditions in $\mathcal{H}_E(E_0, g_0)$ such that $n_p \geq n_q$. Then p and q are incompatible in $\mathcal{H}_E(E_0, g_0)$ iff one of the following happens for $\varrho \in \{<, >\}$ (let $F = F_p \cap F_q$, $F^i = F \cap E_i$ for $i = 0, 1$):*

- (\perp' 1) p and q are incompatible in \mathcal{H}_E , or
- (\perp' 2) there are $a \in F^1$ and $b \in F_q^0$ such that $b \varrho_E a$ but $f_p(a)(i) \varrho g_0(b)(i)$ for some $i \in [n_q, n_p)$, or
- (\perp' 3) for some $b_p \in F_p^0$, $b_q \in F_q^0$ and $a \in F^1$ such that $a \in (b_p, b_q)_E$ we have $n_{b_p b_q} > n_p$.

So in particular if $F_p \cap F_q \subseteq E_0$ then p and q are incomparable in $\mathcal{H}_E(E_0, g_0)$ iff they are incomparable in \mathcal{H}_E .

PROOF. (\Leftarrow) If (\perp' 2) happens, then q forces that $n_{ab} \leq n_q$ but p forces that n_{ab} is at least $i + 1$ for i as in (\perp' 2). So if $r \leq p, q$ then r forces both—a contradiction. If (\perp' 3) happens, then p forces that $n_{b_p a} \leq n_p$, q forces that $n_{b_q a} \leq n_q$, so if p, q were compatible then this would imply that $n_{b_p b_q}$ is at most $n_p = \max\{n_p, n_q\}$ —a contradiction.

(\Rightarrow) Suppose that p and q satisfy the negations of (\perp' 1), (\perp' 2) and (\perp' 3). Without loss of generality $F_q \setminus F_p$ is a singleton $\{c\}$. If $c \in E_1$, then we can prove that p and q are compatible exactly as in the proof of Theorem 4.1. So suppose that $c \in E_0$. Let $n_r = \max\{n_p, n_{bc} : b \in F_p^0\}$. Let \bar{n} be an integer greater than $g_0(b)(i)$ and $f_p(b)(i)$ for all $b \in F_p \cup \{c\}$ and all $i < n_r$. Define $f_r(a)(i)$ for $a \in F_p^1$ and $i \in [n_p, n_r)$ by (letting $\max \emptyset = 0$ and $\min \emptyset = \bar{n}$)

$$(\dagger) \quad f_r(a)(i) = \begin{cases} \max\{g_0(b)(i) : b \in F_p(<_E a)\} & \text{if } c \not<_E a, \\ \min\{g_0(b)(i) : b \in F_p(>_E a)\} & \text{if } c <_E a. \end{cases}$$

By the choice of n_r , the condition r is in $\mathcal{H}_E(E_0, g_0)$. We claim that $r \leq p, q$. To check that $r \leq p$, it is enough to check that ($\mathcal{H}3$) is true for $a, b \in F_p$. This checking splits into cases; pick $i \in [n_p, n_r)$.

Case 1. If $b \in E_0$ then $(\mathcal{H}3)$ follows immediately from $(\mathcal{H}5)$, whether $a \in E_0$ or not.

Case 2. If $a <_E b \in F_p^1$, then we consider subcases.

Case 2.1. If $c <_E a$, then in defining $f_r(a)(i)$ and $f_r(b)(i)$ the first line of (\dagger) applies, but $a <_E b$ implies $F_p^0(<_E a) \subseteq F_p^0(<_E b)$ and the maximum of a bigger set is bigger, so $f_r(a)(i) \leq f_r(b)(i)$.

Case 2.2. If $c \not<_E b$, then in defining $f_r(a)(i)$ and $f_r(b)(i)$ the second line of (\dagger) applies, and the argument is similar to that of Case 2.1, bearing in mind that if $F_p(>_E b) = \emptyset$ then $f_r(b)(i) = \bar{n}$ and \bar{n} is chosen to be large enough.

Case 3. If $a <_E c <_E b$, then $f_r(a)(i) = f_r(a')(i) \leq f_r(b')(i) = f_r(b)(i)$ for some $a' <_E a$ and $b' >_E b$.

So we have proved that r extends p . Now we will assume that r does not extend q , namely that $(\mathcal{H}3)$ fails for $a \in F_p \cap F_q$, c and $i \in [n_q, n_r)$.

Case 4. If $a <_E c$ and $f_r(a)(i) > g_0(c)(i)$, then if $i < n_p$ this is $(\perp'2)$. If $i \geq n_p$ then there is $a' <_E a$ such that $g_0(a')(i) > g_0(c)(i)$, so this is $(\perp'3)$.

Case 5. If $a >_E c$, then the discussion is the same as in Case 4.

So if r does not extend q then one of conditions $(\perp'1)$ – $(\perp'3)$ applies, and the proposition is thus proved. ■

An embedding $\Phi : \mathcal{P} \rightarrow \mathcal{Q}$ is *dense* iff $\Phi''\mathcal{P}$ is a dense subset of \mathcal{Q} .

THEOREM 4.2. *Let $E = E_0 \dot{\cup} E_1$ and let \dot{g}_0 be an \mathcal{H}_{E_0} -name for the generic embedding of E_0 into $({}^\omega\omega, \leq^*)$. Then in a forcing extension by \mathcal{H}_{E_0} the posets $\mathcal{H}_E/\mathcal{H}_{E_0}$ and $\mathcal{H}_E(E_0, \dot{g}_0)$ are equivalent.*

PROOF. By [18, VII.7.11] it is enough to find (working in a ground model) a dense embedding of \mathcal{H}_E into $\mathcal{H}_{E_0} * \check{\mathcal{H}}_E(\check{E}_0, \dot{g}_0)$. Let $p \mapsto \langle p \upharpoonright E_0, \check{p} \rangle$. This mapping is obviously an ordermorphism. The set of all $\langle \bar{p}, \bar{q} \rangle$ such that \bar{q} is “decided” (i.e. it is an element of $\mathcal{H}_E(E_0, \dot{g}_0)$ instead of an \mathcal{H}_{E_0} -name) is dense in the iteration. So we will start from such $\langle \bar{p}, \bar{q} \rangle$ and find p in \mathcal{H}_E such that $\langle p \upharpoonright E_0, p \rangle$ extends $\langle \bar{p}, \bar{q} \rangle$ in the iteration. We claim that \bar{p} and \bar{q} are compatible in \mathcal{H}_E : since \bar{p} forces that \bar{q} is in $\mathcal{H}_E(E_0, \dot{g}_0)$, $f_{\bar{p}} \cup f_{\bar{q}}$ must be a function, and we must also have $F_{\bar{q}}^0 \subseteq F_{\bar{p}}$ and $n_{\bar{q}} \leq n_{\bar{p}}$. [If one of these fails then $f_r \cup f_{\bar{q}}$ is not a function for some $r \leq \bar{p}$ in \mathcal{H}_{E_0} .] So $(\mathcal{H}5)$ for \bar{q} implies that $(\perp'2)$ of Proposition 4.2 fails, so \bar{p} and \bar{q} are compatible in \mathcal{H}_E . Pick $p \in \mathcal{H}_E$ which extends \bar{p} and \bar{q} . But $p \in \mathcal{H}_E$ implies that $p \upharpoonright E_0 \Vdash \check{p} \in \check{\mathcal{H}}_E(\check{E}_0, \dot{g}_0)$, and therefore $p \upharpoonright E_0 \Vdash \check{p} \leq \check{\bar{q}}$ (in $\check{\mathcal{H}}_E(\check{E}_0, \dot{g}_0)$) so $\langle p \upharpoonright E_0, p \rangle$ extends $\langle \bar{p}, \bar{q} \rangle$. ■

If E_0 and E_1 are incomparable then in $\mathcal{H}_E(E_0, g_0)$ side-conditions from E_0 are void so this poset is equivalent to \mathcal{H}_{E_1} . Therefore the following properties of \mathcal{H}_E are immediate consequences of the above statements:

($\mathcal{H}6$) If $E = E_0 \dot{\cup} E_1$ and E_0 and E_1 are incomparable, then \mathcal{H}_E is isomorphic to $\mathcal{H}_{E_0} \times \mathcal{H}_{E_1}$.

($\mathcal{H}7$) If $E = \bigcup_{i \in I} E_i$ and E_i 's are pairwise incomparable, then \mathcal{H}_E is isomorphic to a finite support product of $\{\mathcal{H}_{E_i}\}$.

The following two lemmas will be crucial in the proof of Theorem 9.1.

LEMMA 4.2. *If $X \subseteq E_0$ is such that $X(<_E a)$ is cofinal in $E_0(<_E a)$ and $X(>_E a)$ is cointial in $E_0(>_E a)$ for all $a \in E \setminus E_0$, then in an extension by \mathcal{H}_{E_0} the poset $\mathcal{H}_E(X, \dot{g}_0 \upharpoonright X)$ is a dense subordering of $\mathcal{H}_E(E_0, \dot{g}_0)$.*

PROOF. Let $E' = E_1 \cup X$. Note that the set \mathcal{D} of all $p \in \mathcal{H}_E(E_0, g_0)$ such that $a \varrho b$ implies that $a \varrho c \varrho b$ for some $c \in F_p^0 \cap X$ for $\varrho \in \{\leq_E, \geq_E\}$, $a \in F_p^0$ and $b \in F_p^1$ is dense in $\mathcal{H}_E(E_0, g_0)$.

CLAIM. *Conditions p and q are compatible iff p and $q \upharpoonright E'$ are compatible for all $q \in \mathcal{D}$ and $p \in \mathcal{H}_{E'}(X, \dot{g}_0 \upharpoonright X)$.*

PROOF. We prove only the nontrivial direction, (\Leftarrow). Suppose that p and q are incompatible. Then by Proposition 4.3 one of ($\perp'1$)–($\perp'3$) is true. If ($\perp'1$) applies, then it applies for some $a \in F_q^1 = F_{q \upharpoonright E'}^1$, so p and $q \upharpoonright E'$ are incompatible. If ($\perp'2$) fails for $a \in F_q^1 \cap F_p^1$ and $b \in F_q^0$, then there is $c \in F_q^0 \cap X$ such that ($\perp'2$) fails for a and c ; similarly if ($\perp'3$) fails, and so the claim is verified. ■

So $\mathcal{H}_{E'}(X, \dot{g}_0 \upharpoonright X)$ is a dense subordering of $\mathcal{H}_E(E_0, \dot{g}_0)$. ■

LEMMA 4.3. *If $E = E_0 \dot{\cup} E_1 \dot{\cup} E_2$ and*

(*) $a \varrho b$ implies that $a \varrho c \varrho b$ for some $c \in E_0$

for all $a \in E_1, b \in E_2$ and $\varrho \in \{<_E, >_E\}$, then \mathcal{H}_E is equivalent to

$$\mathcal{H}_{E_0} * (\mathcal{H}_{E_1}(E_0, \dot{g}_0) \times \mathcal{H}_{E_2}(E_0, \dot{g}_0)).$$

PROOF. By Theorem 4.2, \mathcal{H}_E is equivalent to $\mathcal{H}_{E_0} * \mathcal{H}_{E_1}(E_0, \dot{g}_0) * \mathcal{H}_{E_2}(E_0 \cup E_1, \dot{g}_0 \cup \dot{g}_1)$. By (*) and Lemma 4.2, the posets $\mathcal{H}_{E_2}(E_0 \cup E_1, \dot{g}_0 \cup \dot{g}_1)$ and $\mathcal{H}_{E_2}(E_0, \dot{g}_0)$ are equivalent; but the definition of the latter does not depend on the generic object for $\mathcal{H}_{E_1}(E_0, \dot{g}_0)$, so we are in the product situation and the lemma is proved. ■

In the following statement, cf (ci) stands for the cofinality (respectively, cointiality) of a partially ordered set E , namely the smallest size of a set $D \subseteq E$ which is cofinal (resp. cointial) in E .

PROPOSITION 4.4. *If $E = \bigcup_{\xi < \kappa} E_\xi$ is a disjoint union, then we can write \mathcal{H}_E as a finite support ccc iteration $\langle \mathcal{P}_\xi, \dot{Q}_\xi \rangle_{\xi < \kappa}$ such that (\dot{g} is an \mathcal{H}_Q -name for the generic embedding of \mathcal{Q} into $\langle {}^\omega \omega, \leq^* \rangle$):*

- (1) $\Vdash_\xi \check{g} \upharpoonright \check{E}_\xi$ is a \dot{Q}_ξ -name,
- (2) $\Vdash_\xi |\dot{Q}_\xi| \leq |\check{E}_\xi| + |\bigcup_{\eta < \xi} \check{E}_\eta|$, or more precisely

(3) \mathcal{P}_ξ forces that \dot{Q}_ξ is of size $|E_\xi| + \sum_{a \in E_\xi, \eta < \xi} (\text{cf}(E_\eta(<_E a)) + \text{ci}(E_\eta(>_E a)))$.

Proof. Let $E_\xi^+ = \bigcup_{\alpha < \xi} E_\alpha$, let $\mathcal{P}_\alpha = \mathcal{H}_{E_\alpha^+}$, and let \dot{g}_ξ be the \mathcal{P}_ξ -name for the generic embedding of E_ξ^+ into ${}^\omega\omega$ and let $\dot{Q}_\xi = \mathcal{H}_{E_{\xi+1}^+}(E_\xi^+, \dot{g}_\xi)$. Then the conclusion follows immediately from the previous discussion. ■

The following lemma will be used in §6 when proving that gaps in E get mapped into gaps in ${}^\omega\omega$ by \dot{g} .

LEMMA 4.4. *If $E = E_0 \dot{\cup} E_1$, E_1 is at most countable and for all $c \in E_1$ at most one of the sets $E_0(<_E c)$, $E_0(>_E c)$ is nonempty, then \mathcal{H}_{E_0} forces that the poset $\mathcal{H}_E(E_0, \dot{g}_0)$ is σ -centered.*

[Note that some assumption in this lemma is necessary, because if \mathcal{H}_{E_0} generically adds an (ω_1, ω_1) -gap to ${}^\omega\omega$ and $\mathcal{H}_E(E_0, \dot{g}_0)$ fills it, then the ccc-ness of the latter poset is not absolute, so it cannot be σ -centered.]

Proof. In an extension by \mathcal{H}_{E_0} , from Proposition 4.2 it follows that p and q in $\mathcal{H}_E(E_0, \dot{g}_0)$ such that $p \upharpoonright E_1 = q \upharpoonright E_1$ are compatible. There are at most countably many distinct $p \upharpoonright E_1$ for $p \in \mathcal{H}_E(E_0, \dot{g}_0)$, so this poset is σ -centered. ■

Remark. Lemma 4.4 remains true when the assumption that E_1 is countable is replaced by the weaker $|E_1| \leq |E_0|^{\aleph_0} + \mathfrak{c}$, i.e. that E_1 is of size at most continuum in the extension by \mathcal{H}_{E_0} . The proof of this claim is completely analogous to the familiar proof of the fact that the iteration of σ -centered posets of length at most \mathfrak{c} is σ -centered. However, we do not need this extension and find the present formulation of Lemma 4.4 esthetically more pleasing.

We finish this section with two lemmas which are not being used elsewhere in this text, but which shed some light on the poset \mathcal{H}_E . For conditions p, q in a poset \mathcal{P} the *meet* of p and q is the condition $p \wedge q$ which extends both p and q and is minimal with this property, namely every $r \in \mathcal{P}$ which extends both p and q extends $p \wedge q$ as well. Of course, meets always exist in the regular open algebra of \mathcal{P} , but not necessarily in \mathcal{P} .

LEMMA 4.5. *If E is not an antichain, then there are compatible p and q in \mathcal{H}_E such that the meet $p \wedge q$ does not exist in \mathcal{H}_E .*

Proof. Pick $a <_E b$ in E and let $p = \langle \{a\}, 0, \emptyset \rangle$ and $q = \langle \{b\}, 0, \emptyset \rangle$. Then p and q are compatible and $r \leq p, q$ iff $a, b \in F_r$. Also $r \Vdash \dot{g}(\check{a}) \leq^{\check{n}} \dot{g}(\check{b})$ for $r \leq p, q$ and $n = n_r$. But obviously we can pick $r' \leq p, q$ so that $f_{r'}(a)(n_r) > f_{r'}(b)(n_r)$, so r' and r are incompatible. ■

LEMMA 4.6. (a) *If $E = E_0 \dot{\cup} E_1$ and E_1 is infinite, then $\mathcal{H}_E(E_0, \dot{g}_0)$ adds a Cohen real over $V^{\mathcal{H}_{E_0}}$.*

In an extension by \mathcal{H}_E ,

(b) $\text{MA}_{<|E|}$ (countable) is true, and

(c) if E is uncountable then the bounding number \mathfrak{b} is equal to ω_1 .

Proof. (a) Work in $V^{\mathcal{H}_{E_0}}$: For a 1-1 sequence $\vec{a} = \{a_n\}$ in E_1 define the $\mathcal{H}_E(E_0, \dot{g}_0)$ -name for a real $\dot{r}_{\vec{a}}$ by

$$\dot{r}_{\vec{a}}(\check{n}) = \dot{g}(\check{a}_n)(\check{0}) \quad \text{for all } n < \omega.$$

It is easily checked that $\dot{r}_{\vec{a}}$ is Cohen over the intermediate extension.

(b) Since all countable posets are equivalent to \mathcal{C} , it is enough to prove that in an extension by \mathcal{H}_E for every $\kappa < |E|$ and a family $\{\mathcal{D}_\xi\}$ ($\xi < \kappa$) of dense open subsets of \mathbb{R} there is a real in $\bigcap_{\xi < \kappa} \mathcal{D}_\xi$. Each \mathcal{D}_ξ is coded by a single real r_ξ which is added by \mathcal{H}_{A_ξ} for some countable $A_\xi \subseteq E$. Let $E_0 = \bigcup_{\xi < \kappa} A_\xi$. Then $E_1 \setminus E_0$ is infinite and all \mathcal{D}_ξ 's are in the intermediate extension by \mathcal{H}_{E_0} , so an application of (a) finishes the proof.

(c) Pick disjoint countable sequences \vec{a}_ξ ($\xi < \omega_1$) in E and let \dot{f}_ξ be an \mathcal{H}_E -name for a function such that $\dot{f}_\xi \geq^* r_{\vec{a}_\eta}$ for all $\eta < \xi < \omega_1$ and $\dot{f}_\eta \leq^* \dot{f}_\xi$ for all $\eta < \xi$. Let \dot{h} be an \mathcal{H}_E -name for some element of ${}^\omega\omega$; then $\text{supp } \dot{h}$ is disjoint from some \vec{a}_ξ , and $\dot{h} \not\leq r_{\vec{a}_\xi}$ by (a), so the sequence $\{\dot{f}_\xi\}$ is unbounded. ■

5. Five lemmas. This chapter contains a couple of similarly looking lemmas (cf. also [3, Lemma 2.5]). Here \mathcal{P}^0 and \mathcal{P}^1 always stand for posets, and $\mathcal{P}^0 \times \mathcal{P}^1$ -names with superscript $i = 0, 1$ are assumed to be \mathcal{P}^i -names (i.e. \dot{f}^i is a \mathcal{P}^i -name for $i = 0, 1$).

LEMMA 5.1. *If $\mathcal{P} = \mathcal{P}^0 \times \mathcal{P}^1$ forces that $\dot{f}^0 \leq^* \dot{f}^1$ then there are a $\mathcal{P}^0 \times \mathcal{P}^1$ -condition p and a ground-model function f such that $p \Vdash \dot{f}^0 \leq^* \dot{f} \leq^* \dot{f}^1$.*

Proof. Pick $p = \langle p^0, p^1 \rangle$ in \mathcal{P} which decides an integer n such that $\dot{f}^0 \leq^n \dot{f}^1$. For every integer m let k_m be the minimal integer such that $r_m \Vdash_{\mathcal{P}^1} \dot{f}^1(m) = k_m$ for some $r_m \leq p^1$. Then obviously $p \Vdash \dot{f} \leq^n \dot{f}^1$. Suppose that $q \Vdash_{\mathcal{P}^0 \times \mathcal{P}^1} \dot{f}^0(\check{m}) = \check{l}$, for some $q \leq p$ and $l < k_m$. Then $q = \langle q_0, q_1 \rangle$, and $\langle q_0, r_m \rangle \Vdash \dot{f}^0(\check{m}) = \check{l} \wedge \dot{f}^1(\check{m}) = \check{k}_m$, so $m \leq n$ and $p \Vdash \dot{f}^0 \leq^n \dot{f}$. ■

Remark. The conclusion of the above lemma cannot be strengthened to $\Vdash \dot{f}^0 \leq^* \dot{f} \leq^* \dot{f}^1$, namely going below a condition p to decide this statement is necessary, as the following example shows. Let \mathcal{P}^1 be the poset for adding a Cohen real and let \mathcal{P}^0 be any nontrivial ccc poset. \dot{f}^0 is a name for the constant function, such that for every $n \in \omega$ there is $p_n \in \mathcal{P}^0$ forcing that \dot{f}^0 is identically equal to n . If \dot{c} is a \mathcal{P}^0 -name for a Cohen real (in ${}^\omega\omega$), then \dot{f}^1 is defined by $\dot{f}^1(n) = \sum_{i \leq n} \dot{c}(i)$. Since \dot{f}^1 is nondecreasing and unbounded,

the assumptions of Lemma 5.1 are fulfilled, but no ground-model function f is between \dot{f}^0 and \dot{f}^1 with probability one.

The following is a version of Lemma 5.1 specific for \mathcal{H}_E .

LEMMA 5.2. *Let \dot{f}_0 and \dot{f}_1 be \mathcal{H}_E -names such that:*

- (1) $\Vdash \dot{f}_0 \leq^* \dot{f}_1$,
- (2) A, A_0, A_1 are pairwise disjoint countable subsets of E such that
- (3) $\text{supp } \dot{f}_i \subseteq A \cup A_i$, for $i = 0, 1$.

Then there are a finite $F \subseteq A_1$, an $\mathcal{H}_{A \cup F}$ -name \dot{h} for an element of ${}^\omega\omega$ and a condition q in \mathcal{H}_E so that $q \Vdash \dot{f}_0 \leq^ \dot{h} \leq^* \dot{f}_1$.*

PROOF. Pick a condition $q \in \mathcal{H}_E$ and $\bar{n} \in \omega$ so that

$$q \Vdash_E \dot{f}_0 \leq^{\bar{n}} \dot{f}_1.$$

Let $F = F_q \cap A_1$, go to an extension by $\mathcal{H}_{A \cup F}$ below $q \upharpoonright A \cup F$, and let \dot{g} denote the generic embedding of $A \cup F$ into ${}^\omega\omega$. It is enough to prove that in this intermediate extension there is a function h such that in the rest of \mathcal{H}_E (namely, in $\mathcal{H}_E(A \cup F, \dot{g}_0)$) the condition q forces that $\dot{f}_0 \leq^{\bar{n}} \dot{h} \leq^{\bar{n}} \dot{f}_1$. For $m \in \omega$ let k_m be the minimal integer such that for some r_m in $\mathcal{H}_{A \cup A_1}(A \cup F, \dot{g}_0)$ we have

$$r_m \Vdash_{A \cup A_1} \dot{f}_1(\check{m}) = \check{k}_m.$$

Let $h(m) = k_m$. Check that h works: obviously $q \Vdash \dot{h} \leq^{\bar{n}} \dot{f}_1$, so it remains to prove that $q \Vdash \dot{f}_0 \leq^{\bar{n}} \dot{h}$. Notice that q, r_m and p are compatible for every $p \in \mathcal{H}_E(A \cup F, \dot{g}_0)$ such that $F_p \cap A_1 \subseteq F$. [p and r_m are trivially compatible because \dot{f}_p and \dot{f}_{r_m} are compatible on $F_p \cap F_{r_m} \subseteq F_q$, but we have to find $p' \leq p, r_m$ which extends q . Since $F_p \cap F_{r_m} \supseteq F_q$ and $\dot{g} \upharpoonright F_q$ is already decided, every p' extending p and r_m is compatible with q .] Pick $m > \bar{n}$ and $p \in \mathcal{H}_E(A \cup F, \dot{g}_0)$ below q which decides for the value of $\dot{f}_0(m)$, say

$$p \Vdash \check{k} = \dot{f}_0(\check{m}) > \dot{h}(\check{m})$$

for some $k \in \omega$. Notice that we can assume that p is in $\mathcal{H}_{A \cup A_0}(A \cup F, \dot{g}_0)$, therefore p, q and r_m are compatible. So $k \leq k_m$, and therefore $q \Vdash \dot{f}_0 \leq^* \dot{h}$ so \dot{h} and q are as claimed. ■

LEMMA 5.3. *If $\kappa > \min(\mathfrak{c}, |\mathcal{P}^1|)$ is a regular uncountable cardinal, $\mathcal{P}^0 \times \mathcal{P}^1$ is ccc and forces that $\dot{f}_\xi^0 \leq^* \dot{f}_\eta^0 \leq^* \dot{f}_\kappa^1$ for all $\xi < \eta < \kappa$, then $q \Vdash \dot{f}_\xi^0 \leq^* \dot{h} \leq^* \dot{f}_\kappa^1$ for all $\xi < \kappa$, for some $q \in \mathcal{P}^0 \times \mathcal{P}^1$ and some ground-model h .*

PROOF. By Lemma 5.1, for each $\xi < \kappa$ there are $q_\xi = \langle q_\xi^0, q_\xi^1 \rangle$, h_ξ such that $q_\xi \Vdash \dot{f}_\xi^0 \leq^* \dot{h}_\xi \leq^* \dot{f}_\kappa^1$.

CASE 1. If $\kappa > \mathfrak{c}$, then there is h such that the set X of all ξ such that $h_\xi = h$ is of size κ . By ccc-ness, let q be the condition which forces that there

are κ many q_ξ 's in the generic filter \dot{G} . Then in an extension by $\mathcal{P}^0 \times \mathcal{P}^1$ below q , for every η there is $\xi > \eta$ such that q_ξ is in the generic filter \dot{G} , so $\dot{f}_\eta \leq^* \dot{f}_\xi \leq^* \dot{f}_\kappa$, and q, h are as required.

Case 2. If $\kappa > |\mathcal{P}^1|$, then there is $q^1 \in \mathcal{P}^1$ such that the set X of all ξ for which $q^1 = q_\xi^1$ is of size κ . Let $q_0 \in \mathcal{P}^0$ be a condition which forces that q_ξ^0 is in a \mathcal{P}^0 -generic filter for κ many $\xi \in X$. Define a function h as in the proof of Lemma 5.1: let $h(m)$ be the minimal k_m such that $r_m \Vdash_{\mathcal{P}^1} \dot{f}_\kappa^1(\check{m}) = \check{k}_m$. Then $q \Vdash \dot{h} \leq^* \dot{f}_\kappa^1$ and $q_\xi \Vdash \dot{f}_\xi^0 \leq^* \dot{h}$. The proof that q and h are as required is the same as in Case 1. ■

Remark. The assumptions of Lemma 5.3 are in some sense necessary, because it is possible that \mathcal{P}^0 adds a λ -chain $\{\dot{f}_\xi^0\}$ and \mathcal{P}^1 adds \dot{f}_λ^1 above it, but $\mathcal{P}^0 \times \mathcal{P}^1$ forces that no ground-model function is between all \dot{f}_ξ^0 's and \dot{f}_λ^1 . We can even pick \mathcal{P}^1 to be σ -centered and assume that there are no ω_2 -chains in ${}^\omega\omega$ in the ground model. To see this, let our ground model be an extension of a model of CH obtained by adding κ many Cohen reals $\{c_\xi : \xi < \kappa\}$ to ${}^\omega\omega$. Let \mathcal{P}^1 force that $\check{c}_\xi \leq^* \dot{f}_\kappa^1$, and let \mathcal{P}^0 force that $\dot{f}_\xi \leq^* \dot{f}_\eta, \check{c}_\eta$ for all $\xi < \eta < \kappa$. Then by genericity there is no h in $V[\langle c_\xi : \xi < \kappa \rangle]$ as in the conclusion of Lemma 5.3. There is a similar example connected to the result of Lemma 5.5 below, where $\mathcal{P}^0 \times \mathcal{P}^1$ is taken to be $\mathcal{H}_E(E_0, \dot{g}_0)$ for appropriate E and E_0 .

The following lemma will be used in the proof of Theorem 6.1.

LEMMA 5.4. *Let κ be an uncountable regular cardinal. If \mathcal{H}_E -names \dot{f}_ξ ($\xi \leq \kappa$) and $A \subseteq E$ are such that for all $\xi < \eta \leq \kappa$:*

- (1) $\Vdash_{\mathcal{H}_E} \dot{f}_\xi \leq^* \dot{f}_\eta$,
- (2) $\text{supp } \dot{f}_\xi \subseteq A_\xi$, and
- (3) $A_\xi \cap A_\kappa \subseteq A$ for all $\xi < \kappa$,

then there are an \mathcal{H}_E -name \dot{h}_κ and a condition q such that $\text{supp } \dot{h}_\kappa \subseteq^ A$ and $q \Vdash \dot{f}_\xi \leq^* \dot{h}_\kappa \leq^* \dot{f}_\kappa$ for all $\xi < \kappa$.*

Proof. For each $\xi < \kappa$ find q_ξ and n_ξ so that $q_\xi \Vdash \dot{f}_\xi \leq^{n_\xi} \dot{f}_\kappa$. Then we can assume that $\bar{n} = n_\xi$ for all ξ and some \bar{n} , that $\{F_{q_\xi}\}$ is a Δ -system with root F and that $q_\xi \upharpoonright F = q$ for some fixed q in \mathcal{H}_E . Go to an extension by $\mathcal{H}_{A \cup F}$ below q , let \dot{g}_0 denote the generic embedding, and define a function \dot{h}_κ as follows: for all $m \in \omega$ let k_m be the minimal integer such that $r_m \Vdash \dot{f}_\kappa(\check{m}) = \check{k}_m$ for some $r_m \in \mathcal{H}_E(A \cup F, \dot{g}_0)$. This defines \dot{h}_κ , an \mathcal{H}_E -name below a condition q . Obviously $q \Vdash \dot{h}_\kappa \leq^* \dot{f}_\kappa$, and as in Lemma 5.1 we have $q_\xi \Vdash \dot{f}_\xi \leq^{\bar{n}} \dot{h}_\kappa$ for all $\xi < \kappa$. To prove that $q \Vdash \dot{f}_\xi \leq^* \dot{h}_\kappa$ for all $\xi < \kappa$, pick a condition $p \leq q$ and $\xi < \kappa$. Find $\eta > \xi$ such that q_η and p are compatible (this is possible because $\{F_{q_\eta}\}$ is a Δ -system with root F and $p \upharpoonright F$ extends

$q_\eta \upharpoonright F = q$ for all η). Then $p \wedge q_\eta$ forces that $\dot{f}_\xi \leq^* \dot{f}_\eta \leq^{\check{n}} \dot{h}_\kappa$. So the set of all conditions which force that $\dot{f}_\xi \leq^* \dot{h}_\kappa$ is dense below q . ■

The following lemma is used in the proof of Theorem 9.2. For the definition of pregap see Definition 6.1.

LEMMA 5.5. *If $\mathcal{P}^0 \times \mathcal{P}^1$ forces that $\langle \dot{f}_\xi^0, \dot{f}_\eta^1 \rangle_{\xi < \kappa, \eta < \lambda}$ is a pregap, κ, λ are regular and uncountable, $\kappa > |\mathcal{P}^1|$, $\mathcal{P}^0 \times \mathcal{P}^1$ is ccc, and every real in an extension by \mathcal{P}^0 is added by its regular subordering of size less than λ , then $q \Vdash \dot{f}_\xi^0 \leq^* \dot{h} \leq^* \dot{f}_\eta^1$ for all $\xi < \kappa, \eta < \lambda$, for some $q \in \mathcal{P}^0 \times \mathcal{P}^1$ and some ground-model h .*

PROOF. For all $\xi < \kappa$ let \mathcal{P}_ξ^0 be a small regular subordering of \mathcal{P}^0 which adds \dot{f}_ξ^0 . Apply Lemma 5.3 (actually, its dual form) to get a condition $q_\xi = \langle q_\xi^0, q_\xi^1 \rangle$ and a ground-model h_ξ such that $q_\xi \Vdash \dot{f}_\xi^0 \leq^* \check{h}_\xi \leq^* \dot{f}_\eta^1$ for all $\eta < \lambda$. There is a ground-model h such that the set X of all $\xi < \kappa$ such that $h = h_\xi$ is of size κ . Let q be a $\mathcal{P}^0 \times \mathcal{P}^1$ condition which forces that there are κ many $\xi \in X$ such that q_ξ is in a generic filter; then q forces that h splits the pregap. ■

6. ${}^\omega\omega$ in an extension by \mathcal{H}_E . Since $\dot{g}(a)$ is Cohen generic over the universe for all $a \in E$, the set $\dot{g}''E$ is never cofinal in ${}^\omega\omega$ (compare with [13] and [4]), but it does have some unboundedness properties.

LEMMA 6.1. (a) *Let \dot{h} be an \mathcal{H}_E -name for an element of ${}^\omega\omega$ with support A and let $a \in E \setminus A$ be such that $A(\geq_E a)$ is empty. Then $\Vdash \dot{h} \not\leq^* \dot{g}(a)$.*

(b) *$X \subseteq E$ is not countably bounded iff it is forced that $\dot{g}''X$ is unbounded in ${}^\omega\omega$.*

PROOF. (a) Suppose the contrary, i.e. that some condition p forces that $\dot{h} \geq^n \dot{g}(\check{a})$; without loss of generality we have $n_p = n$. Then extend $p \upharpoonright A$ to $q \in \mathcal{H}_A$ which decides that $\dot{h}(\check{n} + 1) = \check{k}$ for some integer k . Now extend p and q to r so that $f_r(a)(n + 1) = k + 1$; this is possible because $a \not\leq_E b$ for all $b \in A$. It is also the desired contradiction.

(b) (\Rightarrow) If \dot{h} is forced to dominate $\dot{g}''X$, then by (a), $\text{supp } \dot{h}$ witnesses that X is countably bounded. (\Leftarrow) If X is countably bounded by $\{a_n\}$, then let \dot{h} be a name such that it is forced that $\dot{g}(a) \leq^* \dot{h}$ for all $n < \omega$. Then \dot{h} bounds $\dot{g}''X$. ■

DEFINITION 6.1. A κ -limit (sometimes also called $\langle \kappa, 1 \rangle$ -gap) is an indexed family $\langle a_\xi, b \rangle_{\xi < \kappa}$ such that $a_\xi <_E a_\eta <_E b$ for all $\xi < \eta < \kappa$, and for all $c \in E$ such that $a_\xi <_E c$ for all $\xi < \kappa$ we have $b \leq_E c$. A gap is an indexed family $\langle a_\xi, b_\eta \rangle_{\xi < \kappa, \eta < \lambda}$ such that:

- (1) $a_\xi <_E a_\eta <_E b_\beta <_E b_\alpha$ for all $\xi < \eta < \kappa$ and all $\alpha < \beta < \lambda$, and
- (2) no $c \in E$ is such that $a_\xi <_E c <_E b_\alpha$ for all $\xi < \kappa$ and all $\alpha < \lambda$.

If a family $\langle a_\xi, b_\eta \rangle_{\xi < \kappa, \eta < \lambda}$ satisfies only condition (1), then we call it a *pregap*; if (2) fails for some c , then we say that c *splits* a pregap. A gap $\langle a_\xi, b_\eta \rangle_{\xi < \kappa, \eta < \lambda}$ is *tight* iff there is no $c \in E$ such that one of the following happens:

- (3) $a_\xi <_E c$ for all ξ but $b_\eta \not<_E c$ for all η ,
- (4) $b_\xi >_E c$ for all η but $a_\xi \not>_E c$ for all ξ .

We always assume that κ and λ are uncountable regular cardinals; this assumption is, in our context of investigating $\langle \kappa, \lambda \rangle$ -gaps, not a loss of generality.

LEMMA 6.2. *Suppose that $\text{supp } h = A$, that $a, b \in E$ are such that the posets $\langle A \cup \{a\}, <_E \rangle$ and $\langle A \cup \{b\}, <_E \rangle$ are isomorphic, and that an isomorphism fixes all elements of A and sends a to b . Then $\Vdash_E \dot{h} \leq^* \dot{g}(\check{a})$ iff $\Vdash_E \dot{h} \leq^* \dot{g}(\check{b})$.*

PROOF. Suppose that some \bar{p} forces that $\dot{h} \leq^* \dot{g}(\check{a})$; we can assume that $\bar{p} \Vdash_E \dot{h} \leq^n \dot{g}(\check{a})$. If $p = \bar{p} \upharpoonright A \cup \{a\}$ then $p \Vdash_{A \cup \{a\}} \dot{h} \leq^n \dot{g}(\check{a})$ by Theorem 4.1. Let p' be a condition isomorphic to p but with a replaced by b . Then $p' \Vdash_{A \cup \{b\}} \dot{h} \leq^n \dot{g}(\check{b})$. ■

THEOREM 6.1. (a) *A \dot{g} -image of a limit $\langle a_\xi, b \rangle_{\xi < \kappa}$ in E is a limit in ${}^\omega \omega$.*
 (b) *A \dot{g} -image of a $\langle \kappa, \omega \rangle$ -gap $\langle a_\xi, b_i \rangle_{\xi < \kappa, i < \omega}$ in E is a gap in ${}^\omega \omega$.*
 (c) *A \dot{g} -image of a gap $\langle a_\xi, b_\eta \rangle_{\xi < \kappa, \eta < \lambda}$ in E is a gap in ${}^\omega \omega$.*
 (d) *A \dot{g} -image of an unbounded chain $\langle a_\xi \rangle_{\xi < \kappa}$ is an unbounded chain in ${}^\omega \omega$.*

PROOF. (a) All we have to prove is that if $\Vdash \dot{h} \not\leq^* \dot{g}(\check{b})$ then there is $\xi < \kappa$ and a condition p such that $p \Vdash \dot{h} \not\leq^* \dot{g}(\check{a}_\xi)$ for every \mathcal{H}_E -name \dot{h} . So assume the contrary, let $A = \text{supp } \dot{h}$. Since for each element c of A either there is $\xi < \kappa$ such that $c \not>_E a_\xi$ or $c >_E b$, we may assume that

- (*) if $a_0 <_E c$ then $b <_E c$ for all $c \in A$.

Also assume that $b \in A$ and that $E = A \cup \{a_\xi : \xi < \kappa\}$ (by Theorem 4.1 we can do this). Go to an extension by \mathcal{H}_A . Let $C = \{n \in \omega : \dot{h}(n) < \dot{g}_A(b_0)(n)\}$. Then C is infinite. We claim that in a further extension by $\mathcal{H}_E(A, \dot{g}_A)$, there are infinitely many $n \in C$ such that $\dot{g}(a_0)(n) = \dot{g}_A(b_0)(n)$. Suppose the contrary, that some q forces that $\dot{g}(a_0)(n) < \dot{g}_A(b_0)(n)$ for all $n \in C \setminus \bar{m}$ for some \bar{m} . We can assume that $b \in F_q$. For all c in $F_p \cap A$ the condition (*) is true, so we can pick $n \geq \bar{m}, n_q$ in C and find $p \leq q$ such that $f_p(a_0)(n) = \dot{g}(b)(n)$; this is a contradiction and it finishes the proof of (a).

(b) Assume \dot{h} is such that $\Vdash \dot{g}(\check{a}_\xi) \leq^* \dot{h}$ for all $\xi < \kappa$. Apply Lemma 5.4 (with $A = \emptyset$) and get q, \dot{h}' such that $q \Vdash \dot{g}(\check{a}_\xi) \leq^* \dot{h}' \leq^* \dot{h}$ for all $\xi < \kappa$ and

$B = \text{supp } \dot{h}'$ is finite. We can assume that $a_0 <_E c$ implies $a_\xi <_E c$ for all $\xi < \kappa$ and all $c \in B$. Go to an extension by \mathcal{H}_B and pick $\bar{m} < \omega$ such that $b_{\bar{m}} \notin F_q \cup B$ and

(**) $c <_E b_{\bar{m}}$ implies $c \not>_E a_0$ for all $c \in B$.

Let $B_0 = \{c \in B : c >_E a_0\}$ and let \dot{h}_0 be defined by $\dot{h}_0(m) = \min\{\dot{g}(c)(m) : c \in B_0\}$. Suppose that $\dot{h}' \not\geq^* \dot{h}_0$, let $C = \{n : \dot{h}'(n) < \dot{h}_0(n)\}$.

CLAIM. *Condition q forces that if \dot{h} splits the pregap then \dot{h}_0 splits the pregap.*

PROOF. We will prove that $q \Vdash \dot{g}(\check{a}_\xi) \leq^* \dot{h}_0 \leq^* \dot{h}'$. Suppose that this fails. The left hand “ \leq^* ” is obvious, so $p \Vdash \dot{h}_0 \not\leq^* \dot{h}'$ for some $p \leq q$ in \mathcal{H}_B . Let $C = \{n \in \omega : \dot{h}'(\check{n}) < \dot{h}_0(\check{n})\}$. We claim that there are infinitely many $m \in C$ such that $\dot{g}(\check{a}_0)(\check{m}) = \dot{h}_0(\check{m})$. Suppose the contrary, so there is a condition $r \leq p$ in $\mathcal{H}_E(B, \dot{g}_B)$ and \bar{n} such that

$$r \Vdash (\forall m > \bar{n})(m \in \dot{C} \rightarrow \dot{g}(\check{a}_0)(\check{m}) < \dot{h}_0(\check{m})).$$

By Theorem 4.1 we can assume that $F_r = \text{supp } \dot{h}_0 \cup \text{supp } \dot{h}' \cup \{a_0\} = B \cup \{a_0\}$. We can assume that $n_p > \bar{n}$ and $n_p \in \dot{C}$. Define a condition $r_1 \in \mathcal{H}_E(B, \dot{g}_B)$ by $r_1 = \langle F_p, n_p + 1, f_r \cup f \rangle$, where $f : F_p \times \{n_p\}$ is defined by

$$f(a)(n_p) = \begin{cases} \min_{c \in B(>_E a)} \dot{g}_B(c)(n_p) & \text{if } B(>_E a) \neq \emptyset, \\ \min_{c \in B_0} \dot{g}_0(c)(n_p) & \text{if } B(>_E a) = \emptyset. \end{cases}$$

Then $r_1 \leq r$ (because $a <_E b$ implies $B(>_E a) \supseteq B(>_E b)$), and $r_1 \Vdash \dot{g}_0(\check{a}_0) = \dot{h}_0(\check{n})$ —a contradiction. So the set $C_1 = \{m \in C : \dot{g}(\check{a}_0)(m) = \dot{h}_0(m)\}$ is forced to be infinite, but $\dot{g}(\check{a}_0)(m) > \dot{h}'(m)$ for all $m \in C_1$, and therefore \dot{h}' does not fill the gap, contrary to our assumptions. ■

But by Lemma 6.1(a) we have $\dot{h}_0 \not\leq^* \dot{g}(b_{\bar{m}})$, so \dot{h}_0 does not fill the pregap, and this proves our claim.

(c) We will first prove the following special case of the theorem:

LEMMA 6.3. *If the gap $\langle a_\xi, b_\eta \rangle_{\xi < \kappa, \eta < \lambda}$ in E is tight, then its \dot{g} -image is a (tight) gap in ${}^\omega\omega$.*

PROOF. Suppose the contrary, that some \mathcal{H}_E -name \dot{h} is forced to fill the pregap. Let $A = \text{supp } \dot{h}$; since no $c \in A$ satisfies (3) or (4) above, by going to end-segments of κ and λ we can assume that each $c \in A$ is either (i) incomparable to all a_ξ, b_η , or (ii) below a_0 , or (iii) above b_0 . Then Lemma 6.2 can be applied to \dot{h} , a_ξ and b_η for all ξ, η , and therefore $\Vdash \dot{h} \geq^* \dot{g}(\check{a}_\xi)$ implies $\Vdash \dot{h} \geq^* \dot{g}(\check{b}_\eta)$ for all ξ, η so \dot{h} cannot fill this gap or even make it nontight. ■

Back to the general case. Let E_0 be the set of all $c \in E$ such that:

- (5) if $c >_E a_\xi$ for all ξ , then $c >_E b_\eta$ for some η , and
- (6) if $c <_E b_\eta$ for all η , then $c <_E a_\xi$ for some ξ .

Then our gap is tight in E_0 , because we are removing all c 's which can make it nontight. Let $E_1 = E \setminus E_0$.

CLAIM. For all $c \in E_1$ at most one of the sets $E_0(<_E c)$ and $E_0(>_E c)$ is nonempty.

PROOF. If $c \in E_1$ fails to satisfy (5) (resp. (6)), so do all elements of $E(>_E c)$ (resp. of $E(<_E c)$). ■

So, by Lemma 4.5, the poset $\mathcal{H}_E(E_0, \dot{g}_0)$ is σ -centered.

LEMMA 6.4. If $\langle a_\xi, b_\eta \rangle_{\xi < \kappa, \eta < \lambda}$ is a gap, then it remains a gap in every forcing extension by a σ -centered poset.

PROOF. By [17, Theorem 4], to every pregap we can associate a poset \mathcal{P} which is σ -centered iff the pregap is split; so our lemma reduces to “If \mathcal{P} is not σ -centered, then it remains so after forcing with a σ -centered poset”, which is obvious. ■

Thus further forcing with $\mathcal{H}_E(E_0, \dot{g}_0)$ will not fill the gap, so the proof is finished.

(d) This follows immediately from Lemma 6.1(b). ■

So we can embed a partially ordered set E into ${}^\omega\omega$ while preserving much of its “nonsaturatedness structure”. In Theorem 9.2 we will prove a partial converse of the above results. The statement “ \mathcal{H}_E forces that there is a $\langle \kappa, \lambda \rangle$ -gap (κ -limit) in ${}^\omega\omega$ iff there is a $\langle \kappa, \lambda \rangle$ -gap (κ -limit) in E ” is false, because of the following classical result of Hausdorff and Rothberger which applies to any of the orderings \leq^* , $<^*$, or $<$:

THEOREM 6.2 ([12], [23]). For every κ there is an unbounded κ -chain in ${}^\omega\omega$ iff there is a $\langle \kappa, \omega \rangle$ -gap in ${}^\omega\omega$. ■

So by Lemma 4.6, if E is uncountable then there is an $\langle \omega_1, \omega \rangle$ -gap in $\langle {}^\omega\omega, \leq^* \rangle$ in an extension by \mathcal{H}_E , and similarly if E has a κ -chain then in an extension by \mathcal{H}_E there is always a κ -limit or a $\langle \kappa, \lambda \rangle$ -gap for some λ . For infinite cardinals κ and λ (not necessarily uncountable) the statement “there is a $\langle \kappa, \lambda^* \rangle$ -gap in ${}^\omega\omega$ ” is not ambiguous, i.e. its truth does not depend on the choice of an ordering; the same is true for unbounded κ -chains.

PROPOSITION 6.1. (a) ([12], [23]) There is a κ -limit in $\langle {}^\omega\omega, < \rangle$ iff there is a $\langle \kappa, \omega \rangle$ -gap in $\langle {}^\omega\omega, < \rangle$.

(b) There are no κ -limits in $\langle {}^\omega\omega, <^* \rangle$.

(c) There is a κ -limit in $\langle {}^\omega\omega, \leq^* \rangle$ iff there is an unbounded chain of length κ in $\langle [\omega]^\omega, \subseteq^* \rangle$.

(d) The existence of a κ -limit in $\langle {}^\omega\omega, \leq^* \rangle$ does not imply the existence of a $\langle \kappa, \omega \rangle$ -gap in ${}^\omega\omega$, and vice versa.

PROOF. (a), (b) See e.g. [24].

(c) (\Rightarrow) Let $\langle a_\xi, b \rangle_{\xi < \kappa}$ be a κ -limit, and define subsets of ω by

$$c_\xi = \{n : a_\xi(n) = b(n)\}.$$

Obviously $\{c_\xi\}$ is a \subseteq^* -increasing sequence in $[\omega]^\omega$, and we claim that its limit is ω : if d is a coinfinite subset of ω such that $c_\xi \subseteq^* d$ for all ξ , then $b' \in {}^\omega\omega$ defined by

$$b'(n) = \begin{cases} b(n) & \text{if } n \in d, \\ b(n) - 1 & \text{if } n \notin d, \end{cases}$$

witnesses that $\langle a_\xi, b \rangle$ is not a κ -limit. (\Leftarrow) Consider the characteristic functions of sets.

(d) By (c), our claim is equivalent to “The existence of an unbounded κ -chain in $[\omega]^\omega$ does not imply the existence of an unbounded κ -chain in ${}^\omega\omega$, and vice versa”, and this is a well-known fact (see [7]). ■

The last lemma of this section (in connection with Proposition 6.1(d)) shows that \mathcal{H}_E sometimes creates “unnecessary” limits in $\langle {}^\omega\omega, \le^* \rangle$.

LEMMA 6.5. *If there is an unbounded κ -chain in E , then in a forcing extension by \mathcal{H}_E there is a κ -limit in $\langle {}^\omega\omega, \le^* \rangle$.*

PROOF. By Proposition 6.1, this reduces to proving that in an extension there is an unbounded κ -chain in $[\omega]^\omega$. Let $\langle a_\xi \rangle_{\xi < \kappa}$ be an unbounded κ -chain in E and define names \dot{c}_ξ for subsets of ω by

$$\dot{c}_\xi = \{n : \dot{g}(a_\xi)(n) \neq 0\}.$$

The proof that $\langle \dot{c}_\xi \rangle_{\xi < \kappa}$ is forced to be an unbounded κ -chain in $[\omega]^\omega$ is similar to the proof of Lemma 6.1. ■

7. Posets with minimal patterns. This chapter is a preparation for the proofs of Theorems 9.1 and 10.1. For definition of $(a, b)_E$ see §0.

THEOREM 7.1. *Let $\langle E, <_E \rangle$ be a poset, $\langle I, <_I \rangle$ a linearly ordered set, and $\{D_\xi\}$ ($\xi \in I$) a family of disjoint finite subsets of E such that for all $\xi < \zeta < \eta$ we have*

(I) $(a, b)_E \cap D_\zeta \neq \emptyset$ whenever $a \in D_\xi$ and $b \in D_\eta$ are $<_E$ -comparable.

Then either (i) there is a chain of type I or I^ in E , or (ii) there are $\xi \neq \eta$ in I such that a and b are $<_E$ -incomparable for all $a \in D_\xi$ and $b \in D_\eta$.*

COROLLARY (Kurepa, [19]). *Every uncountable well-founded poset E having finite levels must have an uncountable chain.*

PROOF. Let $\{D_\xi\}$ ($\xi < \omega_1$) be a decomposition of E into levels. Then $\{D_\xi\}$ satisfy (I) and every $a \in D_\xi$ is comparable with some $b \in D_\eta$, for all $\eta < \xi$. The conditions of Theorem 7.1 are satisfied and there is an ω_1 -chain in E . ■

Proof of Theorem 7.1. Suppose that (ii) fails. We claim that for every k -tuple $\alpha_0 <_I \alpha_1 <_I \dots <_I \alpha_{k-1}$ in I there are distinct $y_j \in D_{\alpha_j}$ ($j < k$) such that either $y_0 \triangleleft y_1 \triangleleft \dots \triangleleft y_{k-1}$ (a chain is *increasing*) or $y_0 \triangleright y_1 \triangleright \dots \triangleright y_{k-1}$ (a chain is *decreasing*). To construct it, pick $y_0 \in D_{\alpha_0}$ and $y_{k-1} \in D_{\alpha_{k-1}}$ such that $y_0 <_E y_{k-1}$ or $y_0 >_E y_{k-1}$ (possible because (ii) fails). Then we can proceed to pick the rest of the chain by using (I).

So for every finite $F \subseteq I$ we fix a chain $C_F = \{y_\xi^F : \xi \in F\}$ such that $y_\xi^F \in D_\xi$ for all $\xi \in F$ and the chain is either strictly increasing or strictly decreasing. For such an F let A_F be the family of all finite subsets of I which include F , and let \mathcal{U} be a uniform ultrafilter on $[I]^{<\omega}$ extending the filter base $\{A_F\}$. Then for all $\xi \in I$ there is $\bar{y}_\xi \in D_\xi$ such that the set

$$\{F : \xi \in F \text{ \& } y_\xi^F = \bar{y}_\xi\}$$

is in \mathcal{U} ; we claim that $\{\bar{y}_\xi\}$ is the desired chain. Assume without loss of generality that for \mathcal{U} many F the chain $\{y_\xi^F\}$ is increasing. We have to prove that all \bar{y}_α are distinct and that $\bar{y}_\alpha \triangleleft \bar{y}_\beta$ whenever $\alpha < \beta$. But the set of all $F \in [I]^{<\omega}$ such that $\alpha, \beta \in F$, $y_\alpha^F = \bar{y}_\alpha$, $y_\beta^F = \bar{y}_\beta$ and the chain $\{y_\xi^F\}$ is increasing is in \mathcal{U} and therefore nonempty, so $\bar{y}_\alpha \triangleleft \bar{y}_\beta$ and they are distinct. ■

Remark. Some assumption like (I) is necessary in Theorem 7.1. To see this let \prec be a linear ordering on ω_2 with no ω_2 - or ω_2^* -chains (e.g. a subordering of $\langle 2^{\omega_1}, <_{\text{Lex}} \rangle$ of appropriate size), and let $E_0 = \langle \omega_2, <_0 \rangle$ and $E_1 = \langle \omega_2, <_1 \rangle$ be defined by

$$\begin{aligned} \alpha <_0 \beta &\text{ iff } \alpha < \beta \text{ and } \alpha \prec \beta, \\ \alpha <_1 \beta &\text{ iff } \alpha < \beta \text{ and } \alpha \succ \beta. \end{aligned}$$

Let E be the disjoint sum of E_0 and E_1 (say, $E_0 \times \{0\} \cup E_1 \times \{1\}$) and let $D_\xi = \{\langle \xi, 0 \rangle, \langle \xi, 1 \rangle\}$ for $\xi < \omega_2$. Then D_ξ and D_η have “vertical and upwards” connections for all ξ and η but there are no ω_2 -chains in E .

In order to make Theorem 7.1 useful in proving our main result we need some definitions. Fix a positive integer n and a poset $\langle E, < \rangle$. If $D = \{x_i^D : i < n\}$ and $F = \{x_i^F : i < n\}$ are two n -tuples of elements in E then let the $<$ -*pattern* τ_{DF} be the isomorphism type of the poset $\langle D \cup F, \leq \rangle$, i.e. the partial ordering $\preceq_{\tau_{DF}}$ on the set $2 \times n$ such that $\langle i, k \rangle \preceq \langle j, l \rangle$ iff $x_k^D \leq x_l^F$ for $k, l < n$ and $i, j < 2$. The sets D and F are *connected* iff there are comparable $a \in D$ and $b \in F$.

We sometimes identify patterns with their graphs and say that a pattern σ_0 is *included* in a pattern σ_1 , denoting this fact by $\sigma_0 \subseteq \sigma_1$. If we consider more than one ordering on the underlying set, then we denote the pattern τ_{DF} corresponding to the ordering \triangleleft by $\tau_{DF}^{\triangleleft}$. A *composition* of two patterns

σ_0 and σ_1 is the pattern $\sigma_0 \circ \sigma_1$ determined by defining $\prec_{\sigma_0 \circ \sigma_1}$ as follows:

$$\begin{aligned} \langle 0, k \rangle \prec_{\sigma_0 \circ \sigma_1} \langle 1, l \rangle & \text{ iff } \langle 0, k \rangle \prec_{\sigma_0} \langle 1, m \rangle \text{ and} \\ & \langle 0, m \rangle \prec_{\sigma_1} \langle 1, l \rangle \text{ for some } m < n, \text{ and} \\ \langle i, k \rangle \prec_{\sigma_0 \circ \sigma_1} \langle i, l \rangle & \text{ iff } \langle i, k \rangle \prec_{\sigma_0} \langle i, l \rangle \text{ for some } i = 0, 1. \end{aligned}$$

In other words, if $\sigma_0 = \tau_{01}^{\triangleleft}$ and $\sigma_1 = \tau_{12}^{\triangleleft}$ then $\sigma_0 \circ \sigma_1$ is the pattern $\tau_{02}^{\triangleleft}$, where “ \triangleleft ” is the transitive closure of “ \triangleleft ” restricted to $D_0 \cup D_1$ and $D_1 \cup D_2$.

We avoid nested indexes so if we have a fixed family $D_\xi = \{x_i^\xi : i < n\}$ ($\xi < \delta$) of finite subsets of a poset $\langle E, < \rangle$ then the pattern $\tau_{D_\xi D_\eta}$ is denoted by $\tau_{\xi\eta}$ instead. Note that in general for every such family and all $\alpha < \beta < \gamma$:

$$(P1) \quad \tau_{\alpha\gamma} \supseteq \tau_{\alpha\beta} \circ \tau_{\beta\gamma}.$$

(P2) An ordering \triangleleft on $\bigcup_{\xi < \delta} D_\xi$ is uniquely determined by its patterns.

We say that the family $\{D_\xi\}$ ($\xi \in I$) has *minimal \triangleleft -patterns* iff $\tau_{\alpha\gamma}^{\triangleleft} = \tau_{\alpha\beta}^{\triangleleft} \circ \tau_{\beta\gamma}^{\triangleleft}$ for all $\alpha <_I \beta <_I \gamma$ in I . So we can reformulate Theorem 7.1:

THEOREM 7.1*. *If $\{D_\xi\}$ is a disjoint family of sets of size n with minimal patterns and all D_ξ, D_η are connected, then there is an I - or an I^* -chain in $\bigcup_{\xi \in I} D_\xi$. ■*

Let us mention a corollary to the proof of Theorem 7.1:

PROPOSITION 7.1. *If $\{D_\xi\}$ is a disjoint family of sets of size n with minimal patterns, then for all $\xi <_I \eta$, either D_ξ and D_η are not connected or there is a $[\xi, \eta]_I$ - or a $[\xi, \eta]^*$ -chain C in $\bigcup_{\xi \in I} D_\xi$ such that $C \cap D_\zeta$ is nonempty for all $\zeta \in [\xi, \eta]_I$. ■*

The next lemma gives sufficient conditions for the existence of an ordering with a given set of patterns.

LEMMA 7.1. *If $\sigma_{\alpha\beta}$ ($\alpha < \beta < \kappa$) is a family of patterns such that $\sigma_{\alpha\beta} \circ \sigma_{\beta\gamma} = \sigma_{\alpha\gamma}$ for all $\alpha < \beta < \gamma$ then there is a unique ordering \triangleleft on $\bigcup_{\xi < \kappa} D_\xi$ such that $\tau_{\alpha\beta}^{\triangleleft} = \sigma_{\alpha\beta}$ for all $\alpha < \beta < \kappa$.*

PROOF. We define the relation \triangleleft in the only possible way, all we have to do is to check that it is transitive; so we pick $a \in D_\alpha, b \in D_\beta$, and $c \in D_\gamma$ such that $a \triangleleft b \triangleleft c$ and prove that $a \triangleleft c$. Essentially the only two interesting cases are: (i) If $\alpha < \beta < \gamma$, then $a \triangleleft c$ by minimality of patterns. (ii) If $\alpha < \gamma < \beta$, then by minimality we can pick $d \in D_\gamma$ such that $a \triangleleft d \triangleleft b$; but then $a \triangleleft c$ because $\sigma_{\alpha\gamma}$ is a pattern. ■

A pattern σ occurs in $\{D_\xi\}$ iff it is equal to some $\tau_{\alpha\beta}$ of this sequence.

LEMMA 7.2. *Let κ be an infinite cardinal. Then every κ -sequence $\{D_\xi\}$ has a subsequence $\{D_{\alpha_\xi}\}$ of the same length such that for every pattern σ that occurs in $\{D_{\alpha_\xi}\}$ there is an infinite increasing ω -sequence $\{\alpha_n\}$ of*

ordinals less than κ such that $\tau_{\alpha_i \alpha_j} = \sigma$ for all $i < j$, and α_0 can be picked to be arbitrarily large.

Proof. If a pattern σ does not satisfy the requirements, pick $\alpha_0 < \kappa$ and $k < \omega$ which witness this and define a partition

$$[\kappa]^2 = K_0 \dot{\cup} K_1$$

by putting $\{\xi, \eta\}$ in K_0 iff $\tau_{\xi\eta} \neq \sigma$. Without loss of generality assume that $\alpha = 0$. Then there are no infinite K_1 -homogeneous sets, so by the partition relation $\kappa \rightarrow (\kappa, \omega)^2$ (see [9, §11]) there is a subfamily of $\{D_\xi\}$ of the size κ in which σ does not occur. We need to repeat this only finitely many times to eliminate all patterns which do not occur often enough. ■

The following lemma plays a crucial role in proving Theorems 9.1, 9.2 and 10.1.

LEMMA 7.3. *If $\langle E, < \rangle$ is a partially ordered set, $\langle I, <_I \rangle$ is a linearly ordered set and D_ξ ($\xi \in I$) is a family of disjoint finite subsets of E , then there is an ordering \triangleleft on E with the following properties:*

- (B1) \triangleleft is coarser than $<$, i.e. $a \triangleleft b$ implies $a < b$.
- (B2) Every $\tau_{\alpha\beta}^{\triangleleft}$ is a composition of finitely many $\tau_{\xi\eta}^<$, i.e.
- (B'2) For all $\alpha < \beta < \kappa$ there is an integer n and ordinals $\alpha = \alpha_0 < \alpha_1 < \dots < \alpha_n = \beta$ such that $\tau_{\alpha\beta}^{\triangleleft} = \tau_{\alpha\alpha_1}^< \circ \tau_{\alpha_1\alpha_2}^< \circ \dots \circ \tau_{\alpha_{n-1}\beta}^<$.
- (B3) D_ξ ($\xi < \kappa$) has minimal \triangleleft -patterns.
- (B4) The orderings \triangleleft and $<$ coincide on every D_ξ .

Proof. For a subset $F = \{\alpha_i : i = 0, \dots, n\}$ of I such that $\alpha_0 <_I \alpha_1 <_I \dots <_I \alpha_n$ let

$$\tau(F) = \tau_{\alpha_0\alpha_1}^< \circ \tau_{\alpha_1\alpha_2}^< \circ \dots \circ \tau_{\alpha_{n-1}\alpha_n}^<.$$

Then by (P1) and the induction one can prove that

$$(*) \quad F \subseteq G \quad \text{implies} \quad \tau(F) \supseteq (G).$$

For $\alpha <_I \beta$ in I let $F_{\alpha\beta}$ be the set $\alpha = \alpha_0 <_I \alpha_1 <_I \dots <_I \alpha_n = \beta$ in I such that the pattern $\tau(F)$ is minimal (with respect to inclusion) among such patterns and let $\tau_{\alpha\beta}^{\triangleleft} = \tau(F_{\alpha\beta})$.

CLAIM. *Patterns in the family $\tau_{\alpha\beta}^{\triangleleft}$ ($\alpha <_I \beta \in I$) are minimal.*

Proof. Fix $\alpha <_I \beta <_I \gamma$. Then $\tau(F_{\alpha\beta}) \subseteq \tau(F_{\alpha\gamma} \cap (\alpha, \beta)_I)$, $\tau(F_{\beta\gamma}) \subseteq \tau(F_{\alpha\gamma} \cap (\beta, \gamma)_I)$, so $\tau(F_{\alpha\beta}) \circ \tau(F_{\beta\gamma}) \subseteq \tau(F_{\alpha\gamma} \cup \{\beta\}) \subseteq \tau(F_{\alpha\gamma})$, and by the choice of $F_{\alpha\gamma}$ we have equality. ■

Therefore the assertion of Lemma 7.3 follows from Lemma 7.1. ■

8. Banach–Mazur game of length ω_1 . Results from this chapter will be used in the proof of Theorem 10.1. We consider 2^{ω_1} as a topological space

with the natural G_δ -topology, where the basic open sets are $[s] = \{x \in 2^{\omega_1} : s \subset x\}$ for $s \in 2^{<\omega_1}$. We will often interchange subsets of ω_1 with their characteristic functions. The poset for adding a *Cohen subset* of ω_1 , \mathcal{C}_{ω_1} , is $2^{<\omega_1}$ ordered by \supseteq .

DEFINITION 8.1 (Banach–Mazur game of length ω_1). Let X be a subset of 2^{ω_1} . The game $\text{BM}(X)$ for two players, I and II, in ω_1 moves is defined as follows: I and II alternately play elements $s_0^I \subset s_0^{II} \subset s_1^I \subset s_1^{II} \subset \dots \subset s_\xi^I \subset s_\xi^{II} \subset \dots$ of $2^{<\omega_1}$, so that in his ξ th move player $*$ plays s_ξ^* which extends the chain of all previously played elements of $2^{<\omega_1}$ ($*$ = I, II). I wins a game iff $\bigcap_{\xi < \omega_1} [s_\xi^I]$ is included in X or if II is the first player to disobey the rules; otherwise II wins.

A *partial play* is a sequence $s_0^I \subset s_0^{II} \subset \dots \subset s_\eta^I \subset s_\eta^{II} \subset \dots \subset s_\xi^I$, $\xi < \omega_1$. (So we consider only partial plays in which II is about to move.) A *strategy for the second player* is a mapping σ from all partial plays into $2^{<\omega_1}$ such that $\sigma(p)$ extends the chain p . Player II *obeys the strategy* σ in a play $\langle s_\xi^I, s_\xi^{II} \rangle_{\xi < \omega_1}$ iff $s_\xi^{II} = \sigma(p_\xi)$ for all ξ , where p_ξ is a partial play ending with the ξ th move of I. A strategy σ is a *winning strategy for II* in $\text{BM}(X)$ iff II wins every game in which he obeys σ . Banach proved the following theorem for the Banach–Mazur game of length ω (see [21]).

THEOREM 8.1. *II has a winning strategy for $X \subseteq 2^{\omega_1}$ iff X is meager.*

PROOF. If X is meager, then there is an ω_1 -sequence F_ξ of nowhere dense subsets of 2^{ω_1} whose union covers X , and it is obvious that II can avoid F_ξ in the ξ th move; this describes the strategy. The nontrivial direction follows from this

CLAIM. *Let σ be a strategy. The set of all outcomes of a Banach–Mazur game in which II uses σ includes a dense G_{\aleph_1} subset of 2^{ω_1} .*

PROOF. The set \mathcal{D}_0^{II} of all $t \in 2^{<\omega_1}$ such that $t = s_0^{II}$ in some valid play (i.e. a play in which both I and II obey the rules) in which II obeys σ is dense in $2^{<\omega_1}$ (because I can enter into any basic $[s]$ in his first move). Pick a maximal antichain (in the \supseteq ordering) \mathcal{A}_0 which is included in \mathcal{D}_0^{II} . Let \mathcal{D}_1^{II} be the set of all $s \in 2^{<\omega_1}$ such that $s = s_1^{II}$ in some valid partial play in which s_0^{II} is in \mathcal{A}_0 . This set is dense in $2^{<\omega_1}$ so let \mathcal{A}_1 be a maximal antichain included in \mathcal{D}_1^{II} . By continuing in this way, we define an ω_1 -sequence $\{\mathcal{A}_\xi\}$ ($\xi < \omega_1$) of maximal antichains such that \mathcal{A}_ξ *refines* \mathcal{A}_η for all $\eta < \xi$ (i.e. for each $s \in \mathcal{A}_\xi$ there is (a unique) $t \in \mathcal{A}_\eta$ such that $t \subset s$). At the successor stages we do the same as above. At the limit stage δ let \mathcal{D}_δ^{II} be the set of all $t \in 2^{<\omega_1}$ such that $t = s_\delta^{II}$ in some valid play in which II obeys σ and in which $s_\xi^{II} \in \mathcal{A}_\xi$ for all $\xi < \delta$. We claim that \mathcal{D}_δ^{II} is dense in $2^{<\omega_1}$: Let \mathcal{A}'_δ be the set of all $t \in 2^{<\omega_1}$ such that $t \upharpoonright \alpha_\xi \in \mathcal{A}_\xi$ for some increasing δ -sequence of ordinals $\{\alpha_\xi\}$ converging to $|t|$. Then \mathcal{A}'_δ is dense and each

element of it corresponds to the first δ moves of a valid play in which II obeys σ , and I is the first one to play after this stage of the game. So I can, by his playing, assure that the set $\mathcal{D}_\delta^{\text{II}}$ is dense as claimed. Let \mathcal{A}_δ be a maximal antichain inside $\mathcal{D}_\delta^{\text{II}}$, so \mathcal{A}_δ refines \mathcal{A}'_δ and all \mathcal{A}_ξ for $\xi < \delta$. This describes the construction of an ω_1 -sequence of antichains $\{\mathcal{A}_\xi\}$. Note that

- (*) if $s \in \mathcal{A}_\eta$, $t \in \mathcal{A}_\xi$ and $s \subset t$, then a partial play witnessing that $s \in \mathcal{A}_\eta$ is an initial segment of a partial play witnessing $t \in \mathcal{A}_\xi$.

Let U_ξ be the family of all $x \in 2^{\omega_1}$ such that $x \upharpoonright \alpha \in \mathcal{A}_\xi$ for some countable ordinal α . Then each U_ξ is a dense open subset of 2^{ω_1} , $G = \bigcap_{\xi < \omega_1} U_\xi$ is a dense G_{\aleph_1} subset of 2^{ω_1} and each element of G is the result of some game in which II obeys σ ; the latter statement follows from (*). ■

This proves the theorem. ■

Fix a large enough θ , say $\theta = (2^{\aleph_1})^+$, and a well-ordering $<_w$ of H_θ which gives rise to Skolem functions. All models that we consider are elementary submodels of H_θ closed under $<_w$, unless otherwise specified. A sequence $\{M_\xi\}$ of countable elementary submodels of H_θ is a *continuous ε -chain* if for all $\alpha < \omega_1$:

- (1) $M_\alpha \prec M_{\alpha+1}$,
- (2) $\langle M_\xi : \xi < \alpha \rangle \in M_{\alpha+1}$, and
- (3) $\bigcup_{\xi < \alpha} M_\xi = M_\alpha$ for α limit.

An elementary submodel M of H_θ is *approachable* iff $M = \bigcup_{\xi < \omega_1} M_\xi$, where $\{M_\xi\}$ is an ε -chain of countable elementary submodels of H_θ . For a function $f : [H_\theta]^{<\omega} \rightarrow H_\theta$ let C_f be the family of all $A \in [H_\theta]^{\aleph_1}$ such that $f''[A]^{<\omega} \subseteq A$ and $\omega_1 \subseteq A$. A family $C \subseteq [H_\theta]^{\aleph_1}$ is *closed unbounded (club)* iff it is equal to C_f for some f . [This is not the standard definition, but by a result of Kueker [15] every “standard” club includes some C_f .] A family $S \subseteq [H_\theta]^{\aleph_1}$ is *stationary* iff it intersects all sets closed unbounded in $[H_\theta]^{\aleph_1}$. Let \mathcal{A} denote the family of all approachable elementary submodels of H_θ . The next lemma shows that \mathcal{A} is a rather large stationary subset of $[H_\theta]^{\aleph_1}$.

LEMMA 8.1. *The union of a \subseteq -chain of approachable models of length ω_1 is approachable.*

PROOF. We denote this chain by $\{M^\alpha\}$ and write $M^\alpha = \bigcup_{\xi < \omega_1} M_\xi^\alpha$. Then by usual bookkeeping we can find ordinals ξ_α for $\alpha < \omega_1$ so that $\{N_{\xi_\alpha}^\alpha\}$ is an ε -chain whose union covers $\bigcup_{\alpha < \omega_1} M^\alpha$. ■

An $x \in 2^{\omega_1}$ is $(M, \mathcal{C}_{\omega_1})$ -*generic* if all its proper initial segments are in M and it is a member of all dense subsets of \mathcal{C}_{ω_1} coded in M . Notice that it is not obvious that there should be an $(M, \mathcal{C}_{\omega_1})$ -generic subset of ω_1 for a given model M of size \aleph_1 (even if we assume that ω_1 is included in M to avoid trivialities).

LEMMA 8.2. *If a model M is approachable, then there is an $(M, \mathcal{C}_{\omega_1})$ -generic subset x of ω_1 .*

PROOF. Let $M = \bigcup_{\xi < \omega_1} M_\xi$. We assume that the sequence is continuous and that $\langle M_\eta \rangle_{\eta \leq \xi}$ is always in $M_{\xi+1}$. We pick an ω_1 -sequence $\{s_\xi\}$ in $2^{<\omega_1}$ so that:

- (1) $s_0 \subset s_1 \subset \dots \subset s_\xi \subset \dots$,
- (2) each s_ξ is generic over M_ξ ; i.e. $s_\xi \in 2^{\delta_\xi}$,
- (3) $[s_\xi]$ avoids all nowhere dense subsets of 2^{ω_1} coded in M_ξ , and
- (4) s_ξ is always a $<_\omega$ -minimal element of $M_{\xi+1}$ which satisfies (1)–(3).

By (4), $\bigcup_{\xi < \delta} s_\xi$ is in $M_{\delta+1}$ for each limit δ , so we can proceed with the construction on limit levels. Then $x = \bigcup_{\xi < \omega_1} s_\xi$ is as required. ■

THEOREM 8.2. *If $X \subseteq 2^{\omega_1}$ is nonmeager, then there is an $(M, \mathcal{C}_{\omega_1})$ -generic $x_M \in X$ for stationary many M in \mathcal{A} .*

PROOF. Suppose the contrary, that there is a club C in $[H_\theta]^{\aleph_1}$ such that there is no $(M, \mathcal{C}_{\omega_1})$ -generic x_M for all $M \in C \cap \mathcal{A}$. Pick $f : [H_\theta]^{<\omega} \rightarrow H_\theta$ such that $C = C_f$. Let Y be the set of all $x \in 2^{\omega_1}$ which are $(M, \mathcal{C}_{\omega_1})$ -generic for some $M \in \mathcal{A} \cap S$; by our assumptions X and Y are disjoint. We will construct a winning strategy σ for II in $\text{BM}(2^{\omega_1} \setminus Y)$, which is therefore a winning strategy for II in $\text{BM}(X)$, thus showing that X is meager (by Theorem 8.1). Along with playing the Banach–Mazur game, II constructs an approachable model $M = \bigcup_{\xi < \omega_1} M_\xi$ in $\mathcal{A} \cap C$ and assures that the outcome x of the game is $(M, \mathcal{C}_{\omega_1})$ -generic. So II plays s_ξ^{II} and M_ξ , so that in each stage α of the game and for all $\xi < \alpha$:

- (1) s_ξ^{II} 's are valid moves in the Banach–Mazur game,
- (2) $\{M_\xi\}$ ($\xi < \alpha$) is a continuous ε -chain,
- (3) each $M_{\xi+1}$ is closed under f and has s_ξ^{II} and $s_{\xi+1}^{\text{I}}$ as elements,
- (4) each s_ξ^{II} is generic over M_ξ ; i.e. $s_\xi^{\text{II}} \in 2^{\delta_\xi}$, and
- (5) $[s_{\xi+1}^{\text{II}}]$ avoids all nowhere dense subsets of 2^{ω_1} coded in M_ξ . [This can be arranged because $M_{\xi+1}$ “knows” that M_ξ is countable.]

If II obeys σ , then after ω_1 many moves of the game he has an approachable model $M = \bigcup_{\xi < \omega_1} M_\xi$ and $x = \bigcup_{\xi < \omega_1} s_\xi^{\text{II}}$ which is $(M, \mathcal{C}_{\omega_1})$ -generic. But M is closed under f and it includes ω_1 , so it is in C , and therefore x is in Y . Since II wins $\text{BM}(X)$, X is meager. ■

LEMMA 8.3. *If S is a stationary subset of \mathcal{A} and x_M is $(M, \mathcal{C}_{\omega_1})$ -generic for all $M \in S$, then the set of all x_M 's is nonmeager.*

PROOF. Suppose the contrary and let F be a code for an F_{\aleph_1} -meager subset of 2^{ω_1} avoiding X . Let $\lambda = (2^\theta)^+$ and let N be an approachable

elementary submodel of H_λ which includes $S, \langle x_M : M \in S \rangle, F, \dots$ and such that $\bar{N} = N \cap H_\theta$ is in S . Then F is in \bar{N} , so $x_{\bar{N}}$ avoids F —a contradiction. ■

9. Proof of the main theorem

THEOREM 9.1. *If $\kappa > \omega_1$ is a regular cardinal then \mathcal{H}_E adds a κ -chain to ${}^\omega\omega$ iff one of the following happens:*

- (†1) E has a κ - or a κ^* -chain, or
- (†2) \mathcal{C} adds a κ -chain to ${}^\omega\omega$.

COROLLARY. *If $\kappa > \mathfrak{c}$ is a regular cardinal, then \mathcal{H}_E adds a κ -chain to ${}^\omega\omega$ iff there is either a κ - or a κ^* -chain in $\langle E, <_E \rangle$. ■*

DEFINITION 9.1. A family $\{A_\xi\}$ forms a *weak Δ -system with root A* iff $A_\xi \cap A_\eta \subseteq A$ for all $\xi \neq \eta$. A family $\{A_\xi\}$ forms a *weak local Δ -system with root A* iff its subfamily $\{A_{\delta+n}\}$ ($n < \omega$) forms a weak Δ -system with root A for all δ .

LEMMA 9.1. *If κ is a regular cardinal larger than \aleph_1 , then every family $\{B_\xi\}_{\xi < \kappa}$ of countable sets has a subfamily $\{A_\xi\}_{\xi < \kappa}$ which forms a weak local Δ -system with a countable root A .*

Proof. Let θ be a large enough cardinal, and let M_ξ ($\xi < \omega_1$) be an ε -chain of countable elementary submodels of H_θ such that the family $\{B_\xi\}$ is an element of M_0 ; let $M = \bigcup_{\xi < \omega_1} M_\xi$. By a counting argument we can find $\bar{\xi} < \omega_1$ such that $B_\eta \cap M = B_\eta \cap M_{\bar{\xi}}$ for κ many η 's; let $A = M_{\bar{\xi}}$. We claim that for every $\alpha < \kappa$ there is a strictly increasing ω -sequence $\{\alpha_n\}$ of ordinals above α which forms a weak Δ -system with root A . It suffices to prove that this is true in M , so note that $A \in M$ and pick $\alpha < \kappa$ in M . Choose δ above $M \cap \kappa$ such that $B_\delta \cap M \subseteq A$. Pick $\alpha_0 > \alpha$ in M such that $A \subseteq B_{\alpha_0}$. Suppose that we have constructed the sequence $\{\alpha_i\}$ ($i < n$) for some integer n . We want to prove that we can continue the construction. We have $B_\delta \cap A_{\alpha_i} \subseteq A$ for all $i < n$, so the family $\{B_{\alpha_i}\}_{i < n} \cup \{B_\delta\}$ is a Δ -system with root A . By elementarity there is an ordinal α_n as required. So our claim is proved, and the lemma follows immediately from it. ■

Proof of Theorem 9.1. Suppose that \dot{f}_ξ ($\xi < \kappa$) is an \mathcal{H}_E -name for a strictly increasing κ -chain in ${}^\omega\omega$. Without loss of generality we can assume that supports of these functions form a weak local Δ -system with a countable root. Now applying Lemma 5.2 to \dot{f}_ξ and $\dot{f}_{\xi+1}$ for all limit ordinals ξ we get \dot{h}_ξ, q_ξ ($\xi < \kappa$) such that $q_\xi \Vdash \dot{f}_\xi \leq^* \dot{h}_\xi \leq^* \dot{f}_{\xi+1}$. Every uncountable family of finite sets includes a Δ -system of the same size; we can assume that supports of \dot{h}_ξ 's form a weak Δ -system with a countable root A and finite tails D_ξ . Since \mathcal{H}_E has κ as a precaliber, without loss of generality we

can assume that the family $\{q_\xi\}$ is centered. Pick a condition \bar{q} which forces that there are κ many q_ξ in \dot{G} .

Now apply Lemma 7.3 to the family $\{D_\xi\}$ and ordering $<_E$ to get a new ordering \triangleleft_0 . If there is a κ - or κ^* -chain in \triangleleft_0 , then by (B1) there is one in $<_E$. Assume that this does not happen; so by Theorem 7.1 there are $\xi < \eta < \kappa$ such that D_ξ and D_η are not connected. Denote the pattern $\tau_{\xi\eta}^{\triangleleft_0}$ by σ . By Lemma 7.2 and further refining we can assume that $\tau_{\alpha,\alpha+m}^{\triangleleft_0} = \sigma$ for all $\alpha < \kappa$ and $m < \omega$, and that $E = A \cup \bigcup_{\xi < \kappa} D_\xi$. Define a relation \triangleleft on E by

$$a \triangleleft b \quad \text{iff} \quad \begin{cases} a, b \in A \cup D_\eta \text{ and } a <_E b \text{ for some } \eta, \text{ or} \\ a, b \in \bigcup_{\xi < \kappa} D_\xi \text{ and } a \triangleleft_0 b, \text{ or} \\ a, b \in \bigcup_{\xi < \kappa} D_\xi \text{ and } a <_E c <_E b \text{ for some } c \in A. \end{cases}$$

So we have

- ($\triangleleft 1$) if $a \in D_\alpha$ and $b \in D_{\alpha+m}$ are \triangleleft -comparable, then there is an element of A which is \triangleleft -between a and b for all $\alpha < \kappa$ and $m < \omega$.

CLAIM 1. *The relation \triangleleft is an ordering on E .*

PROOF. Note first that $a \triangleleft b$ implies $a <_E b$, so we only have to prove that \triangleleft is transitive. Suppose that $a \triangleleft b$ and $b \triangleleft c$; then we have $a <_E c$. Now we consider several cases: (i) If $a \in A$ or $c \in A$ then we are done by $a \triangleleft c$ iff $a <_E c$. (ii) If $b \in A$, then $a \triangleleft c$ by the definition of \triangleleft . (iii) Suppose $a, b, c \in \bigcup_{\xi < \kappa} D_\xi$. If there is $d \in A$ such that $a <_E d <_E b$ or $b <_E d <_E c$, then $a \triangleleft c$. If not, then we have $a \triangleleft_0 b$ and $b \triangleleft_0 c$, so $a \triangleleft_0 c$. This finishes the proof. ■

Note that now

- ($\triangleleft 2$) The relations \triangleleft and $<_E$ coincide on $A \cup D_\eta$ so \dot{h}_η is an $\mathcal{H}_{\langle E, \triangleleft \rangle}$ -name for all η .

CLAIM 2. *The poset $\mathcal{H}_{\langle E, \triangleleft \rangle}$ below $q_\alpha \wedge q_\beta$ forces that $\dot{h}_\alpha \leq^* \dot{h}_\beta$ for all $\alpha < \beta < \kappa$.*

PROOF. Pick $\alpha < \beta$ and suppose that some condition p below $q_\alpha \wedge q_\beta$ forces (in $\mathcal{H}_{\langle E, \triangleleft \rangle}$) that $\dot{h}_\alpha \not\leq^* \dot{h}_\beta$. Find $\alpha = \alpha_0 < \alpha_1 < \dots < \alpha_n = \beta$ as in (B'2) of Lemma 7.3, i.e. such that $\tau_{\alpha_i \alpha_{i+1}}^{\triangleleft_0} = \tau_{\alpha_i \alpha_{i+1}}^{\triangleleft_E}$. Note that this implies that $\tau_{\alpha_i \alpha_{i+1}}^{\triangleleft} = \tau_{\alpha_i \alpha_{i+1}}^{\triangleleft_E}$. Extend p to q to decide $i < n$ such that

$$(1) \quad q \Vdash_{\mathcal{H}_{\langle E, \triangleleft \rangle}} \dot{h}_{\alpha_i} \not\leq^* \dot{h}_{\alpha_{i+1}}.$$

But the condition $q' = q \upharpoonright A \cup D_{\alpha_i} \cup D_{\alpha_{i+1}}$ forces this as well (all involved names are $\mathcal{H}_{\langle A \cup D_{\alpha_i} \cup D_{\alpha_{i+1}}, \triangleleft \rangle}$ -names and q' is the projection of q to this regular subordering). Also $\tau_{\alpha_i \alpha_{i+1}}^{\triangleleft_E} = \tau_{\alpha_i \alpha_{i+1}}^{\triangleleft}$ so the orderings \triangleleft and $<_E$

coincide on $A \cup D_{\alpha_i} \cup D_{\alpha_{i+1}}$, and (1) is true when $\Vdash_{\mathcal{H}_{\langle E, \triangleleft \rangle}}$ is replaced with $\Vdash_{\langle A \cup D_{\alpha_i} \cup D_{\alpha_{i+1}}, <_E \rangle}$ —a contradiction. ■

So the condition \bar{q} still forces that the sequence $\{\dot{h}_\xi\}$ includes a strictly \leq^* -increasing κ -chain in ${}^\omega\omega$. We go to an extension by \mathcal{H}_A below \bar{q} (we can assume that \bar{q} is in \mathcal{H}_A); let \dot{g}_A be the generic embedding of A into ${}^\omega\omega$.

CLAIM 3. *The poset $\mathcal{H}_{A \cup D_\alpha \cup D_{\alpha+m}}(A, \dot{g}_A)$ is equivalent to the product of $\mathcal{H}_{A \cup D_\alpha}(A, \dot{g}_A)$ and $\mathcal{H}_{A \cup D_{\alpha+m}}(A, \dot{g}_A)$ for all $\alpha < \kappa$ and all $m < \omega$.*

PROOF. This follows immediately from ($\triangleleft 1$) and Lemma 4.3. ■

So by Lemma 5.1, we can find an \mathcal{H}_A -name \dot{h}'_ξ and an \mathcal{H}_E -condition $q'_\xi \leq q_\xi \wedge q_{\xi+1}$ for all limit $\xi < \kappa$ such that

$$q'_\xi \Vdash \dot{h}_\xi \leq^* \dot{h}'_\xi \leq^* \dot{h}_{\xi+1}.$$

Without loss of generality the family $\{q'_\xi\}$ is centered and if q' is a condition which forces that there are κ many q'_ξ 's in the generic filter, then it also forces that the family $\{\dot{h}'_\xi\}$ includes a κ -chain which is cofinal in $\{\dot{f}_\xi\}$. So the countable poset \mathcal{H}_A adds a κ -chain, and this finishes the proof. ■

In the following theorem κ and λ are uncountable regular cardinals.

THEOREM 9.2. *If $\kappa > \mathfrak{c}$ and \mathcal{H}_E adds a $\langle \kappa, \lambda \rangle$ -gap to ${}^\omega\omega$, then there is such a gap in E or in E^* .*

PROOF. Let $\langle \dot{f}_{0,\xi}, \dot{f}_{1,\eta} \rangle_{\xi < \kappa, \eta < \lambda}$ be an \mathcal{H}_E -name for a gap. Working as in the proof of Theorem 9.1, we find a countable A and finite $D_{0,\xi}, D_{1,\eta}$ such that $\text{supp } \dot{f}_{i,\xi} \subseteq A \cup D_{i,\xi}$ for all i, ξ . Apply Lemma 7.3 to the family $\{D_{0,\xi}, D_{1,\eta} : \xi < \kappa, \eta < \lambda\}$, where the index set $I = \{0\} \times \kappa \cup \{1\} \times \lambda$ is ordered lexicographically, to get an ordering \triangleleft_0 on $\bigcup_{\langle i,\xi \rangle \in I} D_{i,\xi}$ such that this family has minimal patterns. Let $D^0 = \bigcup_{\xi < \kappa} D_{0,\xi}$ and $D^1 = \bigcup_{\eta < \lambda} D_{1,\eta}$, define \triangleleft on $A \cup D^0 \cup D^1$ and translate $\mathcal{H}_{\langle E, <_E \rangle}$ -names into $\mathcal{H}_{\langle E, \triangleleft \rangle}$ -names as in the proof of Theorem 9.1. In the following considerations “connected” means connected in the ordering \triangleleft_0 , while E stands for the poset $\langle E, \triangleleft \rangle$.

CASE 1: *For all $\xi < \kappa$ and $\eta < \lambda$, $D_{0,\xi}$ and $D_{1,\eta}$ are not connected.* Then \mathcal{H}_E is equivalent to $\mathcal{H}_A * (\mathcal{H}_{A \cup D^0}(A, \dot{g}_A) \times \mathcal{H}_{A \cup D^1}(A, \dot{g}_A))$ by Lemma 4.3. Work in an extension by \mathcal{H}_A : Since A is countable, every real in an extension by $\mathcal{H}_{A \cup D^i}(A, \dot{g}_A)$ ($i = 0, 1$) is added by a countable subordering, so by Lemma 5.5 (applied to $\mathcal{P}^i = \mathcal{H}_{A \cup D^i}(A, \dot{g}_A)$ and $\dot{f}_\xi^0 = \dot{f}_{0,\xi}$ for $i = 0, 1$), there is a ground-model function h which splits this gap.

CASE 2: *$D_{0,\xi}$ and $D_{1,\eta}$ are connected for some ξ, η (without loss assume that $\xi = \eta = 0$).* Then by Proposition 7.1 there is a $\kappa + \lambda^*$ -chain in E or in E^* which intersects all $D_{i,\xi}$'s. If it is a gap in the $<_E$ -ordering, then we are done. So suppose that it is filled by some c_0 , and by symmetry we can

assume that it is a $\kappa + \lambda^*$ -chain in E . Also, all $D_{0,\xi}$'s are of the same fixed size n_0 and all $D_{1,\eta}$'s are of the same fixed size n_1 . We can assume that c_0 is in A . Now our argument splits into a finite binary tree. Let

$$\begin{aligned} D_{0,\xi}^0 &= D_{0,\xi} \setminus D_{0,\xi}(\triangleleft c_0), & D_{1,\eta}^0 &= D_{1,\eta}, \\ D_{0,\xi}^1 &= D_{0,\xi}, & D_{1,\eta}^1 &= D_{1,\eta} \setminus D_{1,\eta}(\triangleright c_0). \end{aligned}$$

Then $|D_{0,\xi}^0| < n_0$ for all ξ , and $|D_{1,\eta}^1| < n_1$ for all η .

If $D_{0,\xi}^0$ and $D_{1,\eta}^0$ are connected for some $\xi < \kappa$ and $\eta < \lambda$, then there is a $\kappa + \lambda^*$ -chain in E or in E^* . The obtained $\kappa + \lambda^*$ -chain is without loss of generality filled by some $c_{00} \in A$, and we can define $D_{0,\xi}^{0i}, D_{1,\eta}^{0i}$ for $i = 0, 1$ as above. Observe that for all $s \in 2^{<\omega}$ and $i = 0, 1$ we have

$$(2) \quad |D_{0,\xi}^{s^i}| + |D_{0,\eta}^{s^i}| < |D_{0,\xi}^s| + |D_{0,\eta}^s| \quad \text{for all } \xi, \eta.$$

Proceeding in this way, we either at some stage obtain a $\langle \kappa, \lambda \rangle$ - or a $\langle \lambda, \kappa \rangle$ -gap in E , or construct a finite (by (2) above) binary subtree T of $2^{<\omega}$ and c_s ($s \in T$) such that

$$(3) \quad \text{for all } a \in D_{0,\xi} \text{ and } b \in D_{1,\eta} \text{ either } a \text{ and } b \text{ are not } \triangleleft\text{-connected or some } c_s \text{ is between them.}$$

[To check (3), let s be the maximal node in T such that $a \in D_{0,\xi}^s$ and $b \in D_{1,\eta}^s$. If a and b are \triangleleft -comparable, then c_{0s} is between them.] By Lemma 4.3, \mathcal{H}_E is equivalent to $\mathcal{H}_{A^*}(\mathcal{H}_{A \cup D^0}(A, \dot{g}_A) \times \mathcal{H}_{A \cup D^1}(A, \dot{g}_A))$, and by Lemma 5.5 there is a condition q in $\mathcal{H}_{\langle E, \triangleleft \rangle}$ which forces that h fills the pregap, i.e.

$$(4) \quad q \Vdash_{\mathcal{H}_{\langle E, \triangleleft \rangle}} \dot{f}_{0,\xi} \leq^* \check{h} \quad \text{and} \quad q \Vdash_{\mathcal{H}_{\langle E, \triangleleft \rangle}} \check{h} \leq^* \dot{f}_{1,\eta} \quad \text{for all } \xi, \eta.$$

What we have to prove is that for some q' ,

$$(5) \quad q' \Vdash_{\mathcal{H}_{\langle E, <_E \rangle}} \dot{f}_{0,\xi} \leq^* \check{h} \quad \text{and} \quad q' \Vdash_{\mathcal{H}_{\langle E, <_E \rangle}} \check{h} \leq^* \dot{f}_{1,\eta} \quad \text{for all } \xi, \eta.$$

By passing to a cofinal subset of κ and λ , we can assume that F_q is included in A' . Since the orderings $<_E$ and \triangleleft coincide on each $A' \cup D_{0,\xi}$ and each $A' \cup D_{1,\eta}$, by Theorem 4.1 formulas (4) and (5) are equivalent. This is a contradiction, so one of the $\kappa + \lambda^*$ -chains obtained in E or E^* during the construction of the tree T was a gap. ■

10. Embedding a dense linearly ordered set by \mathcal{H}_E . By \bar{X} we denote the topological closure of a set X .

THEOREM 10.1. *If \mathcal{H}_E forces that $\langle 2^{\omega_1}, <_{\text{Lex}} \rangle^V$ embeds into $\langle {}^\omega\omega, \leq^* \rangle$, then either a Cohen real forces this or $\langle 2^{\omega_1}, <_{\text{Lex}} \rangle^V$ embeds into E .*

We will prove a stronger result:

THEOREM 10.2. *If X is a nonmeager subset of 2^{ω_1} such that in an extension by \mathcal{H}_E there is an embedding of $\langle X, <_{\text{Lex}} \rangle$ into ${}^\omega\omega$ then there is $s \in 2^{<\omega_1}$ such that one of the following happens:*

- (†1) $\langle X \cap [s], \langle_{\text{Lex}} \rangle$ is embeddable into E or E^* , or
 (†2) $\langle X \cap [s], \langle_{\text{Lex}} \rangle$ is embeddable into ${}^\omega\omega$ after adding a single Cohen real.

Proof of Theorem 10.1. Since $\langle [s], \langle_{\text{Lex}} \rangle$ is isomorphic to $\langle 2^{\omega_1}, \langle_{\text{Lex}} \rangle$, the theorem follows immediately from Theorem 10.2. ■

DEFINITION 10.1. If $\langle L, \langle_L \rangle$ is a linearly ordered set then we say that embeddings $H_0 : \langle L, \langle_L \rangle \rightarrow \langle {}^\omega\omega, \leq^* \rangle$ and $H_1 : \langle L, \langle_L \rangle \rightarrow \langle {}^\omega\omega, \leq^* \rangle$ cohere iff $H_i(x) \leq^* H_j(y)$ for all $x \langle_L y$ in L and $i, j \in \{0, 1\}$.

LEMMA 10.1. For every nonmeager subset X of 2^{ω_1} and a family A_x ($x \in X$) of countable sets there is a countable A such that the set of all $x \in X$ for which the set $\{y \in X : A_y \cap A_x \subseteq A\}$ accumulates to x is nonmeager.

Proof. Let $\theta = (2^{\aleph_1})^+$ and let S be the set of all approachable elementary submodels N of H_θ such that $X \in N$ and there is an $(N, \mathcal{C}_{\omega_1})$ -generic $x_N \in X$. By Theorem 8.2 the set S is stationary. For $N \in S$ fix a continuous ε -chain of countable models converging to N , say $N = \bigcup_{\xi < \omega_1} M_\xi^N$, and let $M_N \prec H_\theta$ be an M_ξ^N such that $A_{x_N} \cap M_N = A_{x_N} \cap N$. By the Pressing Down Lemma there is a stationary $S' \subseteq S$ and a countable $\bar{M} \prec H_\theta$ such that $\bar{M} = M_N$ for all $N \in S'$. By Lemma 8.3 the set $Y = \{x_N : N \in S'\}$ is nonmeager.

CLAIM. For all $x \in Y$ the set $\{y \in X : A_y \cap A_x \subseteq \bar{M}\}$ accumulates to x .

Proof. Pick $x = x_N$ in Y and a countable ordinal α . Then $x \upharpoonright \alpha$ is in N so by elementarity there is a $y \in X \cap [x \upharpoonright \alpha]$ in N such that $A_y \cap A_x \subseteq \bar{M}$. For a fixed x this is true for all countable α , so the claim is proved. ■

This proves the lemma. ■

Proof of Theorem 10.2. Let \dot{f}_x ($x \in X$) be a name for the embedding and $A_x = \text{supp } \dot{f}_x$. Then by Lemma 10.1 there is a countable set $A \subseteq E$ and a nonmeager $X_1 \subseteq X$ such that for each $x \in X_1$ there is an ω_1 -sequence $\{y_\xi\}$ in X converging to x and such that $A_\xi \cap A_x \subseteq A$ for all $\xi < \omega_1$. By passing to an interval $[t]$ of $2^{<\omega_1}$, we can assume that X_1 is dense in $2^{<\omega_1}$.

The idea of the proof is to find an embedding $x \mapsto \dot{h}_x$ which coheres with $x \mapsto \dot{f}_x$ and which is such that \dot{h}_x is an \mathcal{H}_A -name for all $x \in X$ by applying Lemma 5.4 for all $x \in X$. Unfortunately, the name \dot{h}_x this lemma gives does not work with probability one. This is why we will have to go to an extension by \mathcal{H}_E and work there. In this extension set X is not a nonmeager set anymore, but it still retains a largeness property which is sufficient for our needs. Now we go to an extension by \mathcal{H}_E , and let \dot{G} be a generic filter in \mathcal{H}_E . In our following claim all \mathcal{H}_E -names are assumed to be elements of a ground model.

CLAIM 1. *There is an embedding $x \mapsto \dot{h}_x$ of X into ${}^\omega\omega$ which coheres with $x \mapsto \dot{f}_x$ and is such that $\text{supp } \dot{h}_x \subseteq^* A$ for all $x \in X$.*

PROOF. Fix $x \in X_1$ and a sequence $\{y_\xi\}$ which converges to x . We can assume that this sequence is monotonic, say increasing. Note that the proof of Lemma 5.4 (applied in the ground model) shows that the set of conditions q such that there is a name \dot{h} with $\text{supp } \dot{h} \subseteq^* A$ and

$$q \Vdash \dot{f}_{y_\xi} \leq^* \dot{h} \leq^* \dot{f}_x \quad \text{for all } \xi < \omega_1$$

is dense in \mathcal{H}_E ; therefore, in our extension there are $q_x \in \dot{G}$ and \dot{h}_x with these properties. So we have an embedding $x \mapsto \dot{h}_x$ of X_1 into ${}^\omega\omega$ which coheres with $x \mapsto \dot{f}_x$, and we want to extend it to an embedding of X . Let

$$(D) \quad D_x = (\text{supp } \dot{h}_x \cup F_{q_x}) \setminus A \quad \text{for } x \in X_1.$$

For each $x \in X \setminus X_1$ we pick a monotonic ω_1 -sequence $\{y_\xi\}$ inside X_1 converging to x ; then by applying the Δ -system lemma to refine $\{D_{y_\xi}\}$ ($\xi < \omega_1$) we get a finite set Δ such that $D_\xi \cap A_x \subseteq \Delta$ for all ξ . So we can again apply an extension of Lemma 5.4 and get $q_x \in \dot{G}$, and \dot{h}_x with $\text{supp } \dot{h}_x \subseteq^* A$ for all $x \in X$ as required. ■

Let D_x be defined as in (D) above for all $x \in X$. Recall that a poset for adding a Cohen subset of ω_1 , \mathcal{C}_{ω_1} , is $(2^{<\omega_1}, \subseteq)$. It is σ -closed, and after forcing with any ccc poset the intersection of countably many dense subsets of \mathcal{C}_{ω_1} (as computed in the ground model) is dense (by Easton [8]). So $(\mathcal{C}_{\omega_1})^V$ has this property in our extension by \mathcal{H}_E . Recall that by our assumptions $[t] \cap X$ is nonempty for all $t \in (\mathcal{C}_{\omega_1})^V$.

CLAIM 2. *The set of all t such that for some finite $F \subseteq E$,*

$$(*) \quad \bigcap_{x \in I} D_x = F \quad \text{for every nonempty interval } I \text{ of } X \cap [t]$$

is dense in $(\mathcal{C}_{\omega_1})^V$.

PROOF. Let $\mathcal{D}_n = \{s \in (\mathcal{C}_{\omega_1})^V : |\bigcap_{x \in X \cap [s]} D_x| \geq n\}$. Suppose that below some $s \in (\mathcal{C}_{\omega_1})^V$ all \mathcal{D}_n 's are dense; then we can pick $u \in \bigcap_{n=1}^\infty \mathcal{D}_n$. If $x \in [u] \cap X$ then D_x is infinite—a contradiction. Therefore the set of all s such that $[s] \cap \mathcal{D}_n$ is empty for some n is dense in $(\mathcal{C}_{\omega_1})^V$. For such an s pick maximal $m < n$ so that some t extending s is in \mathcal{D}_m , and let $F = \bigcap_{x \in [t]} D_x$; such t and F satisfy (*). ■

By Claim 2 there is a maximal antichain $\mathcal{D}_0 \subseteq (\mathcal{C}_{\omega_1})^V$ and a finite $F_t \subseteq E$ for $t \in \mathcal{D}_0$ such that (*) is true for all t and F_t in \mathcal{D}_0 . Pick $s \in \mathcal{D}_0$ and let $Y = X \cap [s]$, $A' = A \cup F_s$ and $D'_x = D_x \setminus F_s$. Then by (*) we have:

(**) For all $x <_{\text{Lex}} y$ in Y and all $a \in D'_x \cap D'_y$ there is $z \in (x, y)_Y$ such that $a \notin D_z$.

Let $E^+ = A' \cup \bigcup_{x \in Y} \{x\} \times D'_x$ and let $H : E^+ \rightarrow E$ be

$$H(a) = \begin{cases} a & \text{if } a \in E, \\ b & \text{if } a = \langle x, b \rangle \text{ for some } x \in Y. \end{cases}$$

Define an ordering $<_{E^+}$ on E^+ by

$$a \lesssim_{E^+} b \quad \text{iff} \quad H(a) \lesssim_E H(b).$$

Let $D_x^+ = \{x\} \times D'_x$. Then the posets $\langle A' \cup D_x^+, <_{E^+} \rangle$ and $\langle A' \cup D'_x, <_E \rangle$ are isomorphic, so let \dot{h}_x^+ be an $\mathcal{H}_{\langle A' \cup D_x^+, <_{E^+} \rangle}$ -name isomorphic to \dot{h}_x . [Namely, we obtain \dot{h}_x^+ from \dot{h}_x by replacing each occurrence of b with $\langle x, b \rangle$ for all $b \in D'_x$.] Let q_x^+ be the \mathcal{H}_{E^+} -condition isomorphic to q_x . Since all q_x 's are in the same generic filter, the conditions q_x^+ are pairwise compatible.

CLAIM 3. *In the poset $\mathcal{H}_{E^+}(A', \dot{g}_{A'})$ the condition $q_x^+ \wedge q_y^+$ forces that $\dot{h}_x^+ \leq^* \dot{h}_y^+$.*

PROOF. The idea is similar to that in the proof of Claim 2 in Theorem 9.1, but this time we have to construct yet another poset. By $(**)$ pick $x = x_0 <_{\text{Lex}} x_1 <_{\text{Lex}} \dots <_{\text{Lex}} x_n = y$ in Y so that for each $a \in D'_x$ we have $a \notin D'_{x_{i(a)}}$ for some $0 < i(a) \leq n$. For $a \in \bigcup_{1 \leq i \leq n} D'_{x_i} \setminus D'_x$ let $i(a) = 0$. Then define a poset E^{++} by

$$D_i^{++} = \{a : a \in D'_x, i(a) < i\} \cup \{\langle 1, a \rangle : a \in D'_{x_i}, i(a) > i\} \quad \text{for } i \leq n, \text{ and}$$

$$E^{++} = A' \cup \bigcup_{i \leq n} D_i^{++}.$$

The ordering on E^{++} is defined similarly to that on E^+ : let $H : E^{++} \rightarrow E$ be

$$H(a) = \begin{cases} a & \text{if } a \in E, \\ b & \text{if } a = \langle 1, b \rangle \text{ for some } b \in E, \end{cases}$$

and let $a \lesssim_{E^{++}} b$ iff $H(a) \lesssim_E H(b)$. Then:

(1) the posets $\langle A' \cup D_{x_i}^{++}, <_{E^{++}} \rangle$ and $\langle A \cup D'_{x_i}, <_E \rangle$ are isomorphic for all $i \leq n$.

An isomorphism naturally extends to one between $\mathcal{H}_{A \cup D'_{x_i}}$ and $\mathcal{H}_{A \cup D_{x_i}^{++}}$, as well as to an isomorphism between classes of names in these posets. Let q_i^+ be the isomorphic image of q_{x_i} for $i \leq n$. Then we have:

(2) $\langle A \cup D_i^{++} \cup D_{i+1}^{++}, <_{E^{++}} \rangle$ is isomorphic to $\langle A' \cup D'_{x_i} \cup D'_{x_{i+1}}, <_E \rangle$ for all i , and the natural extension of this isomorphism sends q_i^{++}, q_{i+1}^{++} to $q_{x_i}, q_{x_{i+1}}$ respectively and $\dot{h}_i^{++}, \dot{h}_{i+1}^{++}$ to $\dot{h}_{x_i}, \dot{h}_{x_{i+1}}$ respectively.

(3) $\langle A' \cup D_0^{++} \cup D_n^{++}, <_{E^{++}} \rangle$ is isomorphic to $\langle A' \cup D_0^+ \cup D_n^+, <_{E^+} \rangle$, and the natural extension of this isomorphism sends q_0^{++}, q_n^{++} to q_0^+, q_n^+ respectively and $\dot{h}_0^{++}, \dot{h}_n^{++}$ to \dot{h}_0^+, \dot{h}_n^+ respectively.

So $\mathcal{H}_{E^{++}}$ below $\bigwedge_{i \leq n} q_i^{++}$ forces that $\dot{h}_i^{++} \leq^+ \dot{h}_{i+1}^{++}$ for all $i < n$, and hence that $\dot{h}_0^{++} \leq^+ \dot{h}_n^{++}$, therefore $\mathcal{H}_{E^+}(A', \dot{g}_{A'})$ below the condition $\bigwedge_{i \leq n} q_i^+ \upharpoonright (A^+ \cup D_0^+ \cup D_n^+)$ forces that $\dot{h}_0^+ \leq^+ \dot{h}_n^+$. This condition is equivalent to $q_0^+ \wedge q_1^+$. ■

Now apply Lemma 7.3 to $\langle \bigcup_{x \in Y} D_x^+, <_{E^+} \rangle$ to get an ordering \triangleleft_0 on this set. As in the proof of Theorem 9.1, define an ordering \triangleleft on E^+ by

$$a \triangleleft b \quad \text{iff} \quad \begin{cases} a, b \in A' \cup D_\eta^+ \text{ and } a <_{E^+} b \text{ for some } \eta, \text{ or} \\ a, b \in \bigcup_{\xi < \kappa} D_\xi^+ \text{ and } a \triangleleft_0 b, \text{ or} \\ a, b \in \bigcup_{\xi < \kappa} D_\xi^+ \text{ and } a <_{E^+} c <_{E^+} b \text{ for some } c \in A'. \end{cases}$$

The proof that this relation is an ordering is the same as in the proof of Theorem 9.1, as well as the proof that $x \mapsto \dot{h}_x^+$ is forced to be an embedding of Y into ${}^\omega\omega$ coherent with $x \mapsto \dot{f}_x$.

Case 1: *There are $x \neq y$ in Y such that D_x^+ and D_y^+ are connected by \triangleleft .* Then by Proposition 7.1 there is a chain of type $[x, y]_Y$ or $[x, y]_Y^*$ in E^+ and therefore in E , so $(\dagger 1)$ applies with any s such that $[s] \subseteq [x, y]_Y$ (recall that $Y = X \cap [s]$, so $[x, y]_Y = [x, y]_X$).

Case 2: *D_x^+ and D_y^+ are not connected by \triangleleft for all $x \neq y$.* Note that the embedding $x \mapsto \dot{h}'_x$ obtained by Claim 1 is such that it is added only by \mathcal{H}_E (although the image of X consists only of reals added by \mathcal{H}_A). Claim 4 below is stronger because the name for the embedding $x \mapsto \dot{h}''_x$ obtained in it is an $\mathcal{H}_{A'}$ -name.

CLAIM 4. *There are $\mathcal{H}_{A'}$ -names \dot{h}''_x for $x \in Y$ such that $x \mapsto \dot{h}''_x$ is an embedding coherent with $x \mapsto \dot{h}_x$.*

PROOF. By the assumption and Lemma 4.3 the poset \mathcal{H}_{E^+} can be written as

$$\mathcal{H}_{A'} * \prod_{x \in Y} \mathcal{H}_{D_x^+}(A', \dot{g}_{A'}) \quad (\text{product is taken with finite supports}).$$

Work in an extension by \mathcal{H}_{E^+} . For $x \in Y$ pick ω_1 -sequences $\{y_\xi\}$ and $\{z_\xi\}$ in Y converging to x from below (resp. above). The pregap $\langle \dot{h}_{y_\xi}^+, \dot{h}_{z_\xi}^+ \rangle_{\xi < \omega_1}$ is filled by \dot{h}_x^+ . By applying Lemma 5.1 as in the proof of Theorem 9.1 we find a pregap which is cofinal in this one and which is added by $\mathcal{H}_{A'}$. The poset $\mathcal{H}_{A' \cup D_x^+}(A', \dot{g}_{A'})$ is countable and it fills an $\langle \omega_1, \omega_1 \rangle$ -pregap, therefore a pregap is already filled in the intermediate extension by $\mathcal{H}_{A'}$; let \dot{h}''_x be the $\mathcal{H}_{A'}$ -name for the function which fills it. Then the mapping $x \mapsto \dot{h}''_x$ is as required. ■

So in Case 2 a countable poset embeds $\langle Y, <_{\text{Lex}} \rangle$ into ${}^\omega\omega$, so $(\dagger 2)$ applies. ■

11. A preordering on ${}^\omega\omega$. By a Cohen poset \mathcal{C} we mean $\langle 2^{<\omega}, \supseteq \rangle$ and \dot{c} is a name for a \mathcal{C} -generic real. Let \mathcal{N} denote the ideal of nowhere dense subsets of \mathcal{C} , i.e. all sets $A \subseteq \mathcal{C}$ such that the complement $\mathcal{C} \setminus A$ of A includes an open dense set in \mathcal{C} .

DEFINITION 11.1. For $f, g \in {}^\omega\omega$ we define:

- (1) $f =_{\mathcal{N}} g$ iff the set $\{t \in \mathcal{C} : f(t) \neq g(t)\}$ is in \mathcal{N} .
- (2) $f \leq_{\mathcal{N}} g$ iff the set $\{t \in \mathcal{C} : f(t) > g(t)\}$ is in \mathcal{N} .
- (3) $f <_{\mathcal{N}} g$ iff the set $\{t \in \mathcal{C} : f(t) \geq g(t)\}$ is in \mathcal{N} .
- (4) $f =_{\mathcal{N}} g$ iff $f \leq_{\mathcal{N}} g$ and $g \leq_{\mathcal{N}} f$.

[Note that an analog of Proposition 0.1 is true in the case of these orderings.]

THEOREM 11.1. A linearly ordered set $\langle L, <_L \rangle$ embeds into $\langle {}^\omega\omega, <_{\mathcal{N}} \rangle$ iff in a forcing extension by the Cohen algebra, $\langle L, <_L \rangle$ embeds into ${}^\omega\omega$.

For $f \in {}^\omega\omega$ and $r \in 2^\omega$ we define $f \upharpoonright r \in {}^\omega\omega$ by $(f \upharpoonright r)(n) = f(r \upharpoonright n)$. So in particular $f \upharpoonright \dot{c}$ is a Cohen name for an element of ${}^\omega\omega$.

LEMMA 11.1. $f <_{\mathcal{N}} g$ iff $\Vdash_{\mathcal{C}} (\check{f} \upharpoonright \dot{c}) <^* (\check{g} \upharpoonright \dot{c})$.

PROOF. Observe that $t \in \mathcal{C}$ forces that $(f \upharpoonright \dot{c}) <^n (g \upharpoonright \dot{c})$ for some $n \leq |t|$ iff $f(s) < g(s)$ for every $s \in [t]$. ■

Note that for fixed f the function $r \mapsto f \upharpoonright r$ is continuous and that moreover (recall that the metric on 2^ω is defined by $d(r, s) = 1/(\Delta(r, s) + 1)$, where $\Delta(r, s)$ is the minimal n such that $r(n) \neq s(n)$)

$$d(s, r) \leq d(f \upharpoonright r, f \upharpoonright s) \quad \text{for all } r, s \in {}^\omega\omega,$$

i.e. the function $r \mapsto f \upharpoonright r$ is *Lipschitz*. It is well known that for every \mathcal{C} -name \dot{x} for a real there is a ground-model Borel function F such that $F(\dot{c}) = \dot{x}$ with probability one (see e.g. [29, Theorem 2.3]). So Theorem 11.1 above says, among other things, that when investigating \mathcal{C} -names for long chains in ${}^\omega\omega$ we can restrict ourselves to those consisting of Lipschitz functions.

For a \mathcal{C} -name for a function \dot{f} in ${}^\omega\omega$ define $\widehat{f} \in {}^\omega\omega$ by

$$\widehat{f}(t) = \min\{k : \text{some extension of } t \text{ forces that } f \upharpoonright [t] = k\}.$$

The following is an “inverse” of Lemma 11.1.

- LEMMA 11.2. (a) If $\Vdash_{\mathcal{C}} \dot{f} <^* \dot{g}$, then $\widehat{f} <_{\mathcal{N}} \widehat{g}$.
 (b) If $\Vdash_{\mathcal{C}} \dot{f} \leq^* \dot{g}$ and $\dot{f} \neq^* \dot{g}$ then $\widehat{f} \leq_{\mathcal{N}} \widehat{g}$ and $\widehat{f} \neq_{\mathcal{N}} \widehat{g}$.
 (c) For $f \in {}^\omega\omega$ and a \mathcal{C} -name \dot{g} , if $\Vdash_{\mathcal{C}} (\check{f} \upharpoonright \dot{c}) <^* \dot{g}$, then $f <_{\mathcal{N}} \widehat{g}$.

PROOF. (a) Let $\{t_i\}$ be the maximal antichain in \mathcal{C} which decides \overline{m} from which the dominance happens, i.e. there is a sequence $\{m_i\}$ such that

$$t_i \Vdash \dot{f} <^{\overline{m}_i} \dot{g}$$

for all i . We can assume that $m_i < |t_i|$ for all i . We claim that $\widehat{f}(t) < \widehat{g}(t)$ for all $t \in U = \bigcup_{i < \omega} [t_i]$: suppose otherwise, pick t such that $\widehat{f}(t) = k_1 > k_0 = \widehat{g}(t)$ and $t \leq_C t_{\bar{l}}$ for some $l \in \omega$. Let $n = |t|$ and $s \leq_C t$ be such that $s \Vdash \dot{g}(\check{n}) = \check{k}_0$; find $s' \leq_C s$ which decides $\dot{f}(n)$, say $s' \Vdash \dot{f}(\check{n}) = \check{k}$. But $n > m_{\bar{l}}$ and $s' \leq_C t_{\bar{l}}$, so $k < k_0$. On the other hand, by the definition of $\widehat{f}(n)$ we have $k_1 \leq k$ —a contradiction. So we have a dense open subset U of \mathcal{C} which witnesses that $f \leq_{\mathcal{N}} g$, as required.

(b) & (c) These proofs are essentially the same as that of (a). ■

PROOF OF THEOREM 11.1. The theorem follows immediately from the above. ■

12. Concluding remarks. If M is an elementary submodel of a large enough H_θ and $\mathcal{H}_E \in M$, then, by Theorem 4.1, $M \cap \mathcal{H}_E$ is a regular subordering of \mathcal{H}_E , i.e. \mathcal{H}_E is *semicohen* (see [1]). A result of Balcar–Jech–Zapletal ([1, Theorem 3.2]) in our case easily reduces to (see also our Lemma 4.5):

LEMMA 12.1. *The poset \mathcal{H}_E is Cohen iff there is a club $C \subseteq [E]^\omega$ such that for all $A, B \in C$, $a <_E b$ implies that there is $c \in A \cap B$ such that $a <_E c <_E b$ for all $a \in A \setminus B$ and $b \in B \setminus A$. ■*

In particular, \mathcal{H}_{ω_2} is not Cohen, but all its suborderings of smaller size are. [A different example of such a poset is found in [1], where it was remarked that (in our notation) $\mathcal{H}_{\mathfrak{c}^+}$ is not Cohen.] The following example gives a different proof of a recent result of Koppelberg–Shelah ([14]) that there is a regular subalgebra \mathcal{B} of a Cohen algebra for adding \aleph_2 many Cohen reals which is not Cohen. Let $E = \omega_2 \times \{0, 1\} \cup \{a\}$, with the ordering defined by

$$\langle \xi, 0 \rangle <_E a <_E \langle \eta, 1 \rangle \quad \text{for all } \xi, \eta,$$

and let $E_0 = E \setminus \{a\}$. Then \mathcal{H}_{E_0} is not Cohen by the above theorem. To see that \mathcal{H}_E is Cohen, note that it is equivalent to an iteration of $\mathcal{H}_{\{a\}}$ with the poset $\mathcal{H}_E(\{a\}, \dot{g}_{\{a\}})$ and by a version of Lemma 4.3 the former poset is equivalent to a finite support product of countable posets, i.e. the poset for adding \aleph_2 many Cohen reals, \mathcal{C}_{\aleph_2} . So \mathcal{H}_E is $\mathcal{C} * \mathcal{C}_{\aleph_2}$, which is \mathcal{C}_{\aleph_2} . Our example is different from one in [14] because in our case the quotient has the property that a generic object is determined by a single real.

Acknowledgments. I would like to thank: Alan Dow, who remarked that (an old form of) Lemma 5.1 can be made simpler, and otherwise for always being awake on seminars, Fred Galvin, for kindly permitting me to include the proof of his unpublished result (Theorem 3.2) and communicating problems mentioned in §2, Marion Scheepers, for an interesting discussion about the origin of some of the embedding problems solved here, and last but not most, Stevo Todorčević, for initially suggesting me to investigate under

what conditions the poset \mathcal{H}_E adds long chains to ${}^\omega\omega$, for suggesting other problems and possible solutions, for always asking me to find the “right” statement, which greatly improved the shape of this paper, and generally for generously sharing his deep insight.

I would also like to thank the referee who suggested several improvements in the presentation of this text.

References

- [1] B. Balcar, T. Jech and J. Zapletal, *Semicohen Boolean algebras*, preprint, 1995.
- [2] J. Baumgartner and R. Laver, *Iterated perfect-set forcing*, *Ann. Math. Logic* 17 (1979), 271–288.
- [3] J. Brendle and T. LaBerge, *Forcing tightness in products*, preprint, 1994.
- [4] M. Burke, *Notes on embedding partially ordered sets into $\langle {}^\omega\omega, <^* \rangle$* , preprint, 1995.
- [5] J. Cummings, M. Scheepers and S. Shelah, *Type rings*, to appear.
- [6] H. G. Dales and W. H. Woodin, *An Introduction to Independence for Analysts*, London Math. Soc. Lecture Note Ser. 115, Cambridge University Press, 1987.
- [7] P. L. Dordal, *Towers in $[\omega]^\omega$ and ${}^\omega\omega$* , *Ann. Pure Appl. Logic* 45 (1989), 247–277.
- [8] W. B. Easton, *Powers of regular cardinals*, *Ann. Math. Logic* 1 (1970), 139–178.
- [9] P. Erdős, A. Hajnal, A. Maté and R. Rado, *Combinatorial Set Theory—Partition Relations for Cardinals*, North-Holland, 1984.
- [10] F. Galvin, Letter of August 3, 1995.
- [11] —, Letter of August 5, 1995.
- [12] F. Hausdorff, *Die Graduierung nach dem Endverlauf*, *Abh. Königl. Sächs. Gesell. Wiss. Math.-Phys. Kl.* 31 (1909), 296–334.
- [13] S. Hechler, *On the existence of certain cofinal subsets of ${}^\omega\omega$* , in: *Proc. Sympos. Pure Math.* 13, Amer. Math. Soc., 1974, 155–173.
- [14] S. Koppelberg and S. Shelah, *Subalgebras of Cohen algebras need not be Cohen*, preprint, 1995.
- [15] D. W. Kueker, *Countable approximations and Löwenheim–Skolem theorems*, *Ann. Math. Logic* 11 (1977), 77–103.
- [16] K. Kunen, *Inaccessibility properties of cardinals*, Ph.D. thesis, Stanford University, 1968.
- [17] —, $\langle \kappa, \lambda^* \rangle$ -gaps under MA, preprint, 1976.
- [18] —, *Set Theory—An Introduction to Independence Proofs*, North-Holland, 1980.
- [19] G. Kurepa, *L’hypothèse du continu et le problème de Souslin*, *Publ. Inst. Math. Belgrade* 2 (1948), 26–36.
- [20] R. Laver, *Linear orderings in ${}^\omega\omega$ under eventual dominance*, in: *Logic Colloquium ’78*, North-Holland, 1979, 299–302.
- [21] J. C. Oxtoby, *Measure and Category*, Springer, 1970.
- [22] K. Prikry, *Changing measurable into accessible cardinals*, *Dissertationes Math. (Rozprawy Mat.)* 68 (1970).
- [23] F. Rothberger, *Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propriété C*, *Proc. Cambridge Philos. Soc.* 37 (1941), 109–126.

- [24] M. Scheepers, *Gaps in ${}^\omega\omega$* , in: Israel Math. Conf. Proc. 6, Amer. Math. Soc., 1993, 439–561.
- [25] —, *Cardinals of countable cofinality and eventual domination*, Order 11 (1995), 221–235.
- [26] —, *The Boise problem book*, <http://www.unipissing.ca/topology/>.
- [27] R. Solovay, *Discontinuous homomorphisms of Banach algebras*, preprint, 1976.
- [28] S. Todorčević, *Special square sequences*, Proc. Amer. Math. Soc. 105 (1989), 199–205.
- [29] S. Todorčević and I. Farah, *Some Applications of the Method of Forcing*, Mathematical Institute, Belgrade, and Yenisei, Moscow, 1995.

Department of Mathematics
100 St. George Street
University of Toronto
Toronto, Ontario
Canada M5S 3G3
E-mail: ilijas@math.toronto.edu

Received 15 November 1995