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An extension of a theorem of Marcinkiewicz and Zygmund on differentiability

by

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Abstract. Let f be a measurable function such that $\Delta_k(x,h;f) = O(|h|^\lambda)$ at each point x of a set E, where k is a positive integer, $\lambda > 0$ and $\Delta_k(x,h;f)$ is the symmetric difference of f at x of order k. Marcinkiewicz and Zygmund [5] proved that if $\lambda = k$ and if E is measurable then the Peano derivative $f_{(k)}$ exists a.e. on E. Here we prove that if $\lambda > k-1$ then the Peano derivative $f_{([\lambda])}$ exists a.e. on E and that the result is false if $\lambda = k-1$; it is further proved that if λ is any positive integer and if the approximate Peano derivative $f_{(\lambda),\mathrm{a}}$ exists on E then $f_{(\lambda)}$ exists a.e. on E.

1. Introduction. Let f be a real-valued function defined in some neighbourhood of x. Then f is said to have *Peano derivative* (resp. *approximate Peano derivative*) at x of order k if there exist real numbers $\alpha_r,\ 1 \le r \le k$, depending on x and f only such that

$$f(x+h) = f(x) + \sum_{r=1}^{k} \frac{h^r}{r!}\alpha_r + \frac{h^k}{k!}\varepsilon_k(x,h,f),$$

where

$$\lim_{h\to 0} \varepsilon_k(x,h,f) = 0 \qquad (\text{resp. } \operatorname*{lim\,ap}_{h\to 0} \varepsilon_k(x,h,f) = 0).$$

The number α_k is called the *Peano derivative* (resp. *approximate Peano derivative*) of f at x of order k and is denoted by $f_{(k)}(x)$ (resp. $f_{(k),\mathrm{a}}(x)$). For convenience we shall write $\alpha_0 = f(x) = f_{(0)}(x) = f_{(0),\mathrm{a}}(x)$.

Suppose that f has Peano derivative (resp. approximate Peano derivative) at x of order $k-1$. For $h \ne 0$ we write

$$\omega_k(x,h;f) = \omega_k(x,h) = \frac{k!}{h^k}\left[f(x+h) - \sum_{r=0}^{k-1}\frac{h^r}{r!}\alpha_r\right].$$

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The *upper* and *lower Peano derivates* (resp. *approximate Peano derivates*) of f at x of order k are defined by

$$\bar{f}_{(k)}(x) = \limsup_{h\to 0} \omega_k(x,h) \quad (\text{resp. } \bar{f}_{(k),\mathrm{a}}(x) = \limsup_{h\to 0} \operatorname{ap} \omega_k(x,h)),$$
$$\underline{f}_{(k)}(x) = \liminf_{h\to 0} \omega_k(x,h) \quad (\text{resp. } \underline{f}_{(k),\mathrm{a}}(x) = \liminf_{h\to 0} \operatorname{ap} \omega_k(x,h)).$$

The *symmetric difference* of f at x of order k, where k is a positive integer, is defined by

$$\Delta_k(x,h) = \Delta_k(x,h;f) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} f\left(x+ih-\frac{k}{2}h\right).$$

Marcinkiewicz and Zygmund proved in a deep theorem (Theorem 1 of [5]) that if f is measurable and if for a positive integer k,

$$\Delta_k(x,h;f) = O(|h|^k) \quad \text{as } h\to 0,$$

for x in a measurable set E then $f_{(k)}$ exists a.e. on E. For $k=2$ this is proved in [9, II, p. 78, Theorem 4.30]. For general k the proof is very long and involved (it is worth mentioning that the proof offered by Marcinkiewicz and Zygmund has a lacuna filled by Fejzic and Weil [3]).

The purpose of the present paper is to extend this result. In fact we prove in Theorem 3.1 that if f is measurable and if for a positive integer k,

$$\Delta_k(x,h;f) = O(|h|^\lambda) \quad \text{as } h\to 0$$

for x in a set E (not necessarily uniformly), where $\lambda > k-1$, then $f_{([\lambda])}$ exists a.e. on E, $[\lambda]$ denoting the greatest integer not exceeding λ. For $\lambda = k$ this gives the result of Marcinkiewicz and Zygmund cited above. Also we show in Theorem 3.2 that this result is not true for $\lambda = k-1$. Thirdly, in Theorem 3.4 we show that if we further assume that the approximate Peano derivative $f_{(k-1),\mathrm{a}}$ exists on E then the above result is true for $\lambda = k-1$. In fact, we prove in Theorem 3.4 that if f is measurable and if

$$\Delta_k(x,h;f) = O(|h|^p) \quad \text{as } h\to 0$$

for every x in a set E, where k and p are positive integers, and if $f_{(p),\mathrm{a}}$ exists finitely on E then $f_{(p)}$ exists a.e. on E.

We consider the difference

$$\begin{aligned}
&\widetilde{\Delta}_1(x,h) = \widetilde{\Delta}_1(x,h;f) = f(x+h) - f(x),\\
(1.1)\qquad &\widetilde{\Delta}_n(x,h) = \widetilde{\Delta}_n(x,h;f)\\
&\qquad = \widetilde{\Delta}_{n-1}(x,2h;f) - 2^{n-1}\widetilde{\Delta}_{n-1}(x,h;f), \qquad n\ge 2.
\end{aligned}$$

It is known [5] that there are constants $a_j, 0\le j\le k$, depending on j

and k only (with $a_k = 1$) such that

$$(1.2)\qquad \widetilde{\Delta}_k(x,h) = \widetilde{\Delta}_k(x,h;f) = a_0 f(x) + \sum_{j=1}^{k} a_j f(x + 2^{j-1}h), \qquad k \ge 1,$$

the coefficients a_j satisfying

$$\sum_{j=0}^{k} a_j = 0, \quad \sum_{j=1}^{k} 2^{js} a_j = 0, \quad s = 1, \ldots, k-1.$$

Throughout the paper $\mathbb{R}$, $\mathbb{N}$, μ, μ^* will denote the set of reals, the set of positive integers, Lebesgue measure and Lebesgue outer measure respectively.

Theorem MZ1. *Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and let $f_{(k-1)}(x)$ exist for each x in a measurable set $E \subset \mathbb{R}$. If*

$$\omega_k(x,h) = O(1) \quad \text{as } h \to 0 \text{ for } x \in E$$

then $f_{(k)}$ exists a.e. on E.

The above theorem was proved by Denjoy [2] for continuous functions. The theorem in its present form is in Lemma 7 of [5] the proof of which is long and involves the theory of Fourier series and analytic functions. Later a real-variable proof was given by Marcinkiewicz [4] (see also [9, II, p. 76, Theorem 4.24]). A simple and completely different proof is given in [1, p. 54, Corollaries 20 and 21]; see also [6].

Theorem MZ2. *If $f_{(k)}(x)$ exists then there is a number λ_k depending on k only such that*

$$\lambda_k \lim_{u\to 0} \frac{\widetilde{\Delta}_k(x,u;f)}{u^k} = f_{(k)}(x).$$

Theorem MZ3. *There are constants $C_0, C_1, \ldots, C_{2^{k-1}-k}$ such that*

$$\widetilde{\Delta}_k(x,h) = \sum_{i=0}^{2^{k-1}-k} C_i \Delta_k\big(x + \tfrac{1}{2}kh + ih, h\big).$$

Theorems MZ2 and MZ3 are also due to Marcinkiewicz and Zygmund. See Art. 9 and Art. 12 respectively of [5] for the proof.

We need the following definition.

Definition. A function f defined in some neighbourhood of a point x_0 is said to be *smooth* at x_0 if

$$(1.3)\qquad \Delta_2(x_0,h;f) = o(h) \quad \text{as } h \to 0,$$

and f is said to be *uniformly smooth* on a set E if (1.3) holds uniformly on E.

2. Auxiliary results

Lemma 2.1. *Let 0 be a point of outer density of E, let $\alpha,\beta\in\mathbb{R}$ with $\beta\neq 0$ and let $\varepsilon>0$. For each $u>0$ set*

$$B_u=\{v\in[u,2u]:\alpha u+\beta v\in E\}.$$

Then there is a $\delta>0$ such that if $0<u<\delta$ then $\mu^*(B_u)>u(1-\varepsilon)$.

This is Lemma 1 of [3].

Theorem 2.2. *Let $f:\mathbb{R}\to\mathbb{R}$ be measurable and let $f_{(k-1)}$ exist on a set E, $k\in\mathbb{N}$. If*

$$\omega_k(x,h)=O(1)\quad \text{as } h\to 0 \text{ for } x\in E,$$

then $f_{(k)}$ exists finitely a.e. on E.

Proof. Let G be the set of all x such that $f_{(k-1)}$ exists. Then G is measurable and $\overline{f}_{(k)}$ and $\underline{f}_{(k)}$ are measurable on G (see [6]). Hence the set

$$H=\{x\in G:-\infty<\underline{f}_{(k)}(x)\le\overline{f}_{(k)}(x)<\infty\}$$

is measurable. So by Theorem MZ1, $f_{(k)}$ exists finitely a.e. on H. Since $E\subset H$, the result follows.

Lemma 2.3. *Let $k\in\mathbb{N}$ and let $f:\mathbb{R}\to\mathbb{R}$ be measurable. Let*

$$\widetilde{\Delta}_k(x,u;f)=O(1)\quad \text{as } u\to 0$$

for each x in a set $E\subset\mathbb{R}$. Then f is bounded in some neighbourhood of almost every point of E.

Proof. The proof is given in [3, Theorem 2]. We give a proof for completeness.

For each $m\in\mathbb{N}$ let

$$E_m=\{x\in E:|\widetilde{\Delta}_k(x,u)|<m \text{ for } 0<|u|<1/m\},$$
$$F_m=\{x\in E:|f(x)|<m\}.$$

Since $E=\bigcup_m(E_m\cap F_m)$, it suffices to prove that f is bounded on some neighbourhood of every point of outer density of $E_m\cap F_m$. Let x_0 be such a point; suppose $x_0=0$. By Lemma 2.1 there is δ with $0<\delta<1/m$ such that if $0<u<\delta$ then

$$\mu^*(B)>u(1-1/(4k))\quad\text{and}\quad \mu^*(C_r)>u(1-1/(4k)),$$

where

$$B=[u,2u]\cap E_m\cap F_m,$$
$$C_r=\{v\in[u,2u]:v+(u-v)/2^{k-r-1}\in F_m\},\qquad 0\le r\le k-2.$$

Fix $0<u<\delta$. Let

$$D_r=\{v\in[u,2u]:|f(v+(u-v)/2^{k-r-1})|<m\}.$$

Then D_r is measurable and $C_r \subset D_r$ for $0 \le r \le k-2$. Now by the measurability of D_r,

$$\mu^*(B \cap D_r) \ge (1 - 2/(4k))u \quad \text{for } 0 \le r \le k-2,$$

and hence applying this argument repeatedly,

$$\mu^*\Big(B \cap \bigcap_r D_r\Big) \ge (1 - k/(4k))u > 0.$$

Choose $v \in B \cap \bigcap_r D_r$. Since $v \in E_m$ and $|(u-v)/2^{k-1}| < u < \delta < 1/m$,

$$|\widetilde{\Delta}_k(v, (u-v)/2^{k-1})| < m,$$
$$|f(v)| < m, \quad |f(v + (u-v)/2^{k-r-1})| < m \quad \text{for } 0 \le r \le k-2.$$

Hence from (1.2),

$$\begin{aligned} |f(u)| &\le |\widetilde{\Delta}_k(v, (u-v)/2^{k-1})| + |a_0 f(v)| \\ &\quad + \sum_{j=1}^{k-1} |a_j f(v + 2^{j-1}(u-v)/2^{k-1})| \\ &\le m\Big[1 + \sum_{j=0}^{k-1} |a_j|\Big]. \end{aligned}$$

This completes the proof.

Lemma 2.4. *Let $k \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $\lambda > k-1$. Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Let $m \in \mathbb{N}$ and let*

$$E = E_m = \{x : |\Delta_k(x, h)| < m|h|^\lambda \textit{ for } 0 < |h| < 1/m\}.$$

Then

$$\widetilde{\Delta}_k(x, h) = O(|h|^\lambda) \quad \textit{as } h \to 0 \quad \textit{a.e. on } E_m.$$

If $k \ge 2$ then

$$\widetilde{\Delta}_i(x, h) = O(h^i) \quad \textit{as } h \to 0 \quad \textit{a.e. on } E_m,\ 1 \le i \le k-1.$$

Proof. Let $x_0 \in E_m$ be a point of outer density of E_m. We may suppose that $x_0 = 0$. Let $0 < \varepsilon < 1/4^k$. Then by Lemma 2.1 there is δ with $0 < \delta < 1$ such that if $0 < u < \delta$ then

$$\mu^*(B_{ij}) > (1-\varepsilon)u \quad \text{and} \quad \mu^*(C_l) > (1-\varepsilon)u, \tag{2.1}$$

where

$$B_{ij} = \{v \in [u, 2u] : (k/2 + j)(u + i(v-u)/k) \in E\}, \qquad 1 \le i \le k,\ 0 \le j \le 2^{k-1} - k,$$
$$C_l = \{v \in [u, 2u] : 2^l(u+v)/2 \in E\}, \quad 0 \le l \le k-1.$$

Fix $u \in (0, \min[\delta/(2m),\ 1/(m \cdot 2^k)])$. Set

$$S_{ij} = \{v \in [u, 2u] : \\ |\Delta_k((k/2 + j)(u + i(v - u)/k), u + i(v - u)/k)| < m(2u)^\lambda\},$$
$$T_l = \{v \in [u, 2u] : |\Delta_k(2^l(u + v)/2, 2^l(v - u)/k)| < m(2^k u)^\lambda\}.$$

Since f is measurable, the sets S_{ij}, T_l are all measurable. Also $B_{ij} \subset S_{ij}$ and $C_l \subset T_l$. Therefore from (2.1),

$$\mu(S_{ij}) > (1 - \varepsilon)u \quad \text{and} \quad \mu(T_l) > (1 - \varepsilon)u.$$

Since the complement of $\bigcap_i \bigcap_j \bigcap_l (S_{ij} \cap T_l)$ with respect to $[u, 2u]$ has measure $\le 4^k \varepsilon u$, we have

$$\mu\Big(\bigcap_i \bigcap_j \bigcap_l (S_{ij} \cap T_l)\Big) \ge (1 - 4^k \varepsilon)u > 0.$$

Let $v \in \bigcap_i \bigcap_j \bigcap_l (S_{ij} \cap T_l)$. Then since $v \in T_l$,

$$|\Delta_k(2^l(u + v)/2, 2^l(v - u)/k)| < m(2^k u)^\lambda, \qquad 0 \le l \le k - 1,$$

and so

$$\Big|\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(2^l u + 2^l i(v - u)/k)\Big| < m(2^k u)^\lambda.$$

Multiplying by $|a_{l+1}|$ and adding over $l = 0, 1, \ldots, k - 1$ we have

$$\Big|\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \sum_{l=0}^{k-1} a_{l+1} f(2^l u + 2^l i(v - u)/k)\Big| < m_1 u^\lambda,$$

where

$$m_1 = m \sum_{l=0}^{k-1} |a_{l+1}| \cdot 2^{k\lambda}$$

and so by (1.2),

$$\Big|\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \widetilde{\Delta}_k(0, u + i(v - u)/k)\Big| < m_1 u^\lambda. \tag{2.2}$$

Also since $v \in S_{ij}$ for all $1 \le i \le k$ and $0 \le j \le 2^{k-1} - k$,

$$|\Delta_k((k/2 + j)(u + i(v - u)/k), u + i(v - u)/k)| < m(2u)^\lambda \\ \text{for } 1 \le i \le k,\ 0 \le j \le 2^{k-1} - k.$$

Hence from Theorem MZ3,

$$\begin{aligned}(2.3)\quad &|\widetilde{\Delta}_k(0,u+i(v-u)/k)|\\ &\le \sum_{j=0}^{2^{k-1}-k}|C_j|\cdot|\Delta_k((k/2+j)(u+i(v-u)/k),u+i(v-u)/k)|\\ &\le m_2u^{\lambda}\quad \text{for } 1\le i\le k,\end{aligned}$$

where

$$m_2=\sum_{j=0}^{2^{k-1}-k}|C_j|\cdot 2^{\lambda}.$$

From (2.2) and (2.3),

$$|\widetilde{\Delta}_k(0,u)|<Mu^{\lambda},$$

where

$$M=m_1+m_2\sum_{i=1}^{k}\binom{k}{i}.$$

Thus the lemma is proved when $u>0$. The proof is similar when u is negative. This completes the proof of the first part.

By the first part and by Lemma 2.3, f is bounded in some neighbourhood of almost all points of E. Let S be the set of all points $x\in E$ such that f is bounded in some neighbourhood of x and

$$\widetilde{\Delta}_k(x,h)=O(|h|^{\lambda})\quad \text{as } h\to 0. \tag{2.4}$$

Then $\mu^*(S)=\mu^*(E)$. We shall show that for each $x\in S$,

$$\widetilde{\Delta}_i(x,h)=O(h^i)\quad \text{as } h\to 0,\ i=1,\dots,k-1, \tag{2.5}$$

and this will complete the proof.

Let $x\in S$. We may suppose that $x=0$. Then by (2.4) there are $M>0$ and $\delta>0$ such that f is bounded in $[-\delta,\delta]$ and if $0<|u|\le\delta$ then using (1.1),

$$|\widetilde{\Delta}_{k-1}(0,u)-2^{k-1}\widetilde{\Delta}_{k-1}(0,u/2)|<M|u|^{\lambda}.$$

Replacing u successively by $u/2,u/2^2,\dots,u/2^{n-1}$, we have

$$\begin{aligned}&|\widetilde{\Delta}_{k-1}(0,u/2)-2^{k-1}\widetilde{\Delta}_{k-1}(0,u/2^2)|<M|u/2|^{\lambda},\\ &\vdots\\ &|\widetilde{\Delta}_{k-1}(0,u/2^{n-1})-2^{k-1}\widetilde{\Delta}_{k-1}(0,u/2^n)|<M|u/2^{n-1}|^{\lambda}.\end{aligned}$$

Multiplying these inequalities by $1,2^{k-1},2^{2(k-1)},\dots,2^{(n-1)(k-1)}$ respectively and adding we get

$$|\widetilde{\Delta}_{k-1}(0,u)-2^{n(k-1)}\widetilde{\Delta}_{k-1}(0,u/2^n)|<M|u|^{\lambda}\sum_{i=0}^{n-1}(1/2^{\lambda-k+1})^i.$$

Hence

$$(2.6)\qquad |2^{n(k-1)}\widetilde{\Delta}_{k-1}(0,u/2^n)/u^{k-1}| \\ \le M|u|^{\lambda-k+1}\sum_{i=0}^{n-1}(1/2^{\lambda-k+1})^i + |\widetilde{\Delta}_{k-1}(0,u)/u^{k-1}| \quad \text{if } 0<|u|\le\delta.$$

So by (1.2) and (2.6) there is a constant M_2 such that

$$(2.7)\qquad |2^{n(k-1)}\widetilde{\Delta}_{k-1}(0,u/2^n)/u^{k-1}| \le M_2 \quad \text{for } \delta/2^k\le|u|\le\delta/2^{k-1}.$$

Now for each ω satisfying $0<|\omega|\le\delta/2^k$ there is a positive integer n such that $2^n|\omega|\in[\delta/2^k,\delta/2^{k-1}]$ and hence putting $2^n\omega=u$ we get, from (2.7),

$$|\widetilde{\Delta}_{k-1}(0,\omega)/\omega^{k-1}|\le M_2.$$

Thus

$$(2.8)\qquad \widetilde{\Delta}_{k-1}(0,u)=O(u^{k-1}),$$

which proves (2.5) for $i=k-1$. We suppose that

$$(2.9)\qquad \widetilde{\Delta}_j(0,u)=O(u^j) \quad \text{for } 1<j\le k-1.$$

Then there is $L>0$ such that for small $|u|$ we have as above

$$\begin{aligned} &|\widetilde{\Delta}_{j-1}(0,u)-2^{j-1}\Delta_{j-1}(0,u/2)|<L|u|^j,\\ &|\widetilde{\Delta}_{j-1}(0,u/2)-2^{j-1}\widetilde{\Delta}_{j-1}(0,u/2^2)|<L|u/2|^j,\\ &\vdots\\ &|\widetilde{\Delta}_{j-1}(0,u/2^{n-1})-2^{j-1}\widetilde{\Delta}_{j-1}(0,u/2^n)|<L|u/2^{n-1}|^j. \end{aligned}$$

Multiplying these inequalities by $1,2^{j-1},2^{2(j-1)},\ldots,2^{(n-1)(j-1)}$ respectively and adding we get

$$|\widetilde{\Delta}_{j-1}(0,u)-2^{n(j-1)}\widetilde{\Delta}_{j-1}(0,u/2^n)|<2L|u|^j.$$

Hence

$$(2.10)\qquad |2^{n(j-1)}\widetilde{\Delta}_{j-1}(0,u/2^n)/u^{j-1}|<2L|u|+|\widetilde{\Delta}_{j-1}(0,u)/u^{j-1}|.$$

Now just as (2.8) is deduced from (2.6) the following can be deduced from (2.10):

$$(2.11)\qquad \widetilde{\Delta}_{j-1}(0,u)=O(u^{j-1}).$$

Thus if (2.9) holds then (2.11) holds. Since (2.8) holds the proof is complete by induction.

Lemma 2.5. *Under the hypothesis of Lemma* 2.4, $f_{([\lambda])}$ *exists and is finite a.e. on* E, $[\lambda]$ *denoting the greatest integer not exceeding* λ.

Proof. First we consider the case $[\lambda] = k-1$. If $k=1$ then $[\lambda]=0$ and so the result is trivially true. Suppose $k \ge 2$. Then by Lemma 2.4,

$$\widetilde{\Delta}_i(x,u) = O(u^i) \quad \text{as } u \to 0 \text{ for } 1 \le i \le k-1 \tag{2.12}$$

at almost all points of E. So taking $i=1$, by the Denjoy–Young–Saks Theorem [7, p. 271], f' exists and is finite a.e. on E. If $k=2$ then $[\lambda]=1$ and so the result follows. Therefore we suppose $k \ge 3$. Then as above f' exists and is finite a.e. on E. Suppose that $f_{(r)}(x)$ exists and is finite a.e. on E for a fixed r, $1 \le r < k-1$. Let $S \subset E$ be the set of points x such that $f_{(r)}(x)$ exists and (2.12) holds. Then $\mu^*(S) = \mu^*(E)$. Let $x \in S$ be fixed. We may suppose that

$$f_{(i)}(x) = 0 \quad \text{for } i = 0,1,\ldots,r.$$

Then from Theorem MZ2,

$$\lim_{u\to 0} \widetilde{\Delta}_i(x,u)/u^i = 0 \quad \text{for } i = 1,\ldots,r. \tag{2.13}$$

Since $\widetilde{\Delta}_{r+1}(x,u) = O(u^{r+1})$, there are $M>0$ and $\delta>0$ such that

$$|\widetilde{\Delta}_r(x,2u) - 2^r\widetilde{\Delta}_r(x,u)| < M|u|^{r+1} \quad \text{for } 0<|u|<\delta. \tag{2.14}$$

Replacing u by $u/2, u/2^2, \ldots, u/2^n$ successively and then multiplying the terms so obtained by $2^r, 2^{2r}, \ldots, 2^{nr}$ respectively and then adding them with (2.14) we get, as in Lemma 2.4,

$$|\widetilde{\Delta}_r(x,2u) - 2^{r(n+1)}\widetilde{\Delta}_r(x,u/2^n)| < 2M|u|^{r+1}.$$

Dividing by $|u|^r$ and letting $n \to \infty$ gives, by (2.13),

$$|\widetilde{\Delta}_r(x,2u)| \le 2M|u|^{r+1} \quad \text{for } |u| < \delta,$$

that is, $\widetilde{\Delta}_r(x,u) = O(u^{r+1})$ as $u \to 0$. Repeating these arguments we ultimately get $\widetilde{\Delta}_1(x,u) = O(u^{r+1})$ as $u \to 0$, that is,

$$f(x+u) = O(u^{r+1}) \quad \text{as } u \to 0.$$

Since $x \in S$ is arbitrary, by Theorem 2.2, $f_{(r+1)}$ exists a.e. on S, that is, a.e. on E. So by induction $f_{(k-1)}$ exists finitely a.e. on E. Thus the result is true in this case.

To complete the proof we suppose that the result is true for $[\lambda] = k-1+r$, $r \ge 0$. Let $[\lambda] = k+r$. Then $\lambda = k+r+\alpha$, where $0 \le \alpha < 1$. Since

$$|\Delta_k(x,u)| < m|u|^\lambda \quad \text{for } 0<|u|<1/m,\ x \in E,$$

we have

$$|\Delta_k(x,u)| < m|u|^{k-1+r+\alpha} \quad \text{for } 0<|u|<1/m,\ x\in E.$$

Therefore, since the result is true for $[\lambda] = k-1+r$, we conclude that $f_{(k-1+r)}$ exists and is finite a.e. on E. Since $|\Delta_k(x,u)| < m|u|^\lambda$ for $0<|u|<1/m$

and $x \in E$ and since $[\lambda] = k + r$,

$$|\Delta_k(x,u)| < m|u|^{k+r} \quad \text{for } 0 < |u| < 1/m,\ x \in E. \tag{2.15}$$

Therefore proceeding as in Lemma 2.4 we conclude that

$$\widetilde{\Delta}_k(x,u) = O(u^{k+r}) \quad \text{as } u \to 0 \tag{2.16}$$

at almost all points of E. Let S be the set of points x of E such that $f_{(k-1+r)}(x)$ exists and (2.16) holds. Then $\mu^*(S) = \mu^*(E)$. Let $x \in S$; we may suppose that $f_{(i)}(x) = 0$ for $i = 0, 1, \ldots, k-1$. Then from Theorem MZ2,

$$\lim_{u \to 0} \widetilde{\Delta}_i(x,u)/u^i = 0 \quad \text{for } i = 1, \ldots, k-1. \tag{2.17}$$

By (2.16) there are $M > 0$ and $\delta > 0$ such that

$$|\widetilde{\Delta}_{k-1}(x,2u) - 2^{k-1}\widetilde{\Delta}_{k-1}(x,u)| < M|u|^{k+r} \quad \text{for } 0 < |u| < \delta. \tag{2.18}$$

Replacing u by $u/2, u/2^2, \ldots, u/2^n$ successively and then multiplying the inequalities so obtained by $2^{k-1}, 2^{2(k-1)}, \ldots, 2^{n(k-1)}$ respectively and then adding them with (2.18) we get

$$|\widetilde{\Delta}_{k-1}(x,2u) - 2^{(n+1)(k-1)}\widetilde{\Delta}_{k-1}(x,u/2^n)| < 2M|u|^{k+r}.$$

Dividing by $|u|^{k-1}$ and letting $n \to \infty$ we get from this, and from (2.17),

$$|\widetilde{\Delta}_{k-1}(x,2u)| \le 2M|u|^{k+r},$$

that is, $\widetilde{\Delta}_{k-1}(x,u) = O(u^{k+r})$. Repeating these arguments we get $\widetilde{\Delta}_1(x,u) = O(u^{k+r})$, that is, $f(x+u) = O(u^{k+r})$. Since $x \in S$ is arbitrary, by Theorem 2.2, $f_{(k+r)}$ exists a.e. on S, that is, a.e. on E. This shows that the result is true for $[\lambda] = k$+r. This completes the proof of the lemma by induction.

3. Main results

Theorem 3.1. *Let $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ be such that $\lambda > k-1$. Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. If*

$$\Delta_k(x,h;f) = O(|h|^\lambda) \quad \text{as } h \to 0 \tag{3.1}$$

for each point x in a set $E \subset \mathbb{R}$ then $f_{([\lambda])}$ exists and is finite a.e. on E.

Proof. For each positive integer m let

$$E_m = \{x : |\Delta_k(x,u)| < m|u|^\lambda \text{ for } 0 < |u| < 1/m\}.$$

Then $\{E_m\}$ is a non-decreasing sequence and by (3.1), $E \subset \bigcup_{m=1}^\infty E_m$. By Lemma 2.5, $f_{([\lambda])}$ exists and is finite a.e. on E_m for each m. This completes the proof.

The following theorem shows that Theorem 3.1 is not true for $\lambda = k-1$, $k \ge 2$.

THEOREM 3.2. *For each integer $k \geq 2$ there exists a function F such that*

$$\Delta_k(x, h; F) = o(h^{k-1})$$

uniformly for all x, $F^{(k-2)}$ exists and is continuous for all x but $F_{(k-1)}$ can exist at most on a set of measure zero.

To prove the theorem we need the following lemma.

LEMMA 3.3. *Let $k \geq 2$ be an integer, f be locally integrable and uniformly smooth for all x and F be the $(k-2)$th integral of f. Then*

$$\Delta_k(x, 2h; F) = o(h^{k-1})$$

uniformly for all x.

Proof. The case of $k = 2$ is trivial. We assume that $k > 2$ and k is even. The case of k odd is similar. Let $k = 2m$. Since f is uniformly smooth for all x, for every $\varepsilon > 0$ there exists $\delta > 0$, independent of x, such that

$$|(f(x+h) + f(x-h) - 2f(x))/h| < \varepsilon \quad \text{for } 0 < h < \delta \text{ and for all } x.$$

So

$$-\varepsilon t < f(x+t) + f(x-t) - 2f(x) < \varepsilon t \quad \text{for } 0 < t < h < \delta. \tag{3.2}$$

Integrating the inequality (3.2) repeatedly $2m - 2$ times over $[0, h]$ we get

$$\begin{aligned} &-\varepsilon h^{2m-1}/(2m-1)! \\ &\qquad < F(x+h) + F(x-h) - 2\sum_{i=0}^{m-2} \frac{h^{2i}}{(2i)!} F^{(2i)}(x) - 2\frac{h^{2m-2}}{(2m-2)!} f(x) \\ &\qquad < \varepsilon h^{2m-1}/(2m-1)!. \end{aligned}$$

Hence

$$[F(x+h) + F(x-h)]/2 - \sum_{i=0}^{m-2} \frac{h^{2i}}{(2i)!} F^{(2i)}(x) - \frac{h^{2m-2}}{(2m-2)!} f(x) = o(h^{2m-1}), \tag{3.3}$$

uniformly for all x. Now using the relations

$$\sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} i^q = \begin{cases} 0 & \text{if } q = 0, 1, \ldots, p-1, \\ p! & \text{if } q = p, \end{cases} \tag{3.4}$$

from (3.3) we get

$$
\begin{aligned}
&\Delta_{2m}(x,2h;F)\\
&= \sum_{j=0}^{2m}(-1)^{2m-j}\binom{2m}{j}F(x+2jh-2mh)\\
&= \sum_{j=0}^{2m}(-1)^{j}\binom{2m}{j}F(x-2jh+2mh)\\
&= \sum_{j=0}^{2m}(-1)^{j}\binom{2m}{j}\frac{1}{2}[F(x+2(m-j)h)+F(x-2(m-j)h)]\\
&= \sum_{j=0}^{2m}(-1)^{j}\binom{2m}{j}\Big[\sum_{i=0}^{m-2}\frac{[2(m-j)h]^{2i}}{(2i)!}F^{(2i)}(x)+\frac{[2(m-j)h]^{2m-2}}{(2m-2)!}f(x)\Big]\\
&\quad + o(h^{2m-1})\\
&= \sum_{i=0}^{m-2}\frac{h^{2i}}{(2i)!}F^{(2i)}(x)\Big[\sum_{j=0}^{2m}(-1)^{j}\binom{2m}{j}(2m-2j)^{2i}\Big]\\
&\quad + \Big[\frac{(2h)^{2m-2}}{(2m-2)!}f(x)\sum_{j=0}^{2m}(-1)^{j}\binom{2m}{j}(m-j)^{2m-2}\Big] + o(h^{2m-1})\\
&= o(h^{2m-1})
\end{aligned}
$$

uniformly for all x. This completes the proof.

Proof of Theorem 3.2. Let

$$f(x)=\sum_{n=1}^{\infty}n^{-1/2}b^{-n}\cos(b^n x),\qquad b>1 \text{ an integer.}$$

Then f is continuous and uniformly smooth [9, I, p. 47, Theorem 4.10]. For $k=2$, let $F=f$ and for $k>2$ let F be the $(k-2)$th integral of f. We first show that

$$\lim_{h\to 0}\Delta_{k-1}(x,h;F)/h^{k-1} \tag{3.5}$$

can exist finitely at most on a set of measure zero. Let $k=2$. Then

$$
\begin{aligned}
\Delta_1(x,2h;f)/(2h) &= [f(x+h)-f(x-h)]/(2h)\\
&= -\sum_{n=1}^{\infty}n^{-1/2}\sin(b^n x)[\sin(b^n h)/(b^n h)].
\end{aligned}
\tag{3.6}
$$

If

$$\lim_{h\to 0}\Delta_1(x,2h;f)/(2h) \tag{3.7}$$

exists finitely on a set of positive measure then from (3.6) the series

$$-\sum_{n=1}^{\infty} n^{-1/2}\sin(b^n x) \tag{3.8}$$

is Lebesgue summable on a set of positive measure. Since (3.8) is a lacunary series, by [9, I, p. 203, Theorem 6.4], $\sum_{n=1}^{\infty} 1/n$ is convergent, which is a contradiction. So (3.7) exists finitely at most on a set of measure zero.

Next suppose $k > 2$. We prove that (3.5) can exist finitely at most on a set of measure zero. We suppose that k is even. Let $k = 2m$. Now

$$\frac{\Delta_{2m-1}(x,2h;F)}{(2h)^{2m-1}} = -\sum_{n=1}^{\infty} n^{-1/2}\sin(b^n x)(\sin(b^n h)/(b^n h))^{2m-1}. \tag{3.9}$$

If the limit of the left hand side of (3.9) exists on a set of positive measure as $h \to 0$ then the series (3.8) is $(R, 2m-1)$ summable and so as in the case of $k = 2$, $\sum_{n=1}^{\infty} 1/n$ would be convergent, which is a contradiction. Thus the limit of the left hand side of (3.9) as $h \to 0$ can exist at most on a set of measure zero. If k is odd then it can be similarly proved that (3.5) can exist finitely at most on a set of measure zero.

Now from Lemma 3.3 and the construction of the function F we see that

$$\Delta_k(x,h;F) = o(h^{k-1})$$

uniformly for all x. Also it is clear that $F^{(k-2)}$ exists and is continuous for all x. To complete the proof we show that $F_{(k-1)}$ can exist at most on a set of measure zero.

Let, if possible, $F_{(k-1)}$ exist finitely on a set E of positive measure. Then for $x \in E$,

$$F(x+h) = \sum_{j=0}^{k-1} \frac{h^j}{j!} F_{(j)}(x) + o(h^{k-1})$$

and so for $x \in E$, by (3.4),

$$\begin{aligned}
&\Delta_{k-1}(x,2h;F) \\
&\quad = \sum_{i=0}^{k-1} (-1)^{k-1-i}\binom{k-1}{i} F(x+2ih-(k-1)h) \\
&\quad = \sum_{i=0}^{k-1} (-1)^{k-1-i}\binom{k-1}{i}\left[\sum_{j=0}^{k-1} \frac{(2i-k+1)^j h^j}{j!} F_{(j)}(x) + o(h^{k-1})\right]
\end{aligned}$$

$$= \sum_{j=0}^{k-1} \frac{h^j}{j!} F_{(j)}(x) \sum_{i=0}^{k-1} (-1)^{k-1-i} \binom{k-1}{i} (2i-k+1)^j + o(h^{k-1})$$

$$= (2h)^{k-1} F_{(k-1)}(x) + o(h^{k-1}),$$

and so for all $x \in E$,

$$\lim_{h\to 0} \Delta_{k-1}(x, 2h; F)/(2h)^{k-1} = F_{(k-1)}(x),$$

which contradicts the fact that (3.5) can exist at most on a set of measure zero and thus the proof is complete.

Theorem 3.2 shows that in Theorem 3.1 the condition $\lambda > k-1$ is necessary. However, the following theorem shows that this condition can be relaxed if the existence of $f_{([\lambda]),\mathrm{a}}$ is assumed.

THEOREM 3.4. *Let $k \in \mathbb{N}$, $p \in \mathbb{N}$, $p \le k-1$ and let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Let*

$$\Delta_k(x, u) = O(u^p) \quad \text{as } u \to 0,$$

for each point x in a set E. If $f_{(p),\mathrm{a}}$ exists finitely on E then $f_{(p)}$ exists a.e. on E.

We need the following lemma.

LEMMA 3.5. *Let $k \in \mathbb{N}$, $p \in \mathbb{N}$ and let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Let*

$$E = E_m = \{x : f_{(p),\mathrm{a}}(x) \text{ exists finitely and } |\Delta_k(x, u)| < m|u|^p \text{ for } 0 < |u| < 1/m\}.$$

Then $f_{(p)}$ exists a.e. on E.

Proof. Let $x_0 \in E$ be a point of outer density of E. We suppose

$$x_0 = 0 = f(x_0) = f_{(1),\mathrm{a}}(x_0) = \ldots = f_{(p),\mathrm{a}}(x_0).$$

Let $0 < \varepsilon < 1$. Let

$$G = \{x : |f(x)| \le \varepsilon |x|^p/p!\}.$$

Then G is measurable and $0 \in G$ is a point of density of G. Set $H = E \cap G$. Then 0 is a point of outer density of H. Let $0 < \eta < \varepsilon/(2k)$. Then by Lemma 2.1 there is $\delta > 0$ such that if $0 < u < \delta$ then

$$\mu^*(B) > (1-\eta)u, \quad \mu^*(C_j) > (1-\eta)u,$$

where

$$B = \{v \in [u, 2u] : (u+v)/2 \in H\},$$
$$C_j = \{v \in [u, 2u] : v + j(u-v)/k \in H\}, \quad 0 \le j \le k-1.$$

Fix $u \in (0, \min(\delta, 1/m))$. Let

$$S = \{v \in [u, 2u] : |\Delta_k((u+v)/2, (u-v)/k)| < m|(u-v)/k|^p\},$$
$$T_j = \{v \in [u, 2u] : |f(v + j(u-v)/k)| \le \varepsilon |v + j(u-v)/k|^p/p!\}, \quad 0 \le j \le k-1.$$

Since f is measurable, S and T_j are measurable. Also $B \subset S$, $C_j \subset T_j$ and hence

$$\mu(S) > (1-\eta)u, \qquad \mu(T_j) > (1-\eta)u.$$

Therefore

$$\mu\Big(\bigcap_j (S \cap T_j)\Big) > (1 - 2k\eta)u > (1-\varepsilon)u.$$

Hence

$$\Big(\bigcap_j (S \cap T_j)\Big) \cap (u, u + \varepsilon u) \ne \emptyset.$$

Choose $v \in \big(\bigcap_j (S \cap T_j)\big) \cap (u, u + \varepsilon u)$. Then $0 < v - u < \varepsilon u < u < 1/m$ and so

$$|\Delta_k((u+v)/2, (u-v)/k)| < m|(u-v)/k|^p < m(\varepsilon u)^p,$$

which gives

$$\Big|\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f((u+v)/2 + (j - k/2)(u-v)/k)\Big| < m(\varepsilon u)^p.$$

Hence

$$|f(u)| < m(\varepsilon u)^p + \sum_{j=0}^{k-1} \binom{k}{j} |f(v + j(u-v)/k)|.$$

Since $v \in T_j$ for $0 \le j \le k-1$,

$$\begin{aligned} |f(u)| &< m(\varepsilon u)^p + \sum_{j=0}^{k-1} \binom{k}{j} \varepsilon |v + j(u-v)/k|^p/p! \\ &\le m(\varepsilon u)^p + (\varepsilon/p!) \sum_{j=0}^{k-1} \binom{k}{j} (3u)^p \\ &\le \varepsilon \Big[m + (3^p/p!) \sum_{j=0}^{k-1} \binom{k}{j} \Big] u^p. \end{aligned}$$

This shows that $f(u)/u^p \to 0$ as $u \to 0+$.

It can be similarly shown that $f(u)/u^p \to 0$ as $u \to 0-$. This completes the proof of the lemma.

Proof of Theorem 3.4. For each positive integer m, let

$$E_m = \{x : f_{(p),\mathrm{a}}(x) \text{ exists finitely and } |\Delta_k(x,u)| < m|u|^p \text{ for } 0 < |u| < 1/m\}.$$

Then $\{E_m\}$ is a non-decreasing sequence and $E \subset \bigcup_m E_m$. By Lemma 3.5, $f_{(p)}$ exists a.e. on E_m and so the result follows.

COROLLARY 3.6. *Let $p \in \mathbb{N}$, let $f : \mathbb{R} \to \mathbb{R}$ be measurable and let $f(x) = 0$ for $x \in E \subset \mathbb{R}$. If*

$$f(x+u) - f(x-u) = O(u^p)$$

or

$$f(x+u) + f(x-u) = O(u^p)$$

for $x \in E$, then $f_{(p)}$ exists a.e. on E.

Proof. Let

$$E_1 = \{x \in E : f(x+u) - f(x-u) = O(u^p)\},$$
$$E_2 = \{x \in E : f(x+u) + f(x-u) = O(u^p)\}.$$

Then $E = E_1 \cup E_2$. Let D_i be the set of all points of E_i which are also points of outer density of E_i, $i = 1, 2$. Then $f_{(p),\mathrm{a}}(x) = 0$ for $x \in D_1 \cup D_2$. Also

$$\Delta_1(x,u) = O(u^p) \quad \text{as } u \to 0 \text{ for } x \in D_1,$$
$$\Delta_2(x,u) = O(u^p) \quad \text{as } u \to 0 \text{ for } x \in D_2.$$

Hence if $p = 1$ then by Theorem 3.1, f' exists finitely a.e. on D_1 and by Theorem 3.4, f' exists finitely a.e. on D_2 and hence f' exists a.e. on E. If $p \geq 2$ then by Theorem 3.1, $f_{(p)}$ exists finitely a.e. on D_1 and on D_2 and hence $f_{(p)}$ exists finitely a.e. on E.

The above corollary is a generalization of Lemma 11 of [8, p. 268], since we are not assuming the measurability of E.

Theorem 3.4 can further be extended to

THEOREM 3.7. *Let $k \in \mathbb{N}$, $p \in \mathbb{N}$, $p \leq k-1$ and let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Let*

$$\Delta_k(x,u) = O(u^p) \quad \text{as } u \to 0$$

for each point x in a set E. If $f_{(p-1),\mathrm{a}}$ exists and

$$-\infty < \underline{f}_{(p),\mathrm{a}} \leq \overline{f}_{(p),\mathrm{a}} < \infty \quad \text{on } E$$

then $f_{(p-1)}$ exists and

$$-\infty < \underline{f}_{(p)} \leq \overline{f}_{(p)} < \infty \quad \text{a.e. on } E.$$

Proof. The first part follows from Theorem 3.4. The proof of the second part is similar to that of Theorem 3.4. We give a sketch. The corresponding sets in Lemma 3.5 are in this case given by

$$E_m = \{x : f_{(p-1),\mathrm{a}}(x) \text{ exists finitely, } |\Delta_k(x,u)| < m|u|^p \text{ for } 0 < |u| < 1/m \text{ and } -m < \underline{f}_{(p),\mathrm{a}}(x) \le \overline{f}_{(p),\mathrm{a}}(x) < m\}$$

with the assumption that

$$x_0 = 0 = f(x_0) = f_{(1),\mathrm{a}}(x_0) = \ldots = f_{(p-1),\mathrm{a}}(x_0),$$

$$G_m = \{x : |f(x)| \le m|x|^p/p!\},$$

$$T_j = \{v \in [u,2u] : |f(v + j(u-v)/k)| \le m|v + j(u-v)/k|^p/p!\}, \quad 0 \le j \le k-1,$$

and the final step is

$$|f(u)| \le \left[\varepsilon m + m(3^p/p!) \sum_{j=0}^{k-1} \binom{k}{j}\right] u^p$$

showing that $|f(u)| = O(u^p)$ as $u \to 0+$ and similarly $|f(u)| = O(|u|^p)$ as $u \to 0-$.

COROLLARY 3.8. *Under the hypothesis of Theorem* 3.7, *if* $f_{(p-1),\mathrm{a}}$ *exists and*

$$-\infty < \underline{f}_{(p),\mathrm{a}}(x) \le \overline{f}_{(p),\mathrm{a}}(x) < \infty \quad \text{on } E$$

then $f_{(p)}$ *exists a.e. on* E.

The proof follows from Theorems 3.7 and 2.2.

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