An extension of a theorem of Marcinkiewicz and Zygmund on differentiability

by

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Abstract. Let $f$ be a measurable function such that $\Delta_k(x, h; f) = O(|h|^{\lambda})$ at each point $x$ of a set $E$, where $k$ is a positive integer, $\lambda > 0$ and $\Delta_k(x, h; f)$ is the symmetric difference of $f$ at $x$ of order $k$. Marcinkiewicz and Zygmund [5] proved that if $\lambda = k$ and if $E$ is measurable then the Peano derivative $f_{(k)}(x)$ exists a.e. on $E$. Here we prove that if $\lambda > k - 1$ then the Peano derivative $f_{(\lambda)}(x)$ exists a.e. on $E$ and that the result is false if $\lambda = k - 1$; it is further proved that if $\lambda$ is any positive integer and if the approximate Peano derivative $f_{(\lambda), a}$ exists on $E$ then $f_{(\lambda)}(x)$ exists a.e. on $E$.

1. Introduction. Let $f$ be a real-valued function defined in some neighbourhood of $x$. Then $f$ is said to have Peano derivative (resp. approximate Peano derivative) at $x$ of order $k$ if there exist real numbers $\alpha_r$, $1 \leq r \leq k$, depending on $x$ and $f$ only such that

$$f(x + h) = f(x) + \sum_{r=1}^{k} \frac{h^r}{r!} \alpha_r + \frac{h^k}{k!} \omega_k(x, h; f),$$

where

$$\lim_{h \to 0} \omega_k(x, h; f) = 0 \quad \text{(resp. } \lim_{h \to 0} \omega_k(x, h; f; a) = 0).$$

The number $\omega_k(x, h; f)$ is called the Peano derivative (resp. approximate Peano derivative) of $f$ at $x$ of order $k$ and is denoted by $f_{(k)}(x)$ (resp. $f_{(k), a}(x)$). For convenience we shall write $\omega_k(x, h; f) = f_{(0)}(x) = f_{(0), a}(x)$.

Suppose that $f$ has Peano derivative (resp. approximate Peano derivative) at $x$ of order $k - 1$. For $h \neq 0$ we write

$$\omega_k(x, h; f) = \omega_k(x, h) = \frac{k!}{h^k} \left[ f(x + h) - \sum_{r=0}^{k-1} \frac{h^r}{r!} \alpha_r \right].$$

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The upper and lower Peano derivatives (resp. approximate Peano derivatives) of \( f \) at \( x \) of order \( k \) are defined by

\[
\begin{align*}
\tilde{f}^{(k)}(x) &= \limsup_{h \to 0} \omega_k(x, h) \quad \text{(resp. } \tilde{f}^{(k),a}(x) = \limsup_{h \to 0} \text{ap } \omega_k(x, h)),) \\
\tilde{f}^{(k)}(x) &= \liminf_{h \to 0} \omega_k(x, h) \quad \text{(resp. } \tilde{f}^{(k),a}(x) = \liminf_{h \to 0} \text{ap } \omega_k(x, h)).)
\end{align*}
\]

The symmetric difference of \( f \) at \( x \) of order \( k \), where \( k \) is a positive integer, is defined by

\[
\Delta_k(x, h) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + ih - \frac{k}{2}h).
\]

Marcinkiewicz and Zygmund proved in a deep theorem (Theorem 1 of [5]) that if \( f \) is measurable and if for a positive integer \( k \),

\[
\Delta_k(x, h; f) = O(|h|^k) \quad \text{as } h \to 0,
\]

for \( x \) in a measurable set \( E \) then \( \tilde{f}^{(k)} \) exists a.e. on \( E \). For \( k = 2 \) this is proved in [9, II, p. 78, Theorem 4.30]. For general \( k \) the proof is very long and involved (it is worth mentioning that the proof offered by Marcinkiewicz and Zygmund has a lacuna filled by Fejzic and Weil [3]).

The purpose of the present paper is to extend this result. In fact we prove in Theorem 3.1 that if \( f \) is measurable and if for a positive integer \( k \),

\[
\Delta_k(x, h; f) = O(|h|^\lambda) \quad \text{as } h \to 0
\]

for \( x \) in a set \( E \) (not necessarily uniformly), where \( \lambda > k - 1 \), then \( f^{(\lambda)} \) exists a.e. on \( E \). For \( \lambda = k \) this gives the result of Marcinkiewicz and Zygmund cited above. Also we show in Theorem 3.2 that this result is not true for \( \lambda = k - 1 \). Thirdly, in Theorem 3.4 we show that if we further assume that the approximate Peano derivative \( f^{(k-1),a} \) exists on \( E \) then the above result is true for \( \lambda = k - 1 \). In fact, we prove in Theorem 3.4 that if \( f \) is measurable and if

\[
\Delta_k(x, h; f) = O(|h|^p) \quad \text{as } h \to 0
\]

for every \( x \) in a set \( E \), where \( k \) and \( p \) are positive integers, and if \( f^{(p),a} \) exists finitely on \( E \) then \( f^{(p)} \) exists a.e. on \( E \).

We consider the difference

\[
\begin{align*}
\tilde{T}_1(x, h) &= \tilde{T}_1(x, h; f) = f(x + h) - f(x), \\
\tilde{T}_n(x, h) &= \tilde{T}_n(x, h; f) \\
&= \tilde{T}_{n-1}(x, 2h; f) - 2^{n-1} \tilde{T}_{n-1}(x, h; f), \quad n \geq 2.
\end{align*}
\]

It is known [5] that there are constants \( a_j, 0 \leq j \leq k \), depending on \( j \).
and $k$ only (with $a_k = 1$) such that
\begin{equation}
(1.2) \quad \tilde{\Delta}_k(x, h) = \tilde{\Delta}_k(x, h; f) = a_0 f(x) + \sum_{j=1}^{k} a_j f(x + 2^{j-1}h), \quad k \geq 1,
\end{equation}

the coefficients $a_j$ satisfying
\[
\sum_{j=0}^{k} a_j = 0, \quad \sum_{j=1}^{k} 2^{js} a_j = 0, \quad s = 1, \ldots, k - 1.
\]

Throughout the paper $\mathbb{R}$, $\mathbb{N}$, $\mu$, $\mu^*$ will denote the set of reals, the set of positive integers, Lebesgue measure and Lebesgue outer measure respectively.

**Theorem MZ1.** Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and let $f_{(k-1)}(x)$ exist for each $x$ in a measurable set $E \subset \mathbb{R}$. If $\omega_k(x, h) = O(1)$ as $h \to 0$ for $x \in E$
then $f_{(k)}$ exists a.e. on $E$.

The above theorem was proved by Denjoy [2] for continuous functions. The theorem in its present form is in Lemma 7 of [5] the proof of which is long and involves the theory of Fourier series and analytic functions. Later a real-variable proof was given by Marcinkiewicz [4] (see also [9, II, p. 76, Theorem 4.24]). A simple and completely different proof is given in [1, p. 54, Corollaries 20 and 21]; see also [6].

**Theorem MZ2.** If $f_{(k)}(x)$ exists then there is a number $\lambda_k$ depending on $k$ only such that
\[
\lambda_k \lim_{u \to 0} \frac{\tilde{\Delta}_k(x, u; f)}{u^k} = f_{(k)}(x).
\]

**Theorem MZ3.** There are constants $C_0, C_1, \ldots, C_{2^{k-1}-k}$ such that
\[
\tilde{\Delta}_k(x, h) = \sum_{i=0}^{2^{k-1}-k} C_i \Delta_k(x + \frac{1}{2}kh + ih, h).
\]

Theorems MZ2 and MZ3 are also due to Marcinkiewicz and Zygmund. See Art. 9 and Art. 12 respectively of [5] for the proof.

We need the following definition.

**Definition.** A function $f$ defined in some neighbourhood of a point $x_0$ is said to be smooth at $x_0$ if
\begin{equation}
(1.3) \quad \Delta_2(x_0, h; f) = o(h) \quad \text{as } h \to 0,
\end{equation}

and $f$ is said to be uniformly smooth on a set $E$ if (1.3) holds uniformly on $E$. 
2. Auxiliary results

**Lemma 2.1.** Let 0 be a point of outer density of $E$, let $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$ and let $\varepsilon > 0$. For each $u > 0$ set

$$B_u = \{ v \in [u, 2u] : \alpha u + \beta v \in E \}.$$  

Then there is a $\delta > 0$ such that if $0 < u < \delta$ then $\mu^*(B_u) > u(1 - \varepsilon)$. This is Lemma 1 of [3].

**Theorem 2.2.** Let $f : \mathbb{R} \to \mathbb{R}$ be measurable and let $f_{(k-1)}$ exist on a set $E$, $k \in \mathbb{N}$. If

$$\omega_k(x, h) = O(1) \quad \text{as } h \to 0 \quad \text{for } x \in E,$$

then $f_{(k)}$ exists finitely a.e. on $E$.

**Proof.** Let $G$ be the set of all $x$ such that $f_{(k-1)}$ exists. Then $G$ is measurable and $\tilde{f}_{(k)}$ and $\check{f}_{(k)}$ are measurable on $G$ (see [6]). Hence the set

$$H = \{ x \in G : -\infty < \check{f}_{(k)}(x) \leq \tilde{f}_{(k)}(x) < \infty \}$$

is measurable. So by Theorem MZ1, $f_{(k)}$ exists finitely a.e. on $H$. Since $E \subset H$, the result follows.

**Lemma 2.3.** Let $k \in \mathbb{N}$ and let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Let

$$\Delta_k(x,u; f) = O(1) \quad \text{as } u \to 0$$

for each $x$ in a set $E \subset \mathbb{R}$. Then $f$ is bounded in some neighbourhood of almost every point of $E$.

**Proof.** The proof is given in [3, Theorem 2]. We give a proof for completeness.

For each $m \in \mathbb{N}$ let

$$E_m = \{ x \in E : |\Delta_k(x,u)| < m \text{ for } 0 < |u| < 1/m \},$$

$$F_m = \{ x \in E : |f(x)| < m \}.$$  

Since $E = \bigcup_m (E_m \cap F_m)$, it suffices to prove that $f$ is bounded on some neighbourhood of every point of outer density of $E_m \cap F_m$. Let $x_0$ be such a point; suppose $x_0 = 0$. By Lemma 2.1 there is $\delta$ with $0 < \delta < 1/m$ such that if $0 < u < \delta$ then

$$\mu^*(B) > u(1 - 1/(4k)) \quad \text{and} \quad \mu^*(C_r) > u(1 - 1/(4k)),$$

where

$$B = [u, 2u] \cap E_m \cap F_m,$$

$$C_r = \{ v \in [u, 2u] : v + (u - v)/2^{k-r-1} \in F_m \}, \quad 0 \leq r \leq k - 2.$$  

Fix $0 < u < \delta$. Let

$$D_r = \{ v \in [u, 2u] : |f(v + (u - v)/2^{k-r-1})| < m \}.$$
Then $D_r$ is measurable and $C_r \subset D_r$ for $0 \leq r \leq k - 2$. Now by the measurability of $D_r$,
\[
\mu^*(B \cap D_r) \geq (1 - 2/(4k))u \quad \text{for } 0 \leq r \leq k - 2,
\]
and hence applying this argument repeatedly,
\[
\mu^*(B \cap \bigcap_r D_r) \geq (1 - k/(4k))u > 0.
\]
Choose $v \in B \cap \bigcap_r D_r$. Since $v \in E_m$ and $|(u - v)/2^{k-1}| < u < \delta < 1/m$,
\[
|\tilde{\Delta}_k(v, (u - v)/2^{k-1})| < m, \quad |f(v)| < m, \quad |f(v + (u - v)/2^{k-r-1})| < m \quad \text{for } 0 \leq r \leq k - 2.
\]
Hence from (1.2),
\[
|f(u)| \leq |\tilde{\Delta}_k(v, (u - v)/2^{k-1})| + |a_0f(v)|
+ \sum_{j=1}^{k-1} |a_jf(v + 2^{j-1}(u - v)/2^{k-1})|
\leq m\left[1 + \sum_{j=0}^{k-1} |a_j|\right].
\]

This completes the proof.

**Lemma 2.4.** Let $k \in \mathbb{N}$, $\lambda \in \mathbb{R}$ and $\lambda > k - 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. Let $m \in \mathbb{N}$ and let
\[
E = E_m = \{x : |\Delta_k(x, h)| < m|h|^\lambda \text{ for } 0 < |h| < 1/m\}.
\]
Then
\[
\tilde{\Delta}_k(x, h) = O(|h|^\lambda) \quad \text{as } h \to 0 \quad \text{a.e. on } E_m.
\]
If $k \geq 2$ then
\[
\tilde{\Delta}_i(x, h) = O(h^i) \quad \text{as } h \to 0 \quad \text{a.e. on } E_m, \quad 1 \leq i \leq k - 1.
\]

**Proof.** Let $x_0 \in E_m$ be a point of outer density of $E_m$. We may suppose that $x_0 = 0$. Let $0 < \varepsilon < 1/4^k$. Then by Lemma 2.1 there is $\delta$ with $0 < \delta < 1$ such that if $0 < u < \delta$ then
\[
(2.1) \quad \mu^*(B_{ij}) > (1 - \varepsilon)u \quad \text{and} \quad \mu^*(C_l) > (1 - \varepsilon)u,
\]
where
\[
B_{ij} = \{v \in [u, 2u] : (k/2 + j)(u + i(v - u)/k) \in E\},
\]
\[
1 \leq i \leq k, \quad 0 \leq j \leq 2^{k-1} - k,
\]
\[
C_l = \{v \in [u, 2u] : 2^l(u + v)/2 \in E\}, \quad 0 \leq l \leq k - 1.
\]
Fix $u \in (0, \min[\delta/(2m), 1/(m \cdot 2^k)])$. Set

$$S_{ij} = \{ v \in [u, 2u] : |\Delta_k((k/2 + j)(u + i(v - u)/k), u + i(v - u)/k)| < m(2u)^\lambda \},$$

$$T_l = \{ v \in [u, 2u] : |\Delta_k(2^l(u + v)/2, 2^l(v - u)/k)| < m(2^k u)^\lambda \}.$$

Since $f$ is measurable, the sets $S_{ij}, T_l$ are all measurable. Also $B_{ij} \subset S_{ij}$ and $C_l \subset T_l$. Therefore from (2.1),

$$\mu(S_{ij}) > (1 - \varepsilon)u \quad \text{and} \quad \mu(T_l) > (1 - \varepsilon)u.$$

Since the complement of $\bigcap_i \bigcap_j \bigcap_l (S_{ij} \cap T_l)$ with respect to $[u, 2u]$ has measure $\leq 4^k \varepsilon u$, we have

$$\mu\left(\bigcap_i \bigcap_j \bigcap_l (S_{ij} \cap T_l)\right) \geq (1 - 4^k \varepsilon)u > 0.$$

Let $v \in \bigcap_i \bigcap_j \bigcap_l (S_{ij} \cap T_l)$. Then since $v \in T_l$,

$$|\Delta_k(2^l(u + v)/2, 2^l(v - u)/k)| < m(2^k u)^\lambda, \quad 0 \leq l \leq k - 1,$$

and so

$$\left| \sum_{i=0}^k (-1)^{k-i} \frac{k!}{i!} f(2^l u + 2^l i(v - u)/k) \right| < m(2^k u)^\lambda.$$

Multiplying by $|a_{l+1}|$ and adding over $l = 0, 1, \ldots, k - 1$ we have

$$\left| \sum_{i=0}^k (-1)^{k-i} \frac{k!}{i!} \sum_{l=0}^{k-1} a_{l+1} f(2^l u + 2^l i(v - u)/k) \right| < m_1 u^\lambda,$$

where

$$m_1 = m \sum_{l=0}^{k-1} |a_{l+1}| \cdot 2^{k^\lambda}$$

and so by (1.2),

$$\left| \sum_{i=0}^k (-1)^{k-i} \frac{k!}{i!} \tilde{\Delta}_k(0, u + i(v - u)/k) \right| < m_1 u^\lambda.$$

Also since $v \in S_{ij}$ for all $1 \leq i \leq k$ and $0 \leq j \leq 2^{k-1} - k$,

$$|\Delta_k((k/2 + j)(u + i(v - u)/k), u + i(v - u)/k)| < m(2u)^\lambda$$

for $1 \leq i \leq k, 0 \leq j \leq 2^{k-1} - k$.

Hence from Theorem MZ3,
(2.3) \( \left| \tilde{\Delta}_k(0, u + i(v - u)/k) \right| \)
\[
\leq \sum_{j=0}^{2^{k-1}-1} |C_j| \cdot |\Delta_k((k/2 + j)(u + i(v - u)/k), u + i(v - u)/k)|
\]
\[
\leq m_2 u^\lambda \quad \text{for } 1 \leq i \leq k,
\]
where
\[
m_2 = \sum_{j=0}^{2^{k-1}-1} |C_j| \cdot 2^\lambda.
\]

From (2.2) and (2.3),
\[
\left| \tilde{\Delta}_k(0, u) \right| < M u^\lambda,
\]
where
\[
M = m_1 + m_2 \sum_{i=1}^{k} \binom{k}{i}.
\]
Thus the lemma is proved when \( u > 0 \). The proof is similar when \( u \) is negative. This completes the proof of the first part.

By the first part and by Lemma 2.3, \( f \) is bounded in some neighbourhood of almost all points of \( E \). Let \( S \) be the set of all points \( x \in E \) such that \( f \) is bounded in some neighbourhood of \( x \) and

(2.4) \[ \tilde{\Delta}_k(x, h) = O(|h|^\lambda) \quad \text{as } h \to 0. \]

Then \( \mu^*(S) = \mu^*(E) \). We shall show that for each \( x \in S \),

(2.5) \[ \tilde{\Delta}_i(x, h) = O(h^i) \quad \text{as } h \to 0, \quad i = 1, \ldots, k - 1, \]
and this will complete the proof.

Let \( x \in S \). We may suppose that \( x = 0 \). Then by (2.4) there are \( M > 0 \) and \( \delta > 0 \) such that \( f \) is bounded in \([-\delta, \delta]\) and if \( 0 < |u| \leq \delta \) then using (1.1),

\[
\left| \tilde{\Delta}_{k-1}(0, u) - 2^{k-1} \tilde{\Delta}_{k-1}(0, u/2) \right| < M |u|^\lambda.
\]

Replacing \( u \) successively by \( u/2, u/2^2, \ldots, u/2^{n-1} \), we have

\[
\left| \tilde{\Delta}_{k-1}(0, u/2) - 2^{k-1-1} \tilde{\Delta}_{k-1}(0, u/2^2) \right| < M |u/2|^\lambda,
\]

\[
\vdots
\]

\[
\left| \tilde{\Delta}_{k-1}(0, u/2^{n-1}) - 2^{k-1-1} \tilde{\Delta}_{k-1}(0, u/2^n) \right| < M |u/2^{n-1}|^\lambda.
\]

Multiplying these inequalities by \( 1, 2^{k-1}, 2^{2(k-1)}, \ldots, 2^{(n-1)(k-1)} \) respectively and adding we get

\[
\left| \tilde{\Delta}_{k-1}(0, u) - 2^{n(k-1)} \tilde{\Delta}_{k-1}(0, u/2^n) \right| < M |u|^\lambda \sum_{i=0}^{n-1} (1/2^{\lambda(k-1)}).n
\]

Hence
\[ |2^{n(k-1)} \tilde{\Delta}_{k-1}(0, u/2^n)/u^{k-1}| \]
\[ \leq M|u|^\lambda^{k+1} + |\tilde{\Delta}_{k-1}(0, u)/u^{k-1}| \quad \text{if } 0 < |u| \leq \delta. \]

So by (1.2) and (2.6) there is a constant $M_2$ such that
\[ |2^{n(k-1)} \tilde{\Delta}_{k-1}(0, u/2^n)/u^{k-1}| \leq M_2 \quad \text{for } \delta/2^k \leq |u| \leq \delta/2^{k-1}. \]

Now for each $\omega$ satisfying $0 < |\omega| \leq \delta/2^k$ there is a positive integer $n$ such that $2^n|\omega| \in \left[\delta/2^k, \delta/2^{k-1}\right]$ and hence putting $2^n\omega = u$ we get, from (2.7),
\[ |\tilde{\Delta}_{k-1}(0, \omega)/\omega^{k-1}| \leq M_2. \]

Thus
\[ \tilde{\Delta}_{k-1}(0, u) = O(u^{k-1}), \]
which proves (2.5) for $i = k - 1$. We suppose that
\[ \tilde{\Delta}_j(0, u) = O(u^j) \quad \text{for } 1 < j \leq k - 1. \]

Then there is $L > 0$ such that for small $|u|$ we have as above
\[ |\tilde{\Delta}_j(0, u) - 2^{j-1} \Delta_j(0, u/2)| < L|u|^j, \]
\[ |\tilde{\Delta}_j(0, u/2) - 2^{j-1} \tilde{\Delta}_j(0, u/2^2)| < L|u/2|^j, \]
\[ \vdots \]
\[ |\tilde{\Delta}_j(0, u/2^{n-j}) - 2^{j-1} \tilde{\Delta}_j(0, u/2^n)| < L|u/2^{n-1}|^j. \]

Multiplying these inequalities by $1, 2^{j-1}, 2^{2(j-1)}, \ldots, 2^{(n-1)(j-1)}$ respectively and adding we get
\[ |\tilde{\Delta}_j(0, u) - 2^{n(j-1)} \tilde{\Delta}_j(0, u/2^n)| < 2L|u|^j. \]

Hence
\[ |2^{n(j-1)} \tilde{\Delta}_j(0, u/2^n)/u^{j-1}| < 2L|u| + |\tilde{\Delta}_j(0, u)/u^{j-1}|. \]

Now just as (2.8) is deduced from (2.6) the following can be deduced from (2.10):
\[ \tilde{\Delta}_j(0, u) = O(u^{j-1}). \]

Thus if (2.9) holds then (2.11) holds. Since (2.8) holds the proof is complete by induction.

**Lemma 2.5.** Under the hypothesis of Lemma 2.4, $f(\lfloor \lambda \rfloor)$ exists and is finite a.e. on $E$, $\lfloor \lambda \rfloor$ denoting the greatest integer not exceeding $\lambda$. 
Proof. First we consider the case \(|\lambda| = k - 1\). If \(k = 1\) then \(|\lambda| = 0\) and so the result is trivially true. Suppose \(k \geq 2\). Then by Lemma 2.4,

\[
\Delta_i(x, u) = O(u^i) \quad \text{as } u \to 0 \text{ for } 1 \leq i \leq k - 1
\]

at almost all points of \(E\). So taking \(i = 1\), by the Denjoy–Young–Saks Theorem [7, p. 271], \(f'\) exists and is finite a.e. on \(E\). If \(k = 2\) then \(|\lambda| = 1\) and so the result follows. Therefore we suppose \(k \geq 3\). Then as above \(f'\) exists and is finite a.e. on \(E\). Suppose that \(f_{(r)}(x)\) exists and is finite a.e. on \(E\) for a fixed \(r\), \(1 \leq r < k - 1\). Let \(S \subset E\) be the set of points \(x\) such that \(f_{(r)}(x)\) exists and (2.12) holds. Then \(\mu^*(S) = \mu^*(E)\). Let \(x \in S\) be fixed. We may suppose that

\[
f_{(i)}(x) = 0 \quad \text{for } i = 0, 1, \ldots, r.
\]

Then from Theorem MZ2,

\[
\lim_{u \to 0} \Delta_i(x, u)/u^i = 0 \quad \text{for } i = 1, \ldots, r.
\]

Since \(\Delta_{r+1}(x, u) = O(u^{r+1})\), there are \(M > 0\) and \(\delta > 0\) such that

\[
|\Delta_r(x, u) - 2^r \Delta_r(x, u)| < M|u|^{r+1} \quad \text{for } 0 < |u| < \delta.
\]

Replacing \(u\) by \(u/2, u/2^2, \ldots, u/2^n\) successively and then multiplying the terms so obtained by \(2^r, 2^{2r}, \ldots, 2^{nr}\) respectively and then adding them with (2.14) we get, as in Lemma 2.4,

\[
|\Delta_r(x, 2u) - 2^{r(n+1)} \Delta_r(x, u/2^n)| < 2M|u|^{r+1}.
\]

Dividing by \(|u|^r\) and letting \(n \to \infty\) gives, by (2.13),

\[
|\Delta_r(x, 2u)| \leq 2M|u|^{r+1} \quad \text{for } |u| < \delta,
\]

that is, \(\Delta_r(x, u) = O(u^{r+1})\) as \(u \to 0\). Repeating these arguments we ultimately get \(\Delta_1(x, u) = O(u^{r+1})\) as \(u \to 0\), that is,

\[
f(x + u) = O(u^{r+1}) \quad \text{as } u \to 0.
\]

Since \(x \in S\) is arbitrary, by Theorem 2.2, \(f_{(r+1)}(x)\) exists a.e. on \(S\), that is, a.e. on \(E\). So by induction \(f_{(k-1)}(x)\) exists finitely a.e. on \(E\). Thus the result is true in this case.

To complete the proof we suppose that the result is true for \(|\lambda| = k - 1 + r\), \(r \geq 0\). Let \(|\lambda| = k + r\). Then \(\lambda = k + r + \alpha\), where \(0 \leq \alpha < 1\). Since

\[
|\Delta_k(x, u)| < m|u|^\lambda \quad \text{for } 0 < |u| < 1/m, \ x \in E,
\]

we have

\[
|\Delta_k(x, u)| < m|u|^{k-1+r+\alpha} \quad \text{for } 0 < |u| < 1/m, \ x \in E.
\]

Therefore, since the result is true for \(|\lambda| = k - 1 + r\), we conclude that \(f_{(k-1+r)}(x)\) exists and is finite a.e. on \(E\). Since \(|\Delta_k(x, u)| < m|u|^\lambda\) for \(0 < |u| < 1/m\)
and \( x \in E \) and since \( |\lambda| = k + r \),

\[
(2.15) \quad |\Delta_k(x, u)| < m|u|^{k+r} \quad \text{for } 0 < |u| < 1/m, \ x \in E.
\]

Therefore proceeding as in Lemma 2.4 we conclude that

\[
(2.16) \quad \tilde{\Delta}_k(x, u) = O(u^{k+r}) \quad \text{as } u \to 0
\]

at almost all points of \( E \). Let \( S \) be the set of points \( x \) of \( E \) such that \( f_{(k-1+r)}(x) \) exists and (2.16) holds. Then \( \mu^*(S) = \mu^*(E) \). Let \( x \in S \); we may suppose that \( f_i(x) = 0 \) for \( i = 0, 1, \ldots, k-1 \). Then from Theorem \( MZ2 \),

\[
(2.17) \quad \lim_{u \to 0} \tilde{\Delta}_i(x, u)/u^i = 0 \quad \text{for } i = 1, \ldots, k-1.
\]

By (2.16) there are \( M > 0 \) and \( \delta > 0 \) such that

\[
(2.18) \quad |\tilde{\Delta}_{k-1}(x, 2u) - 2^{k-1} \tilde{\Delta}_{k-1}(x, u)| < M|u|^{k+r} \quad \text{for } 0 < |u| < \delta.
\]

Replacing \( u \) by \( u/2, u/2^2, \ldots, u/2^n \) successively and then multiplying the inequalities so obtained by \( 2^{k-1}, 2^{2(k-1)}, \ldots, 2^{n(k-1)} \) respectively and then adding them with (2.18) we get

\[
|\tilde{\Delta}_{k-1}(x, 2u) - 2^{(n+1)(k-1)} \tilde{\Delta}_{k-1}(x, u/2^n)| < 2M|u|^{k+r}.
\]

Dividing by \( |u|^{k-1} \) and letting \( n \to \infty \) we get from this, and from (2.17),

\[
|\tilde{\Delta}_{k-1}(x, 2u)| \leq 2M|u|^{k+r},
\]

that is, \( \tilde{\Delta}_{k-1}(x, u) = O(u^{k+r}) \). Repeating these arguments we get \( \tilde{\Delta}_1(x, u) = O(u^{k+r}) \), that is, \( f(x + u) = O(u^{k+r}) \). Since \( x \in S \) is arbitrary, by Theorem 2.2, \( f_{(k+r)} \) exists a.e. on \( S \), that is, a.e. on \( E \). This shows that the result is true for \( |\lambda| = k+r \). This completes the proof of the lemma by induction.

### 3. Main results

**Theorem 3.1.** Let \( k \in \mathbb{N} \) and \( \lambda \in \mathbb{R} \) be such that \( \lambda > k - 1 \). Let \( f: \mathbb{R} \to \mathbb{R} \) be measurable. If

\[
(3.1) \quad \Delta_k(x, h; f) = O(|h|^\lambda) \quad \text{as } h \to 0
\]

for each point \( x \) in a set \( E \subset \mathbb{R} \) then \( f_{(|\lambda|)} \) exists and is finite a.e. on \( E \).

**Proof.** For each positive integer \( m \) let

\[
E_m = \{ x : |\Delta_k(x, u)| < m|u|^\lambda \text{ for } 0 < |u| < 1/m \}.
\]

Then \( \{E_m\} \) is a non-decreasing sequence and by (3.1), \( E \subset \bigcup_{m=1}^\infty E_m \). By Lemma 2.5, \( f_{(|\lambda|)} \) exists and is finite a.e. on \( E_m \) for each \( m \). This completes the proof.

The following theorem shows that Theorem 3.1 is not true for \( \lambda = k - 1 \), \( k \geq 2 \).
Theorem 3.2. For each integer $k \geq 2$ there exists a function $F$ such that

$$\Delta_k(x, h; F) = o(h^{k-1})$$

uniformly for all $x$, $F^{(k-2)}$ exists and is continuous for all $x$ but $F^{(k-1)}$ can exist at most on a set of measure zero.

To prove the theorem we need the following lemma.

Lemma 3.3. Let $k \geq 2$ be an integer, $f$ be locally integrable and uniformly smooth for all $x$ and $F$ be the $(k-2)$th integral of $f$. Then

$$\Delta_k(x, 2h; F) = o(h^{k-1})$$

uniformly for all $x$.

Proof. The case of $k = 2$ is trivial. We assume that $k > 2$ and $k$ is even. The case of $k$ odd is similar. Let $k = 2m$. Since $f$ is uniformly smooth for all $x$, for every $\varepsilon > 0$ there exists $\delta > 0$, independent of $x$, such that

$$|(f(x+h) + f(x-h) - 2f(x))/h| < \varepsilon$$

for $0 < h < \delta$ and for all $x$. So

(3.2) \[-\varepsilon t < f(x+t) + f(x-t) - 2f(x) < \varepsilon t \quad \text{for} \quad 0 < t < h < \delta.\]

Integrating the inequality (3.2) repeatedly $2m-2$ times over $[0, h]$ we get

$$-\varepsilon h^{2m-1}/(2m-1)! \quad < F(x+h) + F(x-h) - 2 \sum_{i=0}^{m-2} \frac{h^{2i}}{(2i)!} F^{(2i)}(x) - 2 \frac{h^{2m-2}}{(2m-2)!} f(x)$$

$$< \varepsilon h^{2m-1}/(2m-1)!.\]

Hence

(3.3) \[\frac{[F(x+h) + F(x-h)]}{2} - \sum_{i=0}^{m-2} \frac{h^{2i}}{(2i)!} F^{(2i)}(x) - \frac{h^{2m-2}}{(2m-2)!} f(x) = o(h^{2m-1}),\]

uniformly for all $x$. Now using the relations

(3.4) \[\sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} i^q = \begin{cases} 0 & \text{if } q = 0, 1, \ldots, p-1, \\ p! & \text{if } q = p, \end{cases}\]

from (3.3) we get
\[ \Delta_{2m}(x, 2h; F) \]
\[ = \sum_{j=0}^{2m} (-1)^{2m-j} \binom{2m}{j} F(x + 2jh - 2mh) \]
\[ = \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} F(x - 2jh + 2mh) \]
\[ = \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} \left[ \frac{1}{2} [F(x + 2(m - j)h) + F(x - 2(m - j)h)] \right] \]
\[ = \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} \left[ \sum_{i=0}^{m-2} \frac{[2(m - j)h]^{2i}}{(2i)!} F^{(2i)}(x) + \frac{[2(m - j)h]^{2m-2}}{(2m-2)!} f(x) \right] \]
\[ + o(h^{2m-1}) \]
\[ = \sum_{i=0}^{m-2} \frac{h^{2i}}{(2i)!} F^{(2i)}(x) \left[ \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} (2m - 2j)^{2i} \right] \]
\[ + \left[ \frac{(2h)^{2m-2}}{(2m-2)!} f(x) \sum_{j=0}^{2m} (-1)^j \binom{2m}{j} (m - j)^{2m-2} \right] + o(h^{2m-1}) \]
\[ = o(h^{2m-1}) \]

uniformly for all \( x \). This completes the proof.

**Proof of Theorem 3.2.** Let
\[ f(x) = \sum_{n=1}^{\infty} n^{-1/2} b^{-n} \cos(b^n x), \quad b > 1 \text{ an integer}. \]

Then \( f \) is continuous and uniformly smooth [9, I, p. 47, Theorem 4.10]. For \( k = 2 \), let \( F = f \) and for \( k > 2 \) let \( F \) be the \((k - 2)\)th integral of \( f \). We first show that
\[ \lim_{h \to 0} \Delta_{k-1}(x, h; F)/h^{k-1} \]
can exist finitely at most on a set of measure zero. Let \( k = 2 \). Then
\[ \Delta_1(x, 2h; f)/(2h) = [f(x + h) - f(x - h)]/(2h) \]
\[ = - \sum_{n=1}^{\infty} n^{-1/2} \sin(b^n x)[\sin(b^n h)/(b^n h)]. \]

If
\[ \lim_{h \to 0} \Delta_1(x, 2h; f)/(2h) \]
exists finitely on a set of positive measure then from (3.6) the series

\[(3.8) \quad - \sum_{n=1}^{\infty} n^{-1/2} \sin(b^n x)\]

is Lebesgue summable on a set of positive measure. Since (3.8) is a lacunary series, by [9, I, p. 203, Theorem 6.4], \(\sum_{n=1}^{\infty} 1/n\) is convergent, which is a contradiction. So (3.7) exists finitely at most on a set of measure zero.

Next suppose \(k > 2\). We prove that (3.5) can exist finitely at most on a set of measure zero. We suppose that \(k\) is even. Let \(k = 2m\). Now

\[(3.9) \quad \frac{\Delta_{2m-1}(x, 2h; F)}{(2h)^{2m-1}} = - \sum_{n=1}^{\infty} n^{-1/2} \sin(b^n x)(\sin(b^n h)/(b^n h))^{2m-1}.\]

If the limit of the left hand side of (3.9) exists on a set of positive measure as \(h \to 0\) then the series (3.8) is \((R, 2m-1)\) summable and so as in the case of \(k = 2\), \(\sum_{n=1}^{\infty} 1/n\) would be convergent, which is a contradiction. Thus the limit of the left hand side of (3.9) as \(h \to 0\) can exist at most on a set of measure zero. If \(k\) is odd then it can be similarly proved that (3.5) can exist finitely at most on a set of measure zero.

Now from Lemma 3.3 and the construction of the function \(F\) we see that

\[\Delta_k(x, h; F) = o(h^{k-1})\]

uniformly for all \(x\). Also it is clear that \(F^{(k-2)}\) exists and is continuous for all \(x\). To complete the proof we show that \(F_{(k-1)}\) can exist at most on a set of measure zero.

Let, if possible, \(F_{(k-1)}\) exist finitely on a set \(E\) of positive measure. Then for \(x \in E\),

\[F(x + h) = \sum_{j=0}^{k-1} \frac{h^j}{j!} F_{(j)}(x) + o(h^{k-1})\]

and so for \(x \in E\), by (3.4),

\[
\Delta_{k-1}(x, 2h; F) \\
= \sum_{i=0}^{k-1} (-1)^{k-1-i} \binom{k-1}{i} F(x + 2ih - (k - 1)h) \\
= \sum_{i=0}^{k-1} (-1)^{k-1-i} \binom{k-1}{i} \left[ \sum_{j=0}^{k-1} \frac{(2i - k + 1)^j h^j}{j!} F_{(j)}(x) + o(h^{k-1}) \right]
\]
\[
= \sum_{j=0}^{k-1} \frac{h^j}{j!} F(j)(x) \sum_{i=0}^{k-1} (-1)^{k-1-i} \binom{k-1}{i} (2i - k + 1)^j + o(h^{k-1})
\]

\[
= (2h)^{k-1} F_{k-1}(x) + o(h^{k-1}),
\]

and so for all \(x \in E\),

\[
\lim_{h \to 0} \Delta_{k-1}(x, 2h; F)/(2h)^{k-1} = F_{k-1}(x),
\]

which contradicts the fact that (3.5) can exist at most on a set of measure zero and thus the proof is complete.

Theorem 3.2 shows that in Theorem 3.1 the condition \(\lambda > k - 1\) is necessary. However, the following theorem shows that this condition can be relaxed if the existence of \(f(\lambda)\), \(a\) is assumed.

**Theorem 3.4.** Let \(k \in \mathbb{N}\), \(p \in \mathbb{N}\), \(p \leq k - 1\) and let \(f : \mathbb{R} \to \mathbb{R}\) be measurable. Let

\[
\Delta_k(x, u) = O(u^p) \quad \text{as } u \to 0,
\]

for each point \(x\) in a set \(E\). If \(f(p,a)\) exists finitely on \(E\) then \(f(p)\) exists a.e. on \(E\).

We need the following lemma.

**Lemma 3.5.** Let \(k \in \mathbb{N}\), \(p \in \mathbb{N}\) and let \(f : \mathbb{R} \to \mathbb{R}\) be measurable. Let \(E = E_m = \{x : f(p,a)(x) \text{ exists finitely and} \right.\)

\[
|\Delta_k(x, u)| < m|u|^p \quad \text{for } 0 < |u| < 1/m\}.
\]

Then \(f(p)\) exists a.e. on \(E\).

**Proof.** Let \(x_0 \in E\) be a point of outer density of \(E\). We suppose

\[
x_0 = 0 = f(x_0) = f(1,a)(x_0) = \ldots = f(p,a)(x_0).
\]

Let \(0 < \varepsilon < 1\). Let

\[
G = \{x : |f(x)| \leq \varepsilon|x|^p/p!\}.
\]

Then \(G\) is measurable and \(0 \in G\) is a point of density of \(G\). Set \(H = E \cap G\). Then \(0\) is a point of outer density of \(H\). Let \(0 < \eta < \varepsilon/(2k)\). Then by Lemma 2.1 there is \(\delta > 0\) such that if \(0 < u < \delta\) then

\[
\mu^*(B) > (1 - \eta)u, \quad \mu^*(C_j) > (1 - \eta)u,
\]

where

\[
B = \{v \in [u, 2u] : (u + v)/2 \in H\},
\]

\[
C_j = \{v \in [u, 2u] : v + j(u - v)/k \in H\}, \quad 0 \leq j \leq k - 1.
\]
Fix $u \in (0, \min(\delta, 1/m))$. Let
\[ S = \{ v \in [u, 2u] : |\Delta_k((u + v)/2, (u - v)/k)| < m((u - v)/k)^p \}, \]
\[ T_j = \{ v \in [u, 2u] : |f(v + j(u - v)/k)| \leq \varepsilon |v + j(u - v)/k|^p/p! \}, \]
\[ 0 \leq j \leq k - 1. \]
Since $f$ is measurable, $S$ and $T_j$ are measurable. Also $B \subset S$, $C_j \subset T_j$ and hence
\[ \mu(S) > (1 - \eta)u, \quad \mu(T_j) > (1 - \eta)u. \]
Therefore
\[ \mu\left( \bigcap_j (S \cap T_j) \right) > (1 - 2k\eta)u > (1 - \varepsilon)u. \]
Hence
\[ \left( \bigcap_j (S \cap T_j) \right) \cap (u, u + \varepsilon u) \neq \emptyset. \]
Choose $v \in \left( \bigcap_j (S \cap T_j) \right) \cap (u, u + \varepsilon u)$. Then $0 < v - u < \varepsilon u < u < 1/m$ and so
\[ |\Delta_k((u + v)/2, (u - v)/k)| < m((u - v)/k)^p < m(\varepsilon u)^p, \]
which gives
\[ \left| \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f((u + v)/2 + (j - k/2)(u - v)/k) \right| < m(\varepsilon u)^p. \]
Hence
\[ |f(u)| < m(\varepsilon u)^p + \sum_{j=0}^{k-1} \binom{k}{j} |f(v + j(u - v)/k)|. \]
Since $v \in T_j$ for $0 \leq j \leq k - 1$,
\[ |f(u)| < m(\varepsilon u)^p + \sum_{j=0}^{k-1} \binom{k}{j} \varepsilon |v + j(u - v)/k|^p/p! \]
\[ \leq m(\varepsilon u)^p + (\varepsilon/p!) \sum_{j=0}^{k-1} \binom{k}{j} (3u)^p \]
\[ \leq \varepsilon \left[ m + (3p)! \sum_{j=0}^{k-1} \binom{k}{j} \right] u^p. \]
This shows that $f(u)/u^p \to 0$ as $u \to 0^+$. It can be similarly shown that $f(u)/u^p \to 0$ as $u \to 0^-$. This completes the proof of the lemma.
Proof of Theorem 3.4. For each positive integer \( m \), let
\[
E_m = \{ x : f(p)_a(x) \text{ exists finitely and } |\Delta_k(x,u)| < m|u|^p \text{ for } 0 < |u| < 1/m \}.
\]
Then \( \{E_m\} \) is a non-decreasing sequence and \( E \subset \bigcup_m E_m \). By Lemma 3.5, \( f(p)_a \) exists a.e. on \( E_m \) and so the result follows.

Corollary 3.6. Let \( p \in \mathbb{N} \), let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be measurable and let \( f(x) = 0 \) for \( x \in E \subset \mathbb{R} \). If
\[
f(x + u) - f(x - u) = O(u^p)
\]
or
\[
f(x + u) + f(x - u) = O(u^p)
\]
for \( x \in E \), then \( f(p)_a \) exists a.e. on \( E \).

Proof. Let
\[
E_1 = \{ x \in E : f(x + u) - f(x - u) = O(u^p) \},
\]
\[
E_2 = \{ x \in E : f(x + u) + f(x - u) = O(u^p) \}.
\]
Then \( E = E_1 \cup E_2 \). Let \( D_i \) be the set of all points of \( E_i \) which are also points of outer density of \( E_i \), \( i = 1, 2 \). Then \( f(p)_a(x) = 0 \) for \( x \in D_1 \cup D_2 \). Also
\[
\Delta_1(x,u) = O(u^p) \quad \text{as } u \rightarrow 0 \quad \text{for } x \in D_1,
\]
\[
\Delta_2(x,u) = O(u^p) \quad \text{as } u \rightarrow 0 \quad \text{for } x \in D_2.
\]
Hence if \( p = 1 \) then by Theorem 3.1, \( f' \) exists finitely a.e. on \( D_1 \) and by
Theorem 3.4, \( f' \) exists finitely a.e. on \( D_2 \) and hence \( f' \) exists a.e. on \( E \). If \( p \geq 2 \) then by Theorem 3.1, \( f(p)_a \) exists finitely a.e. on \( D_1 \) and on \( D_2 \) and hence \( f(p)_a \) exists finitely a.e. on \( E \).

The above corollary is a generalization of Lemma 11 of [8, p. 268], since
we are not assuming the measurability of \( E \).

Theorem 3.4 can further be extended to

Theorem 3.7. Let \( k \in \mathbb{N} \), \( p \in \mathbb{N} \), \( p \leq k - 1 \) and let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be measurable. Let
\[
\Delta_k(x,u) = O(u^p) \quad \text{as } u \rightarrow 0
\]
for each point \( x \) in a set \( E \). If \( f(p-1)_a \) exists and
\[
-\infty < \underline{f}(p)_a \leq \overline{f}(p)_a < \infty \quad \text{on } E
\]
then \( f(p-1)_a \) exists and
\[
-\infty < \underline{f}(p) \leq \overline{f}(p) < \infty \quad \text{a.e. on } E.
\]

Proof. The first part follows from Theorem 3.4. The proof of the second
part is similar to that of Theorem 3.4. We give a sketch. The corresponding
sets in Lemma 3.5 are in this case given by

\[ E_m = \{ x : f_{(p-1),a}(x) \text{ exists finitely, } |\Delta_k(x,u)| < m|u|^p \] for \( 0 < |u| < 1/m \) and \(-m < f_{(p),a}(x) \leq f_{(p),a}(x) < m\)

with the assumption that

\[ x_0 = 0 = f(x_0) = f_{(1),a}(x_0) = \ldots = f_{(p-1),a}(x_0), \]

\[ G_m = \{ x : |f(x)| \leq m|x|^p/p! \}, \]

\[ T_j = \{ v \in [u,2u] : |f(v + j(u-v)/k)| \leq m|v + j(u-v)/k|^p/p! \}, \]

\( 0 \leq j \leq k - 1, \)

and the final step is

\[ |f(u)| \leq \left[ \varepsilon m + m(3^p/p!) \sum_{j=0}^{k-1} \binom{k}{j} \right] u^p \]

showing that \( |f(u)| = O(u^p) \) as \( u \to 0^+ \) and similarly \( |f(u)| = O(|u|^p) \) as \( u \to 0^- \).

**Corollary 3.8.** Under the hypothesis of Theorem 3.7, if \( f_{(p-1),a} \) exists and

\[ -\infty < f_{(p),a}(x) \leq f_{(p),a}(x) < \infty \quad \text{on } E \]

then \( f_{(p)} \) exists a.e. on \( E \).

The proof follows from Theorems 3.7 and 2.2.

**References**


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