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$\mathcal{M}$-rank and meager groups

by

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Abstract. Assume p^* is a meager type in a superstable theory T. We investigate definability properties of p^*-closure. We prove that if T has $< 2^{\aleph_0}$ countable models then the multiplicity rank $\mathcal{M}$ of every type p is finite. We improve Saffe's conjecture.

0. Introduction. Throughout the paper, T is a superstable theory in a countable language L, and we work within a monster model $\mathfrak{C} = \mathfrak{C}^{\mathrm{eq}}$ of T. The general references are [Ba, Sh, Hru], see also [Ne2]. Suppose p is a regular stationary type. Associated with p is a closure operator cl_p defined by $a \in \mathrm{cl}_p(A)$ iff $\mathrm{stp}(a)$ is hereditarily orthogonal to p. Restricted to $p(\mathfrak{C})$, cl_p induces a pre-geometry, and is equivalent to the closure operator induced by forking dependence. For instance, if p is minimal then cl_p on $p(\mathfrak{C})$ equals acl. So when a, b, c are distinct points on a line in $p(\mathfrak{C})$ (with respect to the cl_p-pregeometry), then $a \in \mathrm{acl}(b, c)$. We show that in fact in many cases $\mathrm{cl}_p(a)$ is definable over $\mathrm{cl}_p(b)$ and $\mathrm{cl}_p(c)$, that is, in the quotient geometry p-closure equals definable closure.

These cases include the case of properly weakly minimal p, and more generally of a meager type p. Now let us recall this and other notions introduced in [Ne2, Ne3].

Suppose $s(x)$ is a partial type over $\mathfrak{C}$. Then $[s]$ denotes the class of partial types over $\mathfrak{C}$, with free variable x, containing s. For any set A let $\mathrm{Tr}_A(s)$ (the trace of s over A) be the set $\{\mathrm{stp}(a/A)\colon a \text{ realizes } s\}$. We denote the set of strong types over A by $\mathrm{Str}(A)$, and identify it with $S(\mathrm{acl}(A))$. $\mathrm{Tr}(s)$ is $\mathrm{Tr}_\emptyset(s)$. We refer the reader to [Ne3] for the properties of Tr. Sometimes, to specify clearly the variable in the types in question, we write e.g. $\mathrm{Str}_x(A)$ to denote the set of strong types over A in variable x. We shall often use the following regularity criterion of Hrushovski [Hru]:

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Suppose p is regular and $q = \mathrm{stp}(a/X)$ is p-simple, of p-weight 1. Then q is regular iff $\mathrm{cl}_p(X) \cap \mathrm{acl}(Xa) \subseteq \mathrm{acl}(X)$.

Now suppose P is a closed subset of $\mathrm{Str}_x(A)$. We say that *forking is meager on* P if for every formula $\varphi(x)$ forking over A, $\mathrm{Tr}_A(\varphi) \cap P$ is nowhere dense in P.

Assume p is a regular stationary type. We say that a formula $\varphi(x)$ over A is a *p-formula* (over A) if the following conditions hold:

(a) φ is p-simple of p-weight 1.

(b) If $a \in \varphi(\mathfrak{C})$ and $w_p(a/A) > 0$ then $\mathrm{stp}(a/A)$ is regular, non-orthogonal to p.

(c) The set $P_\varphi = \{r(x) \in \mathrm{Str}(A) : w_p(r) > 0\}$ is closed.

(d) p-weight 0 is definable on $\varphi(\mathfrak{C})$, that is, if $a \in \varphi(\mathfrak{C})$ and $w_p(a/Ac) = 0$ then for some formula $\psi(x, y)$ over $\mathrm{acl}(A)$, true of (a, c), whenever $\psi(a', c')$ holds then $w_p(a'/Ac') = 0$.

If $\varphi(x)$ satisfies only (a), (c), (d) above, we say that φ is a *weak p-formula* over A. We say that a is *p-proper* over A if $a \in \varphi(\mathfrak{C})$ for some weak p-formula φ over A, and $w_p(a) > 0$. "p-proper" means p-proper over some finite subset of $\mathrm{cl}_p(\emptyset)$.

By [H-S], p-formulas exist over many finite sets A (for non-trivial p). It is quite easy to find them when T is small (see [Ne2]). Notice that if p is weakly minimal, then some weakly minimal $\varphi(x) \in p$ is a p-formula. Also, if φ is a p-formula over A then φ is a p-formula over any A' containing A. The properties of p-formulas ensure that we can work with cl_p there just as with acl in the weakly minimal case.

Remark 0.1. *If $\varphi(x)$ is a weak p-formula over A then for some $A' \subset \mathrm{cl}_p(A)$ with $A \subset A'$ and $A' \setminus A$ finite, over A' there is a p-formula $\varphi'(x) \vdash \varphi(x)$.*

Proof. Choose $\varphi' \vdash \varphi$ over $\mathrm{cl}_p(A)$ with p-weight 1 and minimal ∞-rank. Let φ' be over A'. By Hrushovski's regularity criterion this works.

We say that p is *meager* if for some (equivalently: any) p-formula φ (over some A), forking is meager on P_φ. In [Ne2] we prove that if p is meager then p is locally modular and non-trivial. For example, any properly weakly minimal non-trivial type is meager. Also, the locally modular type obtained in [L-P] by minimizing the ∞-rank of a type with no Morley rank, is meager. In [Ne2, Ne3] we find (in a small T) many meager types.

Suppose d is a closure operator on a subset X of $\mathfrak{C}$ (X may be, for instance, $p(\mathfrak{C})$). We say that $\{a_0, a_1, a_2\} \subset X$ is a *d-triangle* over A if a_0, a_1, a_2 are pairwise independent over A (in the sense of forking), and for $i < 3$, $a_i \in d(A \cup \{a_0, a_1, a_2\} \setminus \{a_i\}) \setminus d(A)$. When $A = \emptyset$, we omit it in this definition.

In this paper we prove that meager types have some of the properties of properly weakly minimal types. This sheds a new light also on the proofs of these properties in the weakly minimal case. Suppose p is a meager type. Since p is non-trivial, we get a cl_p-triangle. Given a cl_p-triangle $\{a_0, a_1, a_2\}$ with a_i p-proper, we find a dcl-triangle $\{a'_0, a'_1, a'_2\}$ with a'_i p-proper and $\mathrm{cl}_p(a_i) = \mathrm{cl}_p(a'_i)$. This in turn provides us with a $\mathrm{cl}_p(\emptyset)$-definable regular group G, with generics non-orthogonal to p (generalizing [Hru, Proposition 5.10]).

In Section 2 we further investigate such groups G and prove (using the results from [Ne2, Ne3]) that if T has $< 2^{\aleph_0}$ countable models then the multiplicity rank $\mathcal{M}(q)$ is finite for every q (this notion is explained in Section 2). This improves a result from [Ne2], where we proved only that $\mathcal{M}(q) < \infty$, and proves a conjecture from [Ne1], saying that under the few models assumption, $\mathcal{M} \leq U$. Also, we improve Saffe's condition from [Ne2].

1. The geometry of a meager type. In this section we assume p is a regular locally modular strong type over $\emptyset$. Also, cl and proper mean cl_p and p-proper here. We are going to investigate geometrical properties of p-closure. So it is natural to work within p-formulas. However, for technical reasons it is convenient to work with p-proper elements, within a broader set-up of weak p-formulas. The proof of the next proposition is technical. Essentially it is parallel to the proof of Proposition 5.10 in [Hru], which deals with weakly minimal types. In the weakly minimal case traces of forking formulas are finite, in the meager case they are merely nowhere dense. So we have to modify the argument from [Hru, 5.10] just as we modified the proof that a properly weakly minimal type is locally modular to show that a meager type is locally modular [Ne2].

PROPOSITION 1.1. *If p is meager and $\{a_0, a_1, a_2\}$ is a* cl*-triangle, with all a_i proper, then for some finite $C \subset \mathrm{cl}(\emptyset)$ there is a* dcl*-triangle $\{a'_0, a'_1, a'_2\}$ over C with all a'_i proper.*

P r o o f. In this proof let $C = \mathrm{cl}(\emptyset)$; however, notice that C can always be replaced by a sufficiently large finite subset. Let $\varphi(x)$ be a weak p-formula over C, true of a_0, a_1 and a_2. Let $a' = a_1 a_2$ and $B = \mathrm{cl}(a_0) \cap \mathrm{acl}(a_0 a')$. Choose a large finite fragment $a'_0 \subset B$ of $Cb(a'/B)$ such that $a' \perp B\,(a'_0)$. Since $w_p(a'/B) = 1$ and $\mathrm{cl}_p(B) \cap \mathrm{acl}(Ba) \subseteq B$, we see that $\mathrm{stp}(a'/B)$ is regular, hence also $\mathrm{stp}(a'/a'_0)$ is regular.

Clearly, a'_0 is cl-interdependent with a_0, hence $w_p(a'_0) = 1$. Again by the regularity criterion, $\mathrm{stp}(a'_0/C)$ is regular, non-orthogonal to p. We shall find a weak p-formula over C, true of a'_0, witnessing that a'_0 is proper. Since $w_p(a_0/a'_0) = 0$, choose a formula $\delta(x, y) \vdash \varphi(x)$ (over C), true of (a_0, a'_0), such that whenever $\delta(c,c')$ holds then $w_p(c/c')=0$. Also, we may say of a_0, a'_0 the following:

(a) for generic $a'_1 \stackrel{\rm s}{\equiv} a_1\,(C)$, there is a'_2 in $\varphi(\mathfrak{C})$ with $w_p(a'_2/a_0a'_1)=0$ and $a'_0 \in \mathrm{acl}(a_0a'_1a'_2)$.

So we may assume that if $\delta(c,c')$ holds then (a) holds with c, c' in place of a_0, a'_0. It follows that $\delta(c,c')$ implies $w_p(c')\le 1$. Indeed, choose a'_1, a'_2 as in (a) (with c, c' in place of a_0, a'_0). We deduce that $w_p(c'/ca'_1)=0$. Since $cc' \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.1ex\hbox{$\mid$}} a'_1\,(C)$, also $w_p(c'/c)=0$, hence $w_p(c)\le 1$ implies $w_p(c')\le 1$.

Let $\psi(y)=\exists x\ \delta(x,y)$. Using δ, we see that p-weight 0 is definable on $\psi(\mathfrak{C})$ and ψ is a weak p-formula over C.

Now we shall prove that $a'_0 \in \mathrm{dcl}(\{a_1,a_2\}\cup \mathrm{acl}(C))$, or else for some $a''_1, a''_2 \in \varphi(\mathfrak{C})$, $\{a'_0,a''_1,a''_2\}$ is a cl-triangle over C with $a'_0 \in \mathrm{dcl}(\{a''_1,a''_2\}\cup \mathrm{acl}(C))$. Suppose not. Then there is a''_0 such that

(b) $$a'_0 \stackrel{\rm s}{\equiv} a''_0\,(C), \quad a'_0 \equiv a''_0\,(Ca') \quad \text{and} \quad a'_0 \ne a''_0.$$

Claim. *Whenever a''_0 satisfies* (b) *then $a'_0 \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.1ex\hbox{$\mid$}} a''_0\,(C)$ and $w_p(a'/a'_0a''_0)=0$. In particular, some $\chi(x_1,x_2,y_0,y_1)$ over* $\mathrm{acl}(C)$, *true of* (a_1,a_2,a'_0,a''_0), *witnesses $w_p(a_1a_2/a'_0a''_0)=0$.*

Proof. If both $a' \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.1ex\hbox{$\mid$}} a'_0\,(a''_0C)$ and $a' \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.1ex\hbox{$\mid$}} a''_0\,(a'_0C)$, then using the fact that a'_0 is a part of $Cb(a'/a'_0C)$, we would get $a'_0=a''_0$, a contradiction. Thus, for example, $a' \not\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.1ex\hbox{$\mid$}} a'_0\,(a''_0C)$, hence $w_p(a'/a'_0a''_0)<w_p(a'/a'_0)$. It follows that $a''_0 \notin \mathrm{cl}(a'_0)$, hence necessarily $a'_0 \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.1ex\hbox{$\mid$}} a''_0\,(C)$. We see that $w_p(a'/a'_0a''_0)=0$. The rest is easy.

By compactness, possibly modifying somewhat χ, we find a formula $\delta(x_1,x_2,y)$ over $\mathrm{acl}(C)$, true of (a_1,a_2,a'_0), such that for every a'_1, a'_2,

(c) whenever $\delta(a'_1,a'_2,a'_0)$, $\delta(a'_1,a'_2,a''_0)$ and $a'_0 \ne a''_0$ hold then $\chi(a'_1,a'_2,a'_0,a''_0)$ holds.

Without loss of generality, $\delta(x_1,x_2,y)$ implies $\varphi(x_1)\wedge\varphi(x_2)$. Let $\delta'(x_1,y)$ be $\exists x_2\ \delta(x_1,x_2,y)$. Since δ' is true of (a_1,a'_0), $\delta'(x_1,a'_0)$ does not fork over C, and is of p-weight 1. By the open mapping theorem, we choose a formula $\delta''(x_1)$ over $\mathrm{acl}(C)$ with

$$\mathrm{stp}(a_1/c)\in[\delta'']\cap \mathrm{Str}(C)\subset \mathrm{Tr}_C(\delta'(x_1,a'_0))\cap[\varphi].$$

By Remark 0.1, choose $\delta^*(x)$ below $\delta''(x)$ which is a p-formula over some finite subset of $\mathrm{cl}(\emptyset)$. So δ^* is a p-formula over C.

Fix $a''_0 \stackrel{\rm s}{\equiv} a'_0\,(C)$ with $a''_0 \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.1ex\hbox{$\mid$}} a'_0(C)$. This determines uniquely $\mathrm{stp}(a'_0a''_0/C)$. Since p is meager, we can choose $r\in P_{\delta^*}\setminus \mathrm{Tr}_C(\exists x_2\ \chi(x_1,x_2,a'_0,a''_0))$. Let a''_1 realize r, and without loss of generality $a''_1 \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.1ex\hbox{$\mid$}} a'_0a''_0\,(C)$.

It follows that $\delta'(a''_1,a'_0)$ holds and $w_p(a''_1a'_0)=2$. Hence we can choose a''_2 so that $\delta(a''_1,a''_2,a'_0)$ holds. If $a'_0\in \mathrm{dcl}(a''_1,a''_2,\mathrm{acl}(C))$, we are done.

Otherwise, we can choose $a_0^* \neq a_0'$ with $a_0^* \stackrel{\mathrm{s}}{\equiv} a_0'\,(C)$ and $a_0^* \equiv a_0'\,(a_1''a_2''C)$. By (c), $\chi(a_1'', a_2'', a_0', a_0^*)$ holds, hence we get $a_0^* \mathop{\smile\hskip-0.9em\vert}\; a_0'\,(C)$ and $a_0'a_0'' \stackrel{\mathrm{s}}{\equiv} a_0'a_0^*\,(C)$. Thus $\exists x_2\ \chi(x_1, x_2, a_0', a_0'')$ is consistent with r, a contradiction.

Notice that if we had e.g. $a_1 \in \mathrm{dcl}(a_0, a_2)$, then choosing $a_0' \subset Cb(a'/b)$ large enough, we might get $a_1 \in \mathrm{dcl}_C(a_0', a_2)$, as well as $a_0' \in \mathrm{dcl}_C(a_1a_2)$, and in the case where $a_0' \in \mathrm{dcl}_C(a_1''a_2'')$, by suitable choice of δ we can ensure that $a_1'' \in \mathrm{dcl}_C(a_0'a_2'')$. Thus, repeating the above construction twice we get what we want.

Examining the proof of Proposition 1.1 we get the following.

R e m a r k 1.2. *Suppose φ is a weak p-formula over $\emptyset$, p is meager and orthogonal to any type with ∞-rank $< R_\infty(\varphi)$. Then there is a* cl_p*-triangle* $\{a_0', a_1', a_2'\}$ *which is also a* dcl*-triangle over some finite $C \subset \mathrm{cl}(\emptyset)$, such that for every $i < 3$, $R_\infty(a_i'/C) = R_\infty(\varphi)$ and a_i' is proper.*

P r o o f. Since every type of ∞-rank $< R_\infty(\varphi)$ is orthogonal to p, we see that in fact φ is a p-formula. Hence by non-triviality of p there is a cl-triangle $\{a_0, a_1, a_2\} \subset \varphi(\mathfrak{C})$. In the proof of Proposition 1.1 we get as an intermediate step a cl-triangle $\{a_0', a_1, a_2\}$ (or $\{a_0', a_1'', a_2''\}$) such that $a_0' \in \mathrm{dcl}_C(a_1a_2)$ (or $a_0' \in \mathrm{dcl}_C(a_1''a_2'')$), and for $i = 1, 2$, $R_\infty(a_i) = R_\infty(\varphi)$ ($R_\infty(a_i'') = R_\infty(\varphi)$ respectively). This implies $R_\infty(a_0'/C) = R_\infty(\varphi)$.

The next corollary generalizes [Hru, 5.10]. As in [Hru], in view of [Hru, 5.7], Proposition 1.1 and the next corollary tell us something new mainly in a very special case when for each a proper over A, $\mathrm{tp}(a/\mathrm{cl}(A))$ is modular. Nevertheless, their proofs illustrate how the meager forking assumption can successfully replace the assumption of algebraicity of forking in the weakly minimal case. We apply this corollary in the next section to prove that when T is superstable with few countable models, then $\mathcal{M}(q)$ is finite for any q.

Corollary 1.3. *If p is meager and $\varphi(x)$ is a weak p-formula over A then in $\varphi(\mathfrak{C})^{\mathrm{eq}}$ there is a* $\mathrm{cl}(A)$*-definable regular group G with generic types non-orthogonal to p. If p is orthogonal to any type with ∞-rank $< R_\infty(\varphi)$, then we can find such a G with $R_\infty(G) = R_\infty(\varphi)$.*

P r o o f. We sketch the proof in the case where p is orthogonal to any type with ∞-rank $< \alpha = R_\infty(\varphi)$, giving a group G of rank α (this case requires more care).

Without loss of generality, A is finite. By Remark 1.2, extending A a little, we find $\{a, b, c\}$ which is a cl_p- and dcl-triangle over A. Moreover, we can assume that $r = \mathrm{stp}(a/A)$, $r' = \mathrm{stp}(b/A)$ and $r'' = \mathrm{stp}(c/A)$ are regular, non-orthogonal to p and have ∞-rank α. So c defines an invertible function mapping $r|Ac$ to $r'|Ac$. Let σ be the germ of this function. So $\sigma \in \mathrm{dcl}(Ac)$, $\mathrm{stp}(\sigma/A)$ is regular and $R_\infty(\sigma/A) = \alpha$.

Choose $\sigma' \stackrel{\mathrm{s}}{\equiv} \sigma\,(A)$ with $\sigma' \mathop{\smile\!\!\!\!\!\!\mid}\, \sigma$ and let $\tau = \sigma^{-1} \circ \sigma'$. By [Hru, 5.2], $q = \mathrm{stp}(\tau/A)$ is regular and q is closed under generic composition. Moreover, since τ, σ, σ' are pairwise A-independent and $\tau \in \mathrm{dcl}_A(\sigma, \sigma')$, we get $R_\infty(\tau/A) = \alpha$. By [Hru, Theorem 1] we get a group G definable over $\mathrm{acl}(A)$ and a definable embedding of τ into the set of generic types of G. Clearly, G satisfies our demands.

In order for the assertion of Proposition 1.1 to be true we do not need meagerness of p. Indeed, below we show that it is sufficient to assume there is a $\mathrm{cl}(\emptyset)$-definable regular group G non-orthogonal to p. By Corollary 1.3, when p is meager then there is such a group. Then Corollary 1.6 below shows that we can strengthen the conclusion of Proposition 1.1 by requiring that $a_i' \in \mathrm{cl}(a_i)$ for $i < 3$.

Suppose a is proper, $a \in \mathrm{cl}(A)$. We say that $\mathrm{cl}(a)$ is *geometrically definable* over A if for some formula $\varphi(x)$ over A of p-weight 0, we have $\varphi(\mathfrak{C}) \subset \mathrm{cl}(a)$ and $\varphi(\mathfrak{C}) \not\subset \mathrm{cl}(\emptyset)$. If moreover a is a generic element of a 0-definable regular group G, then we say that a/G^- is geometrically definable over A if for some φ as above, $\varphi(\mathfrak{C}) \subset a + G^-$. Here $G^- = \mathrm{cl}(\emptyset) \cap G$. Notice that if $\mathrm{cl}(a)$ is geometrically definable over A, then the point corresponding to $\mathrm{cl}(a)$ in the geometry induced by cl is $\mathrm{Aut}(\mathfrak{C}/A)$-invariant.

For the rest of this section assume that there is a 0-definable locally modular abelian group G, and p is the generic type of G^0. Recall that proper means p-proper here. We shall use the following lemma.

Lemma 1.4. *Assume a is proper*, $\mathrm{tp}(a/\mathrm{cl}(\emptyset))$ *is not modular*, $a' \equiv a\,(\mathrm{cl}(\emptyset))$, $a' \mathop{\smile\!\!\!\!\!\!\mid}\, a\,(\mathrm{cl}(\emptyset))$ *and* $c \in \mathrm{cl}(aa') \cap G^0 \setminus \mathrm{cl}(\emptyset)$. *Then there are proper $b \supset a$ and $b' \supset a'$ with $a'b' \equiv ab\,(\mathrm{cl}(\emptyset))$ such that if $c' \equiv c\,(ab')$ then $c - c' \in G^-$ and if $a''c' \equiv ac\,(b')$ and $c - c' \in G^-$ then $a'' \in \mathrm{cl}(a)$.*

Proof. Notice that for any $c' \in G^0 \cap \mathrm{cl}(aa') \setminus \mathrm{cl}(\emptyset)$, $c' \in \mathrm{cl}(c)$ (otherwise $a \in \mathrm{cl}(cc')$, hence $\mathrm{tp}(a/\mathrm{cl}(\emptyset))$ would be modular). Let $b_0 \subset A = \mathrm{cl}(a') \cap \mathrm{acl}(aa'c)$ be so large that $a' \subset b_0$ and $ac \mathop{\smile\!\!\!\!\!\!\mid}\, A\,(b_0)$. Clearly b_0 is proper (see the proof of Proposition 1.1). We shall prove that

$$\text{(a)} \qquad c'a \stackrel{\mathrm{s}}{\equiv} ca\,(b_0) \quad \text{implies} \quad c - c' \in G^-.$$

To prove (a), consider $c^*a^* \stackrel{\mathrm{s}}{\equiv} ca\,(b_0)$ with $c^*a^* \mathop{\smile\!\!\!\!\!\!\mid}\, ca\,(b_0)$. We have $w_p(aca^*c^*) = 3$, $c \mathop{\smile\!\!\!\!\!\!\mid}\, c^*$ and $a \mathop{\smile\!\!\!\!\!\!\mid}\, a^*\,(\mathrm{cl}(\emptyset))$. Hence by modularity there is $c_0 \in \mathrm{cl}(aa^*) \cap G^0 \setminus \mathrm{cl}(\emptyset)$ (p is modular). If $c_0 \notin \mathrm{cl}(cc^*)$, then $a \in \mathrm{cl}(cc^*c_0)$, meaning that $\mathrm{stp}(a/\mathrm{cl}(\emptyset))$ is modular, a contradiction. So $c_0 \in \mathrm{cl}(cc^*) \cap \mathrm{cl}(aa^*)$. Moreover, for each $c_0' \in G^0 \cap \mathrm{cl}(cc^*) \cap \mathrm{cl}(aa^*) \setminus \mathrm{cl}(\emptyset)$, $c_0' \in \mathrm{cl}(c_0)$. Since $c_0 \in \mathrm{cl}(cc^*)$, by [Hru], for some pseudo-endomorphisms α, β of G^0, $c_0 \in \alpha c + \beta c^*$. Clearly α, β are invertible, hence without loss of generality $\alpha = 1$. (One can show that this implies $\beta = -1$, but we will not need this.) Anyway, α, β are definable over $\mathrm{cl}(\emptyset)$, so expanding b_0 a little, we can assume they are definable over b_0.

Now suppose $c'a \stackrel{\mathrm{s}}{\equiv} ca\,(b_0)$, and we must show $c-c' \in G^-$. Since $\mathrm{stp}(ca/b_0)$ is regular, $c'a \not\perp ca\,(b_0)$ and $ca \perp c^*a^*\,(b_0)$, we get $c'a \perp c^*a^*\,(b_0)$ and $c'a \stackrel{\mathrm{s}}{\equiv} c^*a^*\,(b_0)$.

Hence $c' + \beta c^* \in \mathrm{cl}(aa^*)$ and $c + \beta c^* \in \mathrm{cl}(aa^*)$, which gives $c' + \beta c^* \in \mathrm{cl}(c + \beta c^*)$ (this holds because of the remark at the beginning of the proof). Hence for some γ, $c' + \beta c^* =^* \gamma c + \gamma\beta c^*$. So $(1-\gamma)\beta c^* =^* \gamma c - c'$. We could have chosen $c^* \perp cc'$. If $\gamma \neq 1$ then $c^* \in \mathrm{cl}(c, c')$, a contradiction. So $\gamma = 1$, which means that $c - c' \in G^-$. This shows (a).

Next we show

(b) $\qquad a''c' \stackrel{\mathrm{s}}{\equiv} ac\,(b_0)$ and $c - c' \in G^-$ implies $a'' \in \mathrm{cl}(a)$.

We keep the notation of the proof of (a). Suppose $a''c' \stackrel{\mathrm{s}}{\equiv} ac\,(b_0)$ and $c - c' \in G^-$; we must show $a'' \in \mathrm{cl}(a)$. As above we have $aca^*c^* \stackrel{\mathrm{s}}{\equiv} a''c'a^*c^*\,(b_0)$. In particular, $c + \beta c^* =^* c' + \beta c^* \in \mathrm{cl}(aa^*) \cap \mathrm{cl}(a''a^*)$. Hence $a, a'' \in \mathrm{cl}(c + \beta c^*, a^*) \cap \mathrm{cl}(b_0 c)$. Hence $a'' \in \mathrm{cl}(a)$, proving (b).

Properness implies definability of p-weight 0 on a weak p-formula. Hence replacing in (a), (b), b_0 by some b' with $b_0 \subset b' \subset \mathrm{acl}(b_0)$, we may drop "s" in $\stackrel{\mathrm{s}}{\equiv}$ in (a), (b). Choose b with $ab \equiv a'b'\,(\mathrm{cl}(\emptyset))$. Clearly b, b' satisfy our demands.

THEOREM 1.5. *Assume $a_0, \ldots, a_n$ are proper and $a_0 \in \mathrm{cl}(a_1, \ldots, a_n)$. Then there are proper $a'_i \supset a_i$, $i = 1, \ldots, n$, such that $\mathrm{cl}(a_0)$ is geometrically definable over $a'_1, \ldots, a'_n$, and if $a_0 \in G^0$ then a_0/G^- is geometrically definable over $a'_1, \ldots, a'_n$.*

Proof. At each point of the following proof we can replace $\mathrm{cl}(\emptyset)$ by a sufficiently large finite subset. We can assume $A = \{a_1, \ldots, a_n\}$ is cl-independent. We shall prove that $\mathrm{cl}(a_0)$ is geometrically definable over some suitable $a'_1, \ldots, a'_n$, leaving the proof that a_0/G^- is definable over $a'_1, \ldots, a'_n$ to the reader. We say that A is *scattered* if for any $a \in A$, $\mathrm{stp}(a/\mathrm{cl}(A \setminus \{a\}))$ is not modular.

Rearrange the a_i's so that (for some m), $\{a_1, \ldots, a_m\}$ is a maximal subset of A which is scattered. Let $i > m$. First suppose $\mathrm{tp}(a_i/\mathrm{cl}(\emptyset))$ is not modular. Since $\mathrm{tp}(a_i/\mathrm{cl}(a_1, \ldots, a_m))$ is modular, for some $a'_i \in \mathrm{cl}(a_1, \ldots, a_m)$, $a'_i \equiv a_i\,(\mathrm{cl}(\emptyset))$ and cl-independence of A gives $a'_i \perp a_i\,(\mathrm{cl}(\emptyset))$. So choose c_i in $G^0 \cap \mathrm{cl}(a_i, a'_i) \setminus \mathrm{cl}(\emptyset)$.

When $\mathrm{tp}(a_i/\mathrm{cl}(\emptyset))$ is modular, we choose $c_i \in G^0 \cap \mathrm{cl}(a_i) \setminus \mathrm{cl}(\emptyset)$, and replace a_i by $a_i c_i$. We see that

$$a_0 \in \mathrm{cl}(a_1, \ldots, a_m, c_{m+1}, \ldots, c_n).$$

Notice that by Lemma 1.4,

(a) c_i/G^- are geometrically definable over $a_1,\ldots,a_m,a''_{m+1},\ldots,a''_n$ for some proper $a''_i \supset a_i$ $(i>m)$.

If $\mathrm{tp}(a_0/\mathrm{cl}(\emptyset))$ is modular, then necessarily $m<n$. In this case let $c_0 \in G^0 \cap \mathrm{cl}(a_0)\setminus \mathrm{cl}(\emptyset)$, and we see that $c_0 \in \mathrm{cl}(c_{m+1},\ldots,c_n)$ and $a_0 \in \mathrm{cl}(c_0)$, and the rest is easy.

If $\mathrm{tp}(a_0/\mathrm{cl}(\emptyset))$ is not modular, we proceed as follows. Choose $a'_0,\ldots,a'_m$ with $a'_0\ldots a'_m \overset{\mathrm{s}}{\equiv} a_0\ldots a_m$ $(\mathrm{cl}(\emptyset))$ and $a'_0\ldots a'_m \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.1ex\hbox{$|$}} a_0\ldots a_m c_{m+1}\ldots c_n$ $(\mathrm{cl}(\emptyset))$.

For $i\le m$ choose $c_i \in \mathrm{cl}(a_i a'_i)\cap G^0 \setminus \mathrm{cl}(\emptyset)$. Hence

$$c_0 \in \mathrm{cl}(a_1,\ldots,a_m,c_1,\ldots,c_n).$$

Since $\{a_1,\ldots,a_m\}$ is scattered, $c_0\in \mathrm{cl}(c_1,\ldots,c_n)$, that is, $c_0 \in \sum_{i>0}\alpha_i c_i$ for some $\alpha_i \in \mathrm{End}^*(G^0)$. For each $i\le m$ choose proper $b'_i \supset a'_i$ as in Lemma 1.4 (for $a:=a_i$, $a':=a'_i$, $c:=c_i$). Let $q=\mathrm{stp}(b'_0\ldots b'_m/\mathrm{cl}(\emptyset))$. By Lemma 1.4, for each $i\le m$ there is $\varphi_i(x,y,y')$ over $\mathrm{cl}(\emptyset)$, true of (c_i,a_i,b'_i), such that $\varphi_i(x,y,y')$ implies that (x, y, y' are proper and if $\varphi_i(x^*,y^*,y')$ then $y=y^*$ implies $x-x^*\in G^-$, and $x-x^*\in G^-$ implies $y^*\in \mathrm{cl}(y)$).

Thus $a_0,\ldots,a_n$ satisfy a formula $\chi(x_0,\ldots,x_n)$ over $\{c_{m+1},\ldots,c_n\}\cup \mathrm{cl}(\emptyset)$ implying the following: for generic $x'_0,\ldots,x'_n$ in $q(\mathfrak{C})$, and for some z_i in $\varphi_i(\mathfrak{C},x_i,x'_i)$ $(i\le m)$, we have

$$z_0 \in \sum_{0<i\le m}\alpha_i z_i + \sum_{i>m}\alpha_i c_i.$$

We shall show that χ defines geometrically $\mathrm{cl}(a_0)$ over $a_1,\ldots,a_m,c_{m+1}/G^-,\ldots,c_n/G^-$ and $\mathrm{cl}(\emptyset)$.

Suppose $\chi(a''_0,a_1,\ldots,a_m)$. Without loss of generality $b'_0\ldots b'_m \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\lower0.1ex\hbox{$|$}} a''_0 a_1\ldots a_n$ $(\mathrm{cl}(\emptyset))$. Choose c'_i in $\varphi_i(\mathfrak{C},a_i,b'_i)$ for $0<i\le m$ and c'_0 in $\varphi_0(\mathfrak{C},a''_0,b'_0)$ with $c'_0\in\sum_{0<i\le m}\alpha_i c'_i+\sum_{i>m}\alpha_i c_i$. By the choice of φ_i's we have $c'_i-c_i\in G^-$ for $0<i\le m$, hence also $c_0-c'_0\in G^-$. This in turn gives $a''_0\in \mathrm{cl}(a_0)$, hence χ defines geometrically $\mathrm{cl}(a_0)$ over $a_1,\ldots,a_m,c_{m+1},\ldots,c_n$ and $\mathrm{cl}(\emptyset)$. Since χ refers to c_i/G^- rather than to c_i $(i>m)$, we conclude that χ defines geometrically $\mathrm{cl}(a_0)$ over $a_1,\ldots,a_m,c_{m+1}/G^-,\ldots,c_n/G^-$ and $\mathrm{cl}(\emptyset)$. By (a) we are done.

Notice that if $m=n$ then we do not have to enlarge a_i's, it suffices to add some parameters from $\mathrm{cl}(\emptyset)$.

The next corollary shows that when G is present, we can improve Proposition 1.1.

Corollary 1.6. *Assume G is a 0-definable locally modular abelian group, and p is the generic type of G^0. Assume $\{a_0,a_1,a_2\}$ is a* cl-*triangle over* $\mathrm{cl}(\emptyset)$, *with a_i, $i<3$, proper. Then for some proper $a'_i\supset a_i$, $i<3$, $\{a'_0,a'_1,a'_2\}$ is a* dcl-*triangle over* $\mathrm{cl}(\emptyset)$.

Proof. The proof is similar to [Hru, 5.10], only we do not minimize the multiplicity, but rather some local binary rank. By Theorem 1.5 we can assume that $\mathrm{cl}(a_i)$ is definable over $\{a_0, a_1, a_2\} \setminus \{a_i\}$ for each $i < 3$. In other words, for some $\theta(x_0, x_1, x_2)$ over $\mathrm{cl}(\emptyset)$, true of (a_0, a_1, a_2), the following condition (as well as its symmetric variants) holds:

(a) $\theta(x_0, x_1, x_2) \wedge \theta(x_0', x_1, x_2)$ implies $x_0' \in \mathrm{cl}(x_0)$ and $x_0 \in \mathrm{cl}(x_1, x_2)$.

Let $\theta'(x_0, x_1, y)$ be a Boolean combination of formulas of the form $\theta(x_0, x_1, y_i)$, $i < \omega$, and let a_2' be proper and such that $a_2 \subset a_2' \in \mathrm{cl}(a_2)$, $\theta'(x_0, x_1, a_2') \vdash \theta(x_0, x_1, a_2)$, and $\theta'(x_0, a_1, a_2')$ is consistent and has minimal possible binary $\theta(x_0; \bar{z})$-rank (under the previous restrictions). Let $r = \mathrm{tp}(a_1/\mathrm{cl}(\emptyset))$.

Claim. *If* $a_1 \perp a_2'' a_2' \,(\mathrm{cl}(\emptyset))$ *and* $\theta'(x_0, a_1, a_2'')$ *is consistent with* $\theta'(x_0, a_1, a_2')$ *then* $\theta'(x_0, a_1, a_2'')$ *is equivalent to* $\theta'(x_0, a_1, a_2')$.

Proof. Suppose a_0' realizes $\theta'(x, a_1, a_2'') \wedge \theta'(x, a_1, a_2')$. Thus $\{a_0', a_1, a_2'\}$ and $\{a_0', a_1, a_2''\}$ are cl-triangles over $\mathrm{cl}(\emptyset)$. By the choice of θ, a_2' and a_2'' are cl-interdependent. Thus if $\theta'(x, a_1, a_2'')$, $\theta'(x, a_1, a_2'')$ are not equivalent, then one of the formulas $\theta'(x, a_1, a_2')^t \wedge \theta'(x, a_1, a_2'')^v$, $t, v = 0, 1$, is consistent and has smaller $\theta(x, \bar{z})$-rank, where $\varphi^0 = \varphi$ and $\varphi^1 = \neg\varphi$. This contradicts the choice of θ'.

Let a_0' realize $\theta'(x, a_1, a_2')$. So $a_0' \in \mathrm{cl}(a_0)$. Define an equivalence relation E on $\mathrm{tp}(a_2'/\mathrm{cl}(\emptyset))$ by aEa' iff for generic a_1' realizing r, $\theta'(x, a_1', a)$ is equivalent to $\theta'(x, a_1', a')$. Clearly we can replace a_2' by a_2'/E. By the claim, $a_2' \in \mathrm{dcl}(a_0', a_1)$.

To proceed further, replacing a_0 by a_0', a_2 by a_2' and θ by θ', and switching the roles of a_0, a_1, a_2, we can assume that in addition to the properties from the beginning of the proof, $\theta(a_0, a_1, a_2)$ witnesses $a_0 \in \mathrm{dcl}(a_1, a_2)$. Since $a_0 \in \mathrm{dcl}(a_1, a_2)$, we do not have to minimize any rank now, so for some equivalence relations E', E'', replacing a_1 by a_1/E' and a_2 by a_2/E'', we get what we want.

The next corollary generalizes Proposition 1.1. It follows from Corollary 1.6, since for meager p, by Corollary 1.3, a suitable G exists.

Corollary 1.7. *If p is meager and $\{a_0, a_1, a_2\}$ is a* cl-*triangle over* $\mathrm{cl}(\emptyset)$ *with a_i, $i < 3$, proper, then for some proper $a_i' \supset a_i$, $i < 3$, $\{a_0', a_2', a_3'\}$ is a* dcl-*triangle over* $\mathrm{cl}(\emptyset)$.

Despite its simple formulation, the proof of Corollary 1.7 is rather tedious (after tracing it all the way backwards): given a cl-triangle $\{a_1, a_2, a_3\}$ we first find *another* cl-triangle, which is a dcl-triangle (Proposition 1.1). This provides us with a $\mathrm{cl}(\emptyset)$-definable group G (Corollary 1.3). Referring to the very regular structure of forking dependence on G^0, we find that cl itself has some definability properties (Theorem 1.5). This enables us to return to $\{a_0, a_1, a_2\}$, and replace it by an *equivalent* dcl-triangle. It might be interesting to find a more direct proof of Corollary 1.7, without reference to groups.

On the other hand, we know that for any locally modular non-trivial p, after adding some parameters, a group G non-orthogonal to p exists, hence eventually cl has the definability properties listed in Theorem 1.5 and Corollary 1.6. Is Corollary 1.7 true for an arbitrary (also non-meager) locally modular regular type p?

2. $\mathcal{M}$-rank and meager groups. In this section we assume that T is small, superstable and p^* is a meager stationary type over $\emptyset$. We shall prove that if T has few countable models then the $\mathcal{M}$-rank of any type is finite. Also, we shall improve Saffe's condition from [Ne2]. Now we recall the definitions.

$\mathcal{M}$-rank, defined in [Ne1], measures the size of the sets of stationarizations of complete types over finite sets. It is defined by the following conditions. Suppose A is finite and $p \in S(A)$.

(a) $\mathcal{M}(p) \geq 0$.
(b) $\mathcal{M}(p) \geq \alpha + 1$ iff for some finite $B \supset A$ and a non-forking extension q of p over B, $\mathcal{M}(q) \geq \alpha$ and $\mathrm{Tr}_A(q)$ is nowhere dense in $\mathrm{Tr}_A(p)$.
(c) $\mathcal{M}(p) \geq \delta$ for limit δ if $\mathcal{M}(p) \geq \alpha$ for every $\alpha < \delta$.

$\mathcal{M}(a/A)$ abbreviates $\mathcal{M}(\mathrm{tp}(a/A))$. $I(T, \aleph_0) < 2^{\aleph_0}$ implies $\mathcal{M}(p) < \infty$. (In fact, we are going to prove that in this case $\mathcal{M}(p) < \omega$.) We refer the reader to [Ne3] for the basic properties of Tr. By smallness, $\mathcal{M}$ has the following extension property:

If $B \supseteq A$ are finite and $p \in S(A)$, then there is a non-forking extension q of p over B with $\mathcal{M}(p) = \mathcal{M}(q)$.

Also, $\mathcal{M}$ satisfies Lascar's inequality:

$$\mathcal{M}(a/A) \leq \mathcal{M}(ab/A) \leq \mathcal{M}(a/Ab) \oplus \mathcal{M}(b/A).$$

We call a definable regular abelian group *meager* if its generic types are meager. Suppose G is a 0-definable meager group and p^* is the generic type of G^0. We recall some notation from [Ne2, Section 2]. Let $\mathcal{G}$ be the set of generic types of G. So $\mathcal{G} \subset \mathrm{Str}(\emptyset)$. Let $\mathcal{G}m$ be the set of modular types in $\mathcal{G}$. Let Gm be the subgroup of G generated by $\mathcal{G}m$, which is $\bigwedge$-definable over $\emptyset$. For $p, q \in \mathcal{G}$, $p + q = \mathrm{stp}(a + b)$ for any independent realizations a, b of p, q respectively. For any A, $S_{\mathrm{gen}}(A) = \{\mathrm{tp}(a/A) : a \in G \text{ is generic over } A\}$. Notice that G is a p^*-formula. In [Ne2] we prove that $\mathcal{G}m$ is closed nowhere dense and $\mathcal{G} \setminus \mathcal{G}m$ is open in $\mathrm{Str}(\emptyset)$. We restate here Theorems 2.1 and 2.7 (Saffe's condition) from [Ne2] in the following form.

THEOREM 2.1. *Assume T has $< 2^{\aleph_0}$ countable models, A is finite and $p \in S_{\rm gen}(A)$. Then* (1) *or* (2) *below holds.*

(1) $\bigcup\{r + \mathcal{G}m : r \in \mathrm{Tr}(p)\}$ *is open in* $\mathrm{Str}(\emptyset)$.

(2) *The set* $\{r + \mathcal{G}m : r \in \mathrm{Tr}(p)\}$ *is finite.*

When in Theorem 2.1 case (1) holds, we call p *big*. Otherwise we call p *small*. Here we shall improve Theorem 2.1 by showing that if p is big then p is isolated (and for isolated p, (1) holds trivially).

In the proofs we will measure not only complete types, but also some closed subsets of $\mathcal{G}$. Specifically, suppose $X \subset \mathcal{G}$ is closed and A-invariant for some finite set A (that is, for every $f \in \mathrm{Aut}_A(\mathfrak{C})$, $f[X] = X$). Then clearly for every $p \in S_{\rm gen}(A)$, either $\mathrm{Tr}(p) \subset X$ or $\mathrm{Tr}(p) \cap X = \emptyset$.

We define $\mathcal{M}(X)$, the $\mathcal{M}$-*rank* of X, as the first α such that for every $p \in S_{\rm gen}(A)$ with $\mathrm{Tr}(p) \subset X$, $\mathcal{M}(p) \le \alpha$. In the next lemma we collect the basic properties of this notion.

LEMMA 2.2. *Suppose $X, Y \subset \mathcal{G}$ are closed and C-invariant for some finite set C.*

(1) *For $p \in \mathcal{G}$, $\mathcal{M}(X) = \mathcal{M}(p + X)$.*

(2) $\mathcal{M}(X \cup Y) = \max\{\mathcal{M}(X), \mathcal{M}(Y)\}$.

(3) *If G' is a generic subgroup of G $\bigwedge$-definable over C and X is the trace over $\emptyset$ of the set of generic types of G', then $\mathcal{M}(X) = \mathcal{M}(p)$ for any $p \in S_{\rm gen}(C)$ with $\mathrm{Tr}(p)$ open in X.*

Proof. (1) Without loss of generality, p is realized in C by some a. For $q \in S_{\rm gen}(C)$ let $a + q = \mathrm{tp}(a + b/C)$, where b realizes q. Notice that b and $a + b$ are interalgebraic over C. Hence $\mathcal{M}(q) = \mathcal{M}(a + q)$. Let $Y = \{q \in S_{\rm gen}(C) : \mathrm{Tr}(q) \subset X\}$ and $Y' = \{q \in S_{\rm gen}(C) : \mathrm{Tr}(q) \subset p + X\}$. The mapping $q \mapsto a + q$ is a bijection between Y and Y'. So (1) follows.

(2) is easy. For (3), notice that by the smallness of T there is some $p \in S_{\rm gen}(C)$ with $\mathrm{Tr}(p)$ open in X (more precisely, first we find that $\mathrm{Tr}(p)$ is not nowhere dense in X, then by [Ne3, Fact 0.1] it follows that $\mathrm{Tr}(p)$ is open in X). Since finitely many translations of $\mathrm{Tr}(p)$ by generic types of G' cover X, we are done by (2).

For G' and X as in Lemma 2.2(3) we define $\mathcal{M}(G')$ to be $\mathcal{M}(X)$.

THEOREM 2.3. *Assume T is superstable and $I(T, \aleph_0) < 2^{\aleph_0}$. Then for every p, $\mathcal{M}(p)$ is finite.*

Proof. By [Ne2], for every p, $\mathcal{M}(p) < \infty$. So if $\mathcal{M}(p)$ is infinite for some p, then for some p, $\mathcal{M}(p) = \omega$. Let α be the minimal ordinal such that some type p of ∞-rank α has infinite $\mathcal{M}$-rank. By [Ne3, Theorem 1.2], there is a finite set A and a formula $\varphi(x)$ over A of ∞-rank α, isolating a complete type p^0 over A, such that all stationarizations of p^0 are non-orthogonal and meager, $\mathcal{M}(p^0)$ is infinite and φ is a p^0-formula over A. By Corollary 1.3, expanding A a little we get an A-definable meager group G of ∞-rank α, non-orthogonal to p^0. Without loss of generality, we add A to the signature and let p^* be the generic type of G^0. It is easy to see that if $p \in S_{\text{gen}}(\emptyset)$ is isolated then $\mathcal{M}(p)$ is infinite. In fact, [Ne3, Theorem 1.7] implies $\mathcal{M}(p) = \omega + n$ for some finite n. Now we prove that $\mathcal{M}(\mathcal{G}m) < \omega$.

Let $a \in G^0$ be generic. By modularity, for every type $p \in \mathcal{G}m$ there is a formula $\theta_p(x,a)$ such that $p \in \text{Tr}(\theta_p(x,a))$ and $\text{Tr}(\theta_p(x,a)) \subset \mathcal{G}m$. Hence for some p, $\text{Tr}(\theta_p(x,a))$ has non-empty interior in $\mathcal{G}m$. By the smallness of T, for some $r \in S(a) \cap [\theta_p(x,a)]$, $\text{Tr}(r)$ has non-empty interior in $\mathcal{G}m$. By [Ne3, Fact 0.1], $X = \text{Tr}(r)$ is open in $\mathcal{G}m$. So let $\theta(x,a) \in r$ be a formula implying $\theta_p(x,a)$ with $\text{Tr}(r) = \text{Tr}(\theta(x,a))$. By [Ne3, Lemma 1.1], $\mathcal{M}(X) \le \mathcal{M}(r)$. Since $\theta_p(x,a)$ forks over $\emptyset$, we have $R_\infty(r) < \alpha$, hence $\mathcal{M}(r) < \omega$. It follows that $\mathcal{M}(X) < \omega$. Since X is open in $\mathcal{G}m$, there is an $r' \in S_{\text{gen}}(a)$ with $\text{Tr}(r') \subset X$ and $\text{Tr}(r')$ open in $\mathcal{G}m$. So $\mathcal{M}(r') < \omega$. By Lemma 2.2(3), $\mathcal{M}(\mathcal{G}m) < \omega$.

Let $m = \mathcal{M}(\mathcal{G}m)$. Let $q \in S_{\text{gen}}(\emptyset)$ be isolated. Since $\mathcal{M}(q)$ is infinite, there is a finite set B and a non-forking extension $q' \in S(B)$ of q with $\mathcal{M}(q') = m + 1$. By Theorem 2.1 there are 2 cases.

Case 1: *q' is small*. Hence $\text{Tr}(q') \subset \bigcup_i p_i + \mathcal{G}m$ for finitely many $p_1, \ldots, p_n \in \text{Tr}(q')$. By Lemma 2.2, $\mathcal{M}(\bigcup_i p_i + \mathcal{G}m) = m$, hence $\mathcal{M}(q') = \mathcal{M}(\text{Tr}(q')) \le m$, a contradiction.

Case 2: *q' is big*. In this case there is an isolated type $q'' \in S(B)$, which is a non-forking extension of q, such that for $p \in \text{Tr}(q')$, $p + \mathcal{G}m$ meets $\text{Tr}(q'')$. Let a realize p and b realize $r|Ba$ for some $r \in (p + \mathcal{G}m) \cap \text{Tr}(q'')$. Since $\mathcal{M}(a + \mathcal{G}m) = m$, we have $\mathcal{M}(b/aB) \le m$. By the $\mathcal{M}$-rank inequalities,

$$\mathcal{M}(b/B) \le \mathcal{M}(ab/B) \le \mathcal{M}(a/B) \oplus \mathcal{M}(b/aB) \le 2m + 1 < \omega.$$

However, $\mathcal{M}(b/B) = \mathcal{M}(q'')$ is infinite, since q'' is isolated, a contradiction.

The next theorem improves Theorem 2.1.

THEOREM 2.4. *Assume T is superstable, with few countable models, and G is a 0-definable meager group. Then for every finite A and $p \in S_{\text{gen}}(A)$, either p is isolated or $\text{Tr}(p) \subset \bigcup_i r_i + \mathcal{G}m$ for some finitely many $r_1, \ldots, r_n \in \text{Tr}(p)$.*

Proof. Let $k^* = \mathcal{M}(G)$. By Theorem 2.3, k^* is finite. Suppose the theorem is false. Choose k minimal such that for some finite set A there are non-isolated types $p_n \in S_{\text{gen}}(A)$, $n < \omega$, with the following properties:

(a) $\mathcal{M}(p_n) = k$ for every n.

(b) For every r, $r + \mathcal{G}m$ meets $\text{Tr}(p_n)$ for at most finitely many n.

To find k and p_n, $n<\omega$, choose a finite set A' and a big type $p\in S_{\rm gen}(A')$ which is a counterexample to the theorem. So p is non-isolated and $\mathrm{Tr}(p)$ cannot be covered by finitely many sets of the form $r+\mathcal{G}m$. Let a realize p and let $r^*=\mathrm{stp}(a)$. Choose $E_n\in FE(\emptyset)$, $n<\omega$, such that E_{n+1} refines E_n and $\mathrm{stp}(a)\equiv\{E_n(x,a),\ n<\omega\}$. By smallness, choose $p_n\in S_{\rm gen}(A'a)$ with $p_n\in[E_n(x,a)]\setminus[E_{n+1}(x,a)]$ and $\mathrm{Tr}(p_n)$ open in $\mathrm{Tr}(p)$. Notice that

$$\text{(c)}\qquad \bigcap_n \mathrm{cl}\Big(\bigcup_{m>n}\mathrm{Tr}(p_m)\Big)=\{r^*\}.$$

Also, every p_n is big. Otherwise, some open non-empty subset U of $\mathrm{Tr}(p)$ is covered by finitely many sets of the form $r+\mathcal{G}m$. It follows that any $r'\in\mathrm{Tr}(p)$ has an open neighbourhood with this property, and since $\mathrm{Tr}(p)$ is compact, finitely many such sets cover $\mathrm{Tr}(p)$, contradicting the assumption that p is big.

Now, $r^*+\mathcal{G}m$ is disjoint from any $\mathrm{Tr}(p_n)$. Indeed, $r^*+\mathcal{G}m$ is $A'a$-invariant, hence if $r^*+\mathcal{G}m$ meets $\mathrm{Tr}(p_n)$, then $\mathrm{Tr}(p_n)\subset r^*+\mathcal{G}m$, and p_n is small, a contradiction.

Now suppose for some r, $r+\mathcal{G}m$ meets $\mathrm{Tr}(p_n)$ for infinitely many n. By (c) we get $r^*\in r+\mathcal{G}m$, hence $r^*+\mathcal{G}m=r+\mathcal{G}m$, and $r^*+\mathcal{G}m$ meets $\mathrm{Tr}(p_n)$ for some n, a contradiction.

It follows that (b) holds. Since k^* is finite and $\mathcal{M}(p_n)\le k^*$, we see that for some $k\le k^*$, $\mathcal{M}(p_n)=k$ for infinitely many (without loss of generality all) n. Also since p is non-isolated, any p_n is non-isolated. This shows that we can find k and $p_n\in S_{\rm gen}(A)$, $n<\omega$, as required. For every finite set B we have

(d) for at most finitely many n, p_n has infinitely many non-forking extensions over AB,

(e) for at most finitely many n, there is an $r\in\mathcal{G}$ with $r|AB$ modular and $r+\mathcal{G}m$ meeting $\mathrm{Tr}(p_n)$.

Indeed, if p_n has infinitely many non-forking extensions over AB, then there is a $p_n'\in S_{\rm gen}(AB)$, a non-forking extension of p_n, such that $\mathrm{Tr}(p_n')$ is nowhere dense in $\mathrm{Tr}(p_n)$. Consequently, $\mathcal{M}(p_n')<\mathcal{M}(p_n)$. If this happens for every $n\in X$ (for some infinite $X\subset\omega$), then the types p_n', $n\in X$, contradict the minimality of k. This proves (d).

To prove (e), notice that if $q\in S_{\rm gen}(AB)$ and b realizes q with $\mathrm{stp}(b/AB)$ modular, then q is small. (Indeed, otherwise there is an isolated (hence non-modular) $q'\in S_{\rm gen}(AB)$ and b' realizing q' with $b\not\perp b'$; but then q' is modular, a contradiction.) Hence $\mathrm{Tr}(q)$ is covered by finitely many sets of the form $r+\mathcal{G}m$. It follows that Gm is of finite index in the group Gm' generated by realizations of types $r\in\mathcal{G}$ with $r|AB$ modular. Hence in fact there are finitely many types $r_1,\dots,r_l\in\mathcal{G}$ such that every $r\in\mathcal{G}$ with $r|AB$

modular belongs to some $r_i + \mathcal{G}m$ (this is proved also in [Ne2, Lemma 2.4]). By (b), for all but finitely many n, $\mathrm{Tr}(p_n)$ is disjoint from any $r_i + \mathcal{G}m$, hence (e) follows.

Now we choose recursively n_i, $i < \omega$, so that for every $j < \omega$ the following holds:

(f) Whenever a_i realize p_{n_i}, $i < j$, then p_{n_j} has finitely many non-forking extensions over $Aa_{<j}$; also for $r \in \mathcal{G}$, if $r|Aa_{<j}$ is modular then $\mathrm{Tr}(p_{n_j}) \cap (r + \mathcal{G}m) = \emptyset$.

Throughout, $a_{<j}$ denotes $\{a_i : i < j\}$. If (f) holds for every $j' < j$, then clearly there are finitely many possibilities $q_1, \ldots, q_l$ (for some l) for the type $\mathrm{tp}(a_{<j}/A)$. Let b_i realize q_i and $B = \{b_i : i < l\}$. Applying (d), (e) we get the required n_j.

Now suppose a_i realize p_{n_i}, $i < \omega$. We show that $\{a_i : i < \omega\}$ is independent over A. If not, then there is a first a_j which depends on $a_{<j}$ over A. So for $r = \mathrm{stp}(a_j)$, $r|Aa_{<j}$ is modular, contradicting (f).

Since the types p_n, $n < \omega$, are non-isolated, we see that for every j, p_{n_j} is non-isolated over $A \cup \{a_i : i \neq j\}$. Hence by the omitting types theorem, for any $X \subset \omega$ we can find a countable model M of T containing A and a_i, $i \in X$, and omitting the types p_{n_i}, $i \notin X$. This shows that T has $2^{\aleph_0}$-many countable models, a contradiction.

COROLLARY 2.5. *Assume T is superstable, with $< 2^{\aleph_0}$ countable models, and G is a 0-definable locally modular abelian group. Then $\mathcal{M}(\mathcal{G}) = \mathcal{M}(\mathcal{G}m) + 1$ when G is meager, and $\mathcal{M}(\mathcal{G}) = \mathcal{M}(\mathcal{G}m)$ otherwise.*

Proof. For the "otherwise" part notice that if G is not meager then $\mathcal{G}m$ is open in $\mathcal{G}$, and apply Lemma 2.2(3).

I would like to use this opportunity to retreat from Example 1.11 in [Ne3]. The type p appearing there is not regular, and has ∞-rank $\omega + 1$. Moreover, using the "definability lemma" [Ne2, Claim 2.14] and [Ne2, Lemma 2.13] instead of [Ne2, Lemma 2.4], one can prove that Theorem 2.4 is true not only within a meager group, but also within a p-formula for any meager type p. We shall prove even more, improving [Ne2, Corollary 2.15]. We say that a complete type p is *meager* if every stationarization of p is meager.

THEOREM 2.6. *Assume T is superstable with few countable models, A is finite and $p \in S(A)$ is meager. Then exactly one of the following conditions holds.*

(1) *For some $a_1, \ldots, a_n$ realizing p, for every $r \in \mathrm{Tr}_A(p)$, $r|Aa_i$ is modular for some i.*

(2) *p is isolated.*

P r o o f. We can assume that $\mathcal{M}(p) > 0$ and that no $r \in \mathrm{Tr}_A(p)$ is modular (when every $r \in \mathrm{Tr}_A(p)$ is modular, (1) holds). As p is meager, p is regular and non-trivial. We find that

(a) each non-orthogonality class on $\mathrm{Tr}_A(p)$ is open, and there are finitely many of them.

The proof of (a) is similar to [Ne2, Lemma 1.6(1)]. If (a) is false, then each non-orthogonality class on $\mathrm{Tr}_A(p)$ is meager, and there are $2^{\aleph_0}$ of them. As in the proof of [Ne2, Lemma 1.6(1)], varying dimensions we can construct many countable models.

Adding to A elements of p-weight 0 or realizing over A a modular type does not affect (1) nor (2) (we replace p by a non-forking extension over the new A, with the same $\mathcal{M}$-rank). So we can assume that all types in $\mathrm{Tr}_A(p)$ are non-orthogonal. Also, we can assume that there is a meager group G definable over A, with principal generic type p^* non-orthogonal to p. Choose an $r^+ \in \mathrm{Tr}_A(p)$ and c^* realizing r^+. Without loss of generality, there is no $r' \in \mathcal{G} \setminus \mathcal{G}m$ with $r'|Ac^*$ modular (if necessary, we can replace G by a subgroup of finite index in G; the set of $r^* \in \mathcal{G}$ with $r^*|Ac^*$ modular is a finite union of disjoint translates of $\mathcal{G}m$). Since $r^+|Ac^*$ and $p^*|Ac^*$ are modular and non-orthogonal, there are a, b realizing r^+, p^* respectively such that

(b) $\{a, b, c^*\}$ is a forking triangle over A.

Choose a formula $\varphi(x, y, z)$ true of (a, b, c^*), witnessing forking of any of a, b, c^* over the other two, in a definable manner (as in (a) in the proof of the claim in [Ne2, 1.6]). By Theorem 2.4 we know that the theorem is true when p is a generic type of G. Now, $\varphi(x, y, c^*)$ determines a correspondence between some subsets of $\mathrm{Tr}_A(p)$ and $\mathcal{G}$: $r \in \mathrm{Tr}_A(p)$ and $r^* \in \mathcal{G}$ correspond to each other if for some a, b realizing r, r^* respectively, $\varphi(a, b, c^*)$ holds. The main point of the proof is to transfer (via this correspondence) some properties of $\mathcal{G}$ to $\mathrm{Tr}_A(p)$.

Let $X = \{r^* \in \mathcal{G} : r^*$ corresponds (in the above sense) to some $r \in \mathrm{Tr}_A(p)\}$ and $Y = \{r \in \mathrm{Tr}_A(p) : r$ corresponds to some $r^* \in X\}$.

Clearly both X and Y are closed, $r^+ \in Y$ and by the open mapping theorem Y is open in $\mathrm{Tr}_A(p)$. They are (respectively) the range and domain of the correspondence established via φ. Choose $q^* \in S_{\mathrm{gen}}(Ac^*)$ with $\mathrm{Tr}_A(q^*)$ open in X and let $Y' = \{r \in Y : r$ corresponds to some $r^* \in \mathrm{Tr}_A(q^*)\}$. Clearly Y' is clopen in Y. There are two cases, depending on whether q^* is small or big.

C a s e 1: *q^* is small*. So choose finitely many $b_1, \ldots, b_n$ realizing q^* such that for any $r^* \in \mathrm{Tr}_A(q^*)$, $r^*|Ab_i$ is modular for some i. It follows that for any $r \in Y'$, $r|Ab_ic^*$ is modular for some i, that is, there is a b_r realizing

r with $b_r \in \mathrm{cl}_p(Ab_ic^*)$. Since p is meager, the division ring underlying the geometry on $r^+(\mathfrak{C})$ is countable (and even locally finite, see [Ne2, 2.3, 2.5]). Hence as in [Ne2, Lemma 2.13] there are countably many elements c_i, $i<\omega$, realizing types in Y' such that for any $r \in Y'$, $b_r \in \mathrm{cl}_p(Ac_i)$ for some i. It follows that for some i, the set of $r \in Y'$ with $b_r \in \mathrm{cl}_p(Ac_i)$ is not meager in Y'. This easily implies that the set Z of $r \in Y'$ with $r|Ac_i$ modular has non-empty interior in Y'. Since Y' is open in Y and Y is open in $\mathrm{Tr}_A(p)$, we see that Z has non-empty interior in $\mathrm{Tr}_A(p)$. So we can choose finitely many conjugates of c_i over A so that the corresponding conjugates of Z cover $\mathrm{Tr}_A(p)$. Hence (1) holds.

Case 2: *q^* is big*. By Theorem 2.4 in this case q^* is isolated and $\mathrm{Tr}_A(q^*)$ is open in $\mathcal{G}$. In this case we shall prove that p is isolated. Let $A' = Ac^*$ and choose $p' \in S_{p,\mathrm{nf}}(A')$ with $\mathrm{Tr}_A(p')$ open in Y'. Clearly it suffices to prove that p' is isolated. Notice that still every $r \in \mathrm{Tr}_A(p')$ corresponds to some $r^* \in \mathrm{Tr}_A(q^*)$, and every $r^* \in \mathrm{Tr}_A(q^*)$ corresponds to some $r \in \mathrm{Tr}_A(p')$. Also $\mathrm{Tr}_A(p')$ is canonically homeomorphic to $\mathrm{Tr}_{A'}(p')$. Since forking is meager on $\mathrm{Tr}_A(q^*)$, we also find that forking is meager on $\mathrm{Tr}_{A'}(p')$. Similarly we show that

(c) for any a realizing an $r \in \mathrm{Tr}_{A'}(p')$, the set $CL_{A'}(a) = \{r' \in \mathrm{Tr}_{A'}(p') : r'|A'a \text{ is modular}\}$ is closed and nowhere dense in $\mathrm{Tr}_{A'}(p')$,

(d) for any finite B, the set $CL_{A'}(B) = \{r \in \mathrm{Tr}_{A'}(p') : r|A'B \text{ is modular}\}$ is a finite union of sets of the form $CL_{A'}(a)$, $a \models r \in \mathrm{Tr}_{A'}(p')$.

Statement (d) may be also deduced from (c) as in [Ne2, Lemma 2.13]. Using (c) and (d) and assuming that p' is non-isolated, we can proceed as in the proof of Theorem 2.4 (using [Ne2, Claim 2.14]) constructing many countable models of T. Thus p' must be isolated.

It is obvious that conditions (1) and (2) are mutually exclusive.

When a meager type $p \in S(A)$ satisfies (1) of Theorem 2.6, we may regard it small. Hence we can restate Theorem 2.6 by saying that a non-isolated meager type is small. In the following corollary, (1) is basically a variant of Theorem 2.6 and (2) corresponds to Corollary 2.5.

COROLLARY 2.7. *Assume T is superstable with few countable models, A is finite, $p \in S(A)$ is regular and forking is meager on* $\mathrm{Tr}_A(p)$. *Then*

(1) *p is meager and isolated, and*

(2) $\mathcal{M}(p) = 1 + \max\{\mathcal{M}(q) : q \in S_{p,\mathrm{nf}}(AB),\ B \textit{ is finite and } q \textit{ is small}\}$.

Proof. (1) As in [Ne2, Lemma 1.6(1)] we deduce that p is non-trivial (the assumption of almost strong regularity there may be weakened to regularity). By the proof of (a) of Theorem 2.6 we can assume that all stationarizations of p are non-orthogonal. Considering $\Phi = \mathrm{Tr}_A(p)$, by [Ne1, end of

Section 1] we find that p is weakly meager, hence meager. Hence either (1) or (2) of Theorem 2.6 holds. But if condition (1) of Theorem 2.6 holds, then any $r \in \mathrm{Tr}_A(p)$ has a forking extension over $A \cup \{a_1, \ldots, a_n\}$ and forking is not meager on $\mathrm{Tr}_A(p)$. So condition (2) of Theorem 2.6 holds and p is isolated.

(2) follows from Theorem 2.6 just as Corollary 2.5 follows from Theorem 2.4.

3. Between meager and strongly regular types. Suppose T is superstable with few countable models and p is a non-trivial regular stationary type. Meagerness and strong regularity may be regarded as two extreme properties of regular types. In this section we investigate these and some intermediate properties of regular types. In [Ne2] I conjectured that p is either meager or non-orthogonal to a strongly regular type. This is true when T is 1-based. Indeed, in this case, without loss of generality, p is a generic type of a 0-definable locally modular abelian group G, which is a p-formula. If forking is not meager on $\mathcal{G}$, then for some formula $\psi(x)$ forking over $\emptyset$, $\mathrm{Tr}(\psi) \cap \mathcal{G}$ is non-empty and open in $\mathcal{G}$. By [H-P], ψ is a Boolean combination of cosets of $\mathrm{acl}(\emptyset)$-definable subgroups of G. It folows that for some non-generic 0-definable subgroup H of G and for a generic $a \in G^0$, $\mathrm{Tr}((H + a)(x)) \cap \mathcal{G}$ is non-empty and open in $\mathcal{G}$. Thus G/H is a strongly regular group non-orthogonal to p. The next corollary deals with the case where G is meager.

Corollary 3.1. *Assume T is superstable*, 1-*based*, *with few countable models and G is a* 0-*definable meager group. Then for some non-generic* 0-*definable subgroup H of G*, *G/H is a meager group with $\mathcal{M}(G/H) = 1$. Clearly*, *G/H is non-orthogonal to G.*

Proof. 1-basedness implies there is a 0-definable subgroup H of G such that for generic $a \in G^0$, $\mathrm{Tr}((a+H)(x)) \cap \mathcal{G} = \mathcal{G}m$. Let $\mathcal{G}'$ be the set of generic types of G/H and $\mathcal{G}m'$ the set of modular types in $\mathcal{G}'$. Clearly $\mathcal{M}(\mathcal{G}m') = 0$, in fact $\mathcal{G}m'$ is a singleton. Hence by Corollary 2.5, $\mathcal{M}(G/H) = \mathcal{M}(\mathcal{G}') = 1$.

Working within a p-formula, we can refine the argument yielding Corollary 3.1.

Proposition 3.2. *Assume T is superstable*, 1-*based*, *p is a regular stationary type and $\varphi(x)$ is a p-formula over A.*

(1) *If p is not meager then for some $a \in \varphi(\mathfrak{C})$ and $b \in \mathrm{acl}_A(a)$*, $\mathrm{stp}(b/A)$ *is strongly regular*, *non-orthogonal to p.*

(2) *If p is meager then for some $a \in \varphi(\mathfrak{C})$ and $b \in \mathrm{acl}_A(a)$*, $\mathrm{tp}(b/A)$ *is regular*, *isolated*, *non-orthogonal to p and $\mathcal{M}(b/A) = 1$.*

Proof. Without loss of generality $A = \emptyset$.

(1) Suppose $\psi(x, c)$ is a formula forking over $\emptyset$ with $\mathrm{Tr}(\psi(x, c)) \cap P_\varphi$ non-meager in P_φ. We will use only the following (weak) consequence of 1-basedness.

(a) If a realizes $\psi(x, c)$ then $\mathrm{acl}(a) \cap \mathrm{acl}(c) \setminus \mathrm{acl}(\emptyset) \neq \emptyset$.

By (a), for each $a \in \psi(\mathfrak{C}, c)$ we can choose $b_a \in \mathrm{acl}(a) \cap \mathrm{acl}(c) \setminus \mathrm{acl}(\emptyset)$ and a formula $\chi_a(x, y) \vdash \varphi(x)$ true of (a, b_a), witnessing that $b_a \in \mathrm{acl}(a)$ and $a \in \mathrm{cl}_p(b_a)$. Clearly $\chi_a(x, b_a)$ forks over $\emptyset$.

By compactness, there is $a^* \in \psi(\mathfrak{C}, c)$ and $b = b_{a^*}$ such that $\mathrm{Tr}(\chi(x, b)) \cap P_\varphi$ is non-meager, where $\chi = \chi_{a^*}$. Without loss of generality, $\mathrm{Tr}(\chi(x, b))$ is open in P_φ, and so $\mathrm{Tr}(\chi(x, b)) \cap P_\varphi = [\delta] \cap P_\varphi$ for some $\delta(x)$ over $\mathrm{acl}(\emptyset)$. Since for $a \in \chi(x, b)$ we have $b \in \mathrm{acl}(a)$ and $a \in \mathrm{cl}_p(b)$, we see that whenever $\chi(a', b')$ holds and $w_p(a') > 0$ then $\mathrm{stp}(b')$ is regular, non-orthogonal to p and $\chi(x, b') \vdash P_\varphi(x)$, that is, $\mathrm{Tr}(\chi(x, b')) \subset P_\varphi$.

Choose $a' \in \delta(\mathfrak{C}) \cap P_\varphi(\mathfrak{C})$ such that $\mathrm{Tr}(\chi(a', y))$ has maximal size and $r_0 = \mathrm{stp}(b) \in \mathrm{Tr}(\chi(a', y))$. To prove the proposition, we show that r_0 is strongly regular.

Let $\mathrm{Tr}(\chi(a', y)) = \{r_0, \ldots, r_k\}$ for some k. Choose $\chi_i \in r_i$ for $i \leq k$ such that χ_i, $i \leq k$, are pairwise contradictory. So for some $\delta'(x) \in \mathrm{stp}(a')$, $\delta' \vdash \delta$, and if $a'' \in \delta'(\mathfrak{C})$ then $\mathrm{Tr}(\chi(a'', y))$ meets each $[\chi_i]$. Let $\chi'_0(y)$ be

$$\chi_0(y) \wedge \exists x\, (\delta'(x) \wedge \chi(x, y)).$$

By the maximality of k we see that for $a'' \in \delta'(\mathfrak{C}) \cap P_\varphi(\mathfrak{C})$,

$$\mathrm{Tr}(\chi(a'', y)) \cap [\chi'_0] = \{r_0\}.$$

(Clearly, $r_0 \in \mathrm{Tr}(\chi(a'', y))$ since $\delta(a'')$ holds.) If b'' realizes $r'' \in \mathrm{Str}_y(\emptyset) \cap [\chi'_0] \setminus \{r_0\}$, then for some $a^* \in \delta'(\mathfrak{C})$, $\chi(a^*, b'')$ holds. Thus $w_p(a^*) = 0$ and also $w_p(b'') = 0$ (as $b'' \in \mathrm{acl}(a^*)$). Hence χ'_0 witnesses that r_0 is strongly regular.

(2) If p is meager then by [Ne2, Corollary 1.8] the set P_φ has non-empty interior in $S(\mathrm{acl}(A))$. So we can assume that φ isolates p and $A = \emptyset$. Without loss of generality, $p \in S(\emptyset)$ and φ isolates p. By [Ne2, Claim 2.14] choose a formula $\psi(x, y)$ such that for any a realizing p,

$$\mathrm{Tr}(\psi(x, a)) = \{r \in \mathrm{Tr}(p) : r|a \text{ is modular}\}.$$

Fix an a realizing p. As in the proof of (1) we find a formula $\chi(x, y)$ implying $y \in \mathrm{acl}(x)$, and a b such that $\mathrm{Tr}(\chi(x, b))$ is an open subset of $\mathrm{Tr}(\psi(x, a))$. Without loss of generality, $\chi(a, b)$ holds and $\chi(a, y) \vdash \mathrm{tp}(b/a)$. Choose $E \in FE(\emptyset)$ such that

$$\mathrm{Tr}(\chi(x, b)) \cap [E(x, a)] = \mathrm{Tr}(\psi(x, a)) \cap [E(x, a)].$$

Let $q = \mathrm{tp}(b)$. Clearly, q is regular, non-orthogonal to p, hence meager. Also, $q(y)$ is isolated by the formula $\exists x\,(\varphi(x) \wedge \chi(x,y))$. Let

$$E'(y,y') = \forall x\,\exists x'\,((\chi(x,y) \rightarrow \chi(x',y') \wedge E(x,x')) \wedge (\chi(x,y') \rightarrow \chi(x',y) \wedge E(x,x'))).$$

Clearly E' is a finite equivalence relation.

Suppose $r \in \mathrm{Tr}(q) \cap [E'(x,b)]$ and $r|b$ is modular, that is, for some b' realizing r, $b' \not\perp b$. Choose a' with $\chi(a',b')$ and $E(a,a')$. So $a'b' \equiv ab$, $\mathrm{Tr}(\psi(x,a')) = \mathrm{Tr}(\psi(x,a))$ and

$$\mathrm{Tr}(\chi(x,b')) \cap [E(x,a')] = \mathrm{Tr}(\psi(x,a')) \cap [E(x,a')].$$

It follows that

$$\mathrm{Tr}(\chi(x,b)) \cap [E(x,a)] = \mathrm{Tr}(\psi(x,a)) \cap [E(x,a)] = \mathrm{Tr}(\chi(x,b')) \cap [E(x,a)].$$

Since $\mathrm{stp}(a)$ is in this set, there is $a'' \overset{\mathrm{s}}{\equiv} a$ realizing $\chi(x,b')$. In particular, $b' \in \mathrm{acl}(a'')$. It follows that r is realized in $\mathrm{acl}(a)$. This shows that for any finite B and small $q' \in S_{q,\mathrm{nf}}(B)$, $\mathcal{M}(q') = 0$. By Corollary 2.7(2), $\mathcal{M}(q) = 1$.

The theory in Example 1.10 of [Ne3] is not small. However, modifying this example, for any $n > 0$, one can get an example of a 1-based superstable meager group G with $\mathrm{Th}(G)$ having few countable models and $\mathcal{M}(G) = n$. Notice that by Theorem 2.3, for superstable T with few countable models, trivially $\mathcal{M} \le U$ holds (see [Ne1]), answering a question from [Ne2]. Still we do not have any example of a small superstable T with a type p with $\omega \le \mathcal{M}(p) < \infty$.

Conjecture 3.3 ($\mathcal{M}$-gap conjecture). *In a small superstable theory there is no type p with $\omega \le \mathcal{M}(p) < \infty$.*

See [Ne5] for a partial result regarding this conjecture. Regarding Corollary 3.1 and Proposition 3.2, when T is not 1-based, but has few countable models, we can still prove that there is a finite bound m such that every meager type is non-orthogonal to a meager group of $\mathcal{M}$-rank $\le m$. The proof uses some ideas from [T].

Following [T] we say that a regular type p over a finite set A is *eventually strongly non-isolated* (esn, for short) if some non-forking extension p' of p over some finite $A' \supset A$ is strongly non-isolated, that is, for every finite $B \supset A'$, p' is almost orthogonal to any isolated type in $S(B)$. By [T], for regular types, being esn is invariant under non-orthogonality.

Below we shall give a characterization of non-trivial regular esn types in a small superstable T. First we prove the following fact, following easily from [H-S].

Fact 3.4. *Assume T is small, superstable and p is a non-trivial regular type. Then there is a finite set A and a regular type $p' \in S(A)$ which is* *non-orthogonal to* p *and almost strongly regular* (*asr*) *via some* $\varphi(x) \in p'$. *Moreover*, *all stationarizations of* p' *are non-orthogonal.*

Proof. By [H-S] there is a p-formula $\psi(x)$ over some finite set A. By smallness, for some formula $\varphi(x)$ over A, for some type $p' \in S(A)$, $P_\psi(x) \cup \{\varphi(x)\}$ is consistent and implies $p'(x)$. It follows that every stationarization of p' is regular and non-trivial, and p' is asr via φ. By [Ne2, Lemma 1.6], there are finitely many non-orthogonality classes on $\mathrm{Tr}_A(p')$, and all of them are open in $\mathrm{Tr}_A(p')$. Hence expanding A by an element of $\mathrm{acl}(A)$, and modifying φ a little, we can assume that all stationarizations of p' are non-orthogonal to p.

Notice that the formula $\varphi(x)$ from the conclusion of Fact 3.4 is necessarily a p-formula over A (see [Ne2, Lemma 1.6]). The next proposition is a kind of dichotomy theorem for non-trivial regular esn types.

PROPOSITION 3.5. *Assume* T *is small*, *superstable and* p *is a non-trivial regular type. Then* p *is eventually strongly non-isolated iff* (1) *or* (2) *below holds.*

(1) *For some* p' *as in Fact* 3.4, p' *is non-isolated.*
(2) p *is meager.*

Moreover, *conditions* (1) *and* (2) *are mutually exclusive.*

Proof. $\Rightarrow$ Suppose p is a non-trivial regular esn type and (1) fails. We shall prove (2). Choose a p-formula $\varphi(x)$ over a finite set A. It suffices to prove that forking is meager on P_φ. Suppose not. So choose a formula $\psi(x)$ over Ab_0 forking over A, with $\mathrm{Tr}_A(\psi) \cap P_\varphi$ non-empty and open in P_φ. Let $n = w(b_0/A)$. We see that there is no A-independent set $J \subset \psi(\mathfrak{C})$ of size $> n$. So we can choose a maximal finite A-independent set $J' \subset \psi(\mathfrak{C}) \cap P_\varphi(\mathfrak{C})$. In particular, every $r \in P_\varphi \cap \mathrm{Tr}_A(\psi)$ has a forking extension over AJ'. By definability of p-weight 0 on φ, there is a single formula $\psi'(x)$ over AJ', forking over A, with $\mathrm{Tr}_A(\psi') \cap P_\varphi$ non-meager in P_φ. So without loss of generality $b_0 \subset J'$.

Next we can choose an A-independent tuple $b \subset P_\varphi(\mathfrak{C})$ of minimal size such that for some formula $\psi(x)$ over Ab, forking over A, $\mathrm{Tr}_A(\psi) \cap P_\varphi$ is not nowhere dense in P_φ. Say, $b = \langle a, a_1, \ldots, a_n \rangle$, $b' = \langle a_1, \ldots, a_n \rangle$, and let $r = \mathrm{stp}(a/A)$. Since r is esn, for some finite B extending A, with $a \perp B\,(A)$, $\mathrm{tp}(a/B)$ is stationary and strongly non-isolated. Without loss of generality, $b \perp B\,(A)$. Let $A' = Bb'$. Obviously $\varphi(x)$ remains a p-formula over A', the new set P'_φ ($= P_\varphi$ evaluated in $S(\mathrm{acl}(A'))$) is the set of non-forking extensions of the types from P_φ over $\mathrm{acl}(A')$. Since P'_φ is closed in $S(\mathrm{acl}(A'))$, we can regard it as a type over $\mathrm{acl}(A')$, in variable x. First notice that

(a) $$\{\psi(x)\} \cup P'_\varphi(x) \text{ forks over } A'.$$

Suppose not. Choose c realizing $\{\psi(x)\} \cup P'_\varphi(x)$ with $c \mathop{\smile\!\!\!\!\!\mid}\, b\,(A')$ and $c \mathop{\smile\!\!\!\!\!\mid}\, A'\,(A)$. Then $c \mathop{\smile\!\!\!\!\!\mid}\, b\,(A)$, a contradiction.

Also we have

(b) $$\mathrm{Tr}_{A'}(\psi) \cap P'_\varphi \text{ has non-empty interior in } P'_\varphi.$$

Suppose not. Then $\mathrm{Tr}_{A'}(\psi) \cap P'_\varphi$ is nowhere dense in P'_φ. In particular, $\mathrm{Tr}_A(\mathrm{Tr}_{A'}(\psi) \cap P'_\varphi)$ is nowhere dense in P_φ. Let X be the set of $r' \in P_\varphi$ having a forking extension over Ab'. Using the minimality of b and definability of p-weight 0 on φ we see that X is a meager subset of P_φ. On the other hand, clearly

$$\mathrm{Tr}_A(\psi) \cap P_\varphi \subset X \cup \mathrm{Tr}_A(\mathrm{Tr}_{A'}(\psi) \cap P'_\varphi),$$

hence $\mathrm{Tr}_A(\psi)$ is meager in P_φ, a contradiction.

By the proof of Fact 3.4 we find a formula φ' over A' below φ and a type $q \in S(A')$ such that $P'_\varphi \cap [\varphi'] \subset \mathrm{Tr}_{A'}(\varphi) \cap P'_\varphi$ and $P'_\varphi(x) \cup \{\varphi'(x)\}$ is consistent and implies q. We see that q is asr via φ'. As in Fact 3.4 we can assume that all stationarizations of q are non-orthogonal. Since (1) fails, q is isolated. On the other hand, ψ witnesses that q is not almost orthogonal to $\mathrm{tp}(a/A')$, a contradiction.

$\Leftarrow$ Supposing (1) or (2) holds, we must show that p is esn.

Case 1: (1) *holds*. [T, 3.1.3] proves that in this case p' is strongly non-isolated. For the sake of completeness we include a short proof. Say, $p' \in S(A)$ is asr via $\varphi \in p'$. Suppose B is a finite extension of A and $q \in S(B)$ is isolated and not almost orthogonal to a non-forking extension of p' over B. We can choose A, B, p' and q so that $U(q)$ is minimal possible. Let a realize q and choose b realizing p' with $b \mathop{\smile\!\!\!\!\!\mid}\, B\,(A)$ and $a \not\mathop{\smile\!\!\!\!\!\mid}\, b\,(B)$. The dependence of a and b over B is witnessed by a formula $\chi(x,y)$, true of (a,b). Take any c realizing an isolated type in $S(Ba)$ such that $\chi(a,c)$ holds. Since q is isolated, also $\mathrm{tp}(c/B)$ is isolated. It follows that $a \not\mathop{\smile\!\!\!\!\!\mid}\, c\,(B)$ and $\mathrm{tp}(c/B)$ is orthogonal to p'. Hence $b \mathop{\smile\!\!\!\!\!\mid}\, c\,(B)$ and $b \not\mathop{\smile\!\!\!\!\!\mid}\, a\,(Bc)$. Putting $B' = Bc$ we see that $q' = \mathrm{tp}(a/B')$ is isolated, $U(q') < U(q)$ and q' is not almost orthogonal to $\mathrm{tp}(b/B')$, contradicting the minimality of $U(q)$.

Case 2: (2) *holds, so p is meager*. Without loss of generality, p is the principal generic type of a 0-definable meager group G. We will show that p is strongly non-isolated.

Choose a decreasing sequence of formulas φ_n, $n < \omega$, over $\emptyset$, such that $p \equiv \{\varphi_n(x) : n < \omega\}$. We can assume that for each n, $([\varphi_n] \setminus [\varphi_{n+1}]) \cap \mathcal{G}' \neq \emptyset$, where $\mathcal{G}' = \mathcal{G} \setminus \mathcal{G}m$. We proceed as in the proof of Theorem 2.4. Choose isolated $p_n \in S_{\mathrm{gen}}(\emptyset)$, $n < \omega$, with $\varphi_n \wedge \neg\varphi_{n+1} \in p_n$. As in the proof of Theorem 2.4 we see that

(c) for every $r \in \mathcal{G}$, $r + \mathcal{G}m$ meets $\mathrm{Tr}(p_n)$ for at most finitely many n.

Now suppose B is finite and ψ isolates a complete type q over B. For $n < \omega$ let $p'_n \in S_{\rm gen}(B)$ be a non-forking isolated extension of p_n over B. Suppose q is not almost orthogonal to $p|B$. By definability of p-weight 0 on G, for every n large enough, q is not almost orthogonal to p'_n. Without loss of generality, this holds for every n. Let a realize q. Choose a_n realizing p'_n with $a_n \not\downarrow a\,(B)$. In particular, $a_n \not\downarrow aB$. We claim that

(d) there is an infinite $X \subset \omega$ such that $\{a_n, n \in X\}$ is independent.

Suppose not. Then for some finite m, every a_n depends on $a_{<m}$. In other words, for every n, $r_n|a_{<m}$ is modular, where $r_n = \operatorname{stp}(a_n)$. By [Ne2, Lemma 2.4] (or see the proof of Theorem 2.4), there are finitely many types $r^0, \ldots, r^l \in \mathcal{G}$ such that every type r with $r|a_{<m}$ modular belongs to $r^i + \mathcal{G}m$ for some i. This contradicts (c), proving (d).

By (d) we see that $w(aB)$ is infinite, contradicting superstability of T. In fact, this last step does not require superstability, instead we could use smallness (see [Ne4]).

To prove that conditions (1) and (2) are mutually exclusive, suppose (2) holds, that is, p is meager. If (1) holds for some $p' \in S(A)$, which is asr via $\varphi \in p'$, then φ is a p-formula over A and $P_\varphi = \operatorname{Tr}_A(p')$. In this case [Ne2, Corollary 1.8] says that $P'_\varphi = \{r \in P_\varphi : r \text{ is not modular}\}$ is non-empty and open in $S(\operatorname{acl}(A))$, while p' being non-isolated implies that P_φ is nowhere dense in $S(\operatorname{acl}(A))$, a contradiction.

Notice that Proposition 3.5 provides us also with a characterization of a meager type in a small superstable theory: a regular non-trivial type p is meager iff p is esn and every asr type p' non-orthogonal to p is isolated.

COROLLARY 3.6. *Assume T is superstable and $I(T,\aleph_0) < 2^{\aleph_0}$. Then there is a natural number m such that for every meager type p there is a meager group G non-orthogonal to p with $\mathcal{M}(G) \le m$.*

Proof. The proof relies on [T, Lemma 3.3.3]. Suppose the conclusion fails. Then there are meager types p_n, $n < \omega$, such that for each $n \neq k$, p_n is orthogonal to any conjugate of p_k. By Proposition 3.5, the types p_n, $n < \omega$, are esn. In this situation, [T, Lemma 3.3.3] says that T has $2^{\aleph_0}$-many countable models. The proof consists in varying dimensions of types p_n, $n < \omega$, in countable models of T.

QUESTION 3.7. *Is it possible to get $m = 1$ in Corollary 3.6?*

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