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A Ramsey theorem for polyadic spaces

by

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Abstract. A polyadic space is a Hausdorff continuous image of some power of the one-point compactification of a discrete space. We prove a Ramsey-like property for polyadic spaces which for Boolean spaces can be stated as follows: every uncountable clopen collection contains an uncountable subcollection which is either linked or disjoint. One corollary is that $(\alpha\kappa)^\omega$ is not a universal preimage for uniform Eberlein compact spaces of weight at most κ, thus answering a question of Y. Benyamini, M. Rudin and M. Wage. Another consequence is that the property of being polyadic is not a regular closed hereditary property.

1. Introduction. For an infinite cardinal κ, let $\alpha\kappa$ be the Aleksandrov one-point compactification of the discrete space κ and let $\alpha\kappa^\tau$ be the Tikhonov product of τ copies of $\alpha\kappa$. A Hausdorff space X is *polyadic* (Mrówka [Mr70]) if there exist cardinals κ, τ such that X is a continuous image of $\alpha\kappa^\tau$. The much-studied dyadic spaces are precisely the images of $\alpha\omega^\tau$.

Our interest in polyadic spaces began with the problem of whether this property was regular closed hereditary. Theorem 2.1 gives a new Ramsey-like property that all polyadic spaces satisfy. We also use this theorem to solve a problem in function space theory on uniform Eberlein compact spaces; see Corollary 3.3. Corollary 3.5 shows that for hyperspaces $H(X)$, polyadic is equivalent to dyadic.

For $n < \omega$, a collection $\mathcal{O}$ of sets is *n-linked* if for each $\mathcal{O}' \subset \mathcal{O}$ with $|\mathcal{O}'| = n$, $\bigcap \mathcal{O}' \neq \emptyset$. We abbreviate 2-linked by linked. $\mathcal{O}$ is *centered* if for each finite $\mathcal{O}' \subset \mathcal{O}$, $\bigcap \mathcal{O}' \neq \emptyset$. A *$\Delta$-system* is a collection $\mathcal{O}$ of sets for which there exists a set R (called the *root* of the Δ-system) such that if A

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and B are two distinct elements of $\mathcal{O}$, then $A \cap B = R$. A standard fact is the following: if λ is an uncountable regular cardinal and $\langle F_\alpha : \alpha < \lambda \rangle$ is a λ-sequence of finite sets, then there exists $A \subset \lambda$ with $|A| = \lambda$ such that $\{F_\alpha : \alpha \in A\}$ is a Δ-system.

All our spaces are assumed to be Hausdorff. We say that a space Y is an *image* of a space X if there exists a continuous surjection $f : X \to Y$. A topological property P is *imaging* if P is transferred from a space to all of its images, and P is *hyper-extendible* if P is transferred from a space to its Vietoris hyperspace of all non-empty closed subsets.

2. The polyadic Ramsey theorem. Let us set our notation for a standard Sierpiński graph which we will use a couple of times in this paper. Let $\mathbb{R}$ be the set of real numbers, let $A \subset \mathbb{R}$ be of cardinality ω_1, let $<$ denote the usual ordering on A and let $\prec$ denote a well-ordering on A. We say $<$ and $\prec$ *agree* on $\{x, y\}$ if $x < y \Leftrightarrow x \prec y$. Otherwise, we say that they *disagree* on $\{x, y\}$. Define $G \subset [A]^2$ by $\{x, y\} \in G$ iff $<$ and $\prec$ agree on $\{x, y\}$. For $x \in A$, let $J_x = \{y \in A : \{x, y\} \in G\}$. The key property of the Sierpiński graph G is that there exists no uncountable $A' \subset A$ on which either $<$ and $\prec$ agree for all of $[A']^2$ or on which $<$ and $\prec$ disagree for all of $[A']^2$. In addition, we also assume that for each $x \in A$, J_x and $A \setminus J_x$ are both uncountable.

We will also use 2 instances of the partition calculus arrow notation. For an infinite regular cardinal λ, $\lambda \to (\lambda, \omega)$ means that whenever the doubletons of λ, i.e. $[\lambda]^2$, are partitioned into sets A and B, then either there is a subset C of λ with cardinality λ which is homogeneous for A, i.e., $[C]^2 \subset A$, or there is a subset D of λ with cardinality ω which is homogeneous for B. For $n < \omega$, $\omega \to (\omega)^2_n$ means that whenever the doubletons of ω are partitioned into sets $A_1, \ldots, A_n$, then there is an $i < n$ and an infinite $C \subset \omega$ such that C is homogeneous for A_i.

Let λ be an infinite cardinal. We say that a space has *Property* Q_λ if whenever $\langle U_\alpha, V_\alpha \rangle_{\alpha<\lambda}$ is a sequence of pairs of open sets with $\overline{U}_\alpha \subset V_\alpha$ for each $\alpha < \lambda$, then there exists an $A \subset \lambda$ with cardinality λ such that either $\{V_\alpha : \alpha \in A\}$ is linked or $\{U_\alpha : \alpha \in A\}$ is disjoint. We say that a space has *Property* R_λ if every collection of cardinality λ of clopen sets contains a subcollection of cardinality λ which is either linked or disjoint. Since, in a Boolean space, a clopen set B_α can be placed between $\overline{U}_\alpha$ and V_α, these 2 properties are equivalent in Boolean spaces. *Property* Q and *Property* R abbreviate Property Q_{ω_1} and R_{ω_1} respectively.

THEOREM 2.1. *Every polyadic space satisfies Property* Q_λ *for each regular cardinal* $\lambda > \omega$.

Proof. Let λ be a regular cardinal $> \omega$ and let P be a continuous image of $\alpha\kappa^\tau$ for some κ, τ. Since Property Q_λ is an imaging property, it suffices to show that $\alpha\kappa^\tau$ has Property Q_λ. Since $\alpha\kappa^\tau$ is a Boolean space, it suffices to show that $\alpha\kappa^\tau$ has Property R_λ. In exactly the same way that one proves the Noble–Ulmer theorem that a product is ccc iff every finite subproduct is ccc (by using a Δ-system of finite supports) one can prove that a product of Boolean spaces satisfies Property R_λ iff every finite subproduct satisfies Property R_λ. So, it suffices to show that for all $n < \omega$ and all κ, $\alpha\kappa^n$ satisfies Property R_λ. This is Lemma 2.5. ∎

For a brief moment only let us consider *Property* S: every uncountable collection of *open* sets contains an uncountable subcollection which is either linked or disjoint. We mention this only to show why in Properties Q and R we do not just deal with a single family of open sets but with pairs of open sets or clopen sets. A space has *Property* K (the property of Knaster) if every uncountable open collection contains an uncountable linked subcollection. We have the following:

PROPOSITION 2.2. *Property* S *is equivalent to Property* K.

Proof. Assume X has Property S. Let $\mathcal{O}$ be an open family with $|\mathcal{O}| = \omega_1$. We show that $\mathcal{O}$ does not contain an uncountable disjoint subfamily. If so, let $\mathcal{O}'$ be an uncountable disjoint subfamily of $\mathcal{O}$. Let G be a standard Sierpiński graph on $A \subset \mathbb{R}$. Let φ be a bijection $\varphi : [A]^2 \to \mathcal{O}'$. For each $x \in A$, put $U_x = \bigcup_{\{x,y\}\in G} \varphi(\{x,y\})$. Then, for each $x \in A$, U_x is open and, furthermore, $U_x \cap U_y \neq \emptyset$ iff $\{x,y\} \in G$. So, the collection $\{U_x : x \in A\}$ violates Property S. Hence, $\mathcal{O}$ must contain an uncountable linked subfamily and therefore we have shown that X has Property K. ∎

Thus, the polyadic space $\alpha\omega_1$ has Property Q but does not have Property S.

EXAMPLE 2.3. In the definition of Property Q_λ we cannot replace linked by centered (or even 3-linked) and still have Theorem 2.1.

Indeed, in $\alpha\omega_1^2$, for each $\alpha < \omega_1$, put $B_\alpha = (\{\alpha\} \times \alpha\omega_1) \cup (\alpha\omega_1 \times \{\alpha\})$. The collection $\{B_\alpha : \alpha < \omega_1\}$ is linked but contains no uncountable 3-linked subcollections.

$\alpha\omega_1$ satisfies the stronger *Property* T: every uncountable clopen collection contains an uncountable subcollection which is either centered or disjoint. The above B_α's show that Property T is not productive (also that the property gotten from T by replacing centered with 3-linked is not productive). We use special knowledge of $\alpha\kappa$ to get Lemma 2.5 but this leaves open

QUESTION 2.4. *Among Boolean spaces, is Property* R *productive? Or even more strongly, if a Boolean space X has Property* T, *does X^2 have Property* R?

LEMMA 2.5. *For every $n < \omega$, for every cardinal κ, and for every regular cardinal $\lambda > \omega$, $\alpha\kappa^n$ has Property* R_λ.

Proof. Let $\mathcal{B} = \{b_\alpha : \alpha < \lambda\}$ be a clopen family in $\alpha\kappa^n$ of cardinality λ. Let us assume that $\mathcal{B}$ does not contain a linked subfamily of cardinality λ. We now work towards producing a disjoint subfamily of cardinality λ. By thinning to a subfamily of cardinality λ we can assume that there exists $m < \omega$ such that for each $\alpha < \lambda$,

$$b_\alpha = \bigcup_{i<m} r_i^\alpha = \bigcup_{i<m} \prod_{k<n} r_i^\alpha(k),$$

where for all $i < m$ and all $k < n$ either $r_i^\alpha(k)$ is a finite subset of κ of a constant size for every $\alpha < \lambda$ or $r_i^\alpha(k) \cap \kappa$ is a co-finite subset of κ for every $\alpha < \lambda$. Define an indicator function $I : m \times n \to 2$ by $I(i,k) = 0$ iff for all $\alpha < \lambda$, $r_i^\alpha(k)$ is a finite subset of κ. By applying a Δ-system argument for each $i < m$ and each $k < n$, we also assume that R_{ik} is a root for $\{r_i^\alpha(k) : \alpha < \lambda\}$ if $I(i,k) = 0$ and R_{ik} is a root for $\{\kappa \setminus r_i^\alpha(k) : \alpha < \lambda\}$ if $I(i,k) = 1$.

We will show that the following holds:

(∗) for all $i, j < m$ there exists $H \subset \lambda$ with cardinality λ such that $\alpha < \beta$ in H implies $r_i^\alpha \cap r_j^\beta = \emptyset$.

Then, after m^2 successive applications of (∗) we get a $K \subset \lambda$ with cardinality λ such that $\alpha < \beta$ in K implies $b_\alpha \cap b_\beta = \emptyset$. This would complete the proof of the lemma.

Proof of (∗). Fix $i, j < m$. Define a case function ψ with domain n by $\psi(k) = 2^{I(i,k)}3^{I(j,k)}$. We may assume that

(S1) for all $k < n$ with $\psi(k) = 3$,

$$\neg[R_{ik} \subset R_{jk} \,\&\, (\forall\alpha < \lambda) R_{ik} = r_i^\alpha(k)]$$

and that

(S2) for all $k < n$ with $\psi(k) = 2$,

$$\neg[R_{jk} \subset R_{ik} \,\&\, (\forall\alpha < \lambda) R_{jk} = r_j^\alpha(k)]$$

(otherwise we would get (∗) for i, j immediately with $H = \lambda$).

Define a subset P of $[\lambda]^2$ by $\{\alpha < \beta\} \in P$ iff $r_i^\alpha \cap r_j^\beta \neq \emptyset$. Since $\lambda \to (\lambda, \omega)$ and our main overall assumption implies that there is no subset of λ of cardinality λ that is homogeneous for P, we get a countably infinite $A \subset \lambda$

such that $[A]^2 \cap P = \emptyset$. Since $\omega \to (\omega)^2_n$, we get a $k < n$ and an infinite $B \subset A$ such that $\alpha < \beta$ in B implies $r^\alpha_i(k) \cap r^\beta_j(k) = \emptyset$. Clearly, $\psi(k) \neq 6$. If $\psi(k) = 2$, then $r^\beta_j(k) \subset \kappa \setminus r^\alpha_i(k)$ implies $R_{jk} \subset R_{ik}$. Put $r = |R_{ik}|$. Choose a 2-element subset $C \subset B$ and an $(r+1)$-element $D \subset B$ such that $\alpha \in C$ and $\beta \in D \Rightarrow \alpha < \beta$. Since we have two Δ-systems, our inclusion gives us $\bigcup_{\beta \in D} r^\beta_j(k) \subset R_{ik}$. Now (S2) implies $|R_{ik}| \geq r + 1$; a contradiction. In an analogous fashion, using (S1), we get $\psi(k) \neq 3$.

Hence, $\psi(k) = 1$, and thus we get $R_{ik} \cap R_{jk} = \emptyset$. We now apply thinning to complete the proof of $(*)$. There are only finitely many α's in λ such that $r^\alpha_i(k) \cap R_{jk} \neq \emptyset$. Remove these α's. The remaining α's are such that $r^\alpha_i(k) \cap R_{jk} = \emptyset$. For each remaining α there exist only finitely many $\beta > \alpha$ with $r^\alpha_i(k) \cap r^\beta_j(k) \neq \emptyset$. Now we can inductively construct an $H \subset \lambda$ of cardinality λ such that $\alpha < \beta$ in H implies $r^\alpha_i(k) \cap r^\beta_j(k) = \emptyset$. So, we have proved $(*)$ and hence completed the proof of Lemma 2.5. ∎

3. Applications

Example 3.1: An open $U \subset \alpha\omega_1^2$ such that $\overline{U}$ is not polyadic.

Let G be a standard Sierpiński graph on the set ω_1. Put $U = \{(\alpha, \beta) : \{\alpha, \beta\} \in G\}$. For each $\alpha < \omega_1$, put $B_\alpha = (\{\alpha\} \times \alpha\omega_1) \cup (\alpha\omega_1 \times \{\alpha\})$ and put $U_\alpha = B_\alpha \cap \overline{U}$. Then $U_\alpha \cap U_\beta \neq \emptyset \Leftrightarrow \{\alpha, \beta\} \in G$. The collection $\{U_\alpha : \alpha < \omega_1\}$ violates Property R, so $\overline{U}$ is not polyadic.

Our first application relates to the structure of polyadic spaces. The dyadic property is known to be both zeroset and regular closed hereditary. Gerlits [Ge78] has shown that the polyadic property is zeroset hereditary. Thus, we have

Corollary 3.2. *The polyadic property is not regular closed hereditary.*

Problem 4 in Benyamini, Rudin and Wage [BRW77] asks whether $\alpha\kappa^\omega$ is a universal preimage for uniform Eberlein compact spaces of weight at most κ. They prove in this paper that uniform Eberlein compact spaces of weight at most κ are precisely the images of closed subspaces of $\alpha\kappa^\omega$. So, the above example $\overline{U}$ gives us

Corollary 3.3. *$\alpha\kappa^\omega$ is not a universal preimage for uniform Eberlein compact spaces of weight at most κ.*

Problem 3 of [BRW77] of whether there is some closed subspace of $\alpha\kappa^\omega$ which is a universal preimage for uniform Eberlein compact spaces of weight at most κ is still open. We can phrase the negation of this problem as

Question 3.4. *Is it true that for every closed $H \subset \alpha\kappa^\omega$, there exists a closed $K \subset \alpha\kappa^\omega$ such that K is not an image of H?*

Our final application will deal with the hyperspace $H(X)$ of all nonempty closed subsets of a compact space X. We endow $H(X)$ with the Vietoris topology. Mrówka [Mr70] has shown that if X is a compact orderable space with $H(X)$ polyadic, then X must be first countable. From the theorem of Gerlits [Ge78] that character = weight for polyadic spaces, it follows that X must be metrizable. Mrówka uses a property called (K_1) in order to prove his theorem:

(K_1) the closure of a G_δ open set U coincides with the sequential closure of U.

Mrówka proves that every polyadic space satisfies the property (K_1). We will need this result to improve Mrówka's theorem by reducing polyadicity of $H(X)$ to dyadicity of $H(X)$ for any compact space. Our hyperspace notation is as follows: If $\mathcal{O}$ is a collection of subsets of X, then $\langle\mathcal{O}\rangle = \{F \in H(X) : F \subset \bigcup\mathcal{O}$ and $F \cap O \neq \emptyset$ for every $O \in \mathcal{O}\}$. The family $\{\langle\mathcal{O}\rangle : \mathcal{O}$ is a finite family of open subsets of $X\}$ is an open base for $H(X)$.

Corollary 3.5. *If $H(X)$ is polyadic, then $H(X)$ is dyadic.*

Proof. Assume that $H(X)$ is polyadic. We will show that $H(X)$ is ccc and then invoke a theorem of R. Engelking (cf. [Mr70]) that says that a ccc polyadic space is dyadic. We first show that X is ccc. If not, let $\mathcal{M}$ be an uncountable maximal disjoint open family in X. Then X is in the $H(X)$ closure of $\bigcup\{\langle\mathcal{M}'\rangle : \mathcal{M}'$ is a finite subset of $\mathcal{M}\}$. By Mrówka's result, $H(X)$ has the property (K_1) and so we can choose, for $n < \omega$, finite subsets $\mathcal{M}_n \subset \mathcal{M}$ such that $X \in \overline{\bigcup_{n<\omega}\langle\mathcal{M}_n\rangle}$. Pick $M \in \mathcal{M} \setminus \bigcup_{n<\omega}\mathcal{M}_n$. Then $X \in \langle\{M, X\}\rangle$ and $\langle\{M, X\}\rangle \cap \bigcup_{n<\omega}\langle\mathcal{M}_n\rangle = \emptyset$. This contradiction proves that X is ccc.

Theorem 2.1 implies that $H(X)$ has Property Q. It is easily seen that therefore X has Property Q. But a regular ccc space with Property Q has Property K. Property K is hyper-extendible (note that ccc is not hyper-extendible; think of a Souslin continuum), so $H(X)$ has Property K and the proof is complete. ∎

We remark that Shapiro [Sh76] has shown that if $H(X)$ is dyadic, then the weight of X is at most ω_1. Thus it follows that if $H(X)$ is polyadic, then $w(X) \leq \omega_1$.

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