FUNDAMENTA
MATHEMATICAE
151 (1996)

There exists a polyhedron dominating infinitely many different homotopy types

by

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Abstract. We answer a question of K. Borsuk (1968) by showing that there exists a polyhedron dominating infinitely many polyhedra of different homotopy types. This also gives solutions of two other problems in shape theory.

In this paper, every polyhedron is finite. Without loss of generality, we assume every polyhedron and CW-complex to be connected.

In 1979 at the International Topological Conference in Moscow K. Borsuk [B] defined the *capacity* $C(A)$ of a compactum A as the cardinality of the class of the shapes of all compacta X for which $\mathrm{Sh}(X) \leq \mathrm{Sh}(A)$ and asked:

Is the capacity of each polyhedron finite?

Actually this question appeared for the first time in 1968, at the Topological Conference in Herceg-Novi (see also [B_3]).

The problem was stated in both pointed and unpointed case, but by Hastings and Heller's results [HH], one can deduce that the pointed and unpointed capacity of each polyhedron are the same.

For many years the answer to Borsuk's question has been known only in dimension 1. Indeed, every compactum dominated by a 1-dimensional polyhedron has the shape of a movable continuum of dimension 1. Thus, by the result of Trybulec [Tr], it is shape equivalent to some continuum in E^2. Since two continua in E^2 have the same shape if and only if their first Betti numbers coincide (see [B_1], Theorem 7.1, p. 221), the capacity of each 1-dimensional polyhedron is finite.

In the present paper we show that the answer to Borsuk's question is negative:

1991 *Mathematics Subject Classification*: 55P55, 55P15.

There exists a polyhedron (*of dimension* 2) *which homotopy dominates infinitely many polyhedra of different homotopy types*.

We also indicate a few consequences of this result. One of them is a negative solution of some problem of Borsuk and Olędzki on retracts of polyhedra (see [BO], p. 69):

Let $\mathcal{T}$ be a triangulation of a polyhedron P. By a $\mathcal{T}$-retract of P we understand a retract of P which is the union of some simplexes of $\mathcal{T}$. Is it true that for each retract A of P, there is a $\mathcal{T}$-retract of P which is homotopy equivalent to A?

Moreover, we observe that every polyhedron P having $C(P) = \infty$ leads either to a counterexample to the above question or to an example of a polyhedron with infinitely many different shape factors which is still unknown.

We suppose the reader to be familiar with the basic notions and theorems in shape theory; the classical work here is [B_1].

Since every polyhedron is homotopy equivalent to a finite CW-complex of the same dimension, and conversely, we use the terms "polyhedron" and "finite CW-complex" interchangeably.

1. There exists a polyhedron dominating infinitely many different homotopy types. In 1949 J. H. C. Whitehead proved the following theorem:

THEOREM (J. H. C. Whitehead, [Wh_1], [Wh_2]). *Assume that X and Y are finite CW-complexes*, $\dim X = \dim Y = 2$ *and* $\pi_1(X) \cong \pi_1(Y)$. *Then there exist integers m_X and m_Y such that*

$$X \vee \bigvee_{m_X} S^2 \simeq Y \vee \bigvee_{m_Y} S^2 \ (^1).$$

Therefore, we may consider the directed tree whose vertices are the homotopy types of 2-dimensional polyhedra with a given fundamental group and where the homotopy type of X is joined by an edge to the homotopy type of $X \vee S^2$ (this graph clearly contains no circuits).

Each such tree is partitioned into *levels* by the Euler–Poincaré characteristic χ, i.e. the types of X and Y are on the same level in the tree iff $\chi(X) = \chi(Y)$.

By *junctions* in a tree we mean homotopy types with two or more immediate predecessors, i.e. ones which have two or more inequivalent factorizations involving an S^2 summand. Our example of a polyhedron P with infinite $C(P)$ will be obtained as an infinite junction in some tree.

(1) Here $\bigvee_k S^2$, where $k \in \mathbb{N}$, denotes the wedge of k spheres S^2.

Let P be a finite 2-dimensional CW-complex with a single vertex. It is well known that one can assign to P a presentation $\mathcal{P}(P)$ of its fundamental group $\pi_1(P)$ in which the generators correspond to the 1-cells of P and the relators are given by the characteristic maps of the 2-cells of P, where for every sphere S^2, i.e. a 2-cell attached in a trivial way, we add the relation $1 = 1$.

Conversely, if $\mathcal{P}$ is a finite presentation of some group G, then there is a finite 2-dimensional CW-complex $K(\mathcal{P})$ with a single vertex in which the 1-cells correspond to the generators of $\mathcal{P}$ and the attaching maps of the 2-cells are given by the relations of $\mathcal{P}$, i.e. if $\mathcal{P} = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_m \rangle$, then

$$K(\mathcal{P}) = (e^0 \cup e^1_1 \cup \ldots \cup e^1_k) \cup_{r_1} e^2_1 \cup_{r_2} \ldots \cup_{r_m} e^2_m.$$

It is easily seen that $\pi_1(K(\mathcal{P})) \cong G$.

So there is a 1-1 correspondence between finite 2-dimensional CW-complexes with a single vertex with fundamental group isomorphic to G, and finite presentations of this group in which we admit relations $1 = 1$.

From now on we assume every 2-dimensional CW-complex to have only one vertex. This involves no loss of generality, since it is well known that each CW-complex homotopy deforms into a one-vertex CW-complex of the same dimension (see for instance [FF], p. 34).

Let us begin with the remark that, by its proof (see for example [Wh_2], p. 49, Theorem 14), the Whitehead theorem on trees can be strengthened as follows:

THEOREM. *Let X and Y be finite CW-complexes with* $\dim X = \dim Y = 2$ *and* $\pi_1(X) \cong \pi_1(Y)$ *such that the associated presentations* $\mathcal{P}(X)$ *and* $\mathcal{P}(Y)$ *have a generators and b relations*, *and c generators and d relations* (*respectively*). *Then*

$$X \vee \bigvee_{a+d} S^2 \simeq Y \vee \bigvee_{b+c} S^2.$$

Using the above and the result of M. Lustig [Lu], we obtain:

THEOREM 1. *There exists a polyhedron P with* $\dim P = 2$ *and infinitely many polyhedra P_i with* $\dim P_i = 2$, *of different homotopy types*, *such that* $P_i \vee S^2 \simeq P$.

Proof. First observe that from the Whitehead theorem on trees as formulated above, it follows that if there exists a level in some tree containing infinitely many different homotopy types of CW-complexes K_i with $\dim K_i = 2$ for which $\mathcal{P}(K_i)$ have the same number of generators, then there exist infinitely many finite CW-complexes P_i with $\dim P_i = 2$, of different homotopy types, such that $P_i \vee S^2$ are all homotopy equivalent.

Let K_i, for i a prime, be the 2-dimensional CW-complex with a single vertex associated with the presentation

$$\mathcal{P}_i = \langle r, s, t \mid s^2 = t^3,\ [r^2, s^{2i+1}] = 1,\ [r^2, t^{3i+1}] = 1\rangle.$$

M. Lustig [Lu] has shown that the K_i have isomorphic fundamental groups and pairwise different homotopy types.

Since each $\mathcal{P}_i$ has 3 generators and 3 relations, the polyhedron $K = K_1 \vee \bigvee_6 S^2$ has $C(K) = \infty$. Moreover, there exists an infinite junction in the tree of the group $\pi_1(K_i)$, which is the desired conclusion. ∎

Remark 1. In 1979 M. N. Dyer conjectured [Dy] that if X and Y are polyhedra with $\dim X = \dim Y = 2$, then $X \vee S^2 \vee S^2 \simeq Y \vee S^2 \vee S^2$ implies that $X \vee S^2 \simeq Y \vee S^2$. So it is of interest to ask whether the CW-complexes $K_i \vee S^2$, for K_i from the proof of Theorem 1, are all homotopy equivalent.

Remark 2. Note that the CW-complexes K_i are obtained by attaching two 2-cells to the CW-complex associated with the presentation $\langle s, t \mid s^2 = t^3\rangle$ of the trefoil group T. The above-mentioned result of Lustig [Lu] is based on the earlier result of Dunwoody [Du] stating that for each integer i, the pair s^{2i+1}, t^{3i+1} generates T. Moreover, the so-called *relation modules* of T corresponding to these generating systems are pairwise non-isomorphic $\mathbb{Z}T$-modules [BDu]. The corresponding presentations of T have each no more than d relations, for some integer d independent of i (see [MP]). The question arises if there are infinitely many different homotopy types among the associated CW-complexes.

Let us now state some consequences of Theorem 1. The first is an immediate negative solution of the following problem in shape theory [DKN]:

Is it true that for a polyhedron P, the cardinality of the class of the shapes of all movable P-like compacta is finite?

Corollary 1. *There exists a polyhedron P* (*with* $\dim P = 2$) *such that the cardinality of the class of the shapes of all movable P-like compacta is infinite.*

Proof. Indeed, by the known result of Trybulec (see [Kd]), every space dominated by a given polyhedron P has the shape of some P-like compactum. Obviously, every polyhedron is movable. This ends the proof. ∎

Borsuk also asked about the relations between $C(A)$, $C(B)$, $C(A \cap B)$ and $C(A \cup B)$, for two compacta A and B ([B], answered in [K_1]). Our main result implies the following:

Corollary 2. *There exist two finite CW-complexes P and Q with* $\dim P = \dim Q = 2$ *such that* $C(P)$, $C(Q)$ *and* $C(P \cap Q)$ *are finite*, *while* $C(P \cup Q)$ *is infinite.*

Proof. For $1 \leq i \leq 4$, let L_i be the finite 2-dimensional CW-complex associated with the presentation $\mathcal{R}_i$, where

$\mathcal{R}_1 = \langle r, s, t \rangle$,
$\mathcal{R}_2 = \langle r, s, t \mid s^2 = t^3 \rangle$,
$\mathcal{R}_3 = \langle r, s, t \mid s^2 = t^3,\ [r^2, s^3] = 1 \rangle$,
$\mathcal{R}_4 = \langle r, s, t \mid s^2 = t^3,\ [r^2, s^3] = 1,\ [r^2, t^4] = 1 \rangle$,

and let $L_i = L_4 \vee \bigvee_{i-4} S^2$, for $5 \leq i \leq 10$.

Note that for $i \geq 2$, $L_i = L'_{i-1} \cup B_i$, where L'_{i-1} is a finite CW-complex homotopy equivalent to L_{i-1}, B_i is a topological disk, and $L'_{i-1} \cap B_i$ is a circle S^1.

Here $L_{10} = K$, where K is the polyhedron dominating infinitely many different homotopy types of polyhedra from the proof of Theorem 1. Since $C(L_1)$ is finite (we have $C(\bigvee_k S^1) = k + 1$, see the introduction), there exists an integer $2 \leq i \leq 10$ such that $C(L_{i-1})$ is finite while $C(L_i)$ is infinite. Taking $P = L'_{i-1}$ and $Q = B_i$, we finish the proof. ∎

2. An answer to some question concerning retracts of polyhedra. We now answer the question of Borsuk and Olędzki on $\mathcal{T}$-retracts (see the introduction), proving actually more:

THEOREM 2. *There exists a polyhedron P with* $\dim P = 7$ *such that for every triangulation $\mathcal{T}$ of P, there is some retract of P which is not homotopy equivalent to any $\mathcal{T}$-retract of P.*

Proof. Suppose that there exists a polyhedron K which homotopy dominates infinitely many m-dimensional polyhedra of different homotopy types. Then for every triangulation $\mathcal{T}$ of the polyhedron $P = K \times Q^{2m+1}$, where Q^{2m+1} denotes a $(2m+1)$-dimensional cube, there exists some retract of P which is not homotopy equivalent to any $\mathcal{T}$-retract of P.

For the proof, observe that every polyhedron $L \leq K$ with $\dim L = m$ is homeomorphic to a retract of P. Indeed, let $u : L \to K$ be a map converse to the domination $d : K \to L$, i.e. $du \simeq \mathrm{id}_L$. It is well known that every m-dimensional compactum can be embedded in Q^{2m+1}, so let $h : L \to Q^{2m+1}$ be a homeomorphism onto $h(L)$. Then the map $w : L \to P$ defined by $w(l) = (u(l), h(l))$ is also a homeomorphism and $w(L)$ is a retract of P.

We have shown in Theorem 1 that the polyhedron $K = K_1 \vee \bigvee_6 S^2$, where K_1 is the polyhedron associated with the presentation

$$\mathcal{P}_1 = \langle r, s, t \mid s^2 = t^3,\ [r^2, s^3] = 1,\ [r^2, t^4] = 1 \rangle,$$

dominates infinitely many 2-dimensional polyhedra of different homotopy types. Thus the polyhedron $P = K \times Q^5$ satisfies the requirements of the theorem. ∎

Remark 3. It should be noted that in [BO] it was established that for every compactum $Y \in \mathrm{ANR}$ with $\dim Y = n$, the compactum $Z = Y \times Q^{2n+1}$ is *shape regular*, i.e. every compactum X with $\mathrm{Sh}(X) \le \mathrm{Sh}(Z)$ has the shape of some fundamental retract of Z ([BO], Remark on p. 69).

For 1-connected polyhedra the Borsuk problem remains unsolved. It is only known that polyhedra with some special properties of homotopy or homology groups cannot dominate infinitely many different shapes (see [DKN] and [K]). Since every 1-connected polyhedron dominates only polyhedra (see [Wa]), let us formulate:

PROBLEM 1. *Is it true that for every* 1*-connected polyhedron* P *there exists a triangulation* $\mathcal{T}$ *of* P *such that each retract of* P *is homotopy equivalent to some* $\mathcal{T}$*-retract of* P?

One may also state:

PROBLEM 2. *Is it true that for every polyhedron* P *with* $\dim P \le 6$ *there exists a triangulation* $\mathcal{T}$ *of* P *such that each retract of* P *is homotopy equivalent to some* $\mathcal{T}$*-retract of* P?

3. The connections with the Borsuk problem on shape factors of FANR's. Recall that the shapes $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ are said to be *factors* of the shape $\mathrm{Sh}(X \times Y)$ (see [B_1], p. 205). The following question of Borsuk from 1975 remains unanswered [B_3]:

Are there only a finite number of different factors of $\mathrm{Sh}(X)$ *for every* $X \in \mathrm{FANR}$?

For $X \in \mathrm{ANR}$ this problem was posed earlier, in 1971 [B_2]. By the well-known result of J. West (1975, [We]), it is equivalent to the same question for polyhedra.

Let us observe that there is a connection between the Borsuk problem on shape factors of polyhedra and the two main problems considered in this paper. In fact, since for every compactum X shape dominated by a polyhedron, the product $X \times S^1$ has the shape of a polyhedron [M], we get:

COROLLARY 3. *Let* P *be a polyhedron with* $\dim P = n$ *and* $C(P) = \infty$. *Then one of the following conditions is satisfied*:

(i) *There exists a polyhedron* L (*with* $\mathrm{Sh}(L) \le \mathrm{Sh}(P \times S^1)$) *and infinitely many compacta* X_i *of different shapes such that* $\mathrm{Sh}(X_i) \times \mathrm{Sh}(S^1) = \mathrm{Sh}(L)$.

(ii) $P \times S^1$ *homotopy dominates infinitely many polyhedra of different homotopy types.* ∎

The proof of Theorem 2 shows that the condition (ii) of Corollary 3 leads to a counterexample to the question on $\mathcal{T}$-retracts. Indeed, by the known result of Wall (Theorem F of [Wa], p. 66), every polyhedron dominated by an n-dimensional polyhedron is homotopy equivalent to one of dimension $\max(n,3)$. Thus, for the polyhedron $\overline{P} = P \times S^1 \times Q^{2n+3}$, where Q^{2n+3} is a $(2n+3)$-dimensional cube, there is a retract of $\overline{P}$ which is not homotopy equivalent to any $\mathcal{T}$-retract of $\overline{P}$. This gives our claim.

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Received 27 February 1995;
in revised form 6 June and 17 November 1995