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The σ-ideal of closed smooth sets does not have the covering property

by

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Abstract. We prove that the σ-ideal $I(E)$ (of closed smooth sets with respect to a non-smooth Borel equivalence relation E) does not have the covering property. In fact, the same holds for any σ-ideal containing the closed transversals with respect to an equivalence relation generated by a countable group of homeomorphisms. As a consequence we show that $I(E)$ does not have a Borel basis.

1. Introduction. A σ-ideal I of compact subsets of a Polish space X is said to have the *covering property* (see [4, 11], and the notion of I-regularity of [8]) if for every analytic set $A \subseteq X$ such that every closed subset of A is in I there are countably many closed sets F_n in I such that $A \subseteq \bigcup_n F_n$. The covering property for σ-ideals of compact sets is an abstraction of the classical perfect set theorem for analytic sets. In fact, when I is the collection of closed countable subsets of 2^ω (or any compact metric space), the classical perfect set theorem says that I has the covering property. Besides this example, we only know one more non-trivial σ-ideal that has the covering property, namely, the σ-ideal of closed sets of extended uniqueness in the unit circle ([6]). In this article we will be concerned with the σ-ideal of closed smooth sets with respect to a Borel equivalence relation (the definition appears in §3). We will show that it does not have the covering property. Smooth sets appear in the study of Borel equivalence relations ([3]) and are a generalization of the wandering sets studied in ergodic theory ([13]).

It follows from the results in [10] that it suffices to have the covering property for G_δ sets in order to have it for analytic sets. Thus, by a standard argument with the Baire category theorem, the covering property is

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equivalent to the following: Let H be a G_δ set; if for every open set V with $V \cap H \neq \emptyset$ the closure $\overline{V \cap H}$ is not in I, then there is $K \subseteq H$ with K not in I. So, in order to violate the covering property one needs to show that there is a "small" G_δ set (i.e., with every closed subset belonging to I) with "large" closure (i.e., locally not in I). Given an equivalence relation E on X, a subset A of X is called an *E-transversal* (or just a transversal) if for all $x, y \in A$ if $x \neq y$ then $x \not\!E y$. Borel transversals are typical smooth sets. We will construct G_δ transversals with locally non-smooth closure. In §2 we present the basic construction which is similar to the non-Borel basis lemma of [7]. Using this result, in §3 we show that if J is a σ-ideal containing the closed transversals with respect to the equivalence relation generated by a countable group of homeomorphisms of X then J does not have the covering property.

The notion of calibration was introduced in [7] and used there to characterize when a σ-ideal of compact sets can be extended to a σ-ideal of G_δ sets. In fact, given a σ-ideal I of compact sets let $I_{\rm int}$ be the collection of subsets A of X such that every closed subset of A belongs to I. Then I is calibrated iff the collection of G_δ sets in $I_{\rm int}$ forms a σ-ideal ([7]). The σ-ideals defined by measures or capacities are calibrated, but the σ-ideal of meager closed sets is not calibrated. There are not many examples of not calibrated σ-ideals. In fact, calibration is considered a very mild requirement. It is easy to see that calibration follows from the covering property. In §3 we also present some results about $I_{\rm t}(E)$, the σ-ideal of closed sets generated by the collection of closed transversals (where E is a Borel equivalence relation on X). In particular, we show that $I_{\rm t}(E)$ is a $\mathit{\Pi}^1_1$, locally non-Borel and not calibrated σ-ideal. In other words, for $I_{\rm t}(E)$ the failure of the covering property is as strong as it can be.

Our notation is standard as in [9, 5] and concerning σ-ideals we refer the readers to [7] and the references therein. X will always be a compact metric space. We now recall some basic definitions and facts. Analytic sets, denoted by $\boldsymbol{\Sigma}^1_1$, are continuous images of Borel sets. Co-analytic sets, denoted by $\boldsymbol{\Pi}^1_1$, are the complements of analytic sets. The collection of closed subsets of X, which is denoted by $\mathcal{K}(X)$, equipped with the Hausdorff distance is a Polish space. All the notions such as open sets, Borel sets, analytic sets, etc., in $\mathcal{K}(X)$ will refer to the Hausdorff metric (for more details about the topology and descriptive set theory of $\mathcal{K}(X)$ see [5]). A collection I of closed sets is a *σ-ideal* if the following two properties hold: (1) If $K_n \in I$ for all $n \in \omega$ and $K = \bigcup_n K_n$ is closed then $K \in I$. (2) I is *hereditary*, i.e., if $K \in I$ and $F \subseteq K$ is closed then $F \in I$. A $\boldsymbol{\Pi}^1_1$ σ-ideal I satisfies the so-called *dichotomy theorem* ([7]), namely either I is a true $\boldsymbol{\Pi}^1_1$ subset of $\mathcal{K}(X)$ or a G_δ subset. Even more, every $\boldsymbol{\Sigma}^1_1$ σ-ideal is in fact G_δ ([7]). By $I_{\rm int}$ we denote the collection of all subsets A of X such that every closed subset of

A is in I. A σ-ideal I is *calibrated* if whenever for a closed set $F \subseteq X$ there is a sequence (F_n) of closed sets in I with $F - \bigcup_n F_n \in I_{\text{int}}$ then $F \in I$. We say that $B \subset I$ is a *basis* for I if B is hereditary and $I = B_\sigma$, i.e., every $K \in I$ is a countable union of sets in B. We say that I has a *Borel basis* if there is a Borel subset of $\mathcal{K}(X)$ which is a basis for I. I is called *locally non-Borel* if for every closed set $F \notin I$, $I \cap \mathcal{K}(F)$ is not Borel. We say that I is *thin* if every collection of disjoint closed sets not in I is at most countable. These notions were introduced in [7]. A very important criterion known to imply the covering property is the following theorem due to Debs and Saint Raymond ([2]): If I is a calibrated, locally non-Borel $\boldsymbol{\Pi}^1_1$ σ-ideal with a Borel basis then I has the covering property. A proof of this result can be found in [6], p. 208. $\omega^{<\omega}$ denotes the collection of finite sequences of natural numbers. If $s \in \omega^{<\omega}$ and $n \in \mathbb{N}$ then $s^\wedge(n)$ is the concatenation of s with n. 2^ω is the Cantor space with the usual product topology.

2. Basic construction. We say that an equivalence relation E on X is generated by a countable collection (g_n) of homeomorphisms of X if for every x and y in X, xEy if and only if there is n with $g_n(x) = y$. For such equivalence relations we will construct G_δ E-transversals with locally non-smooth closure. The construction given here (Lemma 2.6) is similar to that in §2, Lemma 7 of [7], but we will follow the proof given in [6] (Lemma 7, p. 203); some familiarity with the latter will be helpful.

The following definitions capture what is needed from the construction.

Definition 2.1. Let O be a subset of $X \times X$ and $K \subseteq X$. Put $[K]_O = \{y \in X : \exists z \in K,\ (y,z) \notin O\}$. We will say that a collection (O_n) of subsets of $X \times X$ satisfies $(*)$ if the following conditions hold.

(1) For every n, O_n is symmetric, and $(x,x) \notin O_n$ for all $x \in X$.

(2) For every n, every closed nowhere dense subset $K \subseteq X$ and every non-empty open V in X, there is $y \in V$ such that for all $x \in K$ we have $(y,x) \in O_n$. This is equivalent to saying that for every n, $[K]_{O_n}$ is a meager set.

Remark. The motivation behind the previous definition is the following: Suppose g is a homeomorphism of X and let $O = \{(y,x) : g(x) \neq y \ \&\ g(y) \neq x \ \&\ x \neq y\}$. Then $[K]_O = g[K] \cup g^{-1}[K] \cup K$, which is obviously meager if K is meager. For the case where E is an equivalence relation generated by a collection (g_n) of homeomorphisms of X and O_n is defined as before, A is an E-transversal if and only if for every $x, y \in A$ with $x \neq y$ we have $(x,y) \in \bigcap_n O_n$.

Definition 2.2. Let $\mathcal{F}$ be a collection of closed subsets of X. We will say that a set A is *locally not in* $\mathcal{F}$ (or *$\mathcal{F}$-perfect*) if for every open set V with $V \cap A \neq \emptyset$ there is a non-empty $K \notin \mathcal{F}$ such that $K \subseteq V \cap A$.

Remark. When I is a σ-ideal of closed sets, a subset A of X is said to be *I-perfect* if for all V open with $V \cap A \neq \emptyset$ we have $\overline{V \cap A} \notin I$. For $\mathcal{F} = I$, A is I-perfect if and only if A is locally not in $\mathcal{F}$. When $\mathcal{F}$ is just a hereditary family of closed sets then the usual argument with the Baire category theorem shows that if F is a closed (or even G_δ) set locally not in $\mathcal{F}$ then F cannot be covered by a countable collection of closed sets in $\mathcal{F}$.

We will need the following lemma [6].

LEMMA 2.3. *Let K be a closed nowhere dense set, $D \subseteq X$ a dense set and V an open set with $K \subseteq V$. Then there is a countable discrete set $D_K \subseteq D \cap V$ with $D_K \cap K = \emptyset$ such that $\overline{D}_K = D_K \cup K$.*

Proof. See [6], Lemma 5, p. 202. ∎

LEMMA 2.4. *Let O be an open subset of $X \times X$ such that $(x, x) \notin O$ for every $x \in X$ and L be a closed subset of X such that $[L]_O$ is meager. Let U' and W be open subsets of X with $L \subseteq U'$. Then there are non-empty open subsets V and U such that $\overline{V} \subseteq W$, $\overline{U} \subseteq U'$, $L \subset U$, $\overline{V} \cap \overline{U} = \emptyset$ and $V \times U \subseteq O$.*

Proof. Let $y \in W - [L]_O$, i.e., $\{y\} \times L \subset O$. For every $x \in L$ there are open sets U_x and V_x such that $x \in U_x$, $y \in V_x$, $U_x \cap V_x = \emptyset$ and $V_x \times U_x \subseteq O$. By a standard compactness argument we can find $x_1, \ldots, x_n$ in L such that $U = U_{x_1} \cup \ldots \cup U_{x_n}$ and $V = V_{x_1} \cap \ldots \cap V_{x_n}$ satisfy the conclusion of the lemma. ∎

LEMMA 2.5. *Let O be a symmetric open subset of $X \times X$ such that for every $x \in X$ we have $(x, x) \notin O$ and K be a nowhere dense set such that $[K]_O$ is meager. Let $D = \{x_m\}_{m \geq 0}$ be a countable discrete set with $K \cap D = \emptyset$ such that $\overline{D} = K \cup D$. Let $\{W_m\}$ be a collection of open sets such that $\{x_m\} = D \cap W_m$ and $\overline{W}_m \cap \overline{W}_l = \emptyset$ for $m \neq l$. Then for every m there is an open set V_m such that*:

(a) *$V_m \times V_l \subseteq O$ for $m \neq l$.*
(b) *$\overline{V}_m \subseteq W_m$ and $\overline{V}_m \cap K = \emptyset$.*

Proof. Fix K, D, $\{x_m\}$, $\{W_m\}$ and O as in the hypothesis. Let $D_m = \{x_l : l \geq m\}$ for $m \geq 0$. Notice that $\overline{D}_m = K \cup D_m$. We will define, by induction on m, sequences of open sets V_m and U_m such that:

(1) $V_m \subseteq \overline{V}_m \subseteq W_m$.
(2) $\overline{V}_m \cap \overline{U}_m = \emptyset$.
(3) $V_m \times U_m \subseteq O$.
(4) $\overline{D}_m \subseteq U_m$.
(5) $U_{m+1} \subseteq U_m$ and $V_{m+1} \subseteq U_m$.
(6) $\overline{V}_m \cap K = \emptyset$.

For $m=0$ apply Lemma 2.4 to $L=\overline{D}_0$, $U'=X$ and $W=W_0$ to obtain open sets V_0 and U_0 such that $V_0\subseteq\overline{V}_0\subseteq W_0$, $\overline{D}_0\subseteq U_0\subseteq X$, $\overline{V}_0\cap\overline{U}_0=\emptyset$ and $V_0\times U_0\subset O$. Notice that $\overline{V}_0\cap\overline{K}=\emptyset$.

Suppose we have defined V_m and U_m for $m\le M$ satisfying (1)–(6). Notice first that $K\cup D_{M+1}=\overline{D}_{M+1}\subseteq\overline{D}_M\subseteq U_M$ and $x_M\in U_M\cap W_{M+1}$. Apply then Lemma 2.4 to $L=\overline{D}_{M+1}$, $U'=U_M$ and $W=W_{M+1}\cap U_M$ to get V_{M+1} and U_{M+1} satisfying (1)–(5). Notice that since $K\subseteq U_{M+1}$, we have $\overline{V}_{M+1}\cap K=\emptyset$. The collection $\{V_m\}$ satisfies (a) (here we need the fact that O is symmetric) and (b). ■

Now we are ready to give the proof of the basic construction.

LEMMA 2.6. *Let $\mathcal{F}$ be a collection of non-empty meager closed sets such that the collection of meager closed sets not in $\mathcal{F}$ is dense in $\mathcal{K}(X)$. Let (O_n) be a collection of open subsets of $X\times X$ with property* $(*)$ (*as stated in* 2.1). *Then there is a closed set K locally not in $\mathcal{F}$ and a G_δ set $H\subseteq K$ such that*:

(a) $\overline{H}=K$.

(b) *For every $x,y\in H$ with $x\ne y$ we have $(x,y)\in\bigcap_n O_n$.*

Proof. We will define for each $s\in\omega^{<\omega}$ an open set V_s and a closed set K_s such that:

(1) $V_s\ne\emptyset$, $K_s\subseteq V_s$ and $K_s\notin\mathcal{F}$.

(2) If $n\ne m$ then $\overline{V}_{s^\wedge(n)}\cap\overline{V}_{s^\wedge(m)}=\emptyset$.

(3) $\overline{V}_{s^\wedge(n)}\subseteq V_s$ and $\overline{V}_{s^\wedge(n)}\cap K_s=\emptyset$.

(4) $\operatorname{diam}(\overline{V}_{s^\wedge(n)})\le 2^{-\operatorname{lh}(s)}$.

(5) $\overline{\bigcup_n V_{s^\wedge(n)}}=\bigcup_n\overline{V}_{s^\wedge(n)}\cup K_s$.

(6) $K_s\subset\overline{\bigcup_n K_{s^\wedge(n)}}$.

(7) If $n\ne m$ then $V_{s^\wedge(n)}\times V_{s^\wedge(m)}\subseteq O_{\operatorname{lh}(s)+1}$.

The construction is by induction on $\operatorname{lh}(s)$. We start with $V_\emptyset=X$ and $K_\emptyset$ a meager set not in $\mathcal{F}$. Suppose we have defined K_s and V_s for s with $\operatorname{lh}(s)\le k$ and satisfying (1)–(7). For each s with $\operatorname{lh}(s)=k$ let D_s be a countable discrete set such that $D_s\cap K_s=\emptyset$, $D_s\subseteq V_s$ and $\overline{D}_s=K_s\cup D_s$ as in Lemma 2.3. Let $\{x^s_m\}_{m\ge 0}$ be an enumeration of D_s and pick an open set W^s_n such that $x^s_n\in W^s_n$, $\overline{W^s_n}\cap\overline{W^s_m}=\emptyset$ for $n\ne m$, $\overline{W^s_n}\subseteq V_s$ and $\operatorname{diam}(W^s_n)\le 2^{-n\cdot\operatorname{lh}(s)}$.

Let $D=\bigcup_{\operatorname{lh}(s)=k}D_s$. From (2) we see that D is discrete, $\overline{D}$ is meager and $\overline{D}-D$ is closed. Apply Lemma 2.5 to $O=O_{k+1}$, $K=\overline{D}-D$, D, $\{W^s_n\}$ (for $\operatorname{lh}(s)=k$ and $n\ge 0$) and get open sets $V^s_n\subseteq W^s_n$ satisfying (1)–(6) of Lemma 2.5. Put $V_{s^\wedge(n)}=V^s_n$ and pick $K_{s^\wedge(n)}\subseteq V_{s^\wedge(n)}$ with $K_{s^\wedge(n)}\notin\mathcal{F}$. From the construction it follows that (1), (2), (4) and (7) hold.

To check (3) notice that $\overline{V_n^s} \subseteq W_n^s \subset V_s$ by construction and also $\overline{V_n^s} \cap (\overline{D} - D) = \emptyset$. So if $\overline{V_n^s} \cap K_s \neq \emptyset$, then $\overline{V_n^s} \cap (\overline{D}_s - D_s) \neq \emptyset$. Since $\overline{D}_s - D_s \subseteq \overline{D} - D$, we have $\overline{V_n^s} \cap (\overline{D} - D) \neq \emptyset$, which is a contradiction.

For (5), let $x = \mathrm{Lim}_i\, x_i$ with $x_i \in V_{s^\wedge(n_i)}$. There are two cases to consider: (i) If n_i is eventually equal to k then $x \in \overline{V}_{s^\wedge(k)}$ and we are done. (ii) If n_i is not eventually constant, then $\mathrm{dist}(x_{n_i}^s, x_i) \leq \mathrm{diam}(W_{n_i}^s) \to 0$ as $i \to \infty$. Then from (2) and the fact that $\overline{D}_s = K_s \cup D_s$ we see that $x \in K_s$. The other part is proved analogously.

For (6), let $x \in K_s$. Then $x = \mathrm{Lim}_i\, x_{n_i}^s$ for some subsequence (n_i). Let $y_i \in K_{s^\wedge(n_i)}$. Then $\mathrm{dist}(x_{n_i}^s, y_i) \to 0$ as $i \to \infty$, therefore $x = \mathrm{Lim}_i\, y_i$.

Let $H = \bigcap_n \bigcup_{\mathrm{lh}(s)=n} V_s$ and $K = H \cup \bigcup_s K_s$. As in the proof of Lemma 7 in [6] (p. 203) it can be shown that $K = \bigcap_n \overline{\bigcup_{\mathrm{lh}(s)=n} V_s} = \overline{H}$. As $\mathrm{diam}(K_s) \to 0$, we deduce that K is locally not in $\mathcal{F}$ and from (2) and (7), H satisfies (b). This finishes the proof. ∎

3. Main results. A Borel equivalence relation E on X (Borel as a subset of $X \times X$) is said to be *smooth* if it admits a countable Borel separating family, i.e., a collection (A_n) of E-invariant Borel subsets of X such that for all $x, y \in X$,

$$xEy \quad \text{if and only if} \quad \forall n\ (x \in A_n \leftrightarrow y \in A_n).$$

A fundamental characterization of smooth Borel equivalence relations was proved by Harrington, Kechris and Louveau in [3]: E is smooth if and only if there is no continuous one-to-one function $f : 2^\omega \to X$ such that for all $\alpha, \beta \in 2^\omega$,

$$\alpha E_0 \beta \quad \text{if and only if} \quad f(\alpha) E f(\beta),$$

where E_0 is the equivalence relation defined in 2^ω as follows:

$$\alpha E_0 \beta \quad \text{if and only if} \quad \exists n\ \forall m > n\ \alpha(m) = \beta(m).$$

It is not difficult to see that E_0 is generated by a countable collection of homeomorphisms of 2^ω.

A set $A \subseteq X$ is called *E-smooth* (or just smooth when there is no confusion about E) if there is a Borel set $B \supseteq A$ such that the restriction of E to B is a smooth equivalence relation. A subset of a smooth set is also smooth and a countable union of smooth sets is smooth, i.e., they form a σ-ideal. So, we regard smooth sets as small sets. Every countable set is smooth and in fact every Borel transversal is smooth. A characterization of the analytic smooth sets in terms of measures is as follows: A finite, positive Borel measure μ on X is called *E-ergodic* if for every μ-measurable invariant set A, either $\mu(A) = 0$ or $\mu(X - A) = 0$. It is called *E-non-atomic* if for every $x \in X$, $\mu([x]_E) = 0$. An analytic set $A \subseteq X$ is smooth if and only if $\mu(A) = 0$ for every E-ergodic non-atomic measure μ ([12]).

The following σ-ideal was introduced in [12]:

$$I(E) = \{K \in \mathcal{K}(X) : K \text{ is smooth with respect to } E\}.$$

It was proved there that

THEOREM 3.1. *Let E be a non-smooth Borel equivalence relation on a compact Polish space X. Then $I(E)$ is a calibrated, locally non-Borel, $\boldsymbol{\Pi}_1^1$ non-thin σ-ideal.* ∎

We will use the result of the previous section to show that $I(E)$ does not have the covering property (for E non-smooth). We will also present some results about the σ-ideal generated by the collection of closed transversals.

Let $B = \{F \in \mathcal{K}(X) : F \text{ is a transversal}\}$. Then

$$F \in B \quad \text{if and only if} \quad \forall x, y\ [(x, y \in F \ \&\ xEy) \rightarrow x = y].$$

Consider the following relation:

$$R(x, y, F) \quad \text{if and only if} \quad (x, y \in F \ \&\ xEy) \rightarrow x = y.$$

It is clear that R is Borel and B is Π_1^1. Denote by $I_{\mathrm{t}}(E) = (B)_\sigma$ the σ-ideal generated by B, that is to say, $K \in I_{\mathrm{t}}(E)$ if there are closed transversals $K_n \in B$ such that $K = \bigcup_n K_n$. Then $I_{\mathrm{t}}(E)$ is Π_1^1 (see [7], p. 271), it contains all singletons and therefore it is not Borel (Corollary 5.4 of [12]). If, in particular, E is F_σ (for example E_0) then R is G_δ and therefore B is also G_δ, thus in this case $I_{\mathrm{t}}(E)$ has a Borel basis.

THEOREM 3.2. *Let E be an equivalence relation generated by a countable collection of homeomorphisms of X. Let $\mathcal{F}$ be a collection of non-empty meager closed sets such that the collection of meager closed sets not in $\mathcal{F}$ is dense in $\mathcal{K}(X)$. Then there is a G_δ E-transversal H such that $\overline{H}$ is $\mathcal{F}$-perfect. In particular, if J is a σ-ideal of compact subsets of X such that the collection of meager J-perfect sets is dense in $\mathcal{K}(X)$ and $I_{\mathrm{t}}(E) \subseteq J$, then J does not have the covering property.*

Proof. Let (g_n) be a countable collection of homeomorphisms of X such that xEy if and only if there is n with $g_n(x) = y$. Let $O_n = \{(x, y) \in X \times X : x \neq y \ \&\ g_n(x) \neq y \ \&\ g_n(y) \neq x\}$. Then $[K]_{O_n} = g_n[K] \cup g_n^{-1}[K] \cup K$, which is meager if K is meager. Let $\mathcal{F}$ be a collection of closed sets as in the hypothesis. Then by 2.6 there is an $\mathcal{F}$-perfect closed set K and a G_δ dense subset H of K with H an E-transversal. If $\mathcal{F}$ is hereditary then the Baire category theorem implies that H cannot be covered by countably many sets in $\mathcal{F}$. From this the last claim follows. ∎

COROLLARY 3.3. *Let E be a non-smooth Borel equivalence relation on X. Then $I(E)$ does not have the covering property and does not have a Borel basis.*

P r o o f. First, suppose we have shown that $I(E)$ does not have the covering property. Toward a contradiction, suppose that $I(E)$ has a Borel basis. Then from 3.1 we know that all the hypotheses of the Debs–Saint Raymond theorem (see the introduction) are satisfied and hence $I(E)$ would have the covering property, which contradicts our assumption.

To see that $I(E)$ does not have the covering property, we first observe that if $I(E_0)$ does not have the covering property then neither does $I(E)$ (if f is the function witnessing that E is not smooth and A is a counterexample to the covering property for $I(E_0)$ then $f(A)$ is a counterexample for $I(E)$). Finally, for $I(E_0)$ the required assertion follows from 3.2. ■

R e m a r k. The fact that $I(E)$ does not have the covering property means that there is a G_δ set H that has measure zero with respect to every E-ergodic, non-atomic measure, but it cannot be covered by a countable collection of closed smooth sets. Notice that this is much stronger than just saying there is one such measure μ with $\mu(\overline{H}) > 0$.

Now, we will examine more closely the σ-ideal $I_{\mathrm{t}}(E)$, since it seems to be the cause for the failure of the covering property. In fact, we will show that $I_{\mathrm{t}}(E)$ is not calibrated, i.e., for this σ-ideal the failure of the covering property is as strong as it can be. We will need the following lemma.

LEMMA 3.4. *Let E be an equivalence relation on X and F be a closed set which is locally not a transversal* (*i.e. locally not in the collection B defined above*). *There is a continuous function $f : 2^\omega \to \mathcal{K}(F)$ such that*:

(i) *If γ is eventually zero*, *then $f(\gamma)$ is finite.*
(ii) *If γ is not eventually zero*, *then $f(\gamma)$ is locally not a transversal.*

P r o o f. Let F be a closed set locally not a transversal. We will define a sequence F_s, $s \in 2^{<\omega}$, such that:

(i) F_s is a finite subset of F.
(ii) If $s \prec t$ (i.e., s is an initial segment of t), then $F_s \subseteq F_t$.
(iii) For every $s \in 2^{<\omega}$, $\mathrm{dist}(F_s, F_{s^\wedge(i)}) \le 2^{-\mathrm{lh}(s)}$.
(iv) For every $s \in 2^{<\omega}$, $F_{s^\wedge(0)} = F_s$.
(v) For every $x \in F_s$, $\{y \in F_{s^\wedge(1)} : \mathrm{dist}(x, y) \le 1/2^{\mathrm{lh}(s)+1}\}$ is not a transversal.

We define F_s by induction on the length of s. Pick $x_0 \in F$ and let $F_\emptyset = \{x_0\}$. Suppose for all $s \in 2^n$ we have defined F_s satisfying (i)–(v). Put $F_{s^\wedge(0)} = F_s$. Since F is locally not a transversal, for every $x \in F_s$ we can pick $y, z \in F$ with yEz and such that $\mathrm{dist}(x, y) \le 1/2^{\mathrm{lh}(s)+1}$. Put $y, z \in F_{s^\wedge(1)}$.

Now define f by $f(\gamma) = \overline{\bigcup_n F_{\gamma\lceil n}}$. It is not difficult to check that (ii) and (iii) imply that f is continuous. If γ is eventually zero, then from (i) and (iv) it is clear that $f(\gamma)$ is finite. On the other hand, assume that γ has infinitely many 1's. We will show that $f(\gamma)$ is locally not a transversal. Let V be an open set such that $V \cap f(\gamma) \neq \emptyset$ and let $x \in V \cap f(\gamma)$ and choose n_0 such that $\{y \in Y : \operatorname{dist}(x, y) < 1/2^{n_0}\} \subseteq V$. Let $n > n_0$ be such that $\gamma(n) = 1$ and put $s = \gamma\lceil n$. Then from (v) we see that $V \cap f(\gamma)$ is not a transversal. ■

Next, we show that $I_{\rm t}(E)$ is not calibrated, for E non-smooth.

THEOREM 3.5. *Let E be a non-smooth Borel equivalence relation on X. Then $I_{\rm t}(E)$ is a Π^1_1, locally non-Borel and not calibrated σ-ideal.*

Proof. We have already shown that $I_{\rm t}(E)$ is Π^1_1 and from 3.4 we see that it is locally not G_δ; hence from the dichotomy theorem ([7]), $I_{\rm t}(E)$ is locally not Borel.

We show first that $I_{\rm t}(E_0)$ is not calibrated. Let $x \in 2^\omega$ and $\mathcal{F}$ be the collection of closed meager subsets F of 2^ω such that $[x]_{E_0} \cap F$ is infinite. As E_0 is generated by a countable collection of homeomorphisms of 2^ω, let (O_n) be open subsets of $2^\omega \times 2^\omega$ as in the remark after 2.1. Since the equivalence class of x is dense, so is $\mathcal{K}(X) - \mathcal{F}$. Hence by 2.6 there is a closed set K locally not in $\mathcal{F}$ and a G_δ dense $H \subset K$ such that H is a transversal. Then H and K do not satisfy the condition in the definition of calibration: In fact, notice that $K = H \cup \bigcup_s K_s$ and the sets K_s in the proof of 2.6 can be chosen to be finite subsets of $[x]_{E_0}$, hence they belong to $I_{\rm t}(E_0)$.

For the general case we argue as follows. Let f be a 1-1 continuous function witnessing that E is not smooth. The following facts are easy to verify: (i) a G_δ subset H of 2^ω is in $I_{\rm t}(E_0)_{\rm int}$ iff $f[H]$ is in $I_{\rm t}(E)_{\rm int}$. (ii) K is a E_0-transversal iff $f[K]$ is a E-transversal. Using these facts it is easy to check that the counterexample to the calibration of $I_{\rm t}(E_0)$ is transformed by f into a counterexample to the calibration of $I_{\rm t}(E)$. ■

Remarks. The Debs–Saint Raymond theorem (see the introduction) was used to show that the σ-ideal of closed sets of uniqueness does not have a Borel basis ([6], in contrast to what happens with extended uniqueness) in the same way we did for $I(E)$. We can also use it here to give a proof of the previous result. In fact, suppose toward a contradiction that $I_{\rm t}(E_0)$ is calibrated. Then all hypotheses of the Debs–Saint Raymond theorem are satisfied (recall that in this case the collection of closed transversals is Borel and hence $I_{\rm t}(E_0)$ has a Borel basis) and thus $I_{\rm t}(E_0)$ would have the covering property. We will show that under this assumption $I(E_0)$ would also have the covering property, which contradicts 3.3. For countable Borel equivalence relations we know that every smooth set admits a Borel transversal (see [1]). Let $B \subseteq X$ be an analytic smooth set and T be a Borel transversal for B. Clearly $T \in I_{\rm t}(E_0)_{\rm int}$, hence there are closed transversals F_n such that $T \subseteq \bigcup_n F_n$. It is easy to see then that B is covered by the sets $[F_n]_E$ which are F_σ smooth sets (recall that X is compact and $x \in [F]_E$ if and only if there is $y \in F$ such that xEy, thus $[F]_E$ is an F_σ set).

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References

[1] J. Burgess, *A selection theorem for group actions*, Pacific J. Math. 80 (1979), 333–336.
[2] G. Debs et J. Saint Raymond, *Ensembles d'unicité et d'unicité au sens large*, Ann. Inst. Fourier (Grenoble), 37(3) (1987), 217–239.
[3] L. A. Harrington, A. S. Kechris and A. Louveau, *A Glimm–Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. 3 (1990), 903–928.
[4] A. S. Kechris, *The descriptive set theory of σ-ideals of compact sets*, in: Logic Colloquium'88, R. Ferro, C. Bonotto, S. Valentini and A. Zanardo (eds.), North-Holland, 1989, 117–138.
[5] —, *Classical Descriptive Set Theory*, Springer, 1995.
[6] A. S. Kechris and A. Louveau, *Descriptive Set Theory and the Structure of Sets of Uniqueness*, London Math. Soc. Lecture Note Ser. 128, Cambridge Univ. Press, 1987.
[7] A. S. Kechris, A. Louveau and W. H. Woodin, *The structure of σ-ideals of compact sets*, Trans. Amer. Math. Soc. 301 (1987), 263–288.
[8] A. Louveau, *σ-idéaux engendrés par des ensembles fermés et théorèmes d'approximation*, Trans. Amer. Math. Soc. 257 (1980), 143–169.
[9] Y. N. Moschovakis, *Descriptive Set Theory*, North-Holland, Amsterdam, 1980.
[10] S. Solecki, *Covering analytic sets by families of closed sets*, J. Symbolic Logic 59 (1994), 1022–1031.
[11] C. Uzcátegui, *The covering property for σ-ideals of compact sets*, Fund. Math. 140 (1992), 119–146.
[12] —, *Smooth sets for a Borel equivalence relation*, Trans. Amer. Math. Soc. 347 (1995), 2025–2039.
[13] B. Weiss, *Measurable dynamics*, in: Contemp. Math. 26, Amer. Math. Soc., 1984, 395–421.

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