FUNDAMENTA
MATHEMATICAE
151 (1996)

Bing maps and finite-dimensional maps

by

Michael Levin (Haifa)

Abstract. Let X and Y be compacta and let $f : X \to Y$ be a k-dimensional map. In [5] Pasynkov stated that if Y is finite-dimensional then there exists a map $g : X \to \mathbb{I}^k$ such that $\dim(f \times g) = 0$. The problem that we deal with in this note is whether or not the restriction on the dimension of Y in the Pasynkov theorem can be omitted. This problem is still open.

Without assuming that Y is finite-dimensional Sternfeld [6] proved that there exists a map $g : X \to \mathbb{I}^k$ such that $\dim(f \times g) = 1$. We improve this result of Sternfeld showing that there exists a map $g : X \to \mathbb{I}^{k+1}$ such that $\dim(f \times g) = 0$. The last result is generalized to maps f with weakly infinite-dimensional fibers.

Our proofs are based on so-called Bing maps. A compactum is said to be a Bing compactum if its compact connected subsets are all hereditarily indecomposable, and a map is said to be a Bing map if all its fibers are Bing compacta. Bing maps on finite-dimensional compacta were constructed by Brown [2]. We construct Bing maps for arbitrary compacta. Namely, we prove that for a compactum X the set of all Bing maps from X to $\mathbb{I}$ is a dense G_δ-subset of $C(X, \mathbb{I})$.

1. Introduction. All spaces are assumed to be separable metrizable. $\mathbb{I} = [0, 1]$. By a map we mean a continuous function. In [5] Pasynkov stated:

THEOREM 1.1. *Let $f : X \to Y$ be a k-dimensional map of compacta. Then there exists a map $g : X \to \mathbb{I}^k$ such that $f \times g : X \to Y \times \mathbb{I}^k$ is 0-dimensional.* ■

This theorem is equivalent to

THEOREM 1.2 (Toruńczyk [7]). *Let f, X and Y be as in Theorem 1.1. Then there exists a σ-compact subset A of X such that $\dim A \le k - 1$ and $\dim f|_{X \setminus A} \le 0$.* ■

Now we will prove the equivalence of these theorems. Let $f : X \to Y$ be a map of compacta. The following statements are equivalent:

1991 *Mathematics Subject Classification*: 54F45, 54F15.

Key words and phrases: finite-dimensional maps, hereditarily indecomposable continua.

(i) There exists a σ-compact $(k-1)$-dimensional subset A of X such that $\dim f|_{X\setminus A} \le 0$;

(ii) For almost all maps g in $C(X, \mathbb{I}^k)$, $\dim(f \times g) \le 0$ (where almost all = all but a set of first category);

(iii) There exists a map $g : X \to \mathbb{I}^k$ such that $\dim(f \times g) \le 0$.

Note that in (i)–(iii) we do not assume that Y is finite-dimensional.

(i)⇒(ii) (cf. [7]). Let $A = \bigcup A_i$, where the A_i are compact and $A_i \subset A_{i+1}$. By Hurewicz's theorem [3] almost all maps in $C(X, \mathbb{I}^k)$ are k-to-1 on every A_i. Let g be such a map. Since $A_i \subset A_{i+1}$, g is also k-to-1 on A. Let $y \in Y$ and $a \in \mathbb{I}^k$. Clearly $(f \times g)^{-1}(y, a) \subset (f^{-1}(y) \setminus A) \cup g^{-1}(a)$ and as $g^{-1}(a)$ is finite,

$$\dim(f \times g)^{-1}(y, a) = \dim(f^{-1}(y) \setminus A) \le 0.$$

(ii)⇒(iii) is obvious and for a proof of (iii)⇒(i) see [6].

In this note we study the following problem which is still open.

Problem 1.3. *Do Theorems* 1.1 *and* 1.2 *hold without the finite-dimensionality assumption on* Y?

Sternfeld [6] made a significant progress in solving Problem 1.3.

Theorem 1.4 ([6]). *Let $f : X \to Y$ be a k-dimensional map of compacta. Then for almost all maps $g : X \to \mathbb{I}^k$, $\dim(f \times g) \le 1$.* ∎

Theorem 1.5 ([6]). *Let $f : X \to Y$ be a k-dimensional map of compacta. Then there exists a σ-compact $(k-1)$-dimensional subset A of X such that* $\dim f|_{X\setminus A} \le 1$. ∎

Note that from the proof of the implication (i)⇒(ii) it follows that Theorem 1.4 can be derived from Theorem 1.5.

The approach of [6] does not allow one to reduce the dimension of f to 0 in Theorems 1.4 and 1.5 by removing a σ-compact finite-dimensional subset A. This case is left open in [6]. In this note we prove:

Theorem 1.6. *Let $f : X \to Y$ be a k-dimensional map of compacta. Then there exists a map $g : X \to \mathbb{I}^{k+1}$ such that* $\dim(f \times g) \le 0$. *Equivalently, there exists a σ-compact k-dimensional subset A of X such that* $\dim f|_{X\setminus A} \le 0$.

Theorem 1.7. *Let $f : X \to Y$ be a weakly infinite-dimensional map of compacta. Then there exists a σ-compact weakly infinite-dimensional subset A of X such that $f|_{X\setminus A}$ is* 0*-dimensional.*

The last theorem generalizes the analogous result of [6]. There the dimension of $f|_{X\setminus A}$ is reduced to 1.

Our approach is based on some auxiliary maps which we will call Bing maps. A compactum is said to be a *Bing space* if each of its subcontinua is hereditarily indecomposable. We will say that a map is a *Bing map* if its fibers are Bing spaces. Bing maps on finite-dimensional compacta were constructed by Brown [2]. We construct Bing maps on arbitrary compacta. Namely, we prove:

THEOREM 1.8. *Let X be a compactum. Almost all maps in $C(X,\mathbb{I})$ are Bing maps.*

See [4] for another application of Bing maps.

In the next section we will also use:

THEOREM 1.9 (Bing [1]). *Any two disjoint closed subsets of a compactum can be separated by a Bing compactum.* ∎

THEOREM 1.10 (Bing [1]). *In an n-dimensional* (*strongly infinite-dimensional*) *Bing compactum X there exists a point $x\in X$ such that every nontrivial continuum containing x is n-dimensional* (*strongly infinite-dimensional*). ∎

2. Proofs

Proof of Theorem 1.8. Let $Q = \{(x_1, x_2, \ldots) : x_i \in \mathbb{I}\}$ be the Hilbert cube and let

$$\mathcal{D} = \{(F_0, F_1, V_0, V_1) : F_i, V_i \subset Q,\ F_i \text{ are closed and disjoint, } V_i \text{ are disjoint neighborhoods of } F_i\}.$$

Following [1] we say that $A \subset Q$ is *D-crooked* for $D = (F_0, F_1, V_0, V_1) \in \mathcal{D}$ if there is a neighborhood G of A in Q such that for every $\psi : \mathbb{I} \to G$ with $\psi(0) \in F_0$ and $\psi(1) \in F_1$ there exist $0 < t_0 < t_1 < 1$ such that $\psi(t_0) \in V_1$ and $\psi(t_1) \in V_0$. Clearly

(i) *if A is D-crooked then there exists a neighborhood $A \subset G$ which is also D-crooked.*

Actually, in [1] it is proved that:

(ii) *a compactum $A \subset Q$ is a Bing space if and only if A is D-crooked for every $D \in \mathcal{D}$, and*

(iii) *there exists a sequence $D_1, D_2, \ldots \in \mathcal{D}$ such that for every compactum $A \subset Q$, A is a Bing space if and only if A is D_i-crooked for every D_i.*

We say that a map $g : X \subset Q \to \mathbb{I}$ is *D-crooked* if its fibers are D-crooked.

Let $X \subset Q$ be compact and let $D \in \mathcal{D}$.

(iv) *The set of all D-crooked maps from X to $\mathbb{I}$ is open in $C(X,\mathbb{I})$.*

Let $g : X \to \mathbb{I}$ be D-crooked. By (i) for every $y \in \mathbb{I}$ there is a neighborhood U_y such that $g^{-1}(U_y)$ is also D-crooked. Let $\varepsilon > 0$ be so small that every subset of $\mathbb{I}$ of diameter $\le \varepsilon$ is contained in some U_y. One can show that every map ε-close to g is D-crooked and (iv) follows.

(v) *The set of all D-crooked maps from X to $\mathbb{I}$ is dense in $C(X,\mathbb{I})$.*

Let $g : X \to \mathbb{I}$. We will approximate g by a D-crooked map. By an arbitrary small change of g we may avoid the ends of $\mathbb{I}$ and hence it may be assumed that $g(X)$ does not contain 0 and 1.

Let $\varepsilon > 0$. Take $y_1 = 0 < y_2 < \ldots < y_n = 1$ such that $y_{j+1} - y_j < \varepsilon$. Let $\delta > 0$ be so small that $y_j + \delta < y_{j+1} - \delta$ for every j. By Theorem 1.9 take Bing compacta S_j which separate between $g^{-1}([0, y_j - \delta])$ and $g^{-1}([y_j + \delta, 1])$, $j = 2, \ldots, n-1$ (note that we regard the empty set as a Bing space). Modify g on every $M_j = g^{-1}([y_j - \delta, y_j + \delta])$, $j = 2, \ldots, n-1$, so that the image of M_j is contained in $[y_j - \delta, y_j + \delta]$ and the fibers of $y_j - \delta$, y_j and $y_j + \delta$ are $g^{-1}(y_j - \delta)$, S_j and $g^{-1}(y_j + \delta)$ respectively.

So without loss of generality we may assume that $A_j = g^{-1}(y_j)$ are Bing spaces for all $j = 1, \ldots, n$. Let $A = \bigcup A_j$. Then A is a Bing space. Let $D = (F_0, F_1, V_0, V_1)$. Take disjoint closed neighborhoods F_i' of F_i such that $F_i' \subset V_i$ and define $D' = (F_0', F_1', V_0, V_1)$ and $V_i' = \operatorname{int} F_i'$. By (ii), A is D'-crooked and by (i) we can take a D'-crooked neighborhood B of A in Q.

We claim that $G = B \cup V_0' \cup V_1'$ is D-crooked. Let $\psi : \mathbb{I} \to G$ satisfy $\psi(0) \in F_0$ and $\psi(1) \in F_1$. Clearly there exist $0 \le b_0 < b_1 \le 1$ such that $\psi(b_i) \in \partial V_i' \subset F_i'$ and $\psi([b_0, b_1]) \subset B \setminus (V_1' \cup V_2') \subset B$. Since B is D'-crooked, there exist $b_0 < t_0 < t_1 < b_1$ such that $\psi(t_0) \in V_1$ and $\psi(t_1) \in V_0$ and therefore G is D-crooked.

Clearly $T = X \setminus G$ is D-crooked and since T does not meet A, $A \cup T$ is also D-crooked. Set $X_j = g^{-1}([y_j, y_{j+1}])$ and $T_j = X_j \cap T$. Then T_j does not meet A_j and A_{j+1}. So we can take maps $g_j' : X_j \to [y_j, y_{j+1}]$ such that $g_j'^{-1}(y_j) = A_j \cup T_j$ and $g_j'^{-1}(y_{j+1}) = A_{j+1}$. Define $g' : X \to \mathbb{I}$ by $g'(x) = g_j'(x)$ for $x \in X_j$. Then g' is well-defined and ε-close to g. Every fiber of g' is contained in either $A \cup T$ or G. So g' is D-crooked and (v) follows.

To complete the proof of the theorem we apply the Baire theorem to (iii)–(v). ■

P r o o f o f T h e o r e m 1.6. By Theorem 1.8 take a Bing map $\psi : X \to \mathbb{I}$. Define $p = f \times \psi$ and

$$\mathcal{D}_n = \{D : D \text{ is a continuum contained in a fiber of } p,\ \operatorname{diam} D \ge 1/n\}.$$

Set $B_n = \bigcup_{D \in \mathcal{D}_n} D$ and $B = \bigcup B_n$. Then B_n is compact. Since f is k-dimensional, $\dim D \le k$ for every $D \in \mathcal{D}_n$.

Let us show that $\dim \psi|_{B_n} \le k$. Indeed, for every $a \in \mathbb{I}$, $A = \psi^{-1}(a)$ is a Bing compactum. Clearly $B_n \cap A = \bigcup\{D : D \in \mathcal{D}_n \text{ and } D \subset A\}$. Hence by Theorem 1.10, $\dim(B_n \cap A) \le k$. So $\dim \psi|_{B_n} \le k$.

By Theorem 1.2 and (ii) in the introduction, for every B_n almost all maps φ in $C(X, \mathbb{I}^k)$ satisfy $\dim(\psi \times \varphi)|_{B_n} = 0$ and hence almost all maps φ satisfy $\dim(\psi \times \varphi)|_B = 0$. Let φ be such a map. It is easy to see that for $g = \psi \times \varphi : X \to \mathbb{I}^{k+1}$, $f \times g$ is 0-dimensional and we are done. ∎

Proof of Theorem 1.7. We need the following

LEMMA 2.1. *Let $f : X \to Y$ be a perfect (= closed with compact fibers) map with* $\dim Y = 0$ *and let T be the union of trivial components of X. Then* $\dim T = 0$. ∎

Proof. Let $x \in T$ and let G be a neighborhood of x in X. Take disjoint open sets V_1 and V_2 such that $x \in V_1 \subset G$ and $f^{-1}(y) \subset V$ where $y = f(x)$ and $V = V_1 \cup V_2$. Set $U = Y \setminus f(X \setminus V)$. Then V is open and $y \in U$. Let H be clopen in Y such that $y \in H \subset U$. Then $V' = f^{-1}(H)$ is also clopen in X and $V' \subset V$. Thus $V' = V_1' \cup V_2'$ is a disjoint decomposition of V' with $V_i' = V' \cap V_i$ and therefore the V_i' are clopen in X. Clearly $x \in V_1' \subset G$ and we are done. ∎

Returning to the proof of Theorem 1.7, let ψ, p and B_n be as in the proof of Theorem 1.6. By the same reasoning we see that the B_n are weakly infinite-dimensional. Clearly p is also weakly infinite-dimensional. By [6], Lemma 1, there exists a σ-compact zero-dimensional subset Z of $Y \times \mathbb{I}$ such that for every $y \in Y$, $U_y = (\{y\} \times \mathbb{I}) \setminus Z$ is zero-dimensional. Define $A^1 = p^{-1}(Z)$ and $A^2 = \bigcup_{n \geq 1} B_n$. Set $A = A^1 \cup A^2$ and let us show that A is the desired set.

Obviously A is σ-compact and weakly infinite-dimensional. Let $y \in Y$. Define $V_y = p^{-1}(U_y)$ and let $T_y =$ the union of trivial components of V_y. By Lemma 2.1, $\dim T_y = 0$. Clearly $T_y = V_y \setminus A^2$. Also clearly

$$\begin{aligned} T_y = V_y \setminus A^2 &= p^{-1}(U_y) \setminus A^2 = p^{-1}((\{y\} \times \mathbb{I}) \setminus Z) \setminus A^2 \\ &= (p^{-1}(\{y\} \times \mathbb{I}) \setminus p^{-1}(Z)) \setminus A^2 = (f^{-1}(y) \setminus A^1) \setminus A^2 \\ &= f^{-1}(y) \setminus (A^1 \cup A^2) = f^{-1}(y) \setminus A. \end{aligned}$$

So $f^{-1}(y) \setminus A$ is zero-dimensional and we are done. ∎

References

[1] R. H. Bing, *Higher dimensional hereditarily indecomposable continua*, Trans. Amer. Math. Soc. 71 (1951), 653–663.

[2] N. Brown, *Continuous collections of higher dimensional indecomposable continua*, Ph.D. thesis, University of Wisconsin, 1958.

[3] K. Kuratowski, *Topology II*, PWN, Warszawa, 1968.

[4] M. Levin and Y. Sternfeld, *Atomic maps and the Chogoshvili–Pontrjagin claim*, preprint.

[5] B. A. Pasynkov, *On dimension and geometry of mappings*, Dokl. Akad. Nauk SSSR 221 (1975), 543–546 (in Russian).
[6] Y. Sternfeld, *On finite-dimensional maps and other maps with "small" fibers*, Fund. Math. 147 (1995), 127–133.
[7] H. Toruńczyk, *Finite to one restrictions of continuous functions*, ibid. 75 (1985), 237–249.

Department of Mathematics
Haifa University
Mount Carmel, Haifa 31905, Israel
E-mail: levin@mathcs2.haifa.ac.il

Received 4 January 1996;
in revised form 19 May 1996