FUNDAMENTA
MATHEMATICAE
150 (1996)

On the homotopy category of Moore spaces and the cohomology of the category of abelian groups

by

Hans-Joachim B a u e s (Bonn) and **Manfred H a r t l** (Valenciennes)

Abstract. The homotopy category of Moore spaces in degree 2 represents a nontrivial cohomology class in the cohomology of the category of abelian groups. We describe various properties of this class. We use James–Hopf invariants to obtain explicitly the image category under the functor chain complex of the loop space.

An abelian group A determines the Moore space $M(A) = M(A,2)$ which up to homotopy equivalence is the unique simply connected CW-space X with homology groups $H_2X = A$ and $H_iX = 0$ for $i > 2$. Since $M(A)$ can be chosen to be a suspension, the set of homotopy classes $[M(A), M(B)]$ is a group which is part of a classical central extension of groups

$$\mathrm{Ext}(A, \Gamma B) \rightarrowtail [M(A), M(B)] \twoheadrightarrow \mathrm{Hom}(A, B) \tag{1}$$

due to Barratt. It is known that (1) in general is not split, for example $[M(\mathbb{Z}/2), M(\mathbb{Z}/2)] = \mathbb{Z}/4$. We are not interested here in this additive structure of the sets $[M(A), M(B)]$ but in the multiplicative structure given by the composition of maps, in particular in the extension of groups

$$\mathrm{Ext}(A, \Gamma A) \rightarrowtail \mathcal{E}(M(A)) \twoheadrightarrow \mathrm{Aut}(A), \tag{2}$$

where $\mathcal{E}(M(A))$ is the group of homotopy equivalences of the space $M(A)$. The extension (2) determines the cohomology class

$$\{\mathcal{E}(M(A))\} \in H^2(\mathrm{Aut}(A), \mathrm{Ext}(A, \Gamma A)). \tag{3}$$

Though the group $\mathcal{E}(M(A))$ is defined in an "easy" range of homotopy theory the cohomology class (3) is not yet computed for all abelian groups A.

In this paper we prove a nice algebraic formula for the class (3) if A is a product of cyclic groups and we show that $\{\mathcal{E}(M(A))\}$ is trivial if

1991 *Mathematics Subject Classification*: 55E05, 55E25, 55J.

$\mathrm{Ext}(A,\Gamma A)$ has no 2-torsion; see (3.6) and (5.2). Moreover, we compute for all abelian groups A the image of the class (3) under the surjection of coefficients

$$\mathrm{Ext}(A,\Gamma A) \twoheadrightarrow \mathrm{Ext}(A,H(\Gamma A)). \tag{4}$$

Here $H(\Gamma A)$ is the image of $H : \Gamma A \to A\otimes A$; see (4.2). We do such computations not in the cohomology of groups but more distinctly in the cohomology of categories. In fact, the homotopy category $\mathbf{M}^2$ of Moore spaces $M(A)$ leads to a topological "characteristic class" in the cohomology of the category **Ab** of abelian groups; see (2.2). It is the computation of such topologically defined cohomology classes which motivated the results in this paper. For example the topological James–Hopf invariant on the category $\mathbf{M}^2$ or the "chains on the loop space" functor $C_*\Omega$ on $\mathbf{M}^2$ have interesting interpretations on the level of the cohomology of the category **Ab**; see (4.11). As an application we describe algebraically the image category $(C_*\Omega)(\mathbf{M}^2)$ in the homotopy category of chain algebras showing fundamental differences between the homotopy category of spaces and chain algebras respectively; see (4.12). This implies that the image of the group $\mathcal{E}(M(A))$ under the functor $C_*\Omega$ is part of an extension

$$\mathrm{Ext}(A,H(\Gamma A)) \rightarrowtail (C_*\Omega)\mathcal{E}(M(A)) \twoheadrightarrow \mathrm{Aut}(A), \tag{5}$$

which we compute explicitly in terms of A for all abelian groups A.

1. Linear extensions and cohomology of categories. An *extension* of a group G by a G-module A is a short exact sequence of groups

$$0 \to A \underset{i}{\to} E \underset{p}{\to} G \to 0,$$

where i is compatible with the action of G. Two such extensions E and E' are *equivalent* if there is an isomorphism $\varepsilon : E \cong E'$ of groups with $p'\varepsilon = p$ and $\varepsilon i = i'$. It is well known that the equivalence classes of extensions are classified by the cohomology $H^2(G,A)$.

We now recall from [2] the basic notation of the cohomology of categories. We describe linear extensions of a small category $\mathbf{C}$ by a "natural system" D. The equivalence classes of such extensions are classified by the cohomology $H^2(\mathbf{C},D)$. A natural system D on a category $\mathbf{C}$ is the appropriate generalization of a G-module.

(1.1) Definition. Let $\mathbf{C}$ be a category. The *category of factorizations* in $\mathbf{C}$, denoted by $F\mathbf{C}$, is given as follows. Objects are morphisms $f, g, \ldots$ in $\mathbf{C}$ and morphisms $f \to g$ are pairs (α,β) for which

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ B & \xleftarrow[\beta]{} & B' \end{array}$$

commutes in $\mathbf{C}$. Here $\alpha f \beta$ is a factorization of g. Composition is defined by $(\alpha', \beta')(\alpha, \beta) = (\alpha'\alpha, \beta\beta')$. We clearly have $(\alpha, \beta) = (\alpha, 1)(1, \beta) = (1, \beta)(\alpha, 1)$. A *natural system* (of abelian groups) on $\mathbf{C}$ is a functor $D : F\mathbf{C} \to \mathbf{Ab}$. The functor D carries the object f to $D_f = D(f)$ and carries the morphism $(\alpha, \beta) : f \to g$ to the induced homomorphism

$$D(\alpha, \beta) = \alpha_* \beta^* : D_f \to D_{\alpha f \beta} = D_g.$$

Here we set $D(\alpha, 1) = \alpha_*$, $D(1, \beta) = \beta^*$.

We have a canonical forgetful functor $\pi : F\mathbf{C} \to \mathbf{C}^{\mathrm{op}} \times \mathbf{C}$ so that each *bifunctor* $D : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Ab}$ yields a natural system $D\pi$, also denoted by D. Such a bifunctor is also called a $\mathbf{C}$-*bimodule*. In this case $D_f = D(B, A)$ depends only on the objects A, B for all $f \in \mathbf{C}(B, A)$. Two functors F, G : $\mathbf{Ab} \to \mathbf{Ab}$ yield the $\mathbf{Ab}$-bimodule

$$\mathrm{Hom}(F, G) : \mathbf{Ab}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab}$$

which carries (A, B) to the group of homomorphisms $\mathrm{Hom}(FA, GB)$. If F is the identity functor we write $\mathrm{Hom}(-, G)$. Similarly we define the $\mathbf{Ab}$-bimodule $\mathrm{Ext}(F, G)$.

For a group G and a G-module A the corresponding natural system D on the group G, considered as a category, is given by $D_g = A$ for $g \in G$ and $g_* a = g \cdot a$ for $a \in A$, $g^* a = a$. If we restrict the following notion of a "linear extension" to the case $\mathbf{C} = G$ and $D = A$ we obtain the notion of a group extension above.

(1.2) Definition. Let D be a natural system on $\mathbf{C}$. We say that

$$D \overset{+}{\rightarrowtail} \mathbf{E} \overset{p}{\twoheadrightarrow} \mathbf{C}$$

is a *linear extension of the category* $\mathbf{C}$ by D if (a), (b) and (c) below hold.

(a) $\mathbf{E}$ and $\mathbf{C}$ have the same objects and p is a full functor which is the identity on objects.

(b) For each $f : A \to B$ in $\mathbf{C}$, the abelian group D_f acts transitively and effectively on the subset $p^{-1}(f)$ of morphisms in $\mathbf{E}$. We write $f_0 + \alpha$ for the action of $\alpha \in D_f$ on $f_0 \in p^{-1}(f)$.

(c) The action satisfies the *linear distributivity law*:

$$(f_0 + \alpha)(g_0 + \beta) = f_0 g_0 + f_* \beta + g^* \alpha.$$

Two linear extensions $\mathbf{E}$ and $\mathbf{E}'$ are *equivalent* if there is an isomorphism of categories $\varepsilon : \mathbf{E} \cong \mathbf{E}'$ with $p'\varepsilon = p$ and with $\varepsilon(f_0 + \alpha) = \varepsilon(f_0) + \alpha$

for $f_0 \in \mathrm{Mor}(\mathbf{E})$, $\alpha \in D_{pf_0}$. The extension $\mathbf{E}$ is *split* if there is a functor $s : \mathbf{C} \to \mathbf{E}$ with $ps = 1$. Let $M(\mathbf{C}, D)$ be the set of equivalence classes of linear extensions of $\mathbf{C}$ by $\mathbf{D}$. Then there is a canonical bijection

$$\psi : M(\mathbf{C}, D) \cong H^2(\mathbf{C}, D) \tag{1.3}$$

which maps the split extension to the zero element (see [2] and IV, §6 in [4]). Here $H^n(\mathbf{C}, D)$ denotes the *cohomology* of $\mathbf{C}$ with coefficients in D which is defined below. We obtain a *representing cocycle* Δ_t of the cohomology class $\{\mathbf{E}\} = \psi(\mathbf{E}) \in H^2(\mathbf{C}, D)$ as follows. Let t be a "splitting" function for p which associates with each morphism $f : A \to B$ in $\mathbf{C}$ a morphism $f_0 = t(f)$ in $\mathbf{E}$ with $pf_0 = f$. Then t yields a cocycle Δ_t by the formula

$$t(gf) = t(g)t(f) + \Delta_t(g, f) \tag{1.4}$$

with $\Delta_t(g, f) \in D(gf)$. The cohomology class $\{\mathbf{E}\} = \{\Delta_t\}$ is trivial if and only if $\mathbf{E}$ is a split extension.

(1.5) Definition. Let $\mathbf{C}$ be a small category and let $N_n(\mathbf{C})$ be the set of sequences $(\lambda_1, \ldots, \lambda_n)$ of n composable morphisms in $\mathbf{C}$ (which are the n-simplices of the *nerve* of $\mathbf{C}$). For $n = 0$ let $N_0(\mathbf{C}) = \mathrm{Ob}\,(\mathbf{C})$ be the set of objects in $\mathbf{C}$. The cochain group $F^n = F^n(\mathbf{C}, D)$ is the abelian group of all functions

$$c : N_n(\mathbf{C}) \to \Big(\bigcup_{g \in \mathrm{Mor}(\mathbf{C})} D_g\Big) = D \tag{1}$$

with $c(\lambda_1, \ldots, \lambda_n) \in D_{\lambda_1 \circ \ldots \circ \lambda_n}$. Addition in F^n is given by adding pointwise in the abelian groups D_g. The coboundary $\partial : F^{n-1} \to F^n$ is defined by the formula

$$\begin{aligned}(\partial c)(\lambda_1, \ldots, \lambda_n) = (\lambda_1)_* c(\lambda_2, \ldots, \lambda_n) \\ + \sum_{i=1}^{n-1} (-1)^i c(\lambda_1, \ldots, \lambda_i\lambda_{i+1}, \ldots, \lambda_n) \\ + (-1)^n (\lambda_n)^* c(\lambda_1, \ldots, \lambda_{n-1}).\end{aligned} \tag{2}$$

For $n = 1$ we have $(\partial c)(\lambda) = \lambda_* c(A) - \lambda^* c(B)$ for $\lambda : A \to B \in N_1(\mathbf{C})$. One can check that $\partial c \in F^n$ for $c \in F^{n-1}$ and that $\partial\partial = 0$. Hence the *cohomology groups*

$$H^n(\mathbf{C}, D) = H^n(F^*(\mathbf{C}, D), \partial) \tag{3}$$

are defined for $n \geq 0$. These groups are discussed in [9] and [2]. By change of the universe cohomology groups $H^n(\mathbf{C}, D)$ can also be defined if $\mathbf{C}$ is not a small category. A functor $\phi : \mathbf{C}' \to \mathbf{C}$ induces the homomorphism

$$\phi^* : H^n(\mathbf{C}, D) \to H^n(\mathbf{C}', \phi^* D), \tag{4}$$

where ϕ^*D is the natural system given by $(\phi^*D)_f = D_{\phi(f)}$. On cochains the map ϕ^* is given by the formula

$$(\phi^*f)(\lambda'_1,\ldots,\lambda'_n) = f(\phi\lambda'_1,\ldots,\phi\lambda'_n),$$

where $(\lambda'_1,\ldots,\lambda'_n) \in N_n(\mathbf{C}')$. If ϕ is an equivalence of categories then ϕ^* is an isomorphism. A natural transformation $\tau : D \to D'$ between natural systems induces a homomorphism

$$(5) \qquad \tau_* : H^n(\mathbf{C}, D) \to H^n(\mathbf{C}, D')$$

by $(\tau_*f)(\lambda_1,\ldots,\lambda_n) = \tau_\lambda f(\lambda_1,\ldots,\lambda_n)$ where $\tau_\lambda : D_\lambda \to D'_\lambda$ with $\lambda = \lambda_1 \circ \ldots \circ \lambda_n$ is given by the transformation τ. Now let

$$D'' \overset{l}{\rightarrowtail} D \overset{\tau}{\twoheadrightarrow} D'$$

be a short exact sequence of natural systems on $\mathbf{C}$. Then we obtain as usual the natural long exact sequence

$$(1.6) \qquad \to H^n(\mathbf{C}, D') \xrightarrow{l_*} H^n(\mathbf{C}, D) \xrightarrow{\tau_*} H^n(\mathbf{C}, D'') \xrightarrow{\beta} H^{n+1}(\mathbf{C}, D') \to,$$

where β is the Bockstein homomorphism. For a cocycle c'' representing a class $\{c''\}$ in $H^n(\mathbf{C}, D'')$ we obtain $\beta\{c''\}$ by choosing a cochain c as in (1.5)(1) with $\tau c = c''$. This is possible since τ is surjective. Then $\iota^{-1}\delta c$ is a cocycle which represents $\beta\{c''\}$.

(1.7) Remark. The cohomology (1.5) generalizes the *cohomology of a group*. In fact, let G be a group and let $\mathbf{G}$ be the corresponding category with a single object and with morphisms given by the elements in G. A G-module A yields a natural system D. Then the classical definition of the cohomology $H^n(G, A)$ coincides with the definition of

$$H^n(\mathbf{G}, D) = H^n(G, A)$$

given by (1.5). Further results and applications of the cohomology of categories can be found in [2], [3], [8], [9], [13], [14].

2. The homotopy category $\mathbf{M}^2$ of Moore spaces in degree 2. Let A be an abelian group. A *Moore space* $M(A, n)$, $n \geq 2$, is a simply connected CW-space X with (reduced) homology groups $H_nX = A$ and $H_iX = 0$ for $i \neq n$. An *Eilenberg–MacLane space* $K(A, n)$ is a CW-space Y with homotopy groups $\pi_nY = A$ and $\pi_iY = 0$ for $i \neq n$. Such spaces exist and their homotopy type is well defined by (A, n). The homotopy category of Eilenberg–MacLane spaces $K(A, n)$, $A \in \mathbf{Ab}$, is isomorphic via the functor π_n to the category $\mathbf{Ab}$ of abelian groups. The corresponding result, however, does not hold for the homotopy category $\mathbf{M}^n$ of Moore spaces $M(A, n)$, $A \in \mathbf{Ab}$. This creates the problem to find a suitable algebraic model of the category $\mathbf{M}^n$. For $n \geq 3$ such a model category of $\mathbf{M}^n$ is known (see (V.3a.8) in [2] and (I, §6) in [4]). The category $\mathbf{M}^2$ is not completely understood. We shall use the cohomology of the category **Ab** to describe various properties of the category $\mathbf{M}^2$.

Let $\Gamma : \mathbf{Ab} \to \mathbf{Ab}$ be J. H. C. Whitehead's quadratic functor [15] with

(2.1) $$\Gamma(A) = \pi_3 M(A,2) = H_4 K(A,2).$$

Then we obtain the **Ab**-bimodule

$$\mathrm{Ext}(-,\Gamma) : \mathbf{Ab}^{\mathrm{op}} \times \mathbf{Ab} \to \mathbf{Ab}$$

which carries (A,B) to the group $\mathrm{Ext}(A,\Gamma(B))$.

(2.2) Proposition. *The category* $\mathbf{M}^2$ *is part of a nonsplit linear extension*

$$\mathrm{Ext}(-,\Gamma) \overset{+}{\rightarrowtail} \mathbf{M}^2 \overset{H_2}{\twoheadrightarrow} \mathbf{Ab}$$

and hence $\mathbf{M}^2$, *up to equivalence, is characterized by a cohomology class*

$$\{\mathbf{M}^2\} \in H^2(\mathbf{Ab}, \mathrm{Ext}(-,\Gamma)).$$

Since the extension is nonsplit we have $\{\mathbf{M}^2\} \neq 0$.

Proof. For a free abelian group A_0 with basis Z let

$$M_{A_0} = \bigvee_Z S^1$$

be a one-point union of 1-dimensional spheres S^1 such that $H_1 M_{A_0} = A_0$. For an abelian group A we choose a short exact sequence

$$0 \to A_1 \overset{d_A}{\to} A_0 \to A \to 0,$$

where A_0, A_1 are free abelian. Let $d'_A : M_{A_1} \to M_{A_0}$ be a map which induces d_A in homology and let M_A be the mapping cone of d'_A. Then

$$M(A,2) = \Sigma M_A$$

is the suspension of M_A. The homotopy type of M_A, however, depends on the choice of d'_A and is not determined by A. Using the cofiber sequence for d'_A we obtain the well known exact sequence of groups [11]

$$0 \to \mathrm{Ext}(A, \pi_3 X) \overset{\Delta}{\to} [M(A,2), X] \overset{\mu}{\to} \mathrm{Hom}(A, \pi_2 X) \to 0,$$

where $[Y,X]$ denotes the set of homotopy classes of pointed maps $Y \to X$. We now set $X = M(B,2)$. Then μ is given by the homology functor. We define the action of $\alpha \in \mathrm{Ext}(A, \Gamma B)$ on $\xi \in [M(A,2), M(B,2)]$ by $\xi + \alpha = \xi + \Delta(\alpha)$ where we use the group structure in $[\Sigma M_A, M(B,2)]$. This action satisfies the linear distributivity law so that we obtain the linear extension in (2.2). Compare also (V, §3a) in [2] where we show $\{\mathbf{M}^2\} \neq 0$.

(2.3) Remark. A *Pontryagin map* τ_A for an abelian group A is a map

$$\tau_A : K(A,2) \to K(\Gamma(A),4)$$

which induces the identity of $\Gamma(A)$,

$$\Gamma(A) = H_4K(A,2) \to H_4K(\Gamma(A),4) = \Gamma(A).$$

Such Pontryagin maps exist and are well defined up to homotopy. The map τ_A induces the Pontryagin square which is the cohomology operation [15]

$$H^2(X,A) = [X,K(A,2)] \xrightarrow{(\tau_A)_*} [X,K(\Gamma(A),2)] = H^4(X,\Gamma(A)).$$

The fiber of τ_A is the 3-type of $M(A,2)$. Therefore one gets isomorphisms of categories [7]

$$\mathbf{M}^2 = \mathbf{P}(\mathcal{X}) = \mathrm{Hopair}(\mathcal{X}),$$

where $\mathcal{X}$ is the class of all Pontryagin maps τ_A, $A \in \mathbf{Ab}$. Here $\mathbf{P}(\mathcal{X})$ is the homotopy category of fibers $P(\tau_A)$, $\tau_A \in \mathcal{X}$, and $\mathrm{Hopair}(\mathcal{X})$ is the category of homotopy pairs [10] between Pontryagin maps. We have seen in [9] that via these isomorphisms the class $\{\mathbf{M}^2\}$ is the image of the *universal Toda bracket* $\langle \mathbf{K} \rangle_\Omega \in H^3(\mathbf{K}, D_\Omega)$ where $\mathbf{K}$ is the full subcategory of the homotopy category consisting of $K(A,2)$ and $K(\Gamma(A),4)$, $A \in \mathbf{Ab}$. Hence we get by (2.2):

(2.4) Corollary. $\langle \mathbf{K} \rangle_\Omega \neq 0$.

3. On the cohomology class $\{\mathbf{M}^2\}$. The quadratic functor Γ can also be defined by the universal quadratic map $\gamma : A \to \Gamma(A)$. We have the natural exact sequence in $\mathbf{Ab}$

$$\Gamma(A) \xrightarrow{H} A \otimes A \xrightarrow{q} \Lambda^2 A \to 0, \tag{3.1}$$

where H is defined by $H\gamma(a) = a \otimes a$, $a \in A \in \mathbf{Ab}$, and where $\Lambda^2 A = A \otimes A/\{a \otimes a \sim 0\}$ is the exterior square with quotient map q. We also need the natural homomorphism

$$[1,1] = P : A \otimes A \to \Gamma(A) \tag{3.2}$$

with $P(a \otimes b) = \gamma(a+b) - \gamma(a) - \gamma(b) = [a,b]$. One readily checks that PH is multiplication by 2 on $\Gamma(A)$ and that $HP(a \otimes b) = a \otimes b + b \otimes a$. For $A \in \mathbf{Ab}$ using P and H and q above we obtain the following natural short exact sequences of $\mathbb{Z}/2$-vector spaces:

$$\begin{aligned} &S_1(A) : \Lambda^2(A) \otimes \mathbb{Z}/2 \overset{P}{\rightarrowtail} \Gamma(A) \otimes \mathbb{Z}/2 \overset{\sigma}{\twoheadrightarrow} A \otimes \mathbb{Z}/2, \\ &S_2(A) : \Gamma(A) \otimes \mathbb{Z}/2 \overset{H}{\rightarrowtail} \otimes^2(A) \otimes \mathbb{Z}/2 \overset{q}{\twoheadrightarrow} \Lambda^2(A) \otimes \mathbb{Z}/2. \end{aligned} \tag{3.3}$$

Here σ carries $\gamma(a) \otimes 1$ to $a \otimes 1$, $a \in A$. If we apply the functor $\mathrm{Hom}(-, \Gamma(B) \otimes \mathbb{Z}/2)$ to the exact sequence $S_i(A)$, $i = 1,2$, we get the corresponding exact sequence of $\mathbf{Ab}$-bimodules denoted by $\mathrm{Hom}(S_i(-), \Gamma(-) \otimes \mathbb{Z}/2)$. The associated Bockstein homomorphisms β_i yield thus homomorphisms

$$\begin{array}{c} H^0(\mathbf{Ab}, \operatorname{Hom}(\Gamma(-)\otimes\mathbb{Z}/2, \Gamma(-)\otimes\mathbb{Z}/2)) \\ \downarrow \beta_2 \\ H^1(\mathbf{Ab}, \operatorname{Hom}(\Lambda^2(-)\otimes\mathbb{Z}/2, \Gamma(-)\otimes\mathbb{Z}/2)) \\ \downarrow \beta_1 \\ H^2(\mathbf{Ab}, \operatorname{Hom}(-\otimes\mathbb{Z}/2, \Gamma(-)\otimes\mathbb{Z}/2)) \end{array} \tag{3.4}$$

Moreover, we use the natural homomorphism

$$\chi : \operatorname{Hom}(A\otimes\mathbb{Z}/2, \Gamma(B)\otimes\mathbb{Z}/2) \overset{g}{=} \operatorname{Ext}(A\otimes\mathbb{Z}/2, \Gamma B) \xrightarrow{p^*} \operatorname{Ext}(A, \Gamma B),$$

where g is the natural isomorphism and where $p : A \to A\otimes\mathbb{Z}/2$ is the projection. Let

$$1_\Gamma \in H^0(\mathbf{Ab}, \operatorname{Hom}(\Gamma(-)\otimes\mathbb{Z}/2, \Gamma(-)\otimes\mathbb{Z}/2))$$

be the canonical class which carries the abelian group A to the identity of $\Gamma(A)\otimes\mathbb{Z}/2$. Then one gets the element

$$\chi_*\beta_1\beta_2(1_\Gamma) \in H^2(\mathbf{Ab}, \operatorname{Ext}(-,\Gamma))$$

determined by 1_Γ and the homomorphisms above.

(3.5) Conjecture. $\{\mathbf{M}^2\} = \chi_*\beta_1\beta_2(1_\Gamma)$.

We shall prove various results which support this conjecture.

(3.6) Theorem. *Let* $\mathbf{A}$ *be the full subcategory of* $\mathbf{Ab}$ *consisting of direct sums of cyclic groups and let* $i_{\mathbf{A}} : \mathbf{A} \to \mathbf{Ab}$ *be the inclusion functor. Then we have*

$$i^*_{\mathbf{A}}\{\mathbf{M}^2\} = i^*_{\mathbf{A}}\chi_*\beta_1\beta_2(1_\gamma) \in H^2(\mathbf{A}, \operatorname{Ext}(-,\Gamma)).$$

Proof. We write $C = (\mathbb{Z}/a)\alpha = \alpha(\mathbb{Z}/a)$ if C is a cyclic group isomorphic to $\mathbb{Z}/a$ with generator α, $a \ge 0$. A direct sum of cyclic groups

$$A = \bigoplus_i (\mathbb{Z}/a_i)\alpha_i$$

is *indexed by an ordered set* if the set of generators $\{\alpha_i, <\}$ is a well ordered set. The generator α_i also denotes the inclusion $\alpha_i : \mathbb{Z}/a_i \subset A$ and the corresponding inclusion

$$\alpha_i : \Sigma P_{a_i} \subset \bigvee_i \Sigma P_{a_i} = M(A,2). \tag{3.7}$$

Here $P_n = S^1 \cup_n e^2$ is the *pseudo-projective plane* for $n > 0$ and $P_0 = S^1$ so that $\Sigma P_n = M(\mathbb{Z}/n, 2)$. Let $\alpha^i : A \to \mathbb{Z}/a_i$ be the canonical retraction of α_i with $\alpha^i\alpha_i = 1$ and $\alpha^j\alpha_i = 0$ for $j \ne i$. Let

$$\varphi : A = \bigoplus_i \alpha_i(\mathbb{Z}/a_i) \to B = \bigoplus_j \beta_j(\mathbb{Z}/b_j) \tag{3.8}$$

be a homomorphism. The coordinates $\varphi_{ji} \in \mathbb{Z}$, $\varphi_{ji} : \mathbb{Z}/a_i \to \mathbb{Z}/b_j$, $\mathbf{1} \mapsto \varphi_{ji}\mathbf{1}$, are given by the formula

$$\varphi\alpha_i = \sum \beta_j \, \varphi_{ji}.$$

Let B_2 be the splitting function

$$[\Sigma P_n, \Sigma P_m] \underset{B_2}{\overset{\rightharpoonup}{\leftarrow}} \mathrm{Hom}(\mathbb{Z}/n, \mathbb{Z}/m)$$

obtained in (III, Appendix D) of [3]. We define the map $s\varphi \in [M(A,2), M(B,2)]$ by the *ordered sum*

$$(s\varphi)\alpha_i = \sum_j{}^{<} \beta_j B_2(\varphi_{ji}),$$

where we use the ordering $<$ of the generators in B. Hence we obtain a *splitting function* s:

(3.9) $$[M(A,2), M(B,2)] \underset{s}{\overset{H_2}{\rightleftarrows}} \mathrm{Hom}(A,B)$$

with $H_2 s(\varphi) = \varphi$. Each element $\overline{\varphi} \in [M(A,2), M(B,2)]$ is of the form $\overline{\varphi} = s(\varphi) + \xi$, where $\xi \in \mathrm{Ext}(A, \Gamma B)$. This way we can characterize all elements in $[M(A,2), M(B,2)]$ provided A and B are ordered direct sums of cyclic groups. We use s in (3.9) for the definition of the cocycle Δ_s representing $i^*\{\mathbf{M}^2\}$ in (3.6), that is, by (1.4),

$$s(\psi\varphi) = s(\psi)s(\varphi) + \Delta_s(\psi, \varphi).$$

Below we compute Δ_s. To this end we have to introduce the following groups.

(3.10) Definition. Let A be an abelian group. We have the natural homomorphism between $\mathbb{Z}/2$-vector spaces

(1) $$H : \Gamma(A) \otimes \mathbb{Z}/2 = \Gamma(A \otimes \mathbb{Z}/2) \otimes \mathbb{Z}/2 \to \otimes^2(A \otimes \mathbb{Z}/2)$$

with $H(\gamma(a) \otimes 1) = (a \otimes 1) \otimes (a \otimes 1)$. This homomorphism is injective and hence admits a *retraction homomorphism*

(2) $$r : \otimes^2(A \otimes \mathbb{Z}/2) \to \Gamma(A) \otimes \mathbb{Z}/2$$

with $rH = \mathrm{id}$. For example, given a basis E of the $\mathbb{Z}/2$-vector space $A \otimes \mathbb{Z}/2$ and a well ordering $<$ on E we can define a retraction $r^<$ on the basis elements by the formula ($b, b' \in E$)

(3) $$r^<(b \otimes b') = \begin{cases} \gamma(b) \otimes 1 & \text{for } b = b', \\ [b, b'] \otimes 1 & \text{for } b > b', \\ 0 & \text{for } b < b'. \end{cases}$$

Now let $q \geq 1$ and let

(4) $$j_A : \mathrm{Hom}(\mathbb{Z}/q, A) = A * \mathbb{Z}/q \subset A \overset{p}{\twoheadrightarrow} A \otimes \mathbb{Z}/2$$

be given by the projection p with $p(x) = x \otimes 1$. Also let

(5) $$p_A : \Gamma(A) \otimes \mathbb{Z}/2 \overset{p}{\rightarrowtail} \Gamma(A) \otimes \mathbb{Z}/2 \otimes \mathbb{Z}/q = \mathrm{Ext}(\mathbb{Z}/2 \otimes \mathbb{Z}/q, \Gamma(A)) \overset{p^*}{\to} \mathrm{Ext}(\mathbb{Z}/q, \Gamma(A))$$

be defined by the indicated projections p. Then we obtain the homomorphism

(6) $$\begin{aligned} &\Delta_A : \mathrm{Hom}(\mathbb{Z}/q, A) \otimes \mathrm{Hom}(\mathbb{Z}/q, A) \to \mathrm{Ext}(\mathbb{Z}/q, \Gamma A), \\ &\Delta_A = p_A r(j_A \otimes j_A), \end{aligned}$$

which depends on the choice of the retraction r in (2). Clearly Δ_A is not natural in A since r cannot be chosen to be natural. However, one can easily check that Δ_A is natural for homomorphisms $\varphi : \mathbb{Z}/q \to \mathbb{Z}/t$ between cyclic groups, that is,

(7) $$\Delta_A(\varphi^* \otimes \varphi^*) = \varphi^* \Delta_A.$$

We now define a group

(8) $$G(q, A) = \mathrm{Hom}(\mathbb{Z}/q, A) \times \mathrm{Ext}(\mathbb{Z}/q, \Gamma(A)),$$

where the group law on the right-hand side is given by the *cocycle* Δ_A, that is,

(9) $$(a, b) + (a', b') = (a + a', b + b' + \Delta_A(a \otimes a')).$$

For any abelian group A, by (XII.1.6) of [4] there is an isomorphism

(3.11) $$\varrho : G(q, A) \cong [\Sigma P_q, M(A, 2)]$$

which is natural in $\mathbb{Z}/q$, $q > 1$, and which is compatible with Δ and μ in the proof of (2.2). If A is a direct sum of cyclic groups as above, we obtain maps

$$\overline{\alpha}_i : \Sigma P_{a_i} \to M(A, 2)$$

defined by $\overline{\alpha}_i = \varrho(\alpha_i, 0)$, where $\alpha_i \in \mathrm{Hom}(\mathbb{Z}/a_i, A)$ is the inclusion. These maps yield the homotopy equivalence

$$\bigvee_i \Sigma P_{a_i} \simeq M(A, 2)$$

which we use as an identification. Hence we may assume that ϱ in (3.11) satisfies

(*) $$\varrho(\alpha_i, 0) = \alpha_i,$$

where α_i is the inclusion in (3.7). We need the following function ∇_A, defined for an ordered direct sum A of cyclic groups,

$$\begin{aligned} &\nabla_A : \mathrm{Hom}(\mathbb{Z}/q, A) \to \mathrm{Ext}(\mathbb{Z}/q, \Gamma A), \\ &\nabla_A(x) = \sum_{i<j} \Delta_A(\alpha_i x_i \otimes \alpha_j x_j). \end{aligned} \tag{3.12}$$

Here $x_i \in \mathrm{Hom}(\mathbb{Z}/q, \mathbb{Z}/a_i)$ is the coordinate of $x = \sum_i \alpha_i x_i$. We observe that $\nabla_A = 0$ is trivial if we define Δ_A by $r^<$ in (3.10), where the ordered basis E in $A \otimes \mathbb{Z}/2$ is given by the ordered set of generators in A. Clearly $2\nabla_A(x) = 0$ since $2\Delta_A = 0$. The function ∇_A has the following crucial property:

(3.13) Lemma. *In the group $G(q, A)$ we have the formula*

$$\sum_i{}^{<} x_i^*(\alpha_i, 0) = (x, \nabla_A(x)),$$

where the left-hand side is the ordered sum of the elements $x_i^*(\alpha_i, 0) = (\alpha_i x_i, 0)$ in the group $G(q, A)$.

The lemma is an immediate consequence of the group law (3.10)(9).

For $\varphi \in \mathrm{Hom}(A, B)$ in (3.8) and $q \geq 1$ we define the function

$$\nabla(\varphi) : \mathrm{Hom}(\mathbb{Z}/q, A) \to \mathrm{Ext}(\mathbb{Z}/q, \Gamma(B)) \tag{3.14}$$

via the following commutative diagram, in which $\pi_2(\mathbb{Z}/q, M(A, 2)) = [M(\mathbb{Z}/q, 2), M(A, 2)]$:

$$\begin{array}{ccc}
\pi_2(\mathbb{Z}/q, M(A,2)) & \xrightarrow{(s\varphi)_*} & \pi_2(\mathbb{Z}/q, M(B,2)) \\
\| & & \| \\
G(q, A) & \xrightarrow{(s\varphi)_\sharp} & G(q, B) \\
\| & & \| \\
\mathrm{Hom}(\mathbb{Z}/q, A) \times \mathrm{Ext}(\mathbb{Z}/q, \Gamma A) & & \mathrm{Hom}(\mathbb{Z}/q, B) \times \mathrm{Ext}(\mathbb{Z}/q, \Gamma B)
\end{array}$$

Here the isomorphisms are given as in (3.11). The homomorphism $(s\varphi)_\sharp$, induced by $s\varphi$ in (3.9), determines $\nabla(\varphi)$ by the formula

$$(s\varphi)_\sharp(x, \alpha) = (\varphi_* x, \Gamma(\varphi)_*\alpha + \nabla(\varphi)(x))$$

for $x \in \mathrm{Hom}(\mathbb{Z}/q, A)$ and $\alpha \in \mathrm{Ext}(\mathbb{Z}/q, \Gamma A)$. The function $\nabla(\varphi)$ is not a homomorphism.

(3.15) Lemma. *For $x \in \mathrm{Hom}(\mathbb{Z}/q, A)$ we have*

$$\nabla(\varphi)(x) = \Gamma(\varphi)_*\nabla_A(x) + \sum_i \nabla_B(\varphi\alpha_i x_i) + \sum_{i<t} \Delta_B(\varphi\alpha_i x_i \otimes \varphi\alpha_t x_t).$$

Since all summands are 2-torsion we have $\nabla(\varphi) = 0$ if q is odd.

Proof. For $(\alpha_i, 0) \in G(a_i, A)$ one has the formula

$$(s\varphi)_\sharp(\alpha_i, 0) = \sum_j{}^{<} (\beta_j \varphi_{ji}, 0),$$

as follows from property (3.11)(∗) of the isomorphism χ. Hence by (3.13) we get the following equations:

$$\begin{aligned}
&(s\varphi)_\sharp(x, 0) + (0, \Gamma(\varphi)_* \nabla_A(x)) \\
&= (s\varphi)_\sharp(x, \nabla_A(x)) = (s\varphi)_\sharp\Big(\sum_i{}^{<} x_i^*(\alpha_i, 0)\Big) = \sum_i{}^{<} x_i^*(s\varphi)_\sharp(\alpha_i, 0) \\
&= \sum_i{}^{<} \Big(\sum_j{}^{<} (\beta_j \varphi_{ji} x_i, 0)\Big) = \sum_i{}^{<} (\varphi\alpha_i x_i, \nabla_B(\varphi\alpha_i x_i)).
\end{aligned}$$

Here we have in $G(q, B)$ the equation

$$\sum_i{}^{<} (\varphi\alpha_i x_i, 0) = \Big(\varphi x, \sum_{i<t} \Delta_B(\varphi\alpha_i x_i \otimes \varphi\alpha_t x_t)\Big).$$

This yields the result in (3.15). ∎

We now describe a cocycle δ in the class $\beta_1\beta_2(1_\Gamma)$. For this let A, B, C be ordered direct sums of cyclic groups and consider homomorphisms

(3.16) $$\psi\varphi : A \xrightarrow{\varphi} B \xrightarrow{\psi} C.$$

Let $r_A = r^<$ be the retraction of H in (3.10)(3):

$$\Gamma(A) \otimes \mathbb{Z}/2 \underset{r_A}{\overset{H}{\rightleftarrows}} \otimes^2(A) \otimes \mathbb{Z}/2 \quad \text{(see } S_2(A) \text{ in (3.3))}.$$

Moreover, let s_A be a splitting of σ:

$$\Gamma(A) \otimes \mathbb{Z}/2 \underset{s_A}{\overset{\sigma}{\rightleftarrows}} A \otimes \mathbb{Z}/2 \quad \text{(see } S_1(A) \text{ in (3.3))}$$

defined by

$$s_A\Big(\sum_i x_i\alpha_i \otimes 1\Big) = \sum_i x_i \gamma(\alpha_i) \otimes 1.$$

Here the α_i are the generators of A as in (3.7). We now obtain derivations D_1, D_2 by setting

$$D_2(\psi)q = -\psi_* r_B + \psi^* r_C, \qquad P\,D_1(\varphi) = -\varphi_* s_A + \varphi^* s_B.$$

For this we use the exact sequences $S_i(A)$ in (3.3). We define a 2-cocycle δ which carries (ψ, φ) to the composition

$$\delta(\psi, \varphi) : A \otimes \mathbb{Z}/2 \xrightarrow{D_1(\varphi)} \Lambda^2(B) \otimes \mathbb{Z}/2 \xrightarrow{D_2(\psi)} \Gamma(C) \otimes \mathbb{Z}/2$$

and we observe

(3.17) LEMMA. *We have*

$$\beta_1\beta_2(1_\Gamma) = \{\delta\},$$

where β_1, β_2 *are the Bockstein homomorphisms in* (3.4).

We leave the proof of the lemma as an exercise. The lemma yields a cocycle representing the right-hand side in (3.6).

Next we determine the cocycle Δ_s in (3.9). For this we use the injection

$$g : \mathrm{Ext}(A, \Gamma C) \subset \bigtimes_{q>1} \mathrm{Hom}(\mathrm{Hom}(\mathbb{Z}/q, A), \mathrm{Ext}(\mathbb{Z}/q, \Gamma C)).$$

The element $g\Delta_s(\psi,\varphi)$ is given by the $\mathbb{Z}/q$-natural homomorphism

$$(g\Delta_s(\psi,\varphi))_q : \mathrm{Hom}(\mathbb{Z}/q, A) \to \mathrm{Ext}(\mathbb{Z}/q, \Gamma C)$$

which satisfies

$$(g\Delta_s(\psi,\varphi))_q(x) = \Gamma(\psi)_*\nabla(\varphi)(x) + \nabla(\psi)(\varphi x) - \nabla(\psi\varphi)(x).$$

This equation is an easy consequence of (3.14). As in the remark following (3.12) we may assume that $\nabla_A = \nabla_B = \nabla_C = 0$ are trivial. Moreover, we may assume that q is even since $(g\Delta_s(\psi,\varphi))_q$ is trivial if q is odd. We define a function

$$\varrho_A : A \otimes \mathbb{Z}/2 \to \Lambda^2(A \otimes \mathbb{Z}/2),$$
$$\varrho_A\Big(\sum_i x_i\alpha_i \otimes 1\Big) = \sum_{i<t}(x_i\alpha_i \otimes 1) \wedge (x_t\alpha_t \otimes 1).$$

(3.18) LEMMA. $\nabla(\varphi)(x) = \chi_q\, D_2(\varphi)\varrho_A(x \otimes \mathbb{Z}/2)$.

Here we have $x \in \mathrm{Hom}(\mathbb{Z}/q, A)$ and

$$x \otimes \mathbb{Z}/2 \in \mathrm{Hom}(\mathbb{Z}/q \otimes \mathbb{Z}/2, A \otimes \mathbb{Z}/2) = A \otimes \mathbb{Z}/2$$

since q is even. Moreover, χ_q in Lemma (3.18) is the composition

$$\chi_q : \Gamma(B) \otimes \mathbb{Z}/2 = \mathrm{Ext}(\mathbb{Z}/2, \Gamma B) \to \mathrm{Ext}(\mathbb{Z}/q, \Gamma B)$$

induced by $\mathbb{Z}/q \to \mathbb{Z}/q \otimes \mathbb{Z}/2 = \mathbb{Z}/2$. Lemma (3.18) is a consequence of the formula in (3.15) and the definition of $r_A = r^<$ in (3.10)(3). We apply Lemma (3.18) to the formula for $(g\Delta_s(\psi,\varphi))_q$ above and for $\overline{x} = x \otimes \mathbb{Z}/2$ we get

(3.19) LEMMA. $(g\Delta_s(\psi,\varphi))_q(x) = \chi_q D_2(\psi)(\varrho_B(\varphi\overline{x}) - \varphi_*\varrho_A(\overline{x}))$.

This follows easily from (3.18) since D_1 is a derivation. Finally, we observe:

(3.20) LEMMA. $\varrho_B(\varphi\overline{x}) - \varphi_*\varrho_A(\overline{x}) = D_1(\varphi)(\overline{x})$.

The proof of Lemma (3.20) requires a lengthy computation with the definitions of ϱ_B, ϱ_A and $D_2(\varphi)$. By (3.19) and (3.20) we thus get

$$(g\Delta_s(\psi,\varphi))_q(x) = \chi_q D_2(\psi)D_1(\varphi)(\overline{x}) \tag{3.21}$$

and this yields the formula in (3.6). In fact, (3.21) yields an easy algebraic description of the cocycle Δ_s in terms of the derivations D_1 and D_2 above since g is injective. ∎

4. On the cohomology class {nil} and James–Hopf invariants on $\mathbf{M}^2$. In this section we prove a further formula for the class $\{\mathbf{M}^2\}$, which, however, does not determine $\{\mathbf{M}^2\}$ completely.

For the exterior square $\Lambda^2(B)$ of an abelian group B we have the exact sequence (3.1) which induces the exact sequence

$$\operatorname{Ext}(A, \Gamma B) \xrightarrow{H_*} \operatorname{Ext}(A, \otimes^2 B) \xrightarrow{q_*} \operatorname{Ext}(A, \Lambda^2 B) \to 0$$

and hence we have the binatural short exact sequence

(4.1) $$H_* \operatorname{Ext}(A, \Gamma B) \overset{i}{\rightarrowtail} \operatorname{Ext}(A, \otimes^2 B) \twoheadrightarrow \operatorname{Ext}(A, \Lambda^2 B)$$

together with the surjective map

$$H' : \operatorname{Ext}(A, \Gamma B) \twoheadrightarrow H_* \operatorname{Ext}(A, \Gamma B)$$

induced by H_*. The short exact sequence induces the Bockstein homomorphism

$$\beta : H^1(\mathbf{Ab}, \operatorname{Ext}(-, \Lambda^2)) \to H^2(\mathbf{Ab}, H_* \operatorname{Ext}(-, \Gamma)).$$

(4.2) Theorem. *The algebraic class* $\{\mathrm{nil}\} \in H^1(\mathbf{Ab}, \operatorname{Ext}(-, \Lambda^2))$ *defined below and the class* $\{\mathbf{M}^2\}$ *of the homotopy category of Moore spaces in degree* 2 *satisfy the formula*

$$H'_*\{\mathbf{M}^2\} = \beta\{\mathrm{nil}\} \in H^2(\mathbf{Ab}, H_* \operatorname{Ext}(-, \Gamma)).$$

This result is true in the cohomology of **Ab**. For the algebraic definition of the class {nil} we need the following linear extension **nil**.

(4.3) Definition. Let $\langle Z \rangle$ be the free group generated by the set Z and let $\Gamma_n\langle Z \rangle$ be the subgroup generated by n-fold commutators. Then

(1) $$A = \langle Z \rangle / \Gamma_2 \langle Z \rangle = \bigoplus_Z \mathbb{Z}$$

is the free abelian group generated by Z and

(2) $$E_A = \langle Z \rangle / \Gamma_3 \langle Z \rangle$$

is the *free* nil(2)-*group* generated by Z.

We have the classical central extension of groups

(3) $$\Lambda^2 A \overset{w}{\rightarrowtail} E_A \overset{q}{\twoheadrightarrow} A.$$

The map w is induced by the commutator map with

(4) $$w(qx \wedge qy) = x^{-1} y^{-1} x y.$$

Here the right-hand side denotes the commutator in the group E_A. Using (3) we get the linear extension of categories (compare also [3], [8])

(5) $$\mathrm{Hom}(-,\Lambda^2-)\overset{+}{\rightarrowtail}\mathbf{nil}\overset{\mathrm{ab}}{\twoheadrightarrow}\mathbf{ab}.$$

Here **ab** and **nil** are the full subcategories of the category of groups consisting of free abelian groups and free nil(2)-groups respectively. The functor **ab** in (3) is abelianization and the action $+$ is given by

(6) $$f+\alpha = f + w\alpha q$$

for $f : E_A \to E_B$, $\alpha \in \mathrm{Hom}(A, \Lambda^2 B)$. The right-hand side of (6) is a well defined homomorphism since (3) is central.

(4.4) Definition. We define a derivation

$$\mathrm{nil} : \mathbf{Ab} \to \mathrm{Ext}(-,\Lambda^2)$$

which carries a homomorphism $\varphi : A \to B$ in **Ab** to an element $\mathrm{nil}(\varphi) \in \mathrm{Ext}(A,\Lambda^2 B)$. The cohomology class{nil} represented by the derivation nil is the class used in (4.2). For the definition of nil we choose for each abelian group A a short exact sequence

$$0 \to A_1 \overset{d_A}{\to} A_0 \overset{q}{\to} A \to 0,$$

where A_0, A_1 are free abelian groups. We also choose a homomorphism $\overline{d}_A : E_{A_1} \to E_{A_0}$ between free nil(2)-groups such that the abelianization of $\overline{d}_A$ is d_A. For the homomorphism $\varphi : A \to B$ we choose a commutative diagram in **Ab**

$$\begin{array}{ccccc} A_1 & \xrightarrow{d_A} & A_0 & \xrightarrow{q} & A \\ \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle\varphi} \\ B_1 & \xrightarrow{d_B} & B_0 & \xrightarrow{q} & B \end{array}$$

and we choose a diagram of homomorphisms

$$\begin{array}{ccc} E_{A_1} & \xrightarrow{\bar d_A} & E_{A_0} \\ \downarrow{\scriptstyle\bar\varphi_1} & & \downarrow{\scriptstyle\bar\varphi_0} \\ E_{B_1} & \xrightarrow{\bar d_B} & E_{B_0} \end{array}$$

which by abelianization induces (φ_0, φ_1). This diagram, in general, cannot be chosen to be commutative. Since, however, $\varphi_0 d_A = d_B \varphi_1$ there is a unique element $\alpha \in \mathrm{Hom}(A_1, \Lambda^2 B_0)$ with

$$\overline{\varphi}_0 \overline{d}_A + \alpha = \overline{d}_B \overline{\varphi}_1.$$

Here we use the action in (4.3)(6). Now let

$$\mathrm{nil}(\varphi) \in \mathrm{Ext}(A,\Lambda^2 B) = \mathrm{Hom}(A_1,\Lambda^2 B)/d_A^* \,\mathrm{Hom}(A_0,\Lambda^2 B)$$

be the element represented by the composition

$$(\Lambda^2 q)\alpha : A_1 \to \Lambda^2 B_0 \to \Lambda^2 B.$$

One can check that $\mathrm{nil}(\varphi)$ does not depend on the choice of (φ_0, φ_1) and $(\overline{\varphi}_0, \overline{\varphi}_1)$ and that nil is a derivation, that is, $\mathrm{nil}(\varphi\psi) = \varphi_* \,\mathrm{nil}(\psi) + \psi^* \,\mathrm{nil}(\varphi)$. This completes the definition of the cohomology class $\{\mathrm{nil}\}$.

Next we use the derivation D_1 on $\mathbf{Ab}$ defined as in (3.16). The derivation D_1 carries $\varphi : A \to B$ to

$$D_1(\varphi) \in \mathrm{Hom}(A \otimes \mathbb{Z}/2, \Lambda^2(B) \otimes \mathbb{Z}/2) = \mathrm{Ext}(A \otimes \mathbb{Z}/2, \Lambda^2 B)$$

and hence represents a cohomology class

$$\{D_1\} \in H^1(\mathbf{Ab}, \mathrm{Ext}(- \otimes \mathbb{Z}/2, \Lambda^2)).$$

Let

$$p_2 : \mathrm{Ext}(A \otimes \mathbb{Z}/2, \Lambda^2 B) \to \mathrm{Ext}(A, \Lambda^2 B)$$

be induced by the projection $A \twoheadrightarrow A \otimes \mathbb{Z}/2$.

(4.5) Proposition. *Let* $\mathbf{A}$ *be the full subcategory of* $\mathbf{Ab}$ *consisting of direct sums of cyclic groups. Then*

$$i^*_{\mathbf{A}}(p_2)_*\{D_1\} = i^*_{\mathbf{A}}\{\mathrm{nil}\}$$

in $H^1(\mathbf{A}, \mathrm{Ext}(-, \Lambda^2))$.

We do not know whether this formula also holds if we omit $i^*_{\mathbf{A}}$. Proposition (4.5) implies that the formulas in (4.2) and (3.6) are compatible. For the proof of (4.5) we need the following properties of nil(2)-groups. A group G is a nil(2)-*group* if all triple commutators vanish in G. The commutators in G yield the central homomorphism

(4.6) $$w : \Lambda^2(G^{\mathrm{ab}}) \to G,$$

where $G \to G^{\mathrm{ab}}$, $x \mapsto \{x\}$, is the abelianization of G. We define w by the commutator

$$w(\{x\} \wedge \{y\}) = x^{-1}y^{-1}xy$$

for $x, y \in G$. Let M be a set and let $f : M \to G$ be a function such that only finitely many elements $f(m)$, $m \in M$, are nontrivial and let $<, \ll$ be two total orderings on the set M. Then we have in G the following formula where we write the nonabelian group structure of G additively, the sums are ordered as indicated.

$$\sum_{m\in M}^{\ll} f(m) = \sum_{m\in M}^{<} f(m) + w\Big(\sum_{\substack{m \ll m' \\ m' < m}} \{fm\} \wedge \{fm'\}\Big).$$

For $a \in G$ and $n \in \mathbb{Z}$ let $na = a + \ldots + a$ be the n-fold sum in G in case $n \geq 0$, and let $na = -|n|a$ for $n < 0$. Then one gets in G the formula

$$n \sum_{m \in M}^{<} f(m) = \sum_{m \in M}^{<} nf(m) - w\left(\binom{n}{2} \sum_{m<m'} \{fm\} \wedge \{fm'\}\right),$$

where $\binom{n}{2} = n(n-1)/2$.

Proof of (4.5). Let A and B be direct sums of cyclic groups and let $\varphi : A \to B$ be given by $\varphi_{ji} \in \mathbb{Z}$ as in (3.8). Let A_0 be the free group generated by the set of generators $\{\alpha_i\}$ of A and let A_1 be the free group generated by $\{\alpha_i : a_i \neq 0\}$. Then we choose (see (4.4))

$$\overline{d}_A : E_{A_1} \to E_{A_0}, \quad \overline{d}_A(\alpha_i) = a_i \alpha_i.$$

Similarly we define $\overline{d}_B$. Moreover, we define $\overline{\varphi}_1$ and $\overline{\varphi}_0$ by the ordered sum

$$\overline{\varphi}_0(\alpha_i) = \sum_j^{<} \varphi_{ji}\beta_j \in E_{B_0}, \quad \overline{\varphi}_1(\alpha_i) = \sum_j^{<} (a_i\varphi_{ji}/b_j)\beta_j \in E_{B_1}.$$

Hence we get α in (4.4) by the formula (see (4.6))

$$\begin{aligned}\overline{d}_B\overline{\varphi}_1(\alpha_i) - \overline{\varphi}_0\overline{d}_A(\alpha_i) &= \sum_j^{<} a_i\varphi_{ji}\beta_j - a_i \sum_j^{<} \varphi_{ji}\beta_j \\ &= w\binom{a_i}{2} \sum_{j<t} \{\varphi_{ji}\beta_j\} \wedge \{\varphi_{ti}\beta_t\}.\end{aligned}$$

Hence $\mathrm{nil}(\varphi) \in \mathrm{Ext}(A, \Lambda^2 B)$ is given by the formula ($\alpha_i : \mathbb{Z}/a_i \subset A$ as in (3.7))

$$(\alpha_i)^* \,\mathrm{nil}(\varphi) = \binom{a_i}{2} \sum_{j<t} \varphi_{ji}\varphi_{ti}(1 \otimes \beta_j \wedge \beta_t),$$

where $1 \otimes \beta_j \wedge \beta_t \in \mathbb{Z}/a_i \otimes \Lambda^2 B = \mathrm{Ext}(\mathbb{Z}/a_i, \Lambda^2 B)$. Using the definition of D_1 in the proof of (3.16) it is easy to check that $(\alpha_i)^* p_2 D_1(\varphi)$ coincides with the right-hand side of the formula so that we actually have $\mathrm{nil}(\varphi) = p_2 D_1(\varphi)$. This proves (4.5). ∎

We will need the following element which projects to $\mathrm{nil}(\varphi)$ above.

(4.7) Definition. For φ in the proof above let $\overline{\mathrm{nil}}(\varphi) \in \mathrm{Ext}(A, \otimes^2 B)$ be given by the formula

$$(\alpha_2)^* \overline{\mathrm{nil}}(\varphi) = \binom{a_i}{2} \sum_{j<t} \varphi_{ji}\varphi_{ti}(1 \otimes \beta_j \otimes \beta_t).$$

We clearly have $\mathrm{Ext}(A, p)\overline{\mathrm{nil}}(\varphi) = \mathrm{nil}(\varphi)$ where $p : \otimes^2 B \twoheadrightarrow \Lambda^2 B$ is the projection.

Recall that for the bifunctor $\mathrm{Ext}(-,\otimes^2)$ on **Ab** we have the canonical split linear extension

$$\mathrm{Ext}(-,\otimes^2) \rightarrowtail \mathbf{Ab}\times \mathrm{Ext}(-,\otimes^2) \twoheadrightarrow \mathbf{Ab}.$$

Objects in $\mathbf{Ab}\times \mathrm{Ext}(-,\otimes^2)$ are abelian groups and morphisms $(\varphi,\alpha): A\to B$ are given by $\varphi\in \mathrm{Hom}(A,B)$ and $\alpha\in \mathrm{Ext}(A,\otimes^2 B)$ with composition $(\varphi,\alpha)(\psi,\beta)=(\varphi\psi,\varphi_*\beta+\psi^*\alpha)$. The derivation nil in (4.4) defines a subcategory

(4.8) $$\mathbf{Ab}(\mathrm{nil})\subset \mathbf{Ab}\times \mathrm{Ext}(-,\otimes^2)$$

consisting of all morphisms $(\varphi,\alpha): A\to B$ which satisfy the condition

$$p_*(\alpha)=\mathrm{nil}(\varphi)\in \mathrm{Ext}(A,\Lambda^2 B).$$

Here $p:\otimes^2 B\twoheadrightarrow \Lambda^2 B$ induces $p_*=\mathrm{Ext}(A,p)$. The exact sequence (4.1) shows that we have a commutative diagram of linear extensions of categories

$$\begin{array}{ccccc} H_*\,\mathrm{Ext}(-,\Gamma) & \xrightarrow{+} & \mathbf{Ab}(\mathrm{nil}) & \longrightarrow & \mathbf{Ab} \\ \cap & & \cap & & \| \\ \mathrm{Ext}(-,\otimes^2) & \xrightarrow{+} & \mathbf{Ab}\times \mathrm{Ext}(-,\otimes^2) & \longrightarrow & \mathbf{Ab} \end{array}$$

(4.9) Lemma. *The cohomology class represented by the linear extension for* **Ab**(nil) *satisfies*

$$\{\mathbf{Ab}(\mathrm{nil})\}=\beta\{\mathrm{nil}\}\in H^2(\mathbf{Ab},H_*\,\mathrm{Ext}(-,\Gamma)),$$

where β *is the Bockstein operator in* (4.2).

Proof. Let $s:\mathrm{Ext}(A,\Lambda^2B)\to \mathrm{Ext}(A,\otimes^2 B)$ be a set-theoretic splitting of $\mathrm{Ext}(A,p)=p_*$. Then $\beta\{\mathrm{nil}\}$ is represented by the 2-cocycle $c=i^{-1}\delta(s\,\mathrm{nil})$, where i is the inclusion in (4.1) and where δ is the coboundary in (1.5). Hence c carries the 2-simplex (ψ,φ) in **Ab** to

$$c(\psi,\varphi)=i^{-1}(\psi_* s\,\mathrm{nil}(\varphi)-s\,\mathrm{nil}(\psi\varphi)+\varphi^* s\,\mathrm{nil}(\psi)).$$

On the other hand, we define a set-theoretic section t for the linear extension **Ab**(nil) by $t(\varphi)=(\varphi,s\,\mathrm{nil}(\varphi))$. Then Δ_t in (1.4) is given by

$$s\,\mathrm{nil}(\psi\varphi)=\psi_* s\,\mathrm{nil}(\varphi)+\varphi^* s\,\mathrm{nil}(\psi)+i\Delta_t(\psi,\varphi).$$

Hence $c=-\Delta_t$ yields the lemma. In fact, since the elements in (4.9) are of order 2 we can omit the sign. ■

For Moore spaces $M(A,2)=\Sigma M_A$ and $M(B,2)=\Sigma M_B$ as in (2.2) we have the *James–Hopf invariant* ([12], [5])

(4.10) $$[\Sigma M_A,\Sigma M_B]\xrightarrow{\gamma_2}[\Sigma M_A,\Sigma M_B\wedge M_B]=\mathrm{Ext}(A,B\otimes B),$$

which satisfies for $\alpha\in \mathrm{Ext}(A,\Gamma B)$ the formula

(1) $$\lambda_2(\xi+\alpha)=\lambda_2(\xi)+H_*\alpha.$$

Hence γ_2 induces a well defined function

$$\overline{\gamma}_2 : \mathrm{Hom}(A, B) \to \mathrm{Ext}(A, \Lambda^2 B) \tag{2}$$

defined by $\overline{\gamma}_2(\varphi) = q_* \gamma_2(\xi)$ where ξ induces $H_2(\xi) = \varphi : A \to B$. One can check that $\overline{\gamma}_2$ is a derivation which represents a cohomology class in $H^1(\mathbf{Ab}, \mathrm{Ext}(-, \Lambda^2 B))$. This cohomology class does not depend on the choice of M_A, M_B above.

(4.11) THEOREM. *The cohomology class* $\{\overline{\gamma}_2\}$ *given by the James–Hopf invariant* γ_2 *coincides with*

$$\{\overline{\gamma}_2\} = \{\mathrm{nil}\} \in H^1(\mathbf{Ab}, \mathrm{Ext}(-, \Lambda^2)).$$

Moreover, there is a full functor τ,

$$\mathbf{M}^2 \overset{\tau}{\twoheadrightarrow} \mathbf{Ab}(\mathrm{nil}) \overset{i}{\subset} \mathbf{Ab} \times \mathrm{Ext}(-, \otimes^2),$$

which is the identity on objects and which is defined on morphisms by

$$\tau(\xi) = (H_2 \xi, \gamma_2 \xi).$$

The functor τ *is part of the following commutative diagram of linear extensions*:

$$\begin{array}{ccccc}
\mathrm{Ext}(-, \Gamma) & \xrightarrow{+} & \mathbf{M}^2 & \xrightarrow{H_2} & \mathbf{Ab} \\
\downarrow{\scriptstyle H'} & & \downarrow{\scriptstyle \tau} & & \| \\
H_* \mathrm{Ext}(-, \Gamma) & \xrightarrow{+} & \mathbf{Ab}(\mathrm{nil}) & \longrightarrow & \mathbf{Ab}
\end{array}$$

Proof of (4.2). The existence of the functor τ shows that $H'_*\{\mathbf{M}^2\} = \{\mathbf{Ab}(\mathrm{nil})\}$. Therefore we obtain (4.2) by (4.9). ∎

(4.12) Remark. We can give an alternative description of the functor τ in (4.11) by use of the singular chain complex of a loop space which yields the *Adams–Hilton functor*

$$C_* \Omega : \mathrm{Ho}(\mathbf{Top}^*) \to \mathrm{Ho}(\mathbf{DA})$$

between homotopy categories (compare [1] and also [2]). Here $\mathbf{DA}$ is the category of $\mathbb{Z}$-chain algebras. The functor $C_* \Omega$ restricted to $\mathbf{M}^2$ leads to the following diagram where $\widetilde{\mathbf{M}}^2 \subset \mathrm{Ho}(\mathbf{DA})$ is the full subcategory consisting of $C_* \Omega M(A, 2)$, $A \in \mathbf{Ab}$:

$$\begin{array}{ccc}
\mathbf{M}^2 & \xrightarrow{C_* \Omega} & \widetilde{\mathbf{M}}^2 \subset \mathrm{Ho}(\mathbf{DA}) \\
\downarrow{\scriptstyle \tau} & & \uparrow{\scriptstyle j} \wr \\
\mathbf{Ab}(\mathrm{nil}) & \xrightarrow{i} & \mathbf{Ab} \times \mathrm{Ext}(-, \otimes^2)
\end{array}$$

where j is an equivalence of categories such that $ji\tau$ is naturally isomorphic to $C_*\Omega$.

Proof of (4.11). The image category of the functor

$$\tau : \mathbf{M}^2 \to \mathbf{Ab} \times \mathrm{Ext}(-, \otimes^2)$$

is $\mathbf{Ab}(\mathrm{nil})$ since we show

$$\overline{\gamma}_2 = \mathrm{nil} \tag{1}$$

for compatible choices of $\overline{d}_A, d'_A$ in (4.4) and (2.2). We use the equivalence of linear track extension described in (VI.4.7) of [3]. This shows that a triple $(\overline{\varphi}_0, \overline{\varphi}_1, G)$ with $G \in \mathrm{Hom}(A_1, \otimes^2 B_0)$ satisfying $p_*G = \alpha$ (see (4.4)) corresponds to a diagram

$$\begin{array}{ccc} \Sigma M_{A_1} & \xrightarrow{\Sigma d'_A} & \Sigma M_{A_0} \\ {\scriptstyle \Sigma\varphi'_1}\downarrow & \overset{G'}{\Longrightarrow} & \downarrow{\scriptstyle \Sigma\varphi'_0} \\ \Sigma M_{B_1} & \xrightarrow[\Sigma d'_B]{} & \Sigma M_{B_0} \end{array} \tag{2}$$

Here d'_A and d'_B induce $\overline{d}_A$ and $\overline{d}_B$ respectively and φ'_0, φ'_1 induces $\overline{\varphi}_0$, $\overline{\varphi}_1$ in (4.4). The track G' is determined by G. This track determines a principal map $\overline{\varphi} \in [\Sigma M_A, \Sigma M_B]$ such that $\tau(\overline{\varphi}) = (\varphi, (\otimes^2 q)_*\{G\})$, where $\{G\} \in \mathrm{Ext}(A, \otimes^2 B_0)$ is represented by G. This follows from the bijection (6)–(11) in (VI.4.7) of [3]. Since $p_*G = \alpha$ we get $\overline{\gamma}_2 = \mathrm{nil}$. ■

(4.13) Example. Let A and B be direct sums of cyclic groups as in (3.8) and let $s\varphi \in [M(A,2), M(B,2)]$ be defined as in (3.9). Then the functor τ in (4.11) satisfies

$$\tau(s\varphi) = (\varphi, \overline{\mathrm{nil}}(\varphi)),$$

where $\overline{\mathrm{nil}}(\varphi)$ is defined in (4.7). We obtain this formula by the methods in the proof of (4.11) above. In this case we can also compute the James–Hopf invariant $\gamma_2(s\varphi)$, which actually is $\gamma_2(s\varphi) = \overline{\mathrm{nil}}(\varphi)$.

As a corollary of (4.2) we get:

(4.14) Proposition. *$\{\mathbf{M}^2\}$ is a (nontrivial) element of order 2.*

Proof. We know that multiplication by 2 on $\Gamma(A)$ is the composition

$$2 = PH : \Gamma A \to \otimes^2 A \to \Gamma A,$$

where $P = [1,1]$. Hence also the composition

$$\begin{array}{ccccc} \mathrm{Ext}(A, \Gamma B) & \xrightarrow{H'} & H_* \mathrm{Ext}(A, \Gamma B) & \xrightarrow{P'} & \mathrm{Ext}(A, \Gamma B) \\ \| & & \cap & & \| \\ \mathrm{Ext}(A, \Gamma B) & \xrightarrow{H_*} & \mathrm{Ext}(A, \otimes^2 B) & \xrightarrow{P_*} & \mathrm{Ext}(A, \Gamma B) \end{array}$$

is a multiplication by 2. Therefore by (4.2) we get

$$2\{\mathbf{M}^2\} = (P'\,H')_*\{\mathbf{M}^2\} = P'_*\,H'_*\{\mathbf{M}^2\} = P'_*\,\beta\,\{\mathrm{nil}\}.$$

Here the commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_*\,\mathrm{Ext}(A,\Gamma B) & \longrightarrow & \mathrm{Ext}(A,\otimes^2 B) & \longrightarrow & \mathrm{Ext}(A,\Lambda^2 B) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle P'} & & \downarrow{\scriptstyle P_*} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Ext}(A,\Gamma B) & \longrightarrow & \mathrm{Ext}(A,\Gamma B) & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

shows that $P'_*\beta = 0$. ∎

(4.15) PROPOSITION. *Each element in* $H^1(\mathbf{Ab}, \mathrm{Ext}(-,\Lambda^2))$ *is of order* 2, *in particular*, $2\{\mathrm{nil}\} = 0$.

Proof. Let A, B be abelian groups and let $\varphi \in \mathrm{Hom}(A,B)$. Let $2_A = 2\,\mathrm{id} \in \mathrm{Hom}(A,A)$ be multiplication by 2. Then we have

$$\varphi \circ 2_A = 2\varphi = 2_B \circ \varphi.$$

Now the derivation property of N with $\{N\} \in H^1(\mathbf{Ab}, \mathrm{Ext}(-,\Lambda^2))$ shows

$$N(\varphi \circ 2_A) = \varphi_* N(2_A) + (2_A)^* N(\varphi) = \varphi_*\, N(2_A) + 2N(\varphi),$$
$$N(2_B \circ \varphi) = (2_B)_* N(\varphi) + \varphi^* N(2_B) = 4N(\varphi) + \varphi^* N(2_B).$$

Hence we get $2N(\varphi) = \varphi_* N(2_A) - \varphi^* N(2_B)$, so that $2N$ is an inner derivation. ∎

5. A subcategory of $\mathbf{M}^2$ given by diagonal elements.

Let $\mathbb{Z}/2 * A$ be the 2-torsion of the abelian group A. We construct a subcategory $\mathbf{H}$ of the category of Moore spaces $\mathbf{M}^2$ with the following property.

(5.1) THEOREM. *There exists a subcategory* $\mathbf{H}$ *of* $\mathbf{M}^2$ *together with a commutative diagram of linear extensions*

$$\begin{array}{ccccc}
\mathbb{Z}/2 * \mathrm{Ext}(-,\Gamma) & \xrightarrow{\;+\;} & \mathbf{H} & \longrightarrow & \mathbf{Ab} \\
\cap & & \cap & & \| \\
\mathrm{Ext}(-,\Gamma) & \xrightarrow{\;+\;} & \mathbf{M}^2 & \longrightarrow & \mathbf{Ab}
\end{array}$$

The theorem shows that the class $\{\mathbf{M}^2\}$ is in the image of

$$i_* : H^2(\mathbf{Ab}, \mathbb{Z}/2 * \mathrm{Ext}(-,\Gamma)) \to H^2(\mathbf{Ab}, \mathrm{Ext}(-,\Gamma)),$$

where i is the inclusion $\mathbb{Z}/2 * \mathrm{Ext}(A,\Gamma(B)) \subset \mathrm{Ext}(A,\Gamma(B))$.

(5.2) COROLLARY. *The extension* $\mathbf{M}^2 \to \mathbf{Ab}$ *is split on any full subcategory of* $\mathbf{Ab}$ *consisting of objects* A, B *with* $(\mathbb{Z}/2) * \mathrm{Ext}(A,\Gamma B) = 0$.

(5.3) COROLLARY. *Let* A *be any abelian group for which the* 2*-torsion of* $\mathrm{Ext}(A,\Gamma A)$ *is trivial. Then the group of homotopy equivalences of* $M(A,2)$

is given by the split extension

$$\operatorname{Ext}(A, \Gamma A) \rightarrowtail \mathfrak{E}(M(A,2)) \twoheadrightarrow \operatorname{Aut}(A),$$

where $\varphi \in \operatorname{Aut}(A)$ *acts on* $a \in \operatorname{Ext}(A, \Gamma A)$ *by* $\varphi \cdot a = (\Gamma\varphi)_*(\varphi^{-1})^*(a)$.

Proof of (5.1). For a Moore space $M(A,2) = \Sigma M_A$ we have the *diagonal element*

(1) $$\Delta_A \in [\Sigma M_A, \Sigma M_A \wedge M_A] = \operatorname{Ext}(A, A \otimes A)$$

which is given by the suspension of the reduced diagonal $M_A \to M_A \wedge M_A$. Let $[1_A, 1_A] : \Sigma M_A \wedge M_A \to \Sigma M_A$ be the Whitehead product for the identity 1_A of ΣM_A. Then

(2) $$[1_A, 1_A]\Delta_A = -1_A - 1_A + 1_A + 1_A = 0$$

is the trivial commutator. This implies that also

(3) $$\Delta_A \in \operatorname{Ker}\{[1,1]_* : \operatorname{Ext}(A, A \otimes A) \to \operatorname{Ext}(A, \Gamma A)\}$$

with $[1,1]$ in (3.2). We have the short exact sequences (see (3.3))

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Ext}(A, \Gamma(A) \otimes \mathbb{Z}/2) & \xrightarrow{H_*} & \operatorname{Ext}(A, \otimes^2(A) \otimes \mathbb{Z}/2) & \xrightarrow{q_*} & \operatorname{Ext}(A, \Lambda^2(A) \otimes \mathbb{Z}/2) & \to & 0 \\ & & & & {\scriptstyle [1,1]_*}\searrow & & \swarrow & & \\ & & & & & \operatorname{Ext}(A\Gamma(A) \otimes \mathbb{Z}/2) & & & \end{array}$$

which shows by (3) that for the projection $p : \otimes^2 A \to (\otimes^2 A) \otimes \mathbb{Z}/2$ there is a unique element $\Delta'_A \in \operatorname{Ext}(A, \Gamma(A) \otimes \mathbb{Z}/2)$ with

(4) $$H_*\Delta'_A = p_*\Delta_A.$$

Now, using the surjection $p_* : \operatorname{Ext}(A, \Gamma A) \to \operatorname{Ext}(A, \Gamma(A) \otimes \mathbb{Z}/2)$, we choose an element $\Delta''_A \in \operatorname{Ext}(A, \Gamma A)$ with

(5) $$p_*\Delta''_A = \Delta'_A.$$

We call Δ''_A a *diagonal structure* for A. For the definition of the subcategory $\mathbf{H}$ in $\mathbf{M}^2$ we choose such a diagonal structure for each abelian group A in $\mathbf{Ab}$. We define the set of morphisms in $\mathbf{H}$ with

(6) $$\mathbf{H}(A, B) \subset [\Sigma M_A, \Sigma M_B]$$

by the composition (compare (4.10))

$$[\Sigma M_A, \Sigma M_B] \xrightarrow{\gamma_2} \operatorname{Ext}(A, B \otimes B) \xrightarrow{[1,1]_*} \operatorname{Ext}(A, \Gamma B),$$

and by the diagonal structures Δ''_A, Δ''_B, namely

(7) $$\overline{\varphi} \in \mathbf{H}(A, B) \Leftrightarrow [1,1]_*\gamma_2\overline{\varphi} = -\varphi_*\Delta''_A + \varphi^*\Delta''_B.$$

We show that for $\overline{\varphi} \in \mathbf{H}(A, B)$ and $\overline{\psi} \in \mathbf{H}(B, C)$ we actually have $\overline{\psi}\overline{\varphi} \in \mathbf{H}(A, C)$ so that $\mathbf{H}$ is a well defined subcategory of $\mathbf{M}^2$. For this we need the fact that γ_2 is a derivation, that is,

$$\gamma_2(\overline{\psi}\overline{\varphi}) = \psi_*\gamma_2(\overline{\varphi}) + \varphi^*\gamma_2(\overline{\varphi}).$$

Hence we get

$$\begin{aligned}[1,1]_*\gamma_2(\overline{\psi}\overline{\varphi}) &= [1,1]_*(\psi_*\gamma_2(\overline{\varphi}) + \varphi^*\gamma_2(\overline{\psi})) \\ &= \psi_*[1,1]_*\gamma_2(\overline{\varphi}) + \varphi^*[1,1]_*\gamma_2(\overline{\varphi}) \\ &= \psi_*(-\varphi_*\Delta''_A + \varphi^*\Delta''_B) + \varphi^*(-\psi_*\Delta''_B + \psi^*\Delta''_C) \\ &= -(\psi\varphi)_*\Delta''_A + (\psi\varphi)^*\Delta''_C.\end{aligned}$$

The crucial observation needed for the proof of Theorem (5.1) is the following equation where we use the interchange map $T : B\otimes B \to B\otimes B$ with $T(x\otimes y) = y\otimes x$:

$$(1-T)_*\gamma_2(\overline{\varphi}) = \varphi_*\Delta_A - \varphi^*\Delta_B. \tag{8}$$

This equation follows from the corresponding known property of James–Hopf invariants (Appendix A of [4]) with respect to "cup products" which in our case has the form

$$\overline{\varphi}\cup\overline{\varphi} = \Delta_{1,1}\overline{\varphi} + (1+T_{2,1})\gamma_2(\overline{\varphi}).$$

This equation is equivalent to (8). We now consider the following commutative diagram.

$$\begin{array}{ccccc} \operatorname{Ext}(A,\Gamma B) & = & \operatorname{Ext}(A,\Gamma B) & = & \operatorname{Ext}(A,\Gamma B) \\ \downarrow + & & \downarrow H_* & & \downarrow \cdot 2 \\ [\Sigma M_A,\Sigma M_B] & \xrightarrow{\gamma_2} & \operatorname{Ext}(A,B\otimes B) & \xrightarrow{[1,1]_*} & \operatorname{Ext}(A,\Gamma B) \\ \downarrow \mu & & \downarrow & & \downarrow \\ \operatorname{Hom}(A,B) & \xrightarrow{\bar\gamma_2} & \operatorname{Ext}(A,\Lambda^2 B) & \xrightarrow{[1,1]_*} & \operatorname{Ext}(A,\Gamma(B)\otimes\mathbb{Z}/2) \end{array}$$

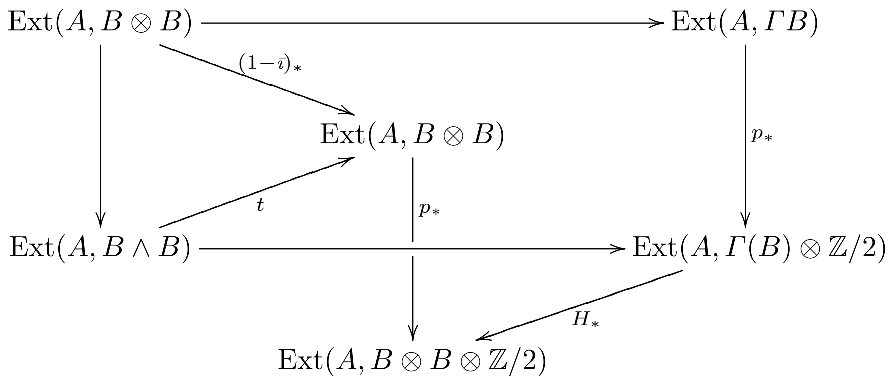
The columns are exact sequences. Here γ_2 is not a homomorphism; since, however, (4.10)(1) holds we see that the induced function $\overline{\gamma}_2$ is well defined. Moreover, we use $[1,1]H = \cdot 2$ so that $[1,1]_*$ in the bottom row is well defined.

We now claim that (8) implies the formula

$$[1,1]_*\overline{\gamma}_2(\varphi) = -\varphi_*\Delta'_A + \varphi^*\Delta'_B. \tag{9}$$

By the diagram above this shows that for any $\varphi\in\operatorname{Hom}(A,B)$ there is an element $\overline{\varphi}$ which satisfies the condition in (7). Thus the functor $\underline{\underline{H}} \to \mathbf{Ab}$ is full, moreover the diagram above shows that $\mathbf{H}$ is part of a linear extension as described in the theorem. In fact, for $\overline{\varphi}\in\mathbf{H}(A,B)$ we have $\overline{\varphi}+\alpha\in\mathbf{H}(A,B)$ if and only if $2\alpha = 0$.

It remains to prove (9). For this consider the commutative diagram



The square in this diagram coincides with the corresponding square in the diagram above. Since for $x \otimes y \in B \otimes B$,

$$H[1,1](x \otimes y) = x \otimes y + y \otimes x \equiv x \otimes y - y \otimes x \bmod 2,$$

we see that the diagram commutes. The homomorphism t is induced by $1 - T$. On the other hand, H_* in the diagram is injective. This shows that (9) holds by the following equations:

$$\begin{aligned} H_*[1,1]_*\overline{\gamma}_2(\varphi) &= H_* p_* [1,1]_* \gamma_2 \overline{\varphi} = p_*(1-T)_*\gamma_2\overline{\varphi} \\ &= p_*(\varphi_*\Delta_A - \varphi^*\Delta_B) = \varphi_*(p_*\Delta_A) - \varphi^*(p_*\Delta_B) \\ &= \varphi_*(H_*\Delta'_A) - \varphi^*(H_*\Delta'_B) = H_*(\varphi_*\Delta'_A - \varphi^*\Delta'_B). \end{aligned}$$

This completes the proof of Theorem (5.1). ∎

Formula (9) in the proof of (5.1) above and (1) in the proof of (4.11) show

$$[1,1]_* \operatorname{nil}(\varphi) = [1,1]_* \overline{\varphi}_2(\varphi) = -\varphi_*\Delta'_A + \varphi^*\Delta'_B.$$

Hence the composition $[1,1]_* \operatorname{nil}$ with

$$[1,1]_* : \operatorname{Ext}(A, \Lambda^2 B) \to \operatorname{Ext}(A, \Gamma B \otimes \mathbb{Z}/2)$$

is an inner derivation. This implies

(5.4) Proposition. *We have*

$$[1,1]_*\{\operatorname{nil}\} = 0$$

in $H^1(\mathbf{Ab}, \operatorname{Ext}(-, \mathbb{Z}/2 \otimes \Gamma))$.

References

[1] J. F. Adams and P. J. Hilton, *On the chain algebra of a loop space*, Comment. Math. Helv. 30 (1956), 305–330.

[2] H.-J. Baues, *Algebraic Homotopy*, Cambridge Stud. Adv. Math. 15, Cambridge University Press, 1988.
[3] —, *Combinatorial Homotopy and 4-Dimensional Complexes*, de Gruyter, Berlin, 1991.
[4] —, *Homotopy Type and Homology*, Oxford Math. Monograph, Oxford University Press, 1996.
[5] —, *Commutator Calculus and Groups of Homotopy Classes*, London Math. Soc. Lecture Note Ser. 50, Cambridge University Press, 1981.
[6] —, *Homotopy types*, in: Handbook of Algebraic Topology, Chapter I, I. M. James (ed.), Elsevier, 1995, 1–72.
[7] —, *On the cohomology of categories, universal Toda brackets, and homotopy pairs*, K-Theory, to appear.
[8] H.-J. Baues and W. Dreckmann, *The cohomology of homotopy categories and the general linear group*, ibid. 3 (1989), 307–338.
[9] H.-J. Baues and G. Wirsching, *The cohomology of small categories*, J. Pure Appl. Algebra 38 (1985), 187–211.
[10] K. A. Hardie, *On the category of homotopy pairs*, Topology Appl. 14 (1982), 59–69.
[11] P. Hilton, *Homotopy Theory and Duality*, Gordon and Breach, 1965.
[12] I. M. James, *Reduced product spaces*, Ann. of Math. 62 (1955), 170–197.
[13] M. Jibladze and T. Pirashvili, *Cohomology of algebraic theories*, J. Algebra 137 (1991), 253–296.
[14] T. Pirashvili and F. Waldhausen, *MacLane homology and topological Hochschild homology*, J. Pure Appl. Algebra 82 (1992), 81–98.
[15] J. H. C. Whitehead, *A certain exact sequence*, Ann. of Math. 52 (1950), 51–110.

Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26
D-53225 Bonn, Germany
E-mail: baues@mpim-bonn.mpg.de

Département de Mathématiques
Université de Valenciennes
B.P. 311
59304 Valenciennes, France
E-mail: hartl@univ-valenciennes.fr

Received 11 September 1995;
in revised form 29 January 1996