Embedding partially ordered sets into $\omega^\omega$

by

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Abstract. We investigate some natural questions about the class of posets which can be embedded into $(\omega, \leq^*)$. Our main tool is a simple ccc forcing notion $\mathcal{H}_E$ which generically embeds a given poset $E$ into $(\omega, \leq^*)$ and does this in a “minimal” way (see Theorems 9.1, 10.1, 6.1 and 9.2).

We describe a simple ccc forcing notion $\mathcal{H}_E$ which embeds a given poset $E$ into $(\omega, \leq^*)$ (see Definition 0.1 and Definition 4.1). It has the property that $\mathcal{H}_{E_0}$ is a regular subordering of $\mathcal{H}_E$ whenever $E_0$ is a subordering of $E$. If $\mathcal{P}$ is a forcing notion, then “$\mathcal{P}$ adds a $\kappa$-chain to $\omega^\omega$” means “In a forcing extension by $\mathcal{P}$ there is a $\kappa$-chain in $\omega^\omega$”, so in particular this phrase applies even if there is already a $\kappa$-chain in $\omega^\omega$. We prove the following results about $\mathcal{H}_E$ (the symbol $\omega^\omega$ stands for the poset $(\omega, \leq^*)$, while $\mathcal{C}$ stands for a poset for adding a single Cohen real):

**Theorem 9.1 (Main Theorem).** If $\kappa > \omega_1$ is a regular cardinal then $\mathcal{H}_E$ adds a $\kappa$-chain to $\omega^\omega$ iff one of the following happens:

†1 $E$ has a $\kappa$- or a $\kappa^*$-chain, or

†2 $\mathcal{C}$ adds a $\kappa$-chain to $\omega^\omega$.

In the case when $E$ is an antichain of size $\kappa$ the poset $\mathcal{H}_E$ reduces to a poset for adding $\kappa$ many Cohen reals, so Theorem 9.1 implies Kunen’s theorem ([16]) that after adding any number of Cohen reals in $\omega^\omega$ there are no well-ordered chains of size larger than the ground-model continuum. In the following two theorems $\kappa$ and $\lambda$ are uncountable regular cardinals; for undefined notions see Definition 6.1 ($\dot{g}$ is a name for the generic embedding of $\mathcal{H}_E$ into $\omega^\omega$).

**Theorem 6.1.** (a) A $\dot{g}$-image of a limit $\langle a_\xi, b_\xi \rangle_{\xi < \kappa}$ in $E$ is a limit in $\omega^\omega$.  
(b) A $\dot{g}$-image of a $\langle \kappa, \omega \rangle$-gap $\langle a_\xi, b_i \rangle_{\xi < \kappa, i < \omega}$ in $E$ is a gap in $\omega^\omega$.

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(c) A $g$-image of a gap $\langle a_\xi, b_\eta \rangle_{\xi, \eta < \kappa, \eta < \lambda}$ in $E$ is a gap in $\omega \omega$.
(d) A $g$-image of an unbounded chain $\langle a_\xi \rangle_{\xi < \kappa}$ is an unbounded chain in $\omega \omega$.

A partial converse of the previous theorem is given in

**Theorem 9.2.** If $\kappa > \varepsilon$ and $\mathcal{H}_E$ adds a $\langle \kappa, \lambda \rangle$-gap to $\omega \omega$, then there is such a gap in $E$ or in $E^*$. 

The following theorem is an attempt at describing which dense linearly ordered sets embed into $\omega \omega$ after forcing with $\mathcal{H}_E$. Note that looking at those linearly ordered sets which are suborderings of $\langle 2^\kappa, <_{\text{Lex}} \rangle$ (the symbol $<_{\text{Lex}}$ stands for the lexicographical ordering) for some $\kappa$ is not a loss of generality. Moreover, the theorem below has interesting applications (see Propositions 1.4 and 1.5).

**Theorem 10.1.** If $\mathcal{H}_E$ forces that $\langle 2^{\omega_1}, <_{\text{Lex}} \rangle^V$ embeds into $\langle \omega \omega, \leq^* \rangle$, then either a Cohen real forces this or $\langle 2^{\omega_1}, <_{\text{Lex}} \rangle^V$ embeds into $E$.

The “Cohen real” alternative in Theorems 9.1 and 10.1 can be avoided if we start from a model of CH (which is the situation where these theorems are most often used), but in general the following question is open:

**Question.** Does forcing with $\mathcal{C}$ add an $\omega_2$-chain to $\omega \omega$ iff there is an $\omega_2$-chain in $\omega \omega$?

We start by presenting applications of Theorems 9.1 and 10.1 in §§1–3. In §1 we answer some questions of Dordal and Scheepers and prove some other related statements. In §2 we use a poset obtained by Todorčević to answer a question of Galvin, proving that the poset $\omega(\omega \omega)$ is not necessarily embeddable into $\omega \omega$. In §3 we use a poset obtained by Galvin to describe a forcing extension of the universe in which an ultrapower $\omega \omega/U$ is not embeddable into $\omega \omega$ for every nonprincipal ultrafilter $U$ on $\omega$. In §4 we define $\mathcal{H}_E$, describe the quotient $\mathcal{H}_E/\mathcal{H}_{E_0}$ for $E_0 \subseteq E$, and prove various properties of these posets. In §5 we prepare for the proofs of the above theorems. §6 is an investigation of gaps and limits in $\omega \omega$ in an extension by $\mathcal{H}_E$. Chapters 7 and 8 include some prerequisites for the proofs of Theorems 9.1 and 10.1 which are independent of the rest of the paper and interesting in their own right: in the former we give a strengthening of an old result of Kurepa that every uncountable well-founded poset with finite levels has an uncountable chain, while in the latter we investigate the Banach–Mazur game of length $\omega_1$. In §9 we state and prove a ZFC, “local”, version of the $\Delta$-system lemma for countable sets (Lemma 9.1) which is, in the absence of CH, used in the proofs of Theorems 9.1 and 10.1. In §11 we show that $\omega \omega$ in an extension by a single Cohen real is reflected in a certain ground-model ordering $\mathcal{C} \omega/N$. In particular, by Theorem 11.1, the question above can be
reformulated as \((\mathcal{N} \text{ is the ideal of nowhere dense subsets of a Cohen poset } \mathcal{C}; \text{ see Definition 11.1})\):

**Question.** Does the existence of an \(\omega_2\)-chain in the poset \(\mathcal{C}/\mathcal{N}\) imply the existence of an \(\omega_2\)-chain in \(\omega_\omega\)?

Our notation is standard, and undefined notions can be found in [18]. If \(\phi\) is a statement of a forcing language, then the phrase “\(\phi\) is true with probability one” is an abbreviation for “every condition of the poset forces \(\phi\)”.

**0. Introduction.** Let \(\langle E, <_E \rangle\) be a partially ordered set. For \(a \in E\) and \(X \subseteq E\) let

\[X(\leq_E a) = \{x \in X : x \leq_E a\};\]

\[X(\geq_E a), X(<_E a)\text{ etc. have similar definitions. A subset } X \text{ of } E \text{ is countably bounded iff there is a countable } A \subseteq E \text{ such that } X = \bigcup_{a \in A} X(\leq_E a).\]

A subset \(X\) of \(E\) is countably bounded from below iff there is a countable \(A \subseteq E\) such that \(X = \bigcup_{a \in A} X(\geq_E a)\). If every \(a \in A\) is \(<_E\)-incomparable with every \(b \in B\) then we say that \(A\) and \(B\) are \(<_E\)-incomparable. If \(a, b\) are elements of a poset \(E\) then the interval of \(E\) with endpoints \(a\) and \(b\) is the set

\((a, b)_E = \{c \in E : a <_E c <_E b \text{ or } b <_E c <_E a\}.\]

In particular, \((a, b)_E = (b, a)_E\) always, and \((a, b)_E\) is empty if \(a\) and \(b\) are incomparable. A mapping \(f : \langle E_0, <_0 \rangle \rightarrow \langle E_1, <_1 \rangle\) is an embedding iff we have \(a <_0 b\) iff \(f(a) <_1 f(b)\) for all \(a, b \in E_0\), as opposed to a strictly increasing mapping which is one such that \(a <_0 b\) implies \(f(a) <_1 f(b)\) for all \(a, b \in E_0\). Of course, in the case when \(E_0\) is linearly ordered these two notions coincide.

**Definition 0.1.** For \(f, g \in \omega_\omega\) we define:

1. \(f \leq^* g\) iff \(\{n : f(n) \leq g(n)\}\) is cofinite.
2. \(f \leq^* g\) iff \(f \leq^* g\) and not \(f \geq^* g\).
3. \(f <^* g\) iff \(\{n : f(n) < g(n)\}\) is cofinite.
4. \(f < g\) iff \(\lim_{n \to \infty}(g(n) - f(n)) = \infty\).
5. \(f \leq^*_n g\) iff \(f(m) \leq g(m)\) for all \(m \geq n\).

Our forcing notion \(\mathcal{H}_E\) generically embeds \(E\) into \(\langle \omega_\omega, \leq^* \rangle\). Similar forcings were used in [13], [27], [20], [4], but the poset \(E\) was usually embedded into the structure \(\langle \omega_\omega, <^* \rangle\). We choose the ordering \(\leq^*\) because it is not clear how we can get the desirable property from Theorem 4.1 with other partial orderings on \(\omega_\omega\). We first prove that the ordering we have chosen is good enough for our purposes.

**Proposition 0.1.** There is an embedding from \(\langle \omega_\omega, \leq^* \rangle\) into \(\langle \omega_\omega, <^* \rangle\).
Proof. Without loss of generality we consider only the subordering consisting of strictly increasing functions. Fix an infinite matrix of positive integers \( a_{mn} \) \((m, n \in \omega)\) such that
\[
a_{mn} > \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} a_{ij}.
\]
for all \( m, n \). For each \( =^* \)-equivalence class pick a representative, and for \( f \in \omega^\omega \) let \( \Psi(f) =^* f \) be the chosen representative. Let \( f \mapsto \hat{f} \) be defined by
\[
\hat{f}(n) = \sum_{i=0}^{n} \sum_{j=0}^{\Psi(f)(i)} a_{ij}
\]
for \( n \in \omega \). Then it is easy to check that for strictly increasing \( f \) and \( g \) we have \( f \leq^* g \) iff \( \hat{f} <^* \hat{g} \). 

Corollary. A linearly ordered set is embeddable into \( \langle \omega^\omega, <^* \rangle \) iff it is embeddable into \( \langle \omega^\omega, \leq^* \rangle \) iff it is embeddable into \( \langle \omega^\omega, \prec \rangle \).

Proof. If a linearly ordered set \( L \) is embeddable into \( E_0 \) and there is a strictly increasing mapping from \( E_0 \) into \( E_1 \), then \( L \) is embeddable into \( E_1 \); so it suffices to define some increasing mappings. Obviously the identity is a strictly increasing mapping from \( \langle \omega^\omega, < \rangle \) into \( \langle \omega^\omega, <^* \rangle \), as well as from \( \langle \omega^\omega, <^* \rangle \) into \( \langle \omega^\omega, \leq^* \rangle \). Finally, the mapping \( f \mapsto \hat{f} \) defined by \( \hat{f}(n) = n + f(n) \) is strictly increasing from \( \langle \omega^\omega, <^* \rangle \) into \( \langle \omega^\omega, < \rangle \). 

So our saying that e.g. “there is an \( \omega_2 \)-chain in \( \omega^\omega \) without specifying an ordering is justified (as long as it is assumed that the ordering is one of the “mod finite” orderings).

1. Applications of the main theorem

Proposition 1.1. There is a forcing extension of \( V \) in which there are no \( \omega_2 \)-chains in \( \omega^\omega \), but there is a poset which adds such a chain without adding new \( \omega_1 \)-sequences of ordinals.

Proof. Start from a model of \( CH \) in which there is an \( \omega_2 \)-Suslin tree \( T \). Go to a forcing extension by \( \mathcal{H}_T \); by Theorem 9.1 there are no \( \omega_2 \)-chains in \( \omega^\omega \), \( T \) remains a Suslin tree (because \( \mathcal{H}_E \) has precaliber \( \aleph_2 \); see Lemma 4.1) and therefore further forcing with \( T \) does not add \( \omega_1 \)-sequences of ordinals while it adds an \( \omega_2 \)-branch to itself, and the image of this branch is an \( \omega_2 \)-chain in \( \omega^\omega \). 

The following answers a question of Dordal [7, Remark 9.5, p. 269] and Scheepers [26, #81]. It is solved independently by Cummings, Scheepers and Shelah in [5].
Proposition 1.2. The existence of an \( \omega_\omega \)-chain in \( 2^\omega \) does not imply the existence of an \( \omega_{\omega+1} \)-chain in \( 2^\omega \).

Proof. We can either start from a model of CH and force with \( \mathcal{H}_{\omega_\omega} \), or we can start from a model with an \( \omega_{\omega+1} \)-Suslin tree \( T \) and force with \( \mathcal{H}_T \). The latter model has the property that there are no \( \omega_{\omega+1} \)-chains but a forcing notion (namely \( T \)) adds one without adding new sequences of ordinals of length \( \omega_\omega \).

Scheepers noticed that \( \omega_{\omega+1} \) embeds into \( 2^\omega \), and therefore in both models constructed in Proposition 1.2 the poset \( 2^\omega \) (see \( \S2 \)) is not embeddable into \( 2^\omega \). So this answers an unpublished question of Galvin which was also asked in [26, \#81]. In the above models the continuum is rather large, and in \( \S2 \) we will prove that this can happen even when the continuum is equal to \( \aleph_2 \) (obviously this is the best possible because CH implies that all posets of size \( \aleph_1 \) embed into \( 2^\omega \)).

Consider a cardinal invariant of the continuum equal to the supremum of all cardinals \( \kappa \) such that there is a \( \kappa \)-chain in \( 2^\omega \). A natural question arises—is this supremum always attained, i.e. can “supremum” be replaced by “maximum” in the above definition? It is easy to show (e.g. by using lemmas from [25]) that if \( \kappa \) is singular and there are \( \lambda \)-chains in \( 2^\omega \) for all \( \lambda < \kappa \), then there is a \( \kappa \)-chain in \( 2^\omega \) as well. Therefore in a model in which the answer to our question is negative this supremum must be a weakly inaccessible cardinal, so the use of an inaccessible cardinal to get a model where sup \( \neq \) max in our next proposition is justified.

Proposition 1.3. If \( \kappa \) is an inaccessible cardinal then there is a cardinal-preserving forcing extension of \( V \) in which there is a \( \lambda \)-chain in \( 2^\omega \) for all \( \lambda < \kappa \) but there is no \( \kappa \)-chain in \( 2^\omega \).

Proof. Let \( E \) be any poset with no \( \kappa \)-chains and with \( \lambda \)-chains for all \( \lambda < \kappa \) and force with \( \mathcal{H}_E \).

Proposition 1.4. There is a forcing extension of the universe in which there is a linearly ordered set \( (L, <_L) \) and a partition \( L = L_0 \cup L_1 \) such that \( L \) is not embeddable into \( 2^\omega \), while both \( L_0 \) and \( L_1 \) are.

Proof. Start from a model of CH, let \( L \) be \( 2^{\omega_1}, <_{\mathrm{Lex}} \) and let \( L_0 \cup L_1 \) be its Bernstein decomposition (i.e. a decomposition such that \( L \) does not embed into \( L_0 \) or into \( L_1 \)). Let \( E \) be \( (L_0, <_{\mathrm{Lex}}) + (L_1, <_{\mathrm{Lex}}) \). Then by Theorem 10.1 the forcing extension by \( \mathcal{H}_E \) is as required.

In some models of Set Theory (e.g. when CH holds; also see [20]) linearly ordered sets which embed into \( 2^\omega \) are exactly those of size at most \( \aleph_1 \), so the statement of Proposition 1.4 fails in such models.
Scheepers observed that under Martin’s Axiom, (a) every ordinal of cardinality at most \( c \) embeds into \( \omega \), and (b) every linearly ordered set of size strictly less than \( c \) embeds into \( \omega \). He asked whether one of these statements implies the other. In the next proposition we show that (a) does not imply (b).

**Proposition 1.5.** It is not provable in ZFC that if \( \omega_3 \) embeds into \( \omega \) then all linearly ordered sets of size \( \aleph_2 \) embed into \( \omega \).

**Proof.** Start from a model of GCH and let \( E = \langle 2^{\omega_1}, \prec_{\text{Lex}} \rangle^V \), so \( E \) is of size \( \aleph_2 \). After adding one Cohen real CH remains true, so \( E \) is not embeddable into \( \omega \) in this extension. Therefore after we force with \( H_{\omega_3} \) (or any other \( H_\kappa \)), by Theorem 10.1 the poset \( E \) is not embeddable into \( \omega \). ■

**Remark.** Our first proof of Proposition 1.5 was to add \( \aleph_2 \) many Cohen subsets of \( \omega_1 \), say \( \langle c_\xi : \xi < \omega_2 \rangle \), and then to force with \( H_{\omega_3} \); in this model the set \( \langle c_\xi : \xi < \omega_2 \rangle \) with the lexicographical ordering is not isomorphic to any \( \langle X, \prec \rangle \), where \( X \) is a set of reals and \( \prec \) is a Borel ordering.

The following extends a result of Brendle–LaBerge, who in [3, Theorems 2.7 and 2.8] proved a special case when \( I \) as below is taken to be the family of all subsets of \( \kappa \) of size smaller than \( \kappa \). The forcing extensions given in [3] are similar to ones obtained by \( H_E \).

**Proposition 1.6.** If \( I \) is a proper \( \sigma \)-ideal on the cardinal \( \kappa \) which includes all countable subsets of \( \kappa \), then there is a forcing extension of \( V \) in which there are no \( (c^+) \) chains in \( \omega \) and there is a set \( \{ x_\xi : \xi < \kappa \} \) in \( \omega \) such that \( \{ x_\xi : \xi \in A \} \) is bounded in \( \omega \) iff \( A \in I \).

**Proof.** Let \( E = \kappa \cup I \) with the ordering \( \xi < A \) iff \( \xi \in \kappa \), \( A \in I \) and \( \xi \in A \). A forcing extension by \( H_E \) satisfies the requirements by Theorem 9.1 and Lemma 6.1. ■

2. A problem of Galvin. On the set \( \langle \omega, \omega \rangle \) of all sequences \( \vec{f} = \langle f_n \rangle \) of elements of \( \langle \omega \rangle \) we define the ordering of eventual dominance, \( \leq^* \), by:

\[
\vec{f} \leq^* \vec{g} \quad \text{iff} \quad f_n \leq^* g_n \text{ for all large enough } n.
\]

[Observe that the symbol \( \leq^* \) in the above line denotes two different orderings on two different sets. The second \( \leq^* \) can be replaced by either \( <^* \) or \( \prec \) (see Definition 1.1), but by Proposition 1.1 a linearly ordered set is embeddable into \( \langle \omega \rangle \) with the ordering that we defined iff it is embeddable into \( \langle \omega \rangle \) with any of these orderings.] We will denote the poset \( \langle \omega, \omega \rangle \) by \( \omega \langle \omega \rangle \).
**Theorem 2.1.** There is a forcing extension of the universe such that

1. There is an $\omega_2$-chain in $\omega(\omega)$.  
2. There are no $\omega_2$-chains in $\omega$.  
3. $\omega(\omega)$ is not embeddable into $\omega$.  
4. Adding a dominating real adds an $\omega_2$-chain to $\omega$. 

Our model will be a forcing extension by $\mathcal{H}_E$, where $E$ is supplied by the following result of Todorčević.

**Theorem 2.2 (28).** (□$_{\omega_1}$) There is a sequence $<_n$ $(n < \omega)$ of tree orderings on $\omega_2$ such that for all $n$,

1. $<_n \subseteq <_{n+1} \subseteq \in$,  
2. $\in = \bigcup_{n<\omega} <^n$, and  
3. no $T^n = \langle \omega_2, <^n \rangle$ has an $\omega_2$-branch. $\blacksquare$

Let $T$ denote the disjoint sum of $T^n$, i.e. $T = \langle \omega_2 \times \omega, <_T \rangle$ and $<_T$ is defined by

$$\langle \xi, m \rangle <_T \langle \eta, n \rangle \iff n = m \text{ and } \xi <^n \eta.$$ 

**Proof of Theorem 2.1.** The model is obtained by forcing with $\mathcal{H}_T$ over a model of CH and $\square_{\omega_1}$.

1. It is enough to provide a sequence $D_\xi = \{ x_{\xi_i} : i < \omega \}$ $(\xi < \omega_2)$ of subsets of $T$ such that for all $\xi < \eta$ and some $n$ we have $x_{\xi_i} <_T x_{\eta_i}$ for all $i \geq n$. Let $D_\xi = \{ (\xi, n) : n < \omega \}$; obviously this family satisfies the requirements.

To prove (2), just notice that $T$ does not have $\omega_2$-chains and apply Theorem 9.1.

3. follows immediately from (1) and (2).

**Claim.** If $d$ is a dominating real, then in $V[d]$ there is a strictly increasing mapping from $(\omega(\omega))^V$ into $\omega$.

**Proof.** Map $\tilde{f} = (f_n : n < \omega)$ to $g$ defined by $g(n) = f_n(d(n))$. To see that this mapping is strictly increasing, note that if $\tilde{f}$ and $\tilde{g}$ are in the ground model, then the function $\Delta_{\tilde{fg}} : \omega \rightarrow \omega$ defined by letting $\Delta_{\tilde{fg}}(n)$ be the least $i$ such that $f_n(j) \geq g_n(j)$ for all $j \geq i$, is dominated by $d$. This shows that our embedding is increasing, and it is strictly increasing by genericity. $\blacksquare$

4. follows immediately from the above claim. $\blacksquare$

**Corollary.** It is not provable in ZFC that there is a strictly increasing mapping from $\omega(\omega)$ into $\omega$. $\blacksquare$

As an application of the above, we mention an unpublished work of Galvin ([11]). Until the end of this section we will adopt Galvin’s original terminology and say that “$E_0$ is embeddable into $E_1$” iff there is a mapping
f : E₀ → E₁ such that a <ₚ E₀ b implies f(a) <ₚ E₁ f(b), i.e. if there is a strictly increasing map from E₀ into E₁ in our terminology. For an indecomposable ordinal α let P(α) be the poset of all f : α → ω, ordered by (otp denotes the order type of a set)

\[ f < g \iff \text{otp}(\{ \xi < \alpha : f(\xi) \geq g(\xi) \}) < \alpha. \]

Galvin observed that P(α) is embeddable into P(β) whenever there is a function g : α → β such that otp(A) = β implies otp(g⁻¹(A)) = α, for all A ⊆ β. So in particular (note that P(ω) here denotes our \( \langle \omega^{\omega}, \prec \rangle \))

(1) P(ω) is embeddable into P(α) for all α.
(2) P(ω²) is embeddable into P(α) for all α ≥ ω².

Galvin asked a general question when P(α) is embeddable into P(β), in particular:

(Q1) Is it provable that P(ω²) is embeddable into P(ω)?
(Q2) Is it provable that P(ω³) is embeddable into P(ω²)?

[“Provable” means “provable in ZFC”; observe that both questions have a positive answer if CH is assumed.] We can reformulate our above Corollary to answer (Q1), namely

**Corollary.** It is not provable in ZFC that P(ω²) is embeddable into P(ω). ■

**Remark.** The tree orderings \( \leq^n \) obtained in [28] have another interesting property:

(T4) the set of \( \leq^n \)-predecessors of α is a closed subset of α + 1 for all \( \alpha < \omega_2 \).

(Note that this implies that \( T^n \) is not Aronszajn.) This easily implies that the natural \( \sigma \)-closed poset \( P_n \) which specializes \( T^n \) has \( \aleph_2 \)-cc. So Theorem 2.2 has another curious consequence: under the assumptions of CH and \( \square_{\omega_1} \) there is a sequence \( P_n (n < \omega_2) \) of \( \sigma \)-closed, \( \aleph_2 \)-cc posets such that every finite product of \( P_n \) is \( \aleph_2 \)-cc, but \( \prod_{n<\omega} P_n \) is not. The fact that \( \prod_{n<\omega} P_n \) is not \( \aleph_2 \)-cc follows from another fact proved in [28]: if the orderings \( \leq^n \) satisfy (T1)–(T4), then one of the trees \( T^n \) is nonspecial.

3. **Ultrapowers of \( \omega^{\omega} \).** Now we construct a model of ZFC in which there are no \( \omega_2 \)-chains in \( \langle \omega^{\omega}, < \rangle \), but for every nonprincipal ultrafilter \( U \) on \( \omega \) there is an \( \omega_2 \)-chain in \( \langle \omega^{\omega}, <_U \rangle \). This scenario is originally used by Solovay in the context of automatic continuity in Banach algebras (see [27]). In fact, in the model of Theorem 3.1 all homomorphisms of Banach algebras are continuous. This is so because the existence of a discontinuous homomorphism implies that there is a strictly increasing mapping from
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$\langle\omega,\leq_U\rangle$ into $\langle\omega,\prec^*\rangle$ for some nonprincipal ultrafilter $U$ on $\omega$ (see [6]). If $U$ is a nonprincipal ultrafilter on $\omega$ then a poset $P_U$ is defined as follows: A typical condition in $P_U$ is $\langle s,A \rangle$, where $s$ is a finite subset of $\omega$, $A \in U$, and $\operatorname{max} s \prec \min A$. The ordering is defined by letting $\langle s,A \rangle \leq \langle t,B \rangle$ iff $t$ is an initial segment of $s$, $A \subseteq B$, and $t \setminus s \subseteq B$. This poset is $\sigma$-centered and it generically adds a subset of $\omega$ (called a Prikry real) which is almost included in all elements of $U$ (see [22]).

**Theorem 3.1. (CH)** Let $\kappa$ be a regular cardinal larger than $\aleph_1$. Then there is a poset $E$ such that in a forcing extension of the universe by $\mathcal{H}_E$, for every nonprincipal ultrafilter $U$ on $\omega$:

1. There are $\kappa$-chains in $\langle\omega/\mathcal{U},\prec_U\rangle$.
2. There are no $\omega_2$-chains in $\langle\omega,\leq^*\rangle$.
3. $\langle\omega,\mathcal{U},\prec_U\rangle$ is not embeddable into $\langle\omega,\leq^*\rangle$.
4. Adding a $\mathcal{U}$-Prikry real adds a $\kappa$-chain to $\langle\omega,\leq^*\rangle$.

The poset $E$ is provided by the following special case of an old unpublished result of Galvin, which is included here with his kind permission.

**Theorem 3.2 ([10]).** If $\kappa$ is a regular cardinal, then there is a poset $\langle G,\prec \rangle$ of size $\kappa$ with no infinite chains but if $\mathcal{E}$ is a linear ordering such that there is a strictly increasing $\Phi : G \to \mathcal{E}$, then $\mathcal{E}$ has a $\kappa$- or a $\kappa^*$-chain.

**Proof.** Let $G$ be $\kappa \times \kappa^*$ with the strict Cartesian ordering $\prec_{sc}$, i.e. $\langle \alpha,\beta \rangle \prec_{sc} \langle \gamma,\delta \rangle$ iff $\alpha < \gamma$ and $\beta > \delta$.

Obviously, every chain in $\kappa \times \kappa^*$ is finite. Suppose that $\langle \mathcal{E},\prec \rangle$ is a linearly ordered set with no $\kappa$- or $\kappa^*$-chains and that $\Phi : \kappa \times \kappa^* \to \mathcal{E}$ is strictly increasing.

**Case 1:** There is a $\beta < \kappa$ such that for all $\alpha < \kappa$ the set $\{ \gamma < \kappa : \Phi(\gamma,\beta) \leq \Phi(\alpha,\beta) \}$ is of size strictly less than $\kappa$. Then we can pick $\alpha_\xi$ ($\xi < \kappa$) such that $\Phi(\alpha_\xi,\beta)$ is an increasing $\kappa$-chain.

**Case 2:** For all $\beta < \kappa$ there is $\alpha_\beta < \kappa$ such that $\{ \gamma < \kappa : \Phi(\gamma,\beta) \leq \Phi(\alpha_\beta,\beta) \}$ is of size $\kappa$. We claim that the chain $\Phi(\alpha_\beta,\beta)$ ($\beta < \kappa$) is strictly decreasing. Suppose the contrary, that $\Phi(\alpha_\beta,\beta) \geq \Phi(\alpha_\gamma,\gamma)$ and $\beta < \gamma$. By the choice of $\alpha_\gamma$ we can pick $\xi > \alpha_\beta$ such that $\Phi(\alpha_\beta,\beta) \geq \Phi(\xi,\gamma)$, but $\langle \alpha_\beta,\beta \rangle \prec_{sc} \langle \xi,\gamma \rangle$—a contradiction.

So there is a $\kappa$- or a $\kappa^*$-chain in $\mathcal{E}$. \(\blacksquare\)

**Proof of Theorem 3.1.** $E$ is $\kappa \times \kappa^*$ ordered by $\prec_{sc}$.

1. By Proposition 0.1, $\kappa \times \kappa^*$ is embeddable into $\langle\omega,\prec^*\rangle$, and $f \mapsto f/\mathcal{U}$ is a strictly increasing mapping from $\langle\omega,\prec^*\rangle$ into $\langle\omega/\mathcal{U},\prec_U\rangle$. So there are $\kappa$-chains in $\langle\omega/\mathcal{U},\prec_U\rangle$ by Theorem 3.2.
(2) follows immediately from Theorem 3.2 and Theorem 4.1.
(3) is a consequence of (1) and (2).

CLAIM. If $x$ is a $\mathcal{U}$-Prikry generic real, then in $V[x]$ there is a Borel strictly increasing mapping from $\langle^{\omega}/\mathcal{U},<_{\mathcal{U}}\rangle$ into $\langle^{\omega},\leq^*\rangle$.

Proof. [The ultrafilter $\mathcal{U}$ restricted to the set $x$ coincides with the Fréchet filter on $x$.] Working in the extension, it is enough to define a Borel mapping $\Phi : \omega \rightarrow \omega$ such that the $\Phi$-image of $[f]_\mathcal{U}$ is included in $[\Phi(f)]_\mathcal{U}$ for all $f$ and $f <_{\mathcal{U}} g$ implies that $\Phi(f) \leq^* \Phi(g)$. Let $e_x$ be the enumeration function of $x$ (i.e. $e_x(n)$ is the $n$th element of the set $x$), and let $\Phi(f)(n) = f(e_x(n))$. This mapping obviously works. ■

(4) follows immediately from the above claim. ■

Stress in Theorem 3.1 is on the fact that (1), (3) and (4) are true for all nonprincipal ultrafilters on $\omega$: namely, it is easy to construct an ultrafilter $\mathcal{U}$ such that there is a $\omega$-chain (or a copy of any given linearly ordered set of size at most $\omega$) in $\langle^{\omega}/\mathcal{U},<_{\mathcal{U}}\rangle$. [Let $<_\omega$ be the ordering on $\omega$ which we want to embed into $\omega/\mathcal{U}$. Start from a family $f_\xi (\xi < \omega)$ in $\omega$ which is independent, i.e. $A_\xi \eta = \{n : f_\xi(n) < f_\eta(n)\}$ is an independent family of subsets of $\omega$ and $f_\xi(n) = f_\eta(n)$ for at most finitely many $n$, for all $\xi \neq \eta$. Then every ultrafilter $\mathcal{U}$ extending the filter base $\mathcal{F} = \{A_\xi \eta : \xi < \omega \} \cup \{\omega \setminus A_\xi \eta : \eta < \omega \}$ works.] Our next example shows that there can be nonprincipal ultrafilters $\mathcal{U}$ such that in $\langle^{\omega}/\mathcal{U},<_{\mathcal{U}}\rangle$ there are no $\omega_2$-chains and the continuum is large.

PROPOSITION 3.1. If we start from a model of CH and add any number of side-by-side Sacks reals with countable supports, then for many ultrafilters $\mathcal{U}$ there are no $\omega_2$-chains in $\langle^{\omega}/\mathcal{U},<_{\mathcal{U}}\rangle$.

Proof. For the undefined notions see [2] or [29, §6.C]. Let $\mathcal{S}_\kappa$ denote the poset for adding $\kappa$ many side-by-side Sacks reals. It is well known that after forcing with $\mathcal{S}_\kappa$ every ground-model selective ultrafilter still generates a selective ultrafilter (see e.g. [29, Theorem 6.8]). Since CH implies that there exists a selective ultrafilter, it will suffice to prove the claim for the case when $\mathcal{U}$ is a ground-model selective ultrafilter. Let $\mathcal{B} = \{B_\alpha : \alpha < \omega_1\}$ be a base for $\mathcal{U}$. Let $\langle \delta_\xi : \xi < \kappa \rangle$ be a name for a sequence of generic Sacks reals. Suppose that $f_\xi (\xi < \omega_2)$ is a name for a strictly increasing chain in $\omega/\mathcal{U}$. By [2] for every $\xi < \kappa$ there is a countable $A_\xi \subseteq \kappa$, a perfect set $P_\xi \subseteq \mathbb{R}^\kappa$, and a continuous function $g_\xi : P_\xi \rightarrow \omega$ such that $P_\xi$ forces $g_\xi (\langle \delta_\alpha : \alpha \in A_\xi \rangle) = f_\xi$. We can assume that $\kappa = \aleph_2$. By CH, we can assume that $A_\xi$'s form a $\Delta$-system, and that there is a partial function $\mathcal{F} : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega$ such that every $g_\xi$ is isomorphic to $\mathcal{F}$. Fix $\xi < \kappa < \omega_2$, and let $p_{\xi \eta} \in \mathcal{S}_\kappa$ and $A \in \mathcal{B}$ be such that

$$p_{\xi \eta} \leq P_{\xi}, P_{\eta} \quad \text{and} \quad p_{\xi \eta} \vDash (\forall n \in \dot{A}) \dot{g}_\xi(n) < \dot{g}_\eta(n).$$
Let $\Phi : S_\kappa \to S_\kappa$ be an automorphism of $S_\kappa$ (compare with paragraph before Definition 4.2) whose extension to $S_\kappa$-names swaps $\dot{g}_\xi$ and $\dot{g}_\eta$. Then $\Phi(p_{\kappa \eta})$ forces $\dot{g}_\eta <_\mathcal{U} \dot{g}_\xi$, a contradiction. 

\section{4. $\mathcal{H}_E$ and its basic properties}

**Definition 4.1.** If $(E, <_E)$ is a partially ordered set, then we define the poset $\mathcal{H}_E$ as follows: A typical condition $p$ is $(F_p, n_p, f_p)$, where

- $(\mathcal{H}1)$ $F_p$ is a finite subset of $E$, $n_p < \omega$, $f_p : F_p \times n_p \to \omega$.
- We say that $p$ extends $q$ iff (as the notation of $(\mathcal{H}1)$ suggests, we will sometimes consider $f_p$ as a mapping from $F_p$ into $\mathcal{F}_n\omega$):
- $(\mathcal{H}2) F_p \supseteq F_q$, $n_p \geq n_q$, $f_p \supseteq f_q$, $f_p(a)(i) \leq f_p(b)(i)$ for all $a <_E b$ in $F_q$ and all $i \in [n_q, n_p]$.
- $(\mathcal{H}3)$ $f_p(a)(i) \leq f_p(b)(i)$ for all $a <_E b$ in $F_p$, and $\forall i \in [n_q, n_p]$.

If $\dot{g}$ is a name for the mapping of $E$ into $\mathcal{F}_n\omega$ defined by $a \mapsto \bigcup_{p \in G} f_p(a)$ ($G$ is a name for the generic filter), then every condition $p$ in $\mathcal{H}_E$ forces that $\dot{g}(a) \leq n_p$ for all $a <_E b \in F_p$. By genericity $\dot{g}(a) \neq^* \dot{g}(b)$ for all distinct $a$ and $b$ in $\mathcal{F}_\omega$. Note that the generic filter $\dot{G}$ is not equal to the set $\{p : f_p(a) \subset \dot{g}(a) \text{ for all } a \in F_p\}$. Instead, we have (let $n_{ab}$ be the least positive integer $n$ such that $g(a) \leq n \land g(b) \leq n$ for all $a, b \in F_p$).

\[
\dot{G} = \{p : f_p(a) \subset \dot{g}(a) \text{ and } n_{ab} \leq n_p \text{ for all } a, b \in F_p\}.
\]

The following useful fact is an immediate consequence of Definition 4.1 (see also Lemma 4.4).

**Proposition 4.1.** If $p, q \in \mathcal{H}_E$ are such that $n_p = n_q$ and $f_p, f_q$ agree on $F_p \cap F_q$, then $p$ and $q$ are compatible, with $(F_p \cup F_q, n_p, f_p \cup f_q)$ extending both. 

The assumption $n_p = n_q$ is not necessary if e.g. $F_p$ and $F_q$ are disjoint, but in general it is (see Proposition 4.2). We will often write $\models \mathcal{H}_E$ instead of $\models \mathcal{H}_{\kappa \eta}$ when this does not lead to confusion. By the above (plus a standard $\Delta$-system argument) we have:

**Lemma 4.1.** $\mathcal{H}_E$ is ccc (moreover, it has precaliber $\kappa$ for every uncountable regular $\kappa$) and $\dot{g}$ is forced to be an embedding of $(E, <_E)$ into $(\mathcal{F}_n\omega, \leq^*)$. 

If $E_0$ is a subordering of $E$ and $p$ is in $\mathcal{H}_E$, then let $p|E_0$ be the condition $p'$ such that $F_p' = F_p \cap E_0$, $n_p' = n_p$, and $f_p' = f_p|F_p' \times n_p$; so in particular $p|\emptyset$ is the maximal condition in $\mathcal{H}_E$. Recall that $\mathcal{P}$ is a regular subordering of $\mathcal{Q}$ (denoted $\mathcal{P} \subset \mathcal{Q}$) iff for every condition $q$ of $\mathcal{Q}$ there is a $q_p \in \mathcal{P}$ (a projection of $q$ to $\mathcal{P}$) such that $p$ is compatible with $q_p$ iff $q$ is, for all $p \in \mathcal{P}$.

[In the terminology of [18], $\mathcal{P}$ is completely embedded into $\mathcal{Q}$.]
THEOREM 4.1. If $E_0$ is any subordering of $E$, then $\mathcal{H}_{E_0} \prec \mathcal{H}_E$. In particular, the projection mapping is $q \mapsto q|_{E_0}$.

Proof. We fix $q \in \mathcal{H}_E$ and $p \in \mathcal{H}_{E_0}$ which extends $q|\mathcal{H}_{E_0}$ and prove that $q$ and $p$ are compatible by finding $r \leq q, p$ such that $F_r = F_q \cup F_p$. It is enough to consider the case when $F_q \setminus E_0$ is a singleton, because the general case follows from this special one by obvious induction. So let $F_q \setminus E_0 = \{c\}$. Let $F_0 = F_q \cap E_0$; if $F_0$ is empty then $p$ and $q$ are by default comparable, so we can assume that $F_0$ is nonempty, and therefore that $n_p \geq n_q$. So by Proposition 4.1 we have to do some work only when $n_p > n_q$, and this work is in defining $f_r(c)|[n_q, n_p)$. If $F_0(< c)$ is nonempty, pick $a_i$ in this last set such that $f_p(a_i)(i)$ is maximal for all $i \in [n_q, n_p)$. If $F_0(< c)$ is empty but $F_0(> c)$ is not, then pick $a_i$ in this last set so that $f_p(a_i)(i)$ is minimal. If no element of $F_0$ is comparable with $c$ then pick $a_i$’s arbitrarily. Let $f_r(c)(i) = f_p(a_i)(i)$ for $i \in [n_q, n_p)$. We then claim that $r = \langle F_p \cup F_q, n_p, f_p \cup f_r \rangle$ extends both $p$ and $q$. To see this, we only have to check if condition (H3) is valid between $q$ and $r$. Suppose first that $F_0(< c) \neq \emptyset$. Pick $i \in [n_q, n_p)$ and $d \in F_q \cap E_0$.

If $d < c$ then $f_r(d)(i) = f_p(d)(i) \leq f_p(a_i)(i) = f_r(c)(i)$, by the choice of $a_i$. If $d \geq c$, then $d > c a_i$, so $f_p(d)(i) \geq f_p(a_i)(i) = f_r(c)(i)$ (because $p$ extends $q|E_0$). The case when $F_0(< c) = \emptyset$ and $F_0(> c) \neq \emptyset$ is handled similarly, and if both sets are empty then the claim is by default true. So $p$ and $q$ are comparable and $q|E_0$ is the projection of $q$ to $\mathcal{H}_{E_0}$. ■

The following gives us an internal characterization of the comparability relation in $\mathcal{H}_E$.

PROPOSITION 4.2. (a) Conditions $p$ and $q$ in $\mathcal{H}_E$ such that $n_p \geq n_q$ are incompatible iff one of the following happens:

(1) $f_p(a)(i) \neq f_q(a)(i)$ for some $a \in F_p \cap F_q$ and some $i \in [n_q, n_p)$.

(2) for $\varphi \in \{<, >\}$: $f_p(a)(i) \varphi f_q(b)(i)$ for some $b \in F_q$ and $a \in F_q$ and $i \in [n_q, n_p)$.

(b) Let $F = F_p \cap F_q$. Then $p$ and $q$ are incompatible iff $q|F$ and $q|F$ are.

Proof. (a) We will prove only the nonobvious direction, so assume that $p \perp q$ and that $f_p \cup f_q$ is a function (i.e. (1) does not apply). If $n_p = n_q$ then $p$ and $q$ are comparable by Proposition 4.1, so we can assume that $n_p > n_q$. But if (2) does not apply, $p$ and $q|F_p$ are comparable, so $p$ and $q$ are comparable by Theorem 4.1.

(b) This follows immediately from Theorem 4.1 applied with $E_0 = F$. ■
By [18, VII.5.12] we can assume that every \( H \)-name \( \tau \) for a real (that is, a subset of \( \omega \)) is in a canonical form, called “nice name” in [18]. Namely, we assume that for a sequence \( \{ A_n^\tau \} \) of antichains we have
\[
\tau = \{(n) \times A_n^\tau : n \in \omega \}.
\]
[So \( p \Vdash \dot{n} \in \tau \) if \( p \in A_n^\tau \), and \( p \Vdash \dot{n} \notin \tau \) iff \( p \) is incompatible with all elements of \( A_n^\tau \).] In particular, \( \tau \) is countable. So we can define a support of a name \( \tau \) by
\[
\text{supp } \tau = \bigcup_{n \in \omega} A_n^\tau.
\]
In particular, \( \text{supp } \tau \) is a countable subset of \( E \).

**Corollary.** (a) For every real \( \dot{x} \) in an extension by \( H \) there is a countable (i.e. Cohen) subordering of \( H \) which adds \( \dot{x} \).

(b) The real \( \dot{g}(a) \) is Cohen over \( V \) for every \( a \in E \).

**Proof.** (a) By the above, \( H_{\text{supp } \dot{x}} \) is a regular subordering of \( H \).

(b) \( H_{\{ \dot{a} \}} \) is a regular subordering of \( H \), and the assertion follows by the definition of \( H_{\{ \dot{a} \}} \).

Observe that if \( \langle E_0, <_0 \rangle \) and \( \langle E_1, <_1 \rangle \) are isomorphic, then every isomorphism naturally extends to an isomorphism between \( H_{E_0} \) and \( H_{E_1} \) and to an isomorphism between the classes of \( H_{E_0} \)- and \( H_{E_1} \)-names. An \( H_{E_0} \)-name \( \dot{f}_0 \) and an \( H_{E_1} \)-name \( \dot{f}_1 \) are isomorphic iff there are \( \text{supp } \dot{f}_1 \subseteq A_i \subseteq E_i \) (\( i = 0, 1 \)) such that the posets \( \langle A_0, <_0 \rangle \) and \( \langle A_1, <_1 \rangle \) are isomorphic and the extension of the isomorphism sends \( \dot{f}_0 \) to \( \dot{f}_1 \).

We will describe the quotient \( H_{E}/H_{E_0} \), after a definition which is slightly more general than we need.

**Definition 4.2.** Let \( E = E_0 \cup E_1 \) and \( g_0 \) be an embedding of \( E_0 \) into \( \langle \omega, \leq^* \rangle \). For \( a, b \in E_0 \) let \( n_{ab} \) be the least positive integer \( n \) such that \( g_0(a) \leq^* n \) \( g_0(b) \) if such an \( n \) exists; otherwise let \( n_{ab} = 0 \). For \( p \in H \) let \( F^0_p = F_p \cap E_0 \) and \( F^1_p = F_p \cap E_1 \). We define the poset \( H_{E}(E_0, g_0) \) as the subordering of \( H \) consisting of all \( p \) such that:

\begin{itemize}
  \item[(H4)] \( F_p | F^0_p \times n_p \subset g_0 \), and
  \item[(H5)] if \( a <_E b \) are in \( F^0_p \), then \( n_{ab} \leq n_p \).
\end{itemize}

The ordering is inherited from \( H \).

So \( H_{E}(E_0, g_0) \) adds a generic \( \dot{g}_1 : (E \setminus E_0) \to \omega \) such that \( g_0 \cup \dot{g}_1 \) is an embedding. For \( p \) in this poset \( p|F^0_p \) is a side-condition and \( p|F^1_p \) is a working part. Note that without requiring \( (H5) \) the set of all conditions \( p \) such that \( n_p \geq n \) would not be dense in \( H_{E}(E_0, g_0) \) for every integer \( n \), and that \( H_{E}(E_0, g_0) \) need not be separative. [E.g. if \( E_0 \) and \( E_1 \) are incomparable
then \( \mathcal{H}_E(E_0, g_0) \) is equivalent to \( \mathcal{H}_E \), because if \( p, q \) in this poset are such that \( p \uparrow E_1 = q \uparrow E_1 \) then for all \( r \) we have \( r \perp p \) iff \( r \perp q \).

**Example 4.1.** An analogous result to Theorem 4.1 fails in the case of \( \mathcal{H}_E(E_0, g_0) \), namely there can be \( X \subseteq E \) such that \( \mathcal{H}_X(E_0 \cap X, g_0|X) \) is not a regular subordering of \( \mathcal{H}_E(E_0, g_0) \). E.g. if \( a \triangleleft b \) are such that \( a \in E_0 \setminus X \) and \( b \in E_1 \cap X \), then a condition \( q \) such that \( a, b \in F_q \) does not have a projection to \( \mathcal{H}_X(E_0 \cap X, g_0|X) \). This is because \( q \) forces that \( n_{ab} \) is at most \( n_q \), while \( \mathcal{H}_X(E_0 \cap X, g_0|X) \) by genericity forces that \( g(b) \) and \( g_0(a) \) are \( \leq \) incomparable.

The fact that the ordering on \( \mathcal{H}_E(E_0, g_0) \) is inherited from \( \mathcal{H}_E \) does not imply that the compatibility relation is inherited from \( \mathcal{H}_E \) as well; compare the following proposition with Proposition 4.2(a).

**Proposition 4.3.** Assume that \( E = E_0 \cup E_1 \) and that \( p, q \) are conditions in \( \mathcal{H}_E(E_0, g_0) \) such that \( n_p \geq n_q \). Then \( p \) and \( q \) are incompatible in \( \mathcal{H}_E(E_0, g_0) \) iff one of the following happens for \( q \in \{<, >\} \) (let \( F = F_p \cap F_q \), \( F^1 = F \cap E_i \) for \( i = 0, 1 \)):

1. \( p \) and \( q \) are incompatible in \( \mathcal{H}_E \), or
2. there are \( a \in F^1 \) and \( b \in F_q^0 \) such that \( b \in E_a, q \) forces \( f_p(a)(i) \) and \( g_0(b)(i) \) for some \( i \in [n_q, n_p] \), or
3. for some \( b_p \in F_p^0, b_q \in F_q^0 \) and \( a \in F^1 \) such that \( a \in (b_p, b_q)_E \) we have \( n_{b_p, b_q} > n_p \).

So in particular if \( F_p \cap F_q \subseteq E_0 \) then \( p \) and \( q \) are incompatible in \( \mathcal{H}_E(E_0, g_0) \) iff they are incomparable in \( \mathcal{H}_E \).

**Proof.** (\( \Rightarrow \)) If (\( \perp' \)) happens, then \( q \) forces that \( n_{ab} \leq n_q \) but \( p \) forces that \( n_{ab} \) is at least \( i + 1 \) for \( i \) as in (\( \perp' \)). So if \( r \leq p, q \) then \( r \) forces both—a contradiction. If (\( \perp' \)) happens, then \( p \) forces that \( n_{b_p, a} \leq n_p \), \( q \) forces that \( n_{b_q, a} \leq n_q \), so if \( p, q \) were compatible then this would imply that \( n_{b_p, b_q} \) is at most \( n_p = \max\{n_p, n_q\} \)—a contradiction.

(\( \Leftarrow \)) Suppose that \( p \) and \( q \) satisfy the negations of (\( \perp' \)), (\( \perp' \)) and (\( \perp' \)). Without loss of generality \( F_q \setminus F_p \) is a singleton \( \{c\} \). If \( c \in E_1 \), then we can prove that \( p \) and \( q \) are compatible exactly as in the proof of Theorem 4.1. So suppose that \( c \in E_0 \). Let \( n_c = \max\{n_p, n_{bc} : b \in F_p^0\} \). Let \( \pi \) be an integer greater than \( q_0(b)(i) \) and \( f_p(b)(i) \) for all \( b \in F_p \cup \{c\} \) and all \( i < n_r \). Define \( f_r(a)(i) \) for all \( a \in F_r^1 \) and \( i \in [n_p, n_r] \) by (letting \( \min\emptyset = 0 \) and \( \min\emptyset = \pi \))

\[
f_r(a)(i) = \begin{cases} \max\{g_0(b)(i) : b \in F_p(<_E a)\} & \text{if } c \notin E a, \\ \min\{g_0(b)(i) : b \in F_p(>_E a)\} & \text{if } c \in E a. \end{cases}
\]

By the choice of \( n_r \), the condition \( r \) is in \( \mathcal{H}_E(E_0, g_0) \). We claim that \( r \leq p, q \).

To check that \( r \leq p \), it is enough to check that (H3) is true for \( a, b \in F_p \). This checking splits into cases: pick \( i \in [n_p, n_r] \).
Case 1. If $b \in E_0$ then ($\mathcal{H}3$) follows immediately from ($\mathcal{H}5$), whether $a \in E_0$ or not.

Case 2. If $a <_E b \in F_p^1$, then we consider subcases.

Case 2.1. If $c <_E a$, then in defining $f_r(a)(i)$ and $f_r(b)(i)$ the first line of (†) applies, but $a <_E b$ implies $F_p^0(<_E a) \subseteq F_p^0(<_E b)$ and the maximum of a bigger set is bigger, so $f_r(a)(i) \leq f_r(b)(i)$.

Case 2.2. If $c \not<_E b$, then in defining $f_r(a)(i)$ and $f_r(b)(i)$ the second line of (†) applies, and the argument is similar to that of Case 2.1, bearing in mind that if $F_p^0(>_E b) = \text{then } f_r(b)(i) = \pi$ and $\pi$ is chosen to be large enough.

Case 3. If $a <_E c <_E b$, then $f_r(a)(i) = f_r(a')(i) \leq f_r(b')(i) = f_r(b)(i)$ for some $a' <_E a$ and $b' >_E b$.

So we have proved that $r$ extends $p$. Now we will assume that $r$ does not extend $q$, namely that ($\mathcal{H}3$) fails for $a \in F_p \cap F_q$, $c$ and $i \in [n_q, n_r]$.

Case 4. If $a <_E c$ and $f_r(a)(i) > g_0(c)(i)$, then if $i < n_p$ this is ($\perp_2$). If $i \geq n_p$ then there is $a' <_E a$ such that $g_0(a')(i) > g_0(c)(i)$, so this is ($\perp_3$).

Case 5. If $a >_E c$, then the discussion is the same as in Case 4.

So if $r$ does not extend $q$ then one of conditions ($\perp_1$)–($\perp_3$) applies, and the proposition is thus proved.

An embedding $\Phi : \mathcal{P} \to Q$ is dense iff $\Phi''\mathcal{P}$ is a dense subset of $Q$.

**Theorem 4.2.** Let $E = E_0 \cup E_1$ and let $\hat{y}_0$ be an $\mathcal{H}_{E_0}$-name for the generic embedding of $E_0$ into $\langle \omega, \leq^* \rangle$. Then in a forcing extension by $\mathcal{H}_{E_0}$ the posets $\mathcal{H}_E/\mathcal{H}_{E_0}$ and $\mathcal{H}_E(E_0, \hat{y}_0)$ are equivalent.

**Proof.** By [18, VII.7.11] it is enough to find (working in a ground model) a dense embedding of $\mathcal{H}_E$ into $\mathcal{H}_{E_0} \ast \mathcal{H}_E(E_0, \hat{y}_0)$. Let $p \mapsto \langle p | E_0, \hat{p} \rangle$. This mapping is obviously an orderrmorphism. The set of all $\langle \bar{p}, \bar{q} \rangle$ such that $\bar{q}$ is “decided” (i.e. it is an element of $\mathcal{H}_{E}(E_0, \hat{y}_0)$ instead of an $\mathcal{H}_{E_0}$-name) is dense in the iteration. So we will start from such $\langle \bar{p}, \bar{q} \rangle$ and find $p$ in $\mathcal{H}_E$ such that $\langle p | E_0, p \rangle$ extends $\langle \bar{p}, \bar{q} \rangle$ in the iteration. We claim that $\bar{p}$ and $\bar{q}$ are compatible in $\mathcal{H}_E$ since $\bar{p}$ forces that $\bar{q}$ is in $\mathcal{H}_E(E_0, \hat{y}_0)$, $f_{\bar{p}} \cup f_{\bar{q}}$ must be a function, and we must also have $F_{\bar{q}} \subseteq F_{\bar{p}}$ and $n_{\bar{q}} \leq n_{\bar{p}}$. [If one of these fails then $f_r \cup f_q$ is not a function for some $r \leq \bar{p}$ in $\mathcal{H}_{E_0}$.] So ($\mathcal{H}5$) for $\bar{q}$ implies that ($\perp_2$) of Proposition 4.2 fails, so $\bar{p}$ and $\bar{q}$ are compatible in $\mathcal{H}_E$. Pick $p \in \mathcal{H}_E$ which extends $\bar{p}$ and $\bar{q}$. But $p \in \mathcal{H}_E$ implies that $p\upharpoonright E_0 \upharpoonright p \in \mathcal{H}_E(E_0, \hat{y}_0)$, and therefore $p\upharpoonright E_0 \upharpoonright p \leq \bar{q}$ (in $\mathcal{H}_E(E_0, \hat{y}_0)$) so $\langle p | E_0, p \rangle$ extends $\langle \bar{p}, \bar{q} \rangle$.]

If $E_0$ and $E_1$ are incomparable then in $\mathcal{H}_E(E_0, \hat{y}_0)$ side-conditions from $E_0$ are void so this poset is equivalent to $\mathcal{H}_{E_1}$. Therefore the following properties of $\mathcal{H}_E$ are immediate consequences of the above statements:
(H6) If \( E = E_0 \cup E_1 \) and \( E_0 \) and \( E_1 \) are incomparable, then \( \mathcal{H}_E \) is isomorphic to \( \mathcal{H}_{E_0} \times \mathcal{H}_{E_1} \).

(\( \mathcal{H}7 \)) If \( E = \bigcup_{i \in I} E_i \) and \( E_i \)'s are pairwise incomparable, then \( \mathcal{H}_E \) is isomorphic to a finite support product of \( \{ \mathcal{H}_E_i \} \).

The following two lemmas will be crucial in the proof of Theorem 9.1.

**Lemma 4.2.** If \( X \subseteq E_0 \) is such that \( X(\prec E \ a) \) is cofinal in \( E_0(\prec E \ a) \) and \( X(\succ E \ a) \) is coinitial in \( E_0(\succ E \ a) \) for all \( a \in E \setminus E_0 \), then in an extension by \( \mathcal{H}_{E_0} \) the poset \( \mathcal{H}_E(X, \dot{g}_0|X) \) is a dense subordering of \( \mathcal{H}_E(E_0, \dot{g}_0) \).

**Proof.** Let \( E' = E_1 \cup X \). Note that the set \( D \) of all \( p \in \mathcal{H}_E(E_0, \dot{g}_0) \) such that \( a \varrho b \) implies that \( a \varrho c \varrho b \) for some \( c \in E_0^0 \cap X \) for all \( \varrho \in \{ \leq E, \geq E \} \), \( a \in F_{\varrho}^0 \) and \( b \in F_{\varrho}^1 \) is dense in \( \mathcal{H}_E(E_0, \dot{g}_0) \).

**Claim.** Conditions \( p \) and \( q \) are compatible iff \( p \) and \( q|E' \) are compatible for all \( q \in D \) and \( p \in \mathcal{H}_{E'}(X, \dot{g}_0|X) \).

**Proof.** We prove only the nontrivial direction, \( (\Leftarrow) \). Suppose that \( p \) and \( q \) are incompatible. Then by Proposition 4.3 one of (\( \Leftarrow' \))–(\( \Leftarrow'3 \)) is true. If (\( \Leftarrow'1 \)) applies, then it applies for some \( a \in E_0^0 = F_{\dot{q}|E'}^1 \), so \( p \) and \( q|E' \) are incompatible. If (\( \Leftarrow'2 \)) fails for \( a \in F_{\dot{q}}^0 \cap F_{\dot{q}}^1 \) and \( b \in F_{\dot{q}}^0 \), then there is \( c \in F_{\dot{q}}^0 \cap X \) such that (\( \Leftarrow'2 \)) fails for \( a \) and \( c \); similarly if (\( \Leftarrow'3 \)) fails, and so the claim is verified.

So \( \mathcal{H}_{E'}(X, \dot{g}_0|X) \) is a dense subordering of \( \mathcal{H}_E(E_0, \dot{g}_0) \).

**Lemma 4.3.** If \( E = E_0 \cup E_1 \cup E_2 \) and

\[
 a \varrho b \quad \text{implies that} \quad a \varrho c \varrho b \quad \text{for some} \quad c \in E_0 
\]

for all \( a \in E_1 \), \( b \in E_2 \) and \( \varrho \in \{ \prec E, \succ E \} \), then \( \mathcal{H}_E \) is equivalent to

\[
 \mathcal{H}_{E_0} \ast (\mathcal{H}_{E_1}(E_0, \dot{g}_0) \times \mathcal{H}_{E_2}(E_0, \dot{g}_0))
\]

**Proof.** By Theorem 4.2, \( \mathcal{H}_E \) is equivalent to \( \mathcal{H}_{E_0} \ast \mathcal{H}_{E_1}(E_0, \dot{g}_0) \ast \mathcal{H}_{E_2}(E_0 \cup E_1, \dot{g}_0 \cup \dot{g}_1) \). By (*) and Lemma 4.2, the posets \( \mathcal{H}_{E_2}(E_0 \cup E_1, \dot{g}_0 \cup \dot{g}_1) \) and \( \mathcal{H}_{E_2}(E_0, \dot{g}_0) \) are equivalent; but the definition of the latter does not depend on the generic object for \( \mathcal{H}_{E_1}(E_0, \dot{g}_0) \), so we are in the product situation and the lemma is proved.

In the following statement, cf (ci) stands for the cofinality (respectively, coinitiality) of a partially ordered set \( E \), namely the smallest size of a set \( D \subseteq E \) which is cofinal (resp. coinitial) in \( E \).

**Proposition 4.4.** If \( E = \bigcup_{\xi \prec \kappa} E_\xi \) is a disjoint union, then we can write \( \mathcal{H}_E \) as a finite support ccc iteration \( \langle P_\xi, \dot{Q}_\xi \rangle_{\xi < \kappa} \) such that \( \dot{g} \) is a \( \dot{Q}_\xi \)-name for the generic embedding of \( Q \) into \( \langle \omega, \leq^* \rangle \):

1. \( \models_\xi \dot{g}|E_\xi \) is a \( \dot{Q}_\xi \)-name”
2. \( \models_\xi |\dot{Q}_\xi| \leq |\dot{E}_\xi| + |\bigcup_{\eta < \xi} \dot{E}_\eta| \), or more precisely
(3) $\mathcal{P}_\xi$ forces that $\hat{Q}_\xi$ is of size $|E_\xi| + \sum_{a \in E_\xi, n < \xi}(c\ell(E_p(<_E a)) + c\ell(E_p(>_E a)))$.

Proof. Let $E^+_\xi = \bigcup_{\alpha < \xi} E_\alpha$, let $\mathcal{P}_\alpha = \mathcal{H}_{E^+_\alpha}$, and let $\hat{g}_\xi$ be the $\mathcal{P}_\xi$-name for the generic embedding of $E^+_\xi$ into $^\omega \omega$ and let $\hat{Q}_\xi = \mathcal{H}_{E^+_{\xi+1}}(E^+_\xi, \hat{g}_\xi)$. Then the conclusion follows immediately from the previous discussion.

The following lemma will be used in §6 when proving that gaps in $E$ get mapped into gaps in $^\omega \omega$ by $\hat{g}$.

Lemma 4.4. If $E = E_0 \cup E_1$, $E_1$ is at most countable and for all $c \in E_1$ at most one of the sets $E_0(<_E c)$, $E_0(>_E c)$ is nonempty, then $\mathcal{H}_{E_0}$ forces that the poset $\mathcal{H}_{E_0}(E_0, \hat{g}_0)$ is $\sigma$-centered.

[Note that some assumption in this lemma is necessary, because if $\mathcal{H}_{E_0}$ generically adds an $\langle \omega_1, \omega_1 \rangle$-gap to $^\omega \omega$ and $\mathcal{H}_{E}(E_0, \hat{g}_0)$ fills it, then the ccc-ness of the latter poset is not absolute, so it cannot be $\sigma$-centered.]

Proof. In an extension by $\mathcal{H}_{E_0}$, from Proposition 4.2 it follows that $p$ and $q$ in $\mathcal{H}_{E}(E_0, \hat{g}_0)$ such that $p|E_1 = q|E_1$ are compatible. There are at most countably many distinct $p|E_1$ for $p \in \mathcal{H}_{E}(E_0, \hat{g}_0)$, so this poset is $\sigma$-centered. ■

Remark. Lemma 4.4 remains true when the assumption that $E_1$ is countable is replaced by the weaker $|E_1| \leq |E_0|^{|\mathbb{N}} + \epsilon$, i.e. that $E_1$ is of size at most continuum in the extension by $\mathcal{H}_{E_0}$. The proof of this claim is completely analogous to the familiar proof of the fact that the iteration of $\sigma$-centered posets of length at most $\epsilon$ is $\sigma$-centered. However, we do not need this extension and find the present formulation of Lemma 4.4 esthetically more pleasing.

We finish this section with two lemmas which are not being used elsewhere in this text, but which shed some light on the poset $\mathcal{H}_{E}$. For conditions $p, q$ in a poset $\mathcal{P}$ the meet of $p$ and $q$ is the condition $p \wedge q$ which extends both $p$ and $q$ and is minimal with this property, namely every $r \in \mathcal{P}$ which extends both $p$ and $q$ extends $p \wedge q$ as well. Of course, meets always exist in the regular open algebra of $\mathcal{P}$, but not necessarily in $\mathcal{P}$.

Lemma 4.5. If $E$ is not an antichain, then there are compatible $p$ and $q$ in $\mathcal{H}_{E}$ such that the meet $p \wedge q$ does not exist in $\mathcal{H}_{E}$.

Proof. Pick $a <_E b$ in $E$ and let $p = \langle \{a\}, 0, \emptyset \rangle$ and $q = \langle \{b\}, 0, \emptyset \rangle$. Then $p$ and $q$ are compatible and $r \leq p, q$ iff $a, b \in F_r$. Also $r \Vdash \hat{g}(\bar{a}) \leq^*_n \hat{g}(\bar{b})$ for $r \leq p, q$ and $n = n_r$. But obviously we can pick $r' \leq p, q$ so that $f_r(a)(n_r) > f_r(b)(n_r)$, so $r'$ and $r$ are incompatible. ■

Lemma 4.6. (a) If $E = E_0 \cup E_1$ and $E_1$ is infinite, then $\mathcal{H}_{E}(E_0, \hat{g}_0)$ adds a Cohen real over $V^{\mathcal{H}_{E_0}}$. 

$\Box$
In an extension by $\mathcal{H}_E$,

(b) $\text{MA}_{<|E|}$ (countable) is true, and

(c) if $E$ is uncountable then the bounding number $\mathfrak{b}$ is equal to $\omega_1$.

Proof. (a) Work in $V^{\mathcal{H}_0}$. For a 1-1 sequence $\vec{a} = \{a_\eta\}$ in $E_1$ define the $\mathcal{H}_E(E_0, \check{g}_0)$-name for a real $\dot{r}_{\vec{a}}$ by

$$\dot{r}_{\vec{a}}(\check{n}) = \check{g}(\check{a}_n)(\check{0})$$

for all $n < \omega$.

It is easily checked that $\dot{r}_{\vec{a}}$ is Cohen over the intermediate extension.

(b) Since all countable posets are equivalent to $\mathcal{C}$, it is enough to prove that in an extension by $\mathcal{H}_E$ for every $\kappa < |E|$ and a family $\{D_\xi\} (\xi \leq \kappa)$ of dense open subsets of $\mathbb{R}$ there is a real in $\bigcap_{\xi < \kappa} D_\xi$. Each $D_\xi$ is coded by a single real $r_\xi$ which is added by $\mathcal{H}_{A_\xi}$ for some countable $A_\xi \subseteq E$. Let $E_0 = \bigcup_{\xi < \kappa} A_\xi$. Then $E_1 \setminus E_0$ is infinite and all $D_\xi$'s are in the intermediate extension by $\mathcal{H}_{E_0}$, so an application of (a) finishes the proof.

(c) Pick disjoint countable sequences $\vec{a}_\xi (\xi < \omega_1)$ in $E$ and let $\dot{f}_\xi$ be an $\mathcal{H}_E$-name for a function such that $\dot{f}_\xi (\check{\eta}) \geq^* \dot{r}_{\vec{a}_\xi}$ for all $\eta < \xi < \omega_1$ and $\dot{f}_\eta \leq^* \dot{f}_\xi$ for all $\eta < \xi$. Let $\dot{h}$ be an $\mathcal{H}_E$-name for some element of $\omega_\omega$; then supp $\dot{h}$ is disjoint from some $\vec{a}_\xi$, and $\dot{h} \not\leq \dot{r}_{\vec{a}_\xi}$ by (a), so the sequence $\{\dot{f}_\xi\}$ is unbounded.

5. Five lemmas. This chapter contains a couple of similarly looking lemmas (cf. also [3, Lemma 2.5]). Here $\mathcal{P}^0$ and $\mathcal{P}^1$ always stand for posets, and $\mathcal{P}^0 \times \mathcal{P}^1$-names with superscript $i = 0, 1$ are assumed to be $\mathcal{P}^i$-names (i.e. $\dot{f}^i$ is a $\mathcal{P}^i$-name for $i = 0, 1$).

Lemma 5.1. If $\mathcal{P} = \mathcal{P}^0 \times \mathcal{P}^1$ forces that $\dot{f}^0 \leq^* \dot{f}^1$ then there are a $\mathcal{P}^0 \times \mathcal{P}^1$-condition $p$ and a ground-model function $f$ such that $p \forces \dot{f}^0 \leq^* \dot{f}^1$.

Proof. Pick $p = \langle p^0, p^1 \rangle$ in $\mathcal{P}$ which decides an integer $n$ such that $\dot{f}^0 \leq^n \dot{f}^1$. For every integer $m$ let $k_m$ be the minimal integer such that $r_m \forces p^1 \dot{f}^1(m) = k_m$ for some $r_m \leq p^1$. Then obviously $p \forces \dot{f} \leq^n \dot{f}^1$. Suppose that $q \forces p^0 \times p^1 \check{f}^0(\check{m}) = l$, for some $q \leq p$ and $l \leq k_m$. Then $q = \langle q_0, q_1 \rangle$, and $\langle q_0, r_m \rangle \forces \check{f}^0(\check{m}) = l \wedge \check{f}^1(\check{m}) = k_m$, so $m \leq n$ and $p \forces \dot{f}^0 \leq^n \dot{f}^1$.

Remark. The conclusion of the above lemma cannot be strengthened to $\forces \dot{f}^0 \leq^* \dot{f} \leq^* \dot{f}^1$, namely going below a condition $p$ to decide this statement is necessary, as the following example shows. Let $\mathcal{P}^1$ be the poset for adding a Cohen real and let $\mathcal{P}^0$ be any nontrivial ccc poset. $\dot{f}^0$ is a name for the constant function, such that for every $n \in \omega$ there is $p_n \in \mathcal{P}^0$ forcing that $\dot{f}^0$ is identically equal to $n$. If $\dot{c}$ is a $\mathcal{P}^0$-name for a Cohen real (in $\omega_\omega$), then $\dot{f}^1$ is defined by $\dot{f}^1(n) = \sum_{i \leq n} \dot{c}(i)$. Since $\dot{f}^1$ is nondecreasing and unbounded,
the assumptions of Lemma 5.1 are fulfilled, but no ground-model function \( f \) is between \( f^0 \) and \( f^1 \) with probability one.

The following is a version of Lemma 5.1 specific for \( \mathcal{H}_E \).

**Lemma 5.2.** Let \( \dot{f}_0 \) and \( \dot{f}_1 \) be \( \mathcal{H}_E \)-names such that:

1. \( \models \dot{f}_0 \leq^* \dot{f}_1 \),
2. \( A, A_0, A_1 \) are pairwise disjoint countable subsets of \( E \) such that \( \supp \dot{f}_i \subseteq A \cup A_i \), for \( i = 0, 1 \).

Then there are a finite \( F \subseteq A_1 \), an \( \mathcal{H}_{A \cup F} \)-name \( \dot{h} \) for an element of \( {}^{\omega} \omega \) and a condition \( q \) in \( \mathcal{H}_E \) so that \( q \models \dot{f}_0 \leq^* \dot{h} \leq^* \dot{f}_1 \).

**Proof.** Pick a condition \( q \in \mathcal{H}_E \) and \( \bar{p}, \bar{q} \in \omega \) so that

\[
q \models E \dot{f}_0 \leq \bar{n} \dot{f}_1.
\]

Let \( F = F_q \cap A_1 \), go to an extension by \( \mathcal{H}_{A \cup F} \) below \( q \upharpoonright (A \cup F) \), and let \( \dot{g} \) denote the generic embedding of \( A \cup F \) into \( {}^{\omega} \omega \). It is enough to prove that in this intermediate extension there is a function \( h \) such that in the rest of \( \mathcal{H}_E \) (namely, in \( \mathcal{H}_E(A \cup F; \dot{g}_0) \)) the condition \( q \) forces that \( \dot{f}_0 \leq \bar{n} \dot{h} \leq \bar{n} \dot{f}_2 \). For \( m \in \omega \) let \( k_m \) be the minimal integer such that for some \( r_m \) in \( \mathcal{H}_{A \cup A_1}(A \cup F; \dot{g}_0) \) we have

\[
\models r_m \models^G \dot{A}_1(\dot{f}_1(\bar{m})) = k_m.
\]

Let \( h(m) = k_m \). Check that \( h \) works: obviously \( q \models \dot{h} \leq^* \dot{f}_1 \), so it remains to prove that \( q \models \dot{f}_0 \leq^* \dot{h} \). Notice that \( q, r_m \) and \( p \) are compatible for every \( p \in \mathcal{H}_E(A \cup F; \dot{g}_0) \) such that \( F_p \cap A_1 \subseteq F \). \( [p] \) and \( r_m \) are trivially compatible because \( f_p \) and \( f_{r_m} \) are compatible on \( F_p \cap F_{r_m} \subseteq F_q \), but we have to find \( p' \leq p, r_m \) which extends \( q \). Since \( F_p \cap F_{r_m} \supseteq F_q \) and \( \dot{g} \upharpoonright F_q \) is already decided, every \( p' \) extending \( p \) and \( r_m \) is compatible with \( q \). Pick \( m > \bar{p} \) and \( p \in \mathcal{H}_E(A \cup F; \dot{g}_0) \) below \( q \) which decides for the value of \( \dot{f}_0(m) \), say

\[
p \models k = \dot{f}_0(\bar{m}) > \dot{h}(\bar{m})
\]

for some \( k \in \omega \). Notice that we can assume that \( p \) is in \( \mathcal{H}_{A \cup A_0}(A \cup F; \dot{g}_0) \), therefore \( p, q, r_m \) are compatible. So \( k \leq k_m \), and therefore \( q \models \dot{f}_0 \leq^* \dot{h} \) so \( h \) and \( q \) are as claimed. \( \blacksquare \)

**Lemma 5.3.** If \( \kappa > \min(\omega, |\mathcal{P}|) \) is a regular uncountable cardinal, \( \mathcal{P}^0 \times \mathcal{P}^1 \) is ccc and forces that \( f^0_\xi \leq^* f^0_\eta \leq^* f^1_\eta \) for all \( \xi < \eta < \kappa \), then \( q \models f^0_\xi \leq^* \dot{h} \leq^* f^1_\xi \) for all \( \xi < \kappa \), for some \( q \in \mathcal{P}^0 \times \mathcal{P}^1 \) and some ground-model \( h \).

**Proof.** By Lemma 5.1, for each \( \xi < \kappa \) there are \( q_\xi = (q^0_\xi, q^1_\xi) \), \( h_\xi \) such that \( q_\xi \models f^0_\xi \leq^* h_\xi \leq^* f^1_\xi \).

**Case 1.** If \( \kappa > \omega \), then there is \( h \) such that the set \( X \) of all \( \xi \) such that \( h_\xi = h \) is of size \( \kappa \). By ccc-ness, let \( q \) be the condition which forces that there
are \( \kappa \) many \( q_\xi \)'s in the generic filter \( \dot{G} \). Then in an extension by \( P^0 \times P^1 \) below \( q \), for every \( \eta \) there is \( \xi > \eta \) such that \( q_\xi \) is in the generic filter \( \dot{G} \), so \( \dot{f}_\xi \leq^* \dot{f}_\kappa \), and \( q, h \) are as required.

**Case 2.** If \( \kappa > |P^1| \), then there is \( q^1 \in P^1 \) such that the set \( X \) of all \( \xi \) for which \( q^1 = q_\xi^1 \) is of size \( \kappa \). Let \( q_0 \in P^0 \) be a condition which forces that \( q_\xi^0 \) is in a \( P^0 \)-generic filter for \( \kappa \) many \( \xi \in X \). Define a function \( h \) as in the proof of Lemma 5.1: let \( h(m) = \kappa \) be the minimal \( k_m \), such that \( r_m \Vdash P^1 \dot{f}_\xi^1(m) = \dot{k}_m \). Then \( q \Vdash \dot{h} \leq^* \dot{f}_\kappa^1 \) and \( q_\xi \Vdash \dot{f}_\xi^0 \leq^* \dot{h} \). The proof that \( q \) and \( h \) are as required is the same as in Case 1. ■

**Remark.** The assumptions of Lemma 5.3 are in some sense necessary, because it is possible that \( P^0 \) adds a \( \lambda \)-chain \( \{ \dot{f}_\xi^0 \} \) and \( P^1 \) adds \( \dot{f}_\lambda^1 \) above it, but \( P^0 \times P^1 \) forces that no ground-model function is between all \( \dot{f}_\xi^0 \)'s and \( \dot{f}_\lambda^1 \). We can even pick \( P^1 \) to be \( \sigma \)-centered and assume that there are no \( \omega_2 \)-chains in \( \omega \) in the ground model. To see this, let our ground model be an extension of a model of \( CH \) obtained by adding \( \kappa \) many Cohen reals \( \{ c_\xi : \xi < \kappa \} \) to \( \omega \). Let \( P^1 \) force that \( \dot{c}_\xi \leq^* \dot{f}_\kappa^1 \), and let \( P^0 \) force that \( \dot{f}_\xi \leq^* \dot{f}_\xi^0, \dot{c}_\eta \) for all \( \xi < \eta < \kappa \). Then by genericity there is no \( h \) in \( V[\{ c_\xi : \xi < \kappa \}] \) as in the conclusion of Lemma 5.3. There is a similar example connected to the result of Lemma 5.5 below, where \( P^0 \times P^1 \) is taken to be \( \mathcal{H}E(E_0, \dot{g}_0) \) for appropriate \( E \) and \( E_0 \).

The following lemma will be used in the proof of Theorem 6.1.

**Lemma 5.4.** Let \( \kappa \) be an uncountable regular cardinal. If \( \mathcal{H}_E \)-names \( \dot{f}_\xi \) \((\xi \leq \kappa)\) and \( A \subseteq E \) are such that for all \( \xi < \eta \leq \kappa \):

1. \( \Vdash_{\mathcal{H}_E} \dot{f}_\xi \leq^* \dot{f}_\eta \),
2. \( \text{supp} \dot{f}_\xi \subseteq A_\xi \), and
3. \( A_\xi \cap A_\kappa \subseteq A \) for all \( \xi < \kappa \),

then there are an \( \mathcal{H}_E \)-name \( \dot{h}_\kappa \) and a condition \( q \) such that \( \text{supp} \dot{h}_\kappa \subseteq^* A \) and \( q \Vdash \dot{f}_\xi \leq^* \dot{h}_\kappa \leq^* \dot{f}_\kappa \) for all \( \xi < \kappa \).

**Proof.** For each \( \xi < \kappa \) find \( q_\xi \) and \( n_\xi \) so that \( q_\xi \Vdash \dot{f}_\xi \leq^{n_\xi} \dot{f}_\kappa \). Then we can assume that \( \pi = n_\kappa \) for all \( \xi \) and some \( \pi \), that \( \{ F_{q_\xi} \} \) is a \( \Delta \)-system with root \( F \) and that \( q_\xi \Vdash F = q \) for some fixed \( q \) in \( \mathcal{H}_E \). Go to an extension by \( \mathcal{H}_{A \cup F} \) below \( q \), let \( \dot{g}_0 \) denote the generic embedding, and define a function \( \dot{h}_\kappa \) as follows: for all \( m \in \omega \) let \( k_m \) be the minimal integer such that \( r_m \Vdash \dot{f}_\kappa^1(m) = \dot{k}_m \) for some \( r_m \in \mathcal{H}_E(A \cup F, \dot{g}_0) \). This defines \( \dot{h}_\kappa \), an \( \mathcal{H}_E \)-name below a condition \( q \). Obviously \( q \Vdash \dot{h}_\kappa \leq^* \dot{f}_\kappa \), and as in Lemma 5.1 we have \( q_\xi \Vdash \dot{f}_\xi \leq^* \dot{h}_\kappa \) for all \( \xi < \kappa \). To prove that \( q \Vdash \dot{f}_\xi \leq^* \dot{h}_\kappa \) for all \( \xi < \kappa \), pick a condition \( p \leq q \) and \( \xi < \kappa \). Find \( \eta > \xi \) such that \( q_\eta \) and \( p \) are compatible (this is possible because \( \{ F_{q_\xi} \} \) is a \( \Delta \)-system with root \( F \) and \( p | F \) extends...
$q_n | F = q$ for all $\eta$). Then $p \land q_n$ forces that $\dot{f}_\xi \leq^* \dot{f}_\eta \leq^* \dot{h}_\kappa$. So the set of all conditions which force that $\dot{f}_\xi \leq^* \dot{h}_\kappa$ is dense below $q$. ■

The following lemma is used in the proof of Theorem 9.2. For the definition of pregap see Definition 6.1.

**Lemma 5.5.** If $\mathcal{P}_0 \times \mathcal{P}^1$ forces that $(\dot{f}_\xi^0, \dot{f}_\eta^1)_{\xi<\kappa, \eta<\lambda}$ is a pregap, $\kappa, \lambda$ are regular and uncountable, $\kappa > |\mathcal{P}^1|$, $\mathcal{P}_0 \times \mathcal{P}^1$ is ccc, and every real in an extension by $\mathcal{P}_0$ is added by its regular subordering of size less than $\lambda$, then $q \Vdash \dot{f}_\xi^0 \leq^* \dot{h} \leq^* \dot{f}_\eta^1$ for all $\xi < \kappa, \eta < \lambda$, for some $q \in \mathcal{P}_0 \times \mathcal{P}^1$ and some ground-model $h$.

**Proof.** For all $\xi < \kappa$ let $\mathcal{P}_0^\xi$ be a small regular subordering of $\mathcal{P}_0$ which adds $\dot{f}_\xi^0$. Apply Lemma 5.3 (actually, its dual form) to get a condition $q_\xi = (q_\xi^0, q_\xi^1)$ and a ground-model $h_\xi$ such that $q_\xi \Vdash \dot{f}_\xi^0 \leq^* h_\xi \leq^* \dot{f}_\eta^1$ for all $\eta < \lambda$. There is a ground-model $h$ such that the set $X$ of all $\xi < \kappa$ such that $h = h_\xi$ is of size $\kappa$. Let $q$ be a $\mathcal{P}_0 \times \mathcal{P}^1$ condition which forces that there are $\kappa$ many $\xi \in X$ such that $q_\xi$ is in a generic filter; then $q$ forces that $h$ splits the pregap. ■

6. $\omega$ in an extension by $\mathcal{H}_E$. Since $\dot{g}(a)$ is Cohen generic over the universe for all $a \in E$, the set $\dot{g}''E$ is never cofinal in $\omega$ (compare with [13] and [4]), but it does have some unboundedness properties.

**Lemma 6.1.** (a) Let $\dot{h}$ be an $\mathcal{H}_E$-name for an element of $\omega$ with support $A$ and let $a \in E \setminus A$ be such that $A(\geq_E a)$ is empty. Then $\Vdash \dot{h} \nleq^* \dot{g}(a)$.

(b) $X \subseteq E$ is not countably bounded if it is forced that $\dot{g}''X$ is unbounded in $\omega$.

**Proof.** (a) Suppose the contrary, i.e. that some condition $p$ forces that $\dot{h} \geq^* \dot{g}(a)$; without loss of generality we have $n_p = n$. Then extend $p \cup A$ to $q \in \mathcal{H}_A$ which decides that $h(\dot{a} + 1) = k$ for some integer $k$. Now extend $p$ and $q$ to $r$ so that $f_r(a)(n + 1) = k + 1$; this is possible because $a \nleq_E b$ for all $b \in A$. It is also the desired contradiction.

(b) ($\Rightarrow$) If $\dot{h}$ is forced to dominate $\dot{g}''X$, then by (a), supp $\dot{h}$ witnesses that $X$ is countably bounded. ($\Leftarrow$) If $X$ is countably bounded by $\{a_\eta\}$, then let $h$ be a name such that it is forced that $\dot{g}(a) \leq^* h$ for all $n < \omega$. Then $h$ bounds $\dot{g}''X$. ■

**Definition 6.1.** A $\kappa$-limit (sometimes also called $\langle \kappa, 1 \rangle$-gap) is an indexed family $\langle a_\xi, b_\xi \rangle_{\xi<\kappa}$ such that $a_\xi <_E a_\eta <_E b_\eta$ for all $\xi < \eta < \kappa$, and for all $c \in E$ such that $a_\xi <_E c$ for all $\xi < \kappa$ we have $b \leq_E c$. A gap is an indexed family $\langle a_\xi, b_\xi \rangle_{\xi<\kappa, \eta<\lambda}$ such that:

1. $a_\xi <_E a_\eta <_E b_\beta <_E b_\alpha$ for all $\xi < \eta < \kappa$ and all $\alpha < \beta < \lambda$, and
2. no $c \in E$ is such that $a_\xi <_E c <_E b_\alpha$ for all $\xi < \kappa$ and all $\alpha < \lambda$. 
If a family \( \langle a_\xi, b_\eta \rangle_{\xi < \kappa, \eta < \lambda} \) satisfies only condition (1), then we call it a \textit{pregap}; if (2) fails for some \( c \), then we say that \( c \text{ splits} \) a pregap. A gap \( \langle a_\xi, b_\eta \rangle_{\xi < \kappa, \eta < \lambda} \) is \textit{tight} if there is no \( c \in E \) such that one of the following happens:

(3) \( a_\xi <_E c \) for all \( \xi \) but \( b_\eta \not<_E c \) for all \( \eta \),

(4) \( b_\xi >_E c \) for all \( \eta \) but \( a_\xi \not>_E c \) for all \( \xi \).

We always assume that \( \kappa \) and \( \lambda \) are uncountable regular cardinals; this assumption is, in our context of investigating \( \langle \kappa, \lambda \rangle \)-gaps, not a loss of generality.

**Lemma 6.2.** Suppose that \( \text{supp } h = A \), that \( a, b \in E \) are such that the posets \( \langle A \cup \{a\}, <_E \rangle \) and \( \langle A \cup \{b\}, <_E \rangle \) are isomorphic, and that an isomorphism fixes all elements of \( A \) and sends \( a \) to \( b \). Then \( \models_E h \leq^* \hat{g}(a) \) iff \( \models_E h \leq^* \hat{g}(b) \).

**Proof.** Suppose that some \( p \) forces that \( h \leq^* \hat{g}(a) \); we can assume that \( p \models_E h \leq^* \hat{g}(a) \). If \( p = p|A \cup \{a\} \) then \( p \models_{A \cup \{\} } h \leq^* \hat{g}(a) \) by Theorem 4.1. Let \( p' \) be a condition isomorphic to \( p \) but with \( a \) replaced by \( b \). Then \( p' \models_{A \cup \{b\} } h \leq^* \hat{g}(b) \).

**Theorem 6.1.** (a) A \( \hat{g} \)-image of a limit \( \langle a_\xi, b_\xi \rangle_{\xi < \kappa} \) in \( E \) is a limit in \( \omega \cdot \omega \).

(b) A \( \hat{g} \)-image of a \( \langle \kappa, \omega \rangle \)-gap \( \langle a_\xi, b_\xi \rangle_{\xi < \kappa, i < \omega} \) in \( E \) is a gap in \( \omega \cdot \omega \).

(c) A \( \hat{g} \)-image of a gap \( \langle a_\xi, b_\xi \rangle_{\xi < \kappa, \eta < \lambda} \) in \( E \) is a gap in \( \omega \cdot \omega \).

(d) A \( \hat{g} \)-image of an unbounded chain \( \langle a_\xi \rangle_{\xi < \kappa} \) is an unbounded chain in \( \omega \cdot \omega \).

**Proof.** (a) All we have to prove is that if \( \models \ hat{h} \not\leq^* \hat{g}(b) \) then there is \( \xi < \kappa \) and a condition \( p \) such that \( p \models \ hat{h} \not\leq^* \hat{g}(a_\xi) \) for every \( \mathcal{H}_E \)-name \( \hat{h} \). So assume the contrary, let \( A = \text{supp } \hat{h} \). Since for each element \( a \in A \) either there is \( \xi < \kappa \) such that \( c \not>_E a_\xi \) or \( c >_E b \), we may assume that

\[ (*) \quad \text{if } a_0 <_E c \text{ then } b <_E c \text{ for all } c \in A. \]

Also assume that \( b \in A \) and that \( E = A \cup \{a_\xi : \xi < \kappa\} \) (by Theorem 4.1 we can do this). Go to an extension by \( \mathcal{H}_A \). Let \( C = \{n \in \omega : \hat{h}(n) < \hat{g}_A(b_0)(n)\} \). Then \( C \) is infinite. We claim that in a further extension by \( \mathcal{H}_E(A, \hat{g}_A) \), there are infinitely many \( n \in C \) such that \( \hat{g}_A(a_0)(n) = \hat{g}_A(b_0)(n) \). Suppose the contrary, that some \( q \) forces that \( \hat{g}_A(a_0)(n) < \hat{g}_A(b_0)(n) \) for all \( n \in C \setminus \overline{m} \) for some \( \overline{m} \). We can assume that \( b \in F_q \). For all \( c \in F_p \cap A \) the condition \( (*) \) is true, so we can pick \( n \geq \overline{m}, n_q \in C \) and find \( p \leq q \) such that \( f_p(a_0)(n) = \hat{g}(b)(n) \); this is a contradiction and it finishes the proof of (a).

(b) Assume \( \hat{h} \) is such that \( \models \hat{g}(a_\xi) \leq^* \hat{h} \) for all \( \xi < \kappa \). Apply Lemma 5.4 (with \( A = \emptyset \)) and get \( q, \hat{h}' \) such that \( q \models \hat{g}(a_\xi) \leq^* \hat{h}' \leq^* \hat{h} \) for all \( \xi < \kappa \) and
\( B = \text{supp} \hat{h} \) is finite. We can assume that \( a_0 <_E c \) implies \( a_\xi <_E c \) for all \( \xi < \kappa \) and all \( c \in B \). Go to an extension by \( \mathcal{H}_B \) and pick \( \overline{\pi} < \omega \) such that \( b_\overline{\pi} \not\in F_q \cup B \) and

\[
(\ast) \quad c <_E b_\overline{\pi} \text{ implies } c \not\in_E a_0 \text{ for all } c \in B.
\]

Let \( B_0 = \{ c \in B : c >_E a_0 \} \) and let \( \hat{h}_0 \) be defined by \( \hat{h}_0(m) = \min\{\hat{g}(c)(m) : c \in B_0 \} \). Suppose that \( h' \not\in^* \hat{h}_0 \), let \( C = \{ n : h'(n) < \hat{h}_0(n) \} \).

**Claim.** Condition \( q \) forces that if \( \hat{h} \) splits the pregap then \( \hat{h}_0 \) splits the pregap.

**Proof.** We will prove that \( q \models \hat{g}(\hat{a}_\xi) \leq^* \hat{h}_0 \leq^* \hat{h}' \). Suppose that this fails. The left hand \( \leq^* \) is obvious, so \( p \models \hat{h}_0 \not\leq^* \hat{h}' \) for some \( p \leq q \) in \( \mathcal{H}_B \).

Let \( C = \{ n \in \omega : \hat{h}'(\check{n}) < \hat{h}_0(\check{n}) \} \). We claim that there are infinitely many \( m \in C \) such that \( \hat{g}(\check{a}_0)(\check{m}) = \hat{h}_0(\check{m}) \). Suppose the contrary, so there is a condition \( r \leq p \) in \( \mathcal{H}_E(B, \hat{g}_B) \) and \( \pi \) such that

\[
r \models (\forall m > \check{n})(m \in C \rightarrow \hat{g}(\check{a}_0)(\check{m}) < \hat{h}_0(\check{m})).
\]

By Theorem 4.1 we can assume that \( F_r = \text{supp} \hat{h}_0 \cup \text{supp} \hat{h}' \cup \{ a_0 \} = B \cup \{ a_0 \} \).

We can assume that \( n_p > \pi \) and \( n_p \in \check{C} \). Define a condition \( r_1 \in \mathcal{H}_E(B, \hat{g}_B) \) by \( r_1 = (F_p, n_p + 1, f_r \cup f) \), where \( f : F_p \times \{ n_p \} \) is defined by

\[
f(a)(n_p) = \begin{cases} 
\min_{c \in B(>E)(a)} \hat{g}_B(c)(n_p) & \text{if } B(>E)(a) \neq \emptyset, \\
\min_{c \in B_0} \hat{g}_0(c)(n_p) & \text{if } B(>E)(a) = \emptyset.
\end{cases}
\]

Then \( r_1 \leq r \) (because \( a <_E b \) implies \( B(>E)(a) \supseteq B(>E)(b) \)), and \( r_1 \models \hat{g}_0(\check{a}_0) = \hat{h}_0(\check{n}) \) — a contradiction. So the set \( C_1 = \{ m \in C : \hat{g}(\check{a}_0)(m) = \hat{h}_0(m) \} \) is forced to be infinite, but \( \hat{g}(a_0)(m) < \hat{h}(m) \) for all \( m \in C_1 \), and therefore \( \hat{h}' \) does not fill the gap, contrary to our assumptions. \( \square \)

But by Lemma 6.1(a) we have \( \hat{h}_0 \not\in^* \hat{g}(b_\overline{\pi}) \), so \( \hat{h}_0 \) does not fill the pregap, and this proves our claim.

(c) We will first prove the following special case of the theorem:

**Lemma 6.3.** If the gap \( \langle a_\xi, b_\eta \rangle_{\xi < \kappa, \eta < \lambda} \) in \( E \) is tight, then its \( \hat{g} \)-image is a (tight) gap in \( \omega^\omega \).

**Proof.** Suppose the contrary, that some \( \mathcal{H}_E \)-name \( \hat{h} \) is forced to fill the pregap. Let \( A = \text{supp} \hat{h} \); since no \( c \in A \) satisfies (3) or (4) above, by going to end-segments of \( \kappa \) and \( \lambda \) we can assume that each \( c \in A \) is either (i) incomparable to all \( a_\xi, b_\eta \), or (ii) below \( a_0 \), or (iii) above \( b_0 \). Then Lemma 6.2 can be applied to \( \hat{h}, a_\xi \) and \( b_\eta \) for all \( \xi, \eta \), and therefore \( \models \hat{h} \geq^* \hat{g}(\check{a}_\xi) \) implies \( \models \hat{h} \geq^* \hat{g}(\check{b}_\eta) \) for all \( \xi, \eta \) so \( \hat{h} \) cannot fill this gap or even make it nontight. \( \square \)

Back to the general case. Let \( E_0 \) be the set of all \( c \in E \) such that:

(5) if \( c >_E a_\xi \) for all \( \xi \), then \( c >_E b_\eta \) for some \( \eta \), and

(6) if \( c <_E b_\eta \) for all \( \eta \), then \( c <_E a_\xi \) for some \( \xi \).
Then our gap is tight in $E_0$, because we are removing all $c$'s which can make it nontight. Let $E_1 = E \setminus E_0$.

Claim. For all $c \in E_1$ at most one of the sets $E_0(<_E c)$ and $E_0(>_E c)$ is nonempty.

Proof. If $c \in E_1$ fails to satisfy (5) (resp. (6)), so do all elements of $E(>_E c)$ (resp. of $E(<_E c)$).

So, by Lemma 4.5, the poset $H_E(E_0, \dot{g}_0)$ is $\sigma$-centered.

Lemma 6.4. If $\langle a_\xi, b_\eta \rangle_{\xi<\kappa, \eta<\lambda}$ is a gap, then it remains a gap in every forcing extension by a $\sigma$-centered poset.

Proof. By [17, Theorem 4], to every pregap we can associate a poset $\mathcal{P}$ which is $\sigma$-centered iff the pregap is split; so our lemma reduces to “If $\mathcal{P}$ is not $\sigma$-centered, then it remains so after forcing with a $\sigma$-centered poset”, which is obvious.

Thus further forcing with $H_E(E_0, \dot{g}_0)$ will not fill the gap, so the proof is finished.

(d) This follows immediately from Lemma 6.1(b).

So we can embed a partially ordered set $E$ into $^{\omega}\omega$ while preserving much of its “nonsaturatedness structure”. In Theorem 9.2 we will prove a partial converse of the above results. The statement “$H_E$ forces that there is a $\langle \kappa, \lambda \rangle$-gap (\kappa-limit) in $^{\omega}\omega$ iff there is a $\langle \kappa, \lambda \rangle$-gap (\kappa-limit) in $E$” is false, because of the following classical result of Hausdorff and Rothberger which applies to any of the orderings $\leq^*, <^*, \text{ or } \prec$:

Theorem 6.2 ([12], [23]). For every $\kappa$ there is an unbounded $\kappa$-chain in $^{\omega}\omega$ iff there is a $\langle \kappa, \omega \rangle$-gap in $^{\omega}\omega$.

So by Lemma 4.6, if $E$ is uncountable then there is an $\langle \omega_1, \omega \rangle$-gap in $^{\omega}\omega$ in an extension by $H_E$, and similarly if $E$ has a $\kappa$-chain then in an extension by $H_E$ there is always a $\kappa$-limit or a $\langle \kappa, \lambda \rangle$-gap for some $\lambda$. For infinite cardinals $\kappa$ and $\lambda$ (not necessarily uncountable) the statement “there is a $\langle \kappa, \lambda^* \rangle$-gap in $^{\omega}\omega$” is not ambiguous, i.e. its truth does not depend on the choice of an ordering: the same is true for unbounded $\kappa$-chains.

Proposition 6.1. (a) ([12], [23]) There is a $\kappa$-limit in $^{\omega}\omega$ iff there is a $\langle \kappa, \omega \rangle$-gap in $^{\omega}\omega$.

(b) There are no $\kappa$-limits in $^{\omega}\omega$.

(c) There is a $\kappa$-limit in $^{\omega}\omega$ iff there is an unbounded chain of length $\kappa$ in $[\omega]^\omega, \subseteq^*$.

(d) The existence of a $\kappa$-limit in $^{\omega}\omega$ does not imply the existence of a $\langle \kappa, \omega \rangle$-gap in $^{\omega}\omega$, and vice versa.

Proof. (a), (b) See e.g. [24].
(c) \( \Rightarrow \) Let \( \langle a_\xi, b_\xi \rangle_{\xi<\kappa} \) be a \( \kappa \)-limit, and define subsets of \( \omega \) by
\[
c_\xi = \{ n : a_\xi(n) = b(n) \}.
\]
Obviously \( \{ c_\xi \} \) is a \( \subseteq^* \)-increasing sequence in \( [\omega]^\omega \), and we claim that its limit is \( \omega \): if \( d \) is a cofinite subset of \( \omega \) such that \( c_\xi \subseteq^* d \) for all \( \xi \), then \( b' \in \omega^\omega \) defined by
\[
b'(n) = \begin{cases} b(n) & \text{if } n \in d, \\ b(n) - 1 & \text{if } n \not\in d, \end{cases}
\]
witnesses that \( \langle a_\xi, b_\xi \rangle \) is not a \( \kappa \)-limit. \( \Leftarrow \) Consider the characteristic functions of sets.

(d) By (c), our claim is equivalent to “The existence of an unbounded \( \kappa \)-chain in \( [\omega]^\omega \) does not imply the existence of an unbounded \( \kappa \)-chain in \( \omega^\omega \), and vice versa”, and this is a well-known fact (see [7]).

The last lemma of this section (in connection with Proposition 6.1(d)) shows that \( \mathcal{H}_E \) sometimes creates “unnecessary” limits in \( \langle \omega^\omega, \leq^* \rangle \).

**Lemma 6.5.** If there is an unbounded \( \kappa \)-chain in \( E \), then in a forcing extension by \( \mathcal{H}_E \) there is a \( \kappa \)-limit in \( \langle \omega^\omega, \leq^* \rangle \).

**Proof.** By Proposition 6.1, this reduces to proving that in an extension there is an unbounded \( \kappa \)-chain in \( [\omega]^\omega \). Let \( \langle a_\xi \rangle_{\xi<\kappa} \) be an unbounded \( \kappa \)-chain in \( E \) and define names \( \check{c}_\xi \) for subsets of \( \omega \) by
\[
\check{c}_\xi = \{ n : \check{g}(a_\xi)(n) \neq 0 \}.
\]
The proof that \( \langle \check{c}_\xi \rangle_{\xi<\kappa} \) is forced to be an unbounded \( \kappa \)-chain in \( [\omega]^\omega \) is similar to the proof of Lemma 6.1.

7. **Posets with minimal patterns.** This chapter is a preparation for the proofs of Theorems 9.1 and 10.1. For definition of \( (a,b)_E \) see §8.

**Theorem 7.1.** Let \( \langle E, <_E \rangle \) be a poset, \( \langle I, <_I \rangle \) a linearly ordered set, and \( \{ D_\xi \} (\xi \in I) \) a family of disjoint finite subsets of \( E \) such that for all \( \xi < \eta < \kappa \) we have
\[(I) \quad (a,b)_E \cap D_\xi \neq \emptyset \text{ whenever } a \in D_\xi \text{ and } b \in D_\eta \text{ are } <_E \text{-comparable}.
\]
Then either (i) there is a chain of type \( I \) or \( I^* \) in \( E \), or (ii) there are \( \xi \neq \eta \) in \( I \) such that \( a \) and \( b \) are \( <_E \)-incomparable for all \( a \in D_\xi \) and \( b \in D_\eta \).

**Corollary** (Kurepa, [19]). Every uncountable well-founded poset \( E \) having finite levels must have an uncountable chain.

**Proof.** Let \( \{ D_\xi \} (\xi < \omega_1) \) be a decomposition of \( E \) into levels. Then \( \{ D_\xi \} \) satisfy (I) and every \( a \in D_\xi \) is comparable with some \( b \in D_\eta \), for all \( \eta < \xi \). The conditions of Theorem 7.1 are satisfied and there is an \( \omega_1 \)-chain in \( E \).
Proof of Theorem 7.1. Suppose that (ii) fails. We claim that for every k-tuple \( \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} \) in \( I \) there are distinct \( y_j \in D_{\alpha_j} \) (\( j < k \)) such that either \( y_0 < \ldots < y_k \) (a chain is increasing) or \( y_0 > \ldots > y_k \) (a chain is decreasing). To construct it, pick \( y_0 \in D_{\alpha_0} \) and \( y_{k-1} \in D_{\alpha_{k-1}} \) such that \( y_0 < E y_k \) or \( y_0 > E y_k \) (possible because (ii) fails). Then we can proceed to pick the rest of the chain by using (I).

So for every finite \( F \subseteq I \) we fix a chain \( C_F = \{ y_F^\xi : \xi \in F \} \) such that \( y_F^\xi \in D_\xi \) for all \( \xi \in F \) and the chain is either strictly increasing or strictly decreasing. For such an \( F \) let \( A_F \) be the family of all finite subsets of \( I \) which include \( F \), and let \( U \) be a uniform ultrafilter on \( [I]^\omega \) extending the filter base \( \{ A_F \} \). Then for all \( \xi \in I \) there is \( \bar{y}_\xi \in D_\xi \) such that the set

\[
\{ F : \xi \in F \& y_F^\xi = \bar{y}_\xi \}
\]

is in \( U \); we claim that \( \{ \bar{y}_\xi \} \) is the desired chain. Assume without loss of generality that for \( U \) many \( F \) the chain \( \{ y_F^\xi \} \) is increasing. We have to prove that all \( \bar{y}_\alpha \) are distinct and that \( \bar{y}_\alpha < \bar{y}_\beta \) whenever \( \alpha < \beta \). But the set of all \( F \in [I]^\omega \) such that \( \alpha, \beta \in F, y_F^\alpha = \bar{y}_\alpha, y_F^\beta = \bar{y}_\beta \) and the chain \( \{ y_F^\xi \} \) is increasing is in \( U \) and therefore nonempty, so \( \bar{y}_\alpha < \bar{y}_\beta \) and they are distinct.

Remark. Some assumption like (I) is necessary in Theorem 7.1. To see this let \( \prec \) be a linear ordering on \( \omega_2 \) with no \( \omega_2 \)- or \( \omega_2^\omega \)-chains (e.g. a subordering of \( \langle 2^{\omega_1}, \leq_{\text{lex}} \rangle \) of appropriate size), and let \( E_0 = \langle \omega_2, <_0 \rangle \) and \( E_1 = \langle \omega_2, <_1 \rangle \) be defined by

\[
\begin{align*}
\alpha <_0 \beta & \iff \alpha < \beta \text{ and } \alpha < \beta, \\
\alpha <_1 \beta & \iff \alpha < \beta \text{ and } \alpha > \beta.
\end{align*}
\]

Let \( E \) be the disjoint sum of \( E_0 \) and \( E_1 \) (say, \( E_0 \times \{ 0 \} \cup E_1 \times \{ 1 \} \)) and let \( D_\xi = \{ \langle \xi, 0 \rangle, \langle \xi, 1 \rangle \} \) for \( \xi < \omega_2 \). Then \( D_\xi \) and \( D_\eta \) have “vertical and upwards” connections for all \( \xi \) and \( \eta \) but there are no \( \omega_2 \)-chains in \( E \).

In order to make Theorem 7.1 useful in proving our main result we need some definitions. Fix a positive integer \( n \) and a poset \( \langle E, \prec \rangle \). If \( D = \{ x_i^D : i < n \} \) and \( F = \{ x_i^F : i < n \} \) are two \( n \)-tuples of elements in \( E \) then let the \( \prec \)-pattern \( \tau_DF \) be the isomorphism type of the poset \( \langle D \cup F, \leq \rangle \), i.e. the partial ordering \( \preceq_{\tau_DF} \) on the set \( 2 \times n \) such that \( \langle i, k \rangle \preceq_{\tau_DF} \langle j, l \rangle \) iff \( x_k^D \leq x_l^F \) for \( k, l < n \) and \( i, j < 2 \). The sets \( D \) and \( F \) are connected iff there are comparable \( a \in D \) and \( b \in F \).

We sometimes identify patterns with their graphs and say that a pattern \( \sigma_0 \) is included in a pattern \( \sigma_1 \), denoting this fact by \( \sigma_0 \subseteq \sigma_1 \). If we consider more than one ordering on the underlying set, then we denote the pattern \( \tau_DF \) corresponding to the ordering \( \prec \) by \( \tau_{DF}^\prec \). A composition of two patterns
σ₀ and σ₁ is the pattern σ₀ ∘ σ₁ determined by defining ≺_{σ₀ ∘ σ₁} as follows:

\[ (0, k) ≺_{σ₀ ∘ σ₁} (1, l) \text{ iff } (0, k) ≺_{σ₀} (1, m) \text{ and } (0, m) ≺_{σ₁} (1, l) \text{ for some } m < n, \text{ and } (i, k) ≺_{σ₀ ∘ σ₁} (i, l) \text{ iff } (i, k) ≺_{σ₀} (i, l) \text{ for some } i = 0, 1. \]

In other words, if σ₀ = τ₁₀ and σ₁ = τ₁₂ then σ₀ ∘ σ₁ is the pattern τ₁₂, where “≺” is the transitive closure of “≺” restricted to D₀ ∪ D₁ and D₁ ∪ D₂.

We avoid nested indexes so if we have a fixed family D = \{x_i : i < n\} \{ξ < δ\} of finite subsets of a poset \(⟨E, <⟩\) then the pattern \(τ_{Dᵦ₁Dᵦ₂} \) is denoted by \(τ_{ξη} \) instead. Note that in general for every such family and all \(α < β < γ\):

1. τ_{αγ} \supseteq τ_{αβ} ∘ τ_{βγ}.
2. An ordering ≺ on \(∪_{ξ < δ} D_ξ\) is uniquely determined by its patterns.

We say that the family \(\{D_ξ : ξ ∈ I\}\) has minimal ≺-patterns iff τ_{αγ} = τ_{αβ} ∘ τ_{βγ} for all \(α < β < γ\) in I. So we can reformulate Theorem 7.1:

**Theorem 7.1**: If \(\{D_ξ : ξ ∈ I\}\) is a disjoint family of sets of size n with minimal patterns and all \(D_ξ, D_η\) are connected, then there is an I- or an I⁺-chain in \(∪_{ξ ∈ I} D_ξ\).

Let us mention a corollary to the proof of Theorem 7.1:

**Proposition 7.1**: If \(\{D_ξ : ξ ∈ I\}\) is a disjoint family of sets of size n with minimal patterns, then for all \(ξ < 1, η\), either \(D_ξ, D_η\) are not connected or there is a \([ξ, η]_I\)- or a \([ξ, η]_I⁺\)-chain \(C\) in \(∪_{ξ ∈ I} D_ξ\) such that \(C ∩ D_ξ\) is nonempty for all \(ξ ∈ [ξ, η]_I\).

The next lemma gives sufficient conditions for the existence of an ordering with a given set of patterns.

**Lemma 7.1**: If \(σ_{αβ} (α < β < κ)\) is a family of patterns such that \(σ_{αβ} ∘ σ_{βγ} = σ_{αγ}\) for all \(α < β < γ\) then there is a unique ordering ≺ on \(∪_{ξ < κ} D_ξ\) such that \(τ_{αβ} = τ_{αγ}\) for all \(α < β < κ\).

**Proof.** We define the relation ≺ in the only possible way; all we have to do is to check that it is transitive; so we pick \(a ∈ D_α, b ∈ D_β,\) and \(c ∈ D_γ\) such that \(a < b < c\) and prove that \(a < c\). Essentially the only two interesting cases are: (i) If \(α < β < γ\), then \(a < c\) by minimality of patterns. (ii) If \(α < γ < β\), then by minimality we can pick \(d ∈ D_γ\) such that \(a < d < b\); but then \(a < c\) because \(σ_{αγ}\) is a pattern.

A pattern σ occurs in \(\{D_ξ\}\) iff it is equal to some \(τ_{αβ}\) of this sequence.

**Lemma 7.2**: Let κ be an infinite cardinal. Then every κ-sequence \(\{D_ξ\}\) has a subsequence \(\{D_{αξ}\}\) of the same length such that for every pattern σ that occurs in \(\{D_{αξ}\}\) there is an infinite increasing ω-sequence \(\{α_n\}\) of
ordinals less than \( \kappa \) such that \( \tau_{\alpha_i \alpha_j} = \sigma \) for all \( i < j \), and \( \alpha_0 \) can be picked to be arbitrarily large.

**Proof.** If a pattern \( \sigma \) does not satisfy the requirements, pick \( \alpha_0 < \kappa \) and \( k < \omega \) which witness this and define a partition

\[ [\kappa]^2 = K_0 \cup K_1 \]

by putting \( \{ \xi, \eta \} \) in \( K_0 \) iff \( \tau_{\xi \eta} \neq \sigma \). Without loss of generality assume that \( \alpha = 0 \). Then there are no infinite \( K_1 \)-homogeneous sets, so by the partition relation \( \kappa \rightarrow (\kappa, \omega)^2 \) (see [9, §11]) there is a subfamily of \( \{ D_\xi \} \) of the size \( \kappa \) in which \( \sigma \) does not occur. We need to repeat this only finitely many times to eliminate all patterns which do not occur often enough.

The following lemma plays a crucial role in proving Theorems 9.1, 9.2 and 10.1.

**Lemma 7.3.** If \( \langle E, \prec \rangle \) is a partially ordered set, \( \langle I, \prec_I \rangle \) is a linearly ordered set and \( D_\xi \) (\( \xi \in I \)) is a family of disjoint finite subsets of \( E \), then there is an ordering \( \prec \) on \( E \) with the following properties:

(B1) \( \prec \) is coarser than \( \prec \), i.e. \( a \prec b \) implies \( a \prec b \).
(B2) Every \( \tau^\prec_{\alpha \beta} \) is a composition of finitely many \( \tau^\prec_{\xi \eta} \), i.e.
(B’2) For all \( \alpha < \beta < \kappa \) there is an integer \( n \) and ordinals \( \alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \beta \) such that \( \tau^\prec_{\alpha \beta} = \tau^\prec_{\alpha_0 \alpha_1} \circ \tau^\prec_{\alpha_1 \alpha_2} \circ \ldots \circ \tau^\prec_{\alpha_n-1 \alpha_n} \).
(B3) \( D_\xi \) (\( \xi < \kappa \)) has minimal \( \prec \)-patterns.
(B4) The orderings \( \prec \) and \( \prec \) coincide on every \( D_\xi \).

**Proof.** For a subset \( F = \{ \alpha_i : i = 0, \ldots, n \} \) of \( I \) such that \( \alpha_0 <_I \alpha_1 <_I \ldots <_I \alpha_n \) let

\[ \tau(F) = \tau^<_E \circ \tau^<_E \circ \ldots \circ \tau^<_E. \]

Then by (P1) and the induction one can prove that

\[ F \subseteq G \quad \text{implies} \quad \tau(F) \supseteq (G). \]

For \( \alpha <_I \beta \) in \( I \) let \( F_{\alpha \beta} \) be the set \( \alpha = \alpha_0 <_I \alpha_1 <_I \ldots <_I \alpha_n = \beta \) in \( I \) such that the pattern \( \tau(F) \) is minimal (with respect to inclusion) among such patterns and let \( \tau^\prec_{\alpha \beta} = \tau(F_{\alpha \beta}) \).

**Claim.** Patterns in the family \( \tau^\prec_{\alpha \beta} \) (\( \alpha <_I \beta \in I \)) are minimal.

**Proof.** Fix \( \alpha <_I \beta <_I \gamma \). Then \( \tau(F_{\alpha \beta}) \subseteq \tau(F_{\alpha \gamma} \cap (\alpha, \beta)_I) \), \( \tau(F_{\beta \gamma}) \subseteq \tau(F_{\alpha \gamma} \cap (\beta, \gamma)_I) \), so \( \tau(F_{\alpha \beta}) \circ \tau(F_{\beta \gamma}) \subseteq \tau(F_{\alpha \gamma} \cup \{ \beta \}) \subseteq \tau(F_{\alpha \gamma}) \), and by the choice of \( F_{\alpha \gamma} \) we have equality.

Therefore the assertion of Lemma 7.3 follows from Lemma 7.1.

8. Banach–Mazur game of length \( \omega_1 \). Results from this chapter will be used in the proof of Theorem 10.1. We consider \( 2^{\omega_1} \) as a topological space
with the natural $G_\delta$-topology, where the basic open sets are $[s] = \{x \in 2^{\omega_1} : s \subseteq x\}$ for $s \in 2^{<\omega_1}$. We will often interchange subsets of $\omega_1$ with their characteristic functions. The poset for adding a Cohen subset of $\omega_1$, $C_{\omega_1}$, is $2^{<\omega_1}$ ordered by $\supseteq$.

**Definition 8.1** (Banach–Mazur game of length $\omega_1$). Let $X$ be a subset of $2^{\omega_1}$. The game BM($X$) for two players, I and II, in $\omega_1$ moves is defined as follows: I and II alternately play elements $s_0^I \subseteq s_0^II \subseteq s_1^I \subseteq s_1^II \subseteq \ldots \subseteq s_\xi^I \subseteq s_\xi^II \subseteq \ldots$ of $2^{<\omega_1}$, so that in his $\xi$th move player * plays $s_\xi^I$ which extends the chain of all previously played elements of $2^{<\omega_1}$ ($* = I, II$). I wins a game iff $\bigcap_{\xi < \omega_1} [s_\xi^I]$ is included in $X$ or if II is the first player to disobey the rules; otherwise II wins.

A *partial play* is a sequence $s_0^I \subseteq s_0^II \subseteq \ldots \subseteq s_\xi^I \subseteq s_\xi^II \subseteq \ldots \subseteq s_\xi^I$, $\xi < \omega_1$. (So we consider only partial plays in which II is about to move.) A *strategy for the second player* is a mapping $\sigma$ from all partial plays into $2^{<\omega_1}$ such that $\sigma(p)$ extends the chain $p$. Player II *obeys the strategy* $\sigma$ in a play $(s_\xi^I, s_\xi^II)_{\xi < \omega_1}$ iff $s_\xi^I = \sigma(p_\xi)$ for all $\xi$, where $p_\xi$ is a partial play ending with the $\xi$th move of I. A strategy $\sigma$ is a *winning strategy for II in BM($X$)* iff II wins every game in which he obeys $\sigma$. Banach proved the following theorem for the Banach–Mazur game of length $\omega$ (see [21]).

**Theorem 8.1.** II has a winning strategy for $X \subseteq 2^{\omega_1}$ iff $X$ is meager.

**Proof.** If $X$ is meager, then there is an $\omega_1$-sequence $F_\xi$ of nowhere dense subsets of $2^{\omega_1}$ whose union covers $X$, and it is obvious that II can avoid $F_\xi$ in the $\xi$th move; this describes the strategy. The nontrivial direction follows from this.

**Claim.** Let $\sigma$ be a strategy. The set of all outcomes of a Banach–Mazur game in which II uses $\sigma$ includes a dense $G_{R_1}$ subset of $2^{\omega_1}$.

**Proof.** The set $D_0^I$ of all $t \in 2^{<\omega_1}$ such that $t = s_0^I$ in some valid play (i.e. a play in which both I and II obey the rules) in which II obeys $\sigma$ is dense in $2^{<\omega_1}$ (because I can enter into any basic $[s]$ in his first move). Pick a maximal antichain (in the $\supseteq$ ordering) $A_0$ which is included in $D_0^I$. Let $D_1^I$ be the set of all $s \in 2^{<\omega_1}$ such that $s = s_1^I$ in some valid partial play in which $s_0^I$ is in $A_0$. This set is dense in $2^{<\omega_1}$ so let $A_1$ be a maximal antichain included in $D_1^I$. By continuing in this way, we define an $\omega_1$-sequence $\{A_\xi\}$ ($\xi < \omega_1$) of maximal antichains such that $A_\xi$ refines $A_\eta$ for all $\eta < \xi$ (i.e. for each $s \in A_\xi$ there is (a unique) $t \in A_\eta$ such that $t \subseteq s$). At the successor stages we do the same as above. At the limit stage $\delta$ let $D_\delta^I$ be the set of all $t \in 2^{<\omega_1}$ such that $t = s_\delta^I$ in some valid play in which II obeys $\sigma$ and in which $s_\xi^I \in A_\xi$ for all $\xi < \delta$. We claim that $D_\delta^I$ is dense in $2^{<\omega_1}$.

Let $A_\delta'$ be the set of all $t \in 2^{<\omega_1}$ such that $t|\alpha_\xi \in A_\xi$ for some increasing $\delta$-sequence of ordinals $\{\alpha_\xi\}$ converging to $|t|$. Then $A_\delta'$ is dense and each
element of it corresponds to the first $\delta$ moves of a valid play in which $\Pi$ obeys $\sigma$, and I is the first one to play after this stage of the game. So I can, by his playing, assure that the set $D^W_{\delta}$ is dense as claimed. Let $A_\delta$ be a maximal antichain inside $D^W_{\delta}$, so $A_\delta$ refines $A_\delta'$ and all $A_\xi$ for $\xi < \delta$. This describes the construction of an $\omega_1$-sequence of antichains $\{A_\xi\}$. Note that

(*) if $s \in A_\eta$, $t \in A_\xi$ and $s \subset t$, then a partial play witnessing that $s \in A_\eta$ is an initial segment of a partial play witnessing $t \in A_\xi$.

Let $U_\xi$ be the family of all $x \in 2^{\aleph_1}$ such that $x[\alpha] \in A_\xi$ for some countable ordinal $\alpha$. Then each $U_\xi$ is a dense open subset of $2^{\aleph_1}$, $G = \bigcap_{\xi < \omega_1} U_\xi$ is a dense $G_{\omega_1}$ subset of $2^{\aleph_1}$ and each element of $G$ is the result of some game in which $\Pi$ obeys $\sigma$; the latter statement follows from (*). ■

This proves the theorem.

Fix a large enough $\theta$, say $\theta = (2^{\aleph_1})^+$, and a well-ordering $<_w$ of $\theta$ which gives rise to Skolem functions. All models that we consider are elementary submodels of $H_\theta$ closed under $<_w$, unless otherwise specified. A sequence $\{M_\xi\}$ of countable elementary submodels of $H_\theta$ is a continuous $\varepsilon$-chain if for all $\alpha < \omega_1$:

1. $M_\alpha < M_{\alpha+1}$,
2. $\langle M_\xi : \xi < \alpha \rangle \in M_{\alpha+1}$, and
3. $\bigcup_{\xi < \alpha} M_\xi = M_\alpha$ for $\alpha$ limit.

An elementary submodel $M$ of $H_\theta$ is approachable iff $M = \bigcup_{\xi < \omega_1} M_\xi$, where $\{M_\xi\}$ is an $\varepsilon$-chain of countable elementary submodels of $H_\theta$. For a function $f : [H_\theta]^{<_w} \rightarrow H_\theta$ let $C_f$ be the family of all $A \in [H_\theta]^{\aleph_1}$ such that $f''[A]^{<_w} \subseteq A$ and $\omega_1 \subseteq A$. A family $C \subseteq [H_\theta]^{\aleph_1}$ is closed unbounded (club) iff it is equal to $C_f$ for some $f$. [This is not the standard definition, but by a result of Kueker [15] every “standard” club includes some $C_f$.] A family $S \subseteq [H_\theta]^{\aleph_1}$ is stationary iff it intersects all sets closed unbounded in $[H_\theta]^{\aleph_1}$. Let $A$ denote the family of all approachable elementary submodels of $H_\theta$. The next lemma shows that $A$ is a rather large stationary subset of $[H_\theta]^{\aleph_1}$.

**Lemma 8.1.** The union of a $\subseteq$-chain of approachable models of length $\omega_1$ is approachable.

**Proof.** We denote this chain by $\{M^\alpha\}$ and write $M^\alpha = \bigcup_{\xi < \omega_1} M^\alpha_\xi$. Then by usual bookkeeping we can find ordinals $\xi_\alpha$ for $\alpha < \omega_1$ so that $\{N^\alpha_{\xi_\alpha}\}$ is an $\varepsilon$-chain whose union covers $\bigcup_{\alpha < \omega_1} M^\alpha$.

An $x \in 2^{\omega_1}$ is $(M, C_{\omega_1})$-generic if all its proper initial segments are in $M$ and it is a member of all dense subsets of $C_{\omega_1}$ coded in $M$. Notice that it is not obvious that there should be an $(M, C_{\omega_1})$-generic subset of $\omega_1$ for a given model $M$ of size $\aleph_1$ (even if we assume that $\omega_1$ is included in $M$ to avoid trivialities).
Lemma 8.2. If a model $M$ is approachable, then there is an $(M, C_{\omega_1})$-generic subset $x$ of $\omega_1$.

Proof. Let $M = \bigcup_{\xi<\omega_1} M_\xi$. We assume that the sequence is continuous and that $\langle M_\eta \rangle_{\eta \leq \xi}$ is always in $M_{\xi+1}$. We pick an $\omega_1$-sequence $\{s_\xi\}$ in $2^{<\omega_1}$ so that:

1. $s_0 \subset s_1 \subset \ldots \subset s_\xi \subset \ldots$,
2. each $s_\xi$ is generic over $M_\xi$; i.e. $s_\xi \in 2^{s_\xi}$,
3. $[s_\xi]$ avoids all nowhere dense subsets of $2^{<\omega_1}$ coded in $M_\xi$, and
4. $s_\xi$ is always a $<_w$-minimal element of $M_{\xi+1}$ which satisfies (1)-(3).

By (4), $\bigcup_{\xi<\delta} s_\xi$ is in $M_{\delta+1}$ for each limit $\delta$, so we can proceed with the construction on limit levels. Then $x = \bigcup_{\xi<\omega_1} s_\xi$ as required. ■

Theorem 8.2. If $X \subseteq 2^{<\omega_1}$ is nonmeager, then there is an $(M, C_{\omega_1})$-generic $x_M \in X$ for stationary many $M$ in $\mathcal{A}$.

Proof. Suppose the contrary, that there is a club $C$ in $[H_\theta]^N_1$ such that there is no $(M, C_{\omega_1})$-generic $x_M$ for all $M \in C \cap \mathcal{A}$. Pick $f : [H_\theta]^{<\omega_1} \rightarrow H_\theta$ such that $C = C_f$. Let $Y$ be the set of all $x \in 2^{<\omega_1}$ which are $(M, C_{\omega_1})$-generic for some $M \in \mathcal{A} \cap S$; by our assumptions $X$ and $Y$ are disjoint. We will construct a winning strategy $\sigma$ for II in BM$(2^{<\omega_1} \setminus Y)$, which is therefore a winning strategy for II in BM$(X)$, thus showing that $X$ is meager (by Theorem 8.1). Along with playing the Banach–Mazur game, II constructs an approachable model $M = \bigcup_{\xi<\omega_1} M_\xi$ in $\mathcal{A} \cap C$ and assures that the outcome $x$ of the game is $(M, C_{\omega_1})$-generic. So II plays $s_\xi^\mathrm{II}$ and $M_\xi$, so that in each stage $\alpha$ of the game and for all $\xi < \alpha$:

1. $s_\xi^\mathrm{II}$’s are valid moves in the Banach–Mazur game,
2. $\{M_\xi\}$ ($\xi < \alpha$) is a continuous $\varepsilon$-chain,
3. each $M_{\xi+1}$ is closed under $f$ and has $s_\xi^\mathrm{II}$ and $s_{\xi+1}^\mathrm{I}$ as elements,
4. each $s_\xi^\mathrm{II}$ is generic over $M_\xi$; i.e. $s_\xi^\mathrm{II} \in 2^{s_\xi^\mathrm{I}}$, and
5. $[s_{\xi+1}^\mathrm{II}]$ avoids all nowhere dense subsets of $2^{<\omega_1}$ coded in $M_\xi$. [This can be arranged because $M_{\xi+1}$ “knows” that $M_\xi$ is countable.]

If II obeys $\sigma$, then after $\omega_1$ many moves of the game he has an approachable model $M = \bigcup_{\xi<\omega_1} M_\xi$ and $x = \bigcup_{\xi<\omega_1} s_\xi^\mathrm{II}$ which is $(M, C_{\omega_1})$-generic. But $M$ is closed under $f$ and it includes $\omega_1$, so it is in $C$, and therefore $x$ is in $Y$. Since II wins BM$(X)$, $X$ is meager. ■

Lemma 8.3. If $S$ is a stationary subset of $\mathcal{A}$ and $x_M$ is $(M, C_{\omega_1})$-generic for all $M \in S$, then the set of all $x_M$’s is nonmeager.

Proof. Suppose the contrary and let $F$ be a code for an $F_{\aleph_1}$-meager subset of $2^{<\omega_1}$ avoiding $X$. Let $\lambda = (2^\theta)^+$ and let $N$ be an approachable
elementary submodel of $H_\lambda$ which includes $S, \langle x_M : M \in S \rangle, F, \ldots$ and such that $N = N \cap H_\theta$ is in $S$. Then $F$ is in $N$, so $x_N$ avoids $F$—a contradiction.

9. Proof of the main theorem

**Theorem 9.1.** If $\kappa > \omega_1$ is a regular cardinal then $H_E$ adds a $\kappa$-chain to $\omega$ if and only if one of the following happens:

1. $E$ has a $\kappa$- or a $\kappa^*$-chain, or
2. $\mathcal{C}$ adds a $\kappa$-chain to $\omega$.

**Corollary.** If $\kappa > \epsilon$ is a regular cardinal, then $H_E$ adds a $\kappa$-chain to $\omega$ if and only if there is either a $\kappa$- or a $\kappa^*$-chain in $\langle E, \langle E \rangle \rangle$.

**Definition 9.1.** A family $\{A_\xi\}$ forms a weak $\Delta$-system with root $A$ iff $A_\xi \cap A_\eta \subseteq A$ for all $\xi \neq \eta$. A family $\{A_\xi\}$ forms a weak local $\Delta$-system with root $A$ iff its subfamily $\{A_\delta \cap n\}$ ($n < \omega$) forms a weak $\Delta$-system with root $A$ for all $\delta$.

**Lemma 9.1.** If $\kappa$ is a regular cardinal larger than $\aleph_1$, then every family $\{B_\xi\}_{\xi < \kappa}$ of countable sets has a subfamily $\{A_\xi\}_{\xi < \kappa}$ which forms a weak local $\Delta$-system with a countable root $A$.

**Proof.** Let $\theta$ be a large enough cardinal, and let $M_\xi$ ($\xi < \omega_1$) be an $\varepsilon$-chain of countable elementary submodels of $H_\theta$ such that the family $\{B_\xi\}$ is an element of $M_\theta$. By a counting argument we can find $\xi < \omega_1$ such that $B_\eta \cap M = B_\eta \cap M_\xi$ for $\kappa$ many $\eta$'s; let $A = M_\xi$. We claim that for every $\alpha < \kappa$ there is a strictly increasing $\omega$-sequence $\{\alpha_n\}$ of ordinals above $\alpha$ which forms a weak $\Delta$-system with root $A$. It suffices to prove that this is true in $M$, so note that $A \in M$ and pick $\alpha < \kappa$ in $M$. Choose $\delta$ above $M \cap \kappa$ such that $B_\delta \cap M \subseteq A$. Pick $\alpha_0 > \alpha$ in $M$ such that $A \subseteq B_{\alpha_0}$. Suppose that we have constructed the sequence $\{\alpha_n\}$ ($i < n$) for some integer $n$. We want to prove that we can continue the construction. We have $B_\delta \cap A_{\alpha_i} \subseteq A$ for all $i < n$, so the family $\{B_{\alpha_i}\}_{i < n} \cup \{B_\delta\}$ is a $\Delta$-system with root $A$. By elementarity there is an ordinal $\alpha_n$ as required. So our claim is proved, and the lemma follows immediately from it.

**Proof of Theorem 9.1.** Suppose that $f_\xi$ ($\xi < \kappa$) is an $H_E$-name for a strictly increasing $\kappa$-chain in $\omega$. Without loss of generality we can assume that supports of these functions form a weak local $\Delta$-system with a countable root. Now applying Lemma 5.2 to $f_\xi$ and $f_{\xi+1}$ for all limit ordinals $\xi$ we get $h_\xi, q_\xi$ ($\xi < \kappa$) such that $q_\xi \models f_\xi \leq^* h_\xi \leq^* f_{\xi+1}$. Every uncountable family of finite sets includes a $\Delta$-system of the same size; we can assume that supports of $h_\xi$'s form a weak $\Delta$-system with a countable root $A$ and finite tails $D_\xi$. Since $H_E$ has $\kappa$ as a precaliber, without loss of generality we
can assume that the family \( \{ q_\xi \} \) is centered. Pick a condition \( \bar{q} \) which forces that there are \( \kappa \) many \( q_\xi \) in \( \mathcal{G} \).

Now apply Lemma 7.3 to the family \( \{ D_\xi \} \) and ordering \( \prec_E \) to get a new ordering \( \prec_0 \). If there is a \( \kappa \)- or \( \kappa^* \)-chain in \( \prec_0 \), then by (B1) there is one in \( \prec_E \). Assume that this does not happen; so by Theorem 7.1 there are \( \xi < \eta < \kappa \) such that \( D_\xi \) and \( D_\eta \) are not connected. Denote the pattern \( \tau^{\xi_0}_{\xi_\eta} \) by \( \sigma \). By Lemma 7.2 and further refining we can assume that \( \tau^{\xi_0}_{\xi_\alpha + m} = \sigma \) for all \( \alpha < \kappa \) and \( m < \omega \), and that \( E = A \cup \bigcup_{\xi < \kappa} D_\xi \). Define a relation \( \prec \) on \( E \) by

\[
a \prec b \quad \text{iff} \quad \begin{cases} a, b \in A \cup D_\eta \text{ and } a \prec_E b \text{ for some } \eta, \text{ or} \\ a, b \in \bigcup_{\xi < \kappa} D_\xi \text{ and } a \prec_0 b, \text{ or} \\ a, b \in \bigcup_{\xi < \kappa} D_\xi \text{ and } a \prec_E c \prec_E b \text{ for some } c \in A.
\end{cases}
\]

So we have

1. the relation \( \prec \) is an ordering on \( E \).

**Claim 1.** The relation \( \prec \) is an ordering on \( E \).

**Proof.** Note first that \( a \prec b \) implies \( a \prec_E b \), so we only have to prove that \( \prec \) is transitive. Suppose that \( a \prec b \) and \( b \prec c \); then we have \( a \prec_E c \). Now we consider several cases: (i) If \( a \in A \) or \( c \in A \) then we are done by \( a \prec c \) iff \( a \prec_E c \). (ii) If \( b \in A \), then \( a \prec c \) by the definition of \( \prec \). (iii) Suppose \( a, b, c \in \bigcup_{\xi \in E} D_\xi \). If there is \( d \in A \) such that \( a \prec_E d \prec_E b \) or \( b \prec_E d \prec_E c \), then \( a \prec c \). If not, then we have \( a \prec_0 b \) and \( b \prec_0 c \), so \( a \prec_0 c \). This finishes the proof. \( \square \)

Note that now

1. The relations \( \prec \) and \( \prec_E \) coincide on \( A \cup D_\eta \) so \( \check{h}_\eta \) is an \( \mathcal{H}(E, \prec) \)-name for all \( \eta \).

**Claim 2.** The poset \( \mathcal{H}(E, \prec) \) below \( q_\alpha \wedge q_\beta \) forces that \( \check{h}_\alpha \leq^* \check{h}_\beta \) for all \( \alpha < \beta < \kappa \).

**Proof.** Pick \( \alpha < \beta \) and suppose that some condition \( p \) below \( q_\alpha \wedge q_\beta \) forces (in \( \mathcal{H}(E, \prec) \)) that \( \check{h}_\alpha \leq^* \check{h}_\beta \). Find \( \alpha = \alpha_0 < \alpha_1 < \ldots < \alpha_n = \beta \) as in (B’2) of Lemma 7.3, i.e. such that \( \tau^{\xi_0}_{\xi_{\alpha_i + 1}} = \tau^{\xi_\alpha}_{\xi_{\alpha_i + 1}} \). Note that this implies that \( \tau^{\xi_\alpha}_{\xi_{\alpha_i + 1}} = \tau^{\xi_\alpha}_{\xi_{\alpha_i + 1}} \). Extend \( p \) to \( q \) to decide \( i < n \) such that

\[
q \models \mathcal{H}(E, \prec) \quad \check{h}_{\alpha_i} \leq^* \check{h}_{\alpha_{i+1}}.
\]

But the condition \( q' = q \upharpoonright (A \cup D_{\alpha_i} \cup D_{\alpha_{i+1}}) \) forces this as well (all involved names are \( \mathcal{H}(A \cup D_{\alpha_i} \cup D_{\alpha_{i+1}}, \prec) \)-names and \( q' \) is the projection of \( q \) to this regular subordering). Also \( \tau^{\xi_\alpha}_{\xi_{\alpha_i + 1}} = \tau^{\xi_\alpha}_{\xi_{\alpha_{i+1}}} \) so the orderings \( \prec \) and \( \prec_E \)
coincide on \( A \cup D_{\alpha_1} \cup D_{\alpha_1+1} \), and (1) is true when \( \models \mathcal{H}_{(E,<\omega)} \) is replaced with \( \models \mathcal{H}_{(A,g)} \) — a contradiction. 

So the condition \( \overline{\gamma} \) still forces that the sequence \( \{ h_\xi \} \) includes a strictly \( \leq^* \)-increasing \( \kappa \)-chain in \( \omega^\omega \). We go to an extension by \( \mathcal{H}_A \) below \( \overline{\gamma} \) (we can assume that \( \overline{\gamma} \) is in \( \mathcal{H}_A \)); let \( \dot{g}_A \) be the generic embedding of \( A \) into \( \omega^\omega \).

**Claim 3.** The poset \( \mathcal{H}_{A \cup D_\alpha} \) is equivalent to the product of \( \mathcal{H}_{A \cup D_\alpha} \) and \( \mathcal{H}_{A \cup D_{\alpha+m}} \) for all \( \alpha < \kappa \) and all \( m < \omega \).

**Proof.** This follows immediately from \( (<1) \) and Lemma 4.3.

So by Lemma 5.1, we can find an \( \mathcal{H}_A \)-name \( \dot{h}_\xi' \) and an \( \mathcal{H}_E \)-condition \( q_\xi' \leq q_\xi \land q_{\xi+1} \) for all limit \( \xi < \kappa \) such that

\[
q_\xi' \models \dot{h}_\xi \leq^* \dot{h}_\xi' \leq^* \dot{h}_{\xi+1}.
\]

Without loss of generality the family \( \{ q_\xi' \} \) is centered and if \( q' \) is a condition which forces that there are many \( q_\xi' \)'s in the generic filter, then it also forces that the family \( \{ h_\xi' \} \) includes a \( \kappa \)-chain which is cofinal in \( \{ f_\xi \} \). So the countable poset \( \mathcal{H}_A \) adds a \( \kappa \)-chain, and this finishes the proof.

In the following theorem \( \kappa \) and \( \lambda \) are uncountable regular cardinals.

**Theorem 9.2.** If \( \kappa > \epsilon \) and \( \mathcal{H}_E \) adds a \( (\kappa, \lambda) \)-gap to \( \omega^\omega \), then there is such a gap in \( E \) or in \( E^* \).

**Proof.** Let \( \langle f_0, \ldots, f_{1,\eta} \rangle_{\xi<\kappa, \eta<\lambda} \) be an \( \mathcal{H}_E \)-name for a gap. Working as in the proof of Theorem 9.1, we find a countable \( A \) and finite \( D_{0,\xi}, D_{1,\eta} \) such that \( \text{supp} \ f_{i,\xi} \subseteq A \cup D_{i,\xi} \) for all \( i, \xi \). Apply Lemma 7.3 to the family \( \{ D_{0,\xi}, D_{1,\eta} : \xi < \kappa, \eta < \lambda \} \), where the index set \( I = \{ 0 \} \times \kappa \cup \{ 1 \} \times \lambda \) is ordered lexicographically, to get an ordering \( <_0 \) on \( \bigcup_{(i,\xi) \in I} D_{i,\xi} \) such that this family has minimal patterns. Let \( D^0 = \bigcup_{\xi<\kappa} D_{0,\xi} \) and \( D^1 = \bigcup_{\eta<\lambda} D_{1,\eta} \), define \( < \) on \( A \cup D^0 \cup D^1 \) and translate \( \mathcal{H}_{(E,<\omega)} \)-names into \( \mathcal{H}_{(E,<\omega)} \)-names as in the proof of Theorem 9.1. In the following considerations “connected” means connected in the ordering \( <_0 \), while \( E \) stands for the poset \( (E,<) \).

**Case 1:** For all \( \xi < \kappa \) and \( \eta < \lambda \), \( D_{0,\xi} \) and \( D_{1,\eta} \) are not connected. Then \( \mathcal{H}_E \) is equivalent to \( \mathcal{H}_A \ast (\mathcal{H}_{A \cup D^0(A,\dot{g}_A)} \times \mathcal{H}_{A \cup D^1(A,\dot{g}_A)}) \) by Lemma 4.3. Work in an extension by \( \mathcal{H}_A \): Since \( A \) is countable, every real in an extension by \( \mathcal{H}_{A \cup D^0(A,\dot{g}_A)} \) (\( i = 0, 1 \)) is added by a countable subordering, so by Lemma 5.5 (applied to \( \mathcal{P}^1 = \mathcal{H}_{A \cup D^1(A,\dot{g}_A)} \) and \( f^0_\xi = f_{0,\xi} \) for \( i = 0, 1 \)), there is a ground-model function \( h \) which splits this gap.

**Case 2:** \( D_{0,\xi} \) and \( D_{1,\eta} \) are connected for some \( \xi, \eta \) (without loss assume that \( \xi = \eta = 0 \)). Then by Proposition 7.1 there is a \( \kappa + \lambda^* \)-chain in \( E \) or in \( E^* \) which intersects all \( D_{i,\xi} \)'s. If it is a gap in the \( <_E \)-ordering, then we are done. So suppose that it is filled by some \( c_0 \), and by symmetry we can
assume that it is a $\kappa + \lambda^*$-chain in $E$. Also, all $D_{0,\xi}$’s are of the same fixed size $n_0$ and all $D_{1,\eta}$’s are of the same fixed size $n_1$. We can assume that $c_0$ is in $A$. Now our argument splits into a finite binary tree. Let

$$
D^0_{0,\xi} = D_{0,\xi} \setminus D_{0,\xi}(\vartriangleleft c_0), \quad D^0_{1,\eta} = D_{1,\eta}, \\
D^1_{0,\xi} = D_{0,\xi}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \Quad
(1) $\langle X \cap [s], <_{\text{Lex}} \rangle$ is embeddable into $E$ or $E^*$, or
(2) $\langle X \cap [s], <_{\text{Lex}} \rangle$ is embeddable into $\omega$ after adding a single Cohen real.

**Proof of Theorem 10.1.** Since $\langle [s], <_{\text{Lex}} \rangle$ is isomorphic to $\langle 2^{\omega_1}, <_{\text{Lex}} \rangle$, the theorem follows immediately from Theorem 10.2. □

**Definition 10.1.** If $\langle L, <_L \rangle$ is a linearly ordered set then we say that embeddings $H_0 : \langle L, <_L \rangle \rightarrow \langle \omega, \leq^* \rangle$ and $H_1 : \langle L, <_L \rangle \rightarrow \langle \omega, \leq^* \rangle$ cohere if $H_i(x) \leq^* H_j(y)$ for all $x <_L y$ in $L$ and $i, j \in \{0, 1\}$.

**Lemma 10.1.** For every nonmeager subset $X$ of $2^{\omega_1}$ and a family $A_x$ ($x \in X$) of countable sets there is a countable $A$ such that the set of all $x \in X$ for which the set $\{y \in X : A_y \cap A_x \subseteq A\}$ accumulates to $x$ is nonmeager.

**Proof.** Let $\theta = (2^{\aleph_1})^+$ and let $S$ be the set of all approachable elementary submodels $N$ of $H_\theta$ such that $X \in N$ and there is an $(N, C_\omega)$-generic $x_N \in X$. By Theorem 8.2 the set $S$ is stationary. For $N \in S$ fix a continuous $\varepsilon$-chain of countable models converging to $N$, say $N = \bigcup_{\xi < \omega_1} M^N_\xi$, and let $M_N \prec H_\theta$ be an $M^N_\xi$ such that $A_{x_N} \cap M_N = A_{x_N} \cap N$. By the Pressing Down Lemma there is a stationary $S' \subseteq S$ and a countable $M = M_N$ for all $N \in S'$. By Lemma 8.3 the set $Y = \{x_N : N \in S'\}$ is nonmeager.

**Claim.** For all $x \in Y$ the set $\{y \in X : A_y \cap A_x \subseteq M\}$ accumulates to $x$.

**Proof.** Pick $x = x_N$ in $Y$ and a countable ordinal $\alpha$. Then $x|\alpha$ is in $N$ so by elementarity there is a $y \in X \cap [x|\alpha]$ in $N$ such that $A_y \cap A_x \subseteq M$. For a fixed $x$ this is true for all countable $\alpha$, so the claim is proved. □

This proves the lemma. □

**Proof of Theorem 10.2.** Let $f_x$ ($x \in X$) be a name for the embedding and $A_x = \text{supp } f_x$. Then by Lemma 10.1 there is a countable set $A \subseteq E$ and a nonmeager $X_1 \subseteq X$ such that for each $x \in X_1$ there is a $\omega_1$-sequence $\{y_\xi\}$ in $X$ converging to $x$ and such that $A_\xi \cap A_x \subseteq A$ for all $\xi < \omega_1$. By passing to an interval $[t]$ of $2^{<\omega_1}$, we can assume that $X_1$ is dense in $2^{<\omega_1}$.

The idea of the proof is to find an embedding $x \mapsto h_x$ which coheres with $x \mapsto f_x$ and which is such that $h_x$ is an $\mathcal{H}_A$-name for all $x \in X$ by applying Lemma 5.4 for all $x \in X$. Unfortunately, the name $h_x$ this lemma gives does not work with probability one. This is why we will have to go to an extension by $\mathcal{H}_E$ and work there. In this extension set $X$ is not a nonmeager set anymore, but it still retains a largeness property which is sufficient for our needs. Now we go to an extension by $\mathcal{H}_E$, and let $G$ be a generic filter in $\mathcal{H}_E$. In our following claim all $\mathcal{H}_E$-names are assumed to be elements of a ground model.
Claim 1. There is an embedding $x \mapsto \hat{h}_x$ of $X$ into $\omega \omega$ which coheres with $x \mapsto \hat{f}_x$ and is such that $\text{supp} \hat{h}_x \subseteq^* A$ for all $x \in X$.

Proof. Fix $x \in X_1$ and a sequence $\{y_\xi\}$ which converges to $x$. We can assume that this sequence is monotonic, say increasing. Note that the proof of Lemma 5.4 (applied in the ground model) shows that the set of conditions $q$ such that there is a name $\dot{h}$ with $\text{supp} \hat{h} \subseteq^* A$ and $q \models \dot{f}_{y_\xi} \leq^* \hat{h} \leq^* \dot{f}_x$ for all $\xi < \omega_1$

is dense in $\mathcal{H}_E$; therefore, in our extension there are $q_x \in \dot{G}$ and $\hat{h}_x$ with these properties. So we have an embedding $x \mapsto \hat{h}_x$ of $X_1$ into $\omega \omega$ which coheres with $x \mapsto \hat{f}_x$, and we want to extend it to an embedding of $X$. Let

$$D_x = (\text{supp} \hat{h}_x \cup F_{y_\xi}) \setminus A \quad \text{for } x \in X_1.$$  

For each $x \in X \setminus X_1$ we pick a monotonic $\omega_1$-sequence $\{y_\xi\}$ inside $X_1$ converging to $x$; then by applying the $\Delta$-system lemma to refine $\{D_{y_\xi}\}$ ($\xi < \omega_1$) we get a finite set $\Delta$ such that $D_\xi \cap A_x \subseteq \Delta$ for all $\xi$. So we can again apply an extension of Lemma 5.4 and get $q_x \in \dot{G}$, and $\hat{h}_x$ with $\text{supp} \hat{h}_x \subseteq^* A$ for all $x \in X$ as required. ■

Let $D_x$ be defined as in (D) above for all $x \in X$. Recall that a poset for adding a Cohen subset of $\omega_1$, $\mathcal{C}_{\omega_1}$, is $(2^{<\omega_1}, \subseteq)$. It is $\sigma$-closed, and after forcing with any ccc poset the intersection of countably many dense subsets of $\mathcal{C}_{\omega_1}$ (as computed in the ground model) is dense (by Easton [8]). So $(\mathcal{C}_{\omega_1})^V$ has this property in our extension by $\mathcal{H}_E$. Recall that by our assumptions $[t] \cap X$ is nonempty for all $t \in (\mathcal{C}_{\omega_1})^V$.

Claim 2. The set of all $t$ such that for some finite $F \subseteq E$,

$$(*) \quad \bigcap_{x \in I} D_x = F \quad \text{for every nonempty interval } I \text{ of } X \cap [t]$$

is dense in $(\mathcal{C}_{\omega_1})^V$.

Proof. Let $D_n = \{s \in (\mathcal{C}_{\omega_1})^V : |\bigcap_{x \in X \cap [s]} D_x| \geq n\}$. Suppose that below some $s \in (\mathcal{C}_{\omega_1})^V$ all $D_n$’s are dense; then we can pick $u \in \bigcap_{n=1}^\infty D_n$.

If $x \in [u] \cap X$ then $D_x$ is infinite—a contradiction. Therefore the set of all $s$ such that $[s] \cap D_n$ is empty for some $n$ is dense in $(\mathcal{C}_{\omega_1})^V$. For such an $s$ pick maximal $m < n$ so that some $t$ extending $s$ is in $D_m$, and let $F = \bigcap_{x \in [t]} D_x$; such $t$ and $F$ satisfy $(*)$. ■

By Claim 2 there is a maximal antichain $D_0 \subseteq (\mathcal{C}_{\omega_1})^V$ and a finite $F_t \subseteq E$ for $t \in D_0$ such that $(*)$ is true for all $t$ and $F_t$ in $D_0$. Pick $s \in D_0$ and let $Y = X \cap [s]$, $A' = A \cup F_s$ and $D'_x = D_x \setminus F_s$. Then by $(*)$ we have:

$$(**) \quad \text{For all } x <_{\text{Lex}} y \text{ in } Y \text{ and all } a \in D'_x \cap D'_y \text{ there is } z \in (x,y)_V \text{ such that } a \not\in D_z.$$
Let $E^+ = A' \cup \bigcup_{x \in Y} \{x\} \times D'_x$ and let $H : E^+ \rightarrow E$ be

\[ H(a) = \begin{cases} a & \text{if } a \in E, \\ b & \text{if } a = \langle x, b \rangle \text{ for some } x \in Y. \end{cases} \]

Then define a poset $x \in E$ for some $x \in Y$. Define an ordering $<_{E^+}$ on $E^+$ by

\[ a \leq_{E^+} b \iff H(a) \leq_E H(b). \]

Let $D^+ = \{x\} \times D'_x$. Then the posets $\langle A' \cup D^+_x, <_{E^+} \rangle$ and $\langle A' \cup D'_x, <_{E} \rangle$ are isomorphic, so let $h^+_x$ be an $H_{\langle A' \cup D^+_x, <_{E^+} \rangle}$-name isomorphic to $h_x$. Namely, we obtain $h^+_x$ from $h_x$ by replacing each occurrence of $b$ with $\langle x, b \rangle$ for all $b \in D'_x$. Let $q^+_x$ be the $H_{E^+}$-condition isomorphic to $q_x$. Since all $q_x$’s are in the same generic filter, the conditions $q^+_x$ are pairwise compatible.

**Claim 3.** In the poset $H_{E^+}(A', g_{A'})$ the condition $q^+_x \wedge q^+_y$ forces that $h^+_x \leq^* h^+_y$.

**Proof.** The idea is similar to that in the proof of Claim 2 in Theorem 9.1, but this time we have to construct yet another poset. By (**) pick $x = x_0 <_{\text{Lex}} x_1 <_{\text{Lex}} \ldots <_{\text{Lex}} x_n = y$ in $Y$ so that for each $a \in D'_x$ we have $a \notin D'_{x(a)}$ for some $0 < i(a) \leq n$. For $a \in \bigcup_{1 \leq i \leq n} D'_{x_i} \setminus D'_x$ let $i(a) = 0$. Then define a poset $E^{++}$ by

\[ D^{++}_i = \{a : a \in D'_x, i(a) < i\} \cup \{(1, a) : a \in D'_x, i(a) > i\} \text{ for } i \leq n, \]

\[ E^{++} = A' \cup \bigcup_{i \leq n} D^{++}_i. \]

The ordering on $E^{++}$ is defined similarly to that on $E^+$: let $H : E^{++} \rightarrow E$ be

\[ H(a) = \begin{cases} a & \text{if } a \in E, \\ b & \text{if } a = (1, b) \text{ for some } b \in E, \end{cases} \]

and let $a \leq_{E^{++}} b$ iff $H(a) \leq_E H(b)$. Then:

1. the posets $\langle A' \cup D^{++}_{x_i}, <_{E^{++}} \rangle$ and $\langle A \cup D'_{x_i}, <_E \rangle$ are isomorphic for all $i \leq n$.

An isomorphism naturally extends to one between $\mathcal{H}_{A \cup D'_{x_i}}$ and $\mathcal{H}_{A' \cup D^{++}_{x_i}}$, as well as to an isomorphism between classes of names in these posets. Let $q^+_i$ be the isomorphic image of $q_{x_i}$ for $i \leq n$. Then we have:

2. $\langle A \cup D^{++}_{x_i} \cup D^{++}_{x_{i+1}}, <_{E^{++}} \rangle$ is isomorphic to $\langle A' \cup D'_x \cup D'_{x_{i+1}}, <_E \rangle$ for all $i$, and the natural extension of this isomorphism sends $q^+_i, q^+_{i+1}$ to $q_{x_i}, q_{x_{i+1}}$, respectively and $h^+_i, h^+_{i+1}$ to $h_{x_i}, h_{x_{i+1}}$, respectively.

3. $\langle A' \cup D^{++}_0 \cup D^{++}_n, <_{E^{++}} \rangle$ is isomorphic to $\langle A' \cup D^+_0 \cup D^+_n, <_{E^+} \rangle$, and the natural extension of this isomorphism sends $q^+_0, q^+_n$ to $q^+_0, q^+_n$, respectively and $h^+_0, h^+_n$ to $h^+_0, h^+_n$, respectively.


So $\mathcal{H}_{E^+}$ below $\bigwedge_{i \leq n} q_i^+$ forces that $\dot{h}_{i^+} \leq_+ \dot{h}_{i+1}^+$ for all $i < n$, and hence that $\dot{h}_0^+ \leq_+ \dot{h}_1^+$, therefore $\mathcal{H}_{E^+}(A', \dot{g}_{A'})$ below the condition $\bigwedge_{i \leq n} q_i^+ (A^+ \cup D^+_0 \cup D^+_n)$ forces that $\dot{h}_0^+ \leq_+ \dot{h}_1^+$. This condition is equivalent to $q_0^+ \wedge q_1^+$. ■

Now apply Lemma 7.3 to $\langle \bigcup_{x \in Y} D^+_x, <_{E^+} \rangle$ to get an ordering $<_0$ on this set. As in the proof of Theorem 9.1, define an ordering $\prec$ on $E^+$ by

$$a \prec b \iff \begin{cases} a, b \in A' \cup D^+_\eta \text{ and } a <_{E^+} b \text{ for some } \eta, \text{ or } \\ a, b \in \bigcup_{\xi < \kappa} D^+_\xi \text{ and } a <_0 b, \text{ or } \\ a, b \in \bigcup_{\xi < \kappa} D^-_\xi \text{ and } a <_{E^+} c <_{E^+} b \text{ for some } c \in A'. \end{cases}$$

The proof that this relation is an ordering is the same as in the proof of Theorem 9.1, as well as the proof that $x \mapsto \dot{h}_{x^+}$ is forced to be an embedding of $Y$ into $\omega_\omega$ coherent with $x \mapsto \dot{f}_x$.

**Case 1:** There are $x \neq y \in Y$ such that $D^+_x$ and $D^+_y$ are connected by $\prec$. Then by Proposition 7.1 there is a chain of type $[x, y]_Y$ or $[x, y]_Y^*$ in $E^+$ and therefore in $E$, so $(\dagger 1)$ applies with any $s$ such that $[s] \subseteq [x, y]_Y$ (recall that $Y = X \cap [s]$, so $[x, y]_Y = [x, y]_X$).

**Case 2:** $D^+_x$ and $D^+_y$ are not connected by $\prec$ for all $x \neq y$. Note that the embedding $x \mapsto \dot{h}_x$ obtained by Claim 1 is such that it is added only by $\mathcal{H}_E$ (although the image of $X$ consists only of reals added by $\mathcal{H}_A$). Claim 4 below is stronger because the name for the embedding $x \mapsto \dot{h}_x$ obtained in it is an $\mathcal{H}_{A'}$-name.

**Claim 4.** There are $\mathcal{H}_{A'}$-names $\dot{h}_x''$ for $x \in Y$ such that $x \mapsto \dot{h}_x''$ is an embedding coherent with $x \mapsto \dot{h}_x$.

**Proof.** By the assumption and Lemma 4.3 the poset $\mathcal{H}_{E^+}$ can be written as

$$\mathcal{H}_{A'} \ast \prod_{x \in Y} \mathcal{H}_{D^+_x}(A', \dot{g}_{A'}) \quad (\text{product is taken with finite supports}).$$

Work in an extension by $\mathcal{H}_{E^+}$. For $x \in Y$ pick $\omega_1$-sequences $\{y_\xi\}$ and $\{z_\zeta\}$ in $Y$ converging to $x$ from below (resp. above). The pregap $\langle \dot{h}_{y_\xi}, \dot{h}_{z_\zeta} \rangle_{\xi < \omega_1}$ is filled by $\dot{h}_x^+$. By applying Lemma 5.1 as in the proof of Theorem 9.1 we find a pregap which is cofinal in this one and which is added by $\mathcal{H}_{A'}$. The poset $\mathcal{H}_{A' \cup D^+_x}(A', \dot{g}_{A'})$ is countable and it fills an $\langle \omega_1, \omega_1 \rangle$-pregap, therefore a pregap is already filled in the intermediate extension by $\mathcal{H}_{A'}$: let $\dot{h}_x''$ be the $\mathcal{H}_{A'}$-name for the function which fills it. Then the mapping $x \mapsto \dot{h}_x''$ is as required. ■

So in Case 2 a countable poset embeds $\langle Y, <_{L_X} \rangle$ into $\omega_\omega$, so $(\dagger 2)$ applies. ■
11. A preordering on $\omega$. By a Cohen poset $\mathcal{C}$ we mean $\langle 2^{<\omega}, \supseteq \rangle$ and $\dot{c}$ is a name for a $\mathcal{C}$-generic real. Let $\mathcal{N}$ denote the ideal of nowhere dense subsets of $\mathcal{C}$, i.e. all sets $A \subseteq \mathcal{C}$ such that the complement $\mathcal{C} \setminus A$ of $A$ includes an open dense set in $\mathcal{C}$.

**Definition 11.1.** For $f, g \in \omega$ we define:

1. $f =_\mathcal{N} g$ iff the set $\{ t \in \mathcal{C} : f(t) \neq g(t) \}$ is in $\mathcal{N}$.
2. $f \leq_\mathcal{N} g$ iff the set $\{ t \in \mathcal{C} : f(t) > g(t) \}$ is in $\mathcal{N}$.
3. $f <_\mathcal{N} g$ iff the set $\{ t \in \mathcal{C} : f(t) \geq g(t) \}$ is in $\mathcal{N}$.
4. $f =_\mathcal{N} g$ iff $f \leq_\mathcal{N} g$ and $g \leq_\mathcal{N} f$.

[Note that an analog of Proposition 0.1 is true in the case of these orderings.]

**Theorem 11.1.** A linearly ordered set $\langle L, <_L \rangle$ embeds into $\langle \omega, <_\mathcal{N} \rangle$ iff in a forcing extension by the Cohen algebra, $\langle L, <_L \rangle$ embeds into $\omega$.

For $f \in \omega$ and $r \in 2^\omega$ we define $f[r] \in \omega$ by $(f[r])(n) = f(r|n)$. So in particular $f[\dot{c}]$ is a Cohen name for an element of $\omega$.

**Lemma 11.1.** $f <_\mathcal{N} g \iff \models_c (\dot{f}[\dot{c}) <^* (\dot{g}[\dot{c})$.

**Proof.** Observe that $t \in \mathcal{C}$ forces that $(f[\dot{c}) <^n (g[\dot{c})$ for some $n \leq |t|$ iff $f(s) < g(s)$ for every $s \in [t]$. ♦

Note that for fixed $f$ the function $r \mapsto f[r]$ is continuous and that moreover (recall that the metric on $2^\omega$ is defined by $d(r, s) = 1/(\Delta(r, s) + 1)$, where $\Delta(r, s)$ is the minimal $n$ such that $r(n) \neq s(n)$)

$$d(s, r) \leq d(f[r], f[s]) \quad \text{for all } r, s \in \omega,$$

i.e. the function $r \mapsto f[r]$ is Lipschitz. It is well known that for every $C$-name $\dot{x}$ for a real there is a ground-model Borel function $F$ such that $F(\dot{c}) = \dot{x}$ with probability one (see e.g. [29, Theorem 2.3]). So Theorem 11.1 above says, among other things, that when investigating $C$-names for long chains in $\omega$ we can restrict ourselves to those consisting of Lipschitz functions.

For a $C$-name for a function $f$ in $\omega$ define $\hat{f} \in \omega$ by

$$\hat{f}(t) = \min \{ k : \text{some extension of } t \text{ forces that } f(|t|) = k \}.$$

The following is an “inverse” of Lemma 11.1.

**Lemma 11.2.** (a) If $\models_c \dot{f} <^* \dot{g}$, then $\hat{f} <_\mathcal{N} \hat{g}$.
(b) If $\models_c \dot{f} \leq^* \dot{g}$ and $\dot{f} \neq^* \dot{g}$ then $\hat{f} \leq_\mathcal{N} \hat{g}$ and $\hat{f} \neq_\mathcal{N} \hat{g}$.
(c) For $f \in \omega$ and a $C$-name $\dot{g}$, if $\models_c (\dot{f}[\dot{c}) <^* \dot{g}$, then $f <_\mathcal{N} \dot{g}$.

**Proof.** (a) Let $\{ t_i \}$ be the maximal antichain in $\mathcal{C}$ which decides $\mathfrak{m}$ from which the dominance happens, i.e. there is a sequence $\{ m_i \}$ such that

$$t_i \models \dot{f} <^* m_i \dot{g}$$
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for all $i$. We can assume that $m_i < |t_i|$ for all $i$. We claim that $\hat{f}(t) < \hat{g}(t)$ for all $t \in U = \bigcup_{i<\omega} [t_i]$; suppose otherwise, pick $t$ such that $\hat{f}(t) = k_1 > k_0 = \hat{g}(t)$ and $t <_C t_l$ for some $l \in \omega$. Let $n = |t|$ and $s <_C t$ be such that $s \Vdash \hat{g}(\bar{n}) = \bar{k}_0$; find $s' <_C s$ which decides $\hat{f}(n)$, say $s' \Vdash \hat{f}(\bar{n}) = \bar{k}$. But $n > m_l$ and $s' <_C \bar{t}_l$, so $k < k_0$. On the other hand, by the definition of $\hat{f}(n)$ we have $k_1 < k_0$—a contradiction. So we have a dense open subset $U$ of $\mathcal{C}$ which witnesses that $f <_{\mathcal{N}} g$, as required.

(b) & (c) These proofs are essentially the same as that of (a). ■

12. Concluding remarks. If $M$ is an elementary submodel of a large enough $\mathcal{H}_0$ and $\mathcal{H}_E \in M$, then, by Theorem 4.1, $M \cap \mathcal{H}_E$ is a regular subordering of $\mathcal{H}_E$, i.e. $\mathcal{H}_E$ is semicothen (see [1]). A result of Balcar–Jech–Zapletal ([1, Theorem 3.2]) in our case easily reduces to (see also our Lemma 4.5):

**Lemma 12.1.** The poset $\mathcal{H}_E$ is Cohen iff there is a club $C \subseteq [E]^\omega$ such that for all $A,B \in C$, $a <_E b$ implies that there is $c \in A \cap B$ such that $a <_E c <_E b$ for all $a \in A \setminus B$ and $b \in B \setminus A$. ■

In particular, $\mathcal{H}_{\omega_2}$ is not Cohen, but all its suborderings of smaller size are. [A different example of such a poset is found in [1], where it was remarked that (in our notation) $\mathcal{H}_{\omega_1}$ is not Cohen.] The following example gives a different proof of a recent result of Koppelberg–Shelah ([14]) that there is a regular subalgebra $\mathcal{B}$ of a Cohen algebra for adding $\aleph_2$ many Cohen reals which is not Cohen. Let $E = \omega_2 \times \{0,1\} \cup \{a\}$, with the ordering defined by

$$\langle \xi, 0 \rangle <_E a <_E \langle \eta, 1 \rangle$$

for all $\xi, \eta$, and let $E_0 = E \setminus \{a\}$. Then $\mathcal{H}_{E_0}$ is not Cohen by the above theorem. To see that $\mathcal{H}_{E_0}$ is Cohen, note that it is equivalent to an iteration of $\mathcal{H}_{\{a\}}$ with the poset $\mathcal{H}_{E_0}(\{a\}, \hat{g}_{\{a\}})$ and by a version of Lemma 4.3 the former poset is equivalent to a finite support product of countable posets, i.e. the poset for adding $\aleph_2$ many Cohen reals, $\mathcal{C}_{\aleph_2}$. So $\mathcal{H}_{E_0}$ is $\mathcal{C} \ast \mathcal{C}_{\aleph_2}$, which is $\mathcal{C}_{\aleph_2}$. Our example is different from one in [14] because in our case the quotient has the property that a generic object is determined by a single real.

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what conditions the poset \( \mathcal{H}_E \) adds long chains to “\( \omega \), for suggesting other problems and possible solutions, for always asking me to find the “right” statement, which greatly improved the shape of this paper, and generally for generously sharing his deep insight.

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References

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