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On automorphisms of Boolean algebras embedded in $P(\omega)/\mathrm{fin}$

by

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Abstract. We prove that, under CH, for each Boolean algebra A of cardinality at most the continuum there is an embedding of A into $P(\omega)/\mathrm{fin}$ such that each automorphism of A can be extended to an automorphism of $P(\omega)/\mathrm{fin}$. We also describe a model of $\mathrm{ZFC} + \mathrm{MA}(\sigma\text{-linked})$ in which the continuum is arbitrarily large and the above assertion holds true.

It is well known that, under CH (the continuum hypothesis), each Boolean algebra of cardinality at most 2^ω can be embedded in $P(\omega)/\mathrm{fin}$ (see e.g. [5]). This implication cannot be reversed: there is a model of set theory in which $2^\omega > \omega_1$ and the above conclusion still holds ([1]). It is also known that CH is equivalent to the following condition: each Parovičenko algebra (i.e. algebra of cardinality 2^ω, atomless and having neither countable limits nor countable unfilled gaps) is isomorphic to $P(\omega)/\mathrm{fin}$. We begin by proving the following.

Proposition 1. *If* CH *holds*, *then for every Boolean algebra A of cardinality at most the continuum there is an embedding $i : A \to P(\omega)/\mathrm{fin}$ such that each automorphism of $i(A)$ can be extended to an automorphism of $P(\omega)/\mathrm{fin}$.*

Proof. Assume CH. Let A be a Boolean algebra of cardinality at most the continuum. We will construct an extension $A^\star$ of A such that:

1. $A^\star$ is a Parovičenko algebra;

2. $\bigcup_{\alpha<\omega_1} A_\alpha = A^\star$, where $(A_\alpha : \alpha < \omega_1)$ is an increasing sequence of algebras satisfying the following two conditions:

($\star$) $\operatorname{card} A_\alpha \le 2^\omega$, $A_0 = A$, $A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$ for every limit $\lambda < \omega_1$,

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($\star\star$) for every automorphism ϕ_α of A_α there exists an automorphism $\phi_{\alpha+1}$ of $A_{\alpha+1}$ such that $\phi_\alpha \subseteq \phi_{\alpha+1}$.

It is clear that if an algebra $A^\star$ satisfies the above conditions then each automorphism of A can be extended to an automorphism of $A^\star$. Thus to prove our theorem it suffices to construct $A^\star$.

Fix a pairing $k : \omega_1 \times \omega_1 \to \omega_1$ (one-to-one and onto) such that $k(\zeta, \xi) \geq \zeta$ for all $\zeta, \xi < \omega_1$. It remains to describe the successor step from α to $\alpha + 1$. Suppose that we have defined a sequence $(A_\gamma : \gamma \leq \alpha)$ satisfying (inductive) conditions ($\star$) and ($\star\star$).

Let $\mathcal{E}_\gamma(\phi, a, c)$ abbreviate the statement: *ϕ is an automorphism of A_γ such that $\phi(a) = c$.*

Assume that at each stage $\gamma \leq \alpha$ we chose an enumeration $(x^\gamma_\xi : \xi < \omega_1)$ of the collection of the following families:

$$\{(c_i, d_j : i, j < \omega) : \exists\phi\ \forall i, j < \omega\ [\mathcal{E}_\gamma(\phi, a_i, c_i) \wedge \mathcal{E}_\gamma(\phi, b_i, d_i)]\},$$

where $(a_i, b_j : i, j < \omega)$ is a countable ordered gap $a_0 < a_1 < \ldots < b_1 < b_0$ of elements of A_γ, $\{(b_i : i < \omega) : \exists\phi\ \forall i < \omega\ \mathcal{E}_\gamma(\phi, a_i, b_i)\}$, where $(a_i : i < \omega)$ is a decreasing chain of elements of A_γ, and the set of all atoms of the algebra A_γ. (Since we assumed CH, we have at most ω_1 objects to enumerate.)

We identify A_α with the field $\mathcal{B}(X_\alpha)$ of open-closed subsets of the associated Stone space X_α. The ordinal α determines a certain object, namely x^ζ_ξ, where ξ and ζ are ordinals such that $k(\zeta, \xi) = \alpha$ ($\zeta < \alpha$). If x^ζ_ξ is a family of chains or gaps, we take $A_{\alpha+1}$ to be the subfield of $P(X_\alpha)$ generated by $A_\alpha = \mathcal{B}(X_\alpha)$ and by

$$\Big\{b = \bigcap_{i<\omega} b_i : (a_j, b_i : i, j < \omega) \in x^\zeta_\xi\Big\}$$

when x^ζ_ξ is a collection of gaps, or by

$$\Big\{b = \bigcap_{i<\omega} b_i : (b_i : i < \omega) \in x^\zeta_\xi\Big\}$$

when x^ζ_ξ consists of countable chains. Using the Sikorski theorem (on extending homomorphisms, see e.g. [7], [5]) we extend each automorphism of A_α to an automorphism of $A_{\alpha+1}$ and therefore ($\star$), ($\star\star$) hold.

Now, suppose that x^ζ_ξ is a set of nonzero elements of A_α. Each element of the family is an atom of A_ζ but it need not remain an atom in A_α. If there are at least countably many elements $e_i < a$, $a \in x^\zeta_\xi$, then we put $A_{\alpha+1} = A_\alpha$. (Note that the property ($\star\star$) implies that if some element of x^ζ_ξ is an atom then all elements of the set are atoms.) Suppose that each $a \in x^\zeta_\xi$ is a finite sum of atoms $a = e_1 + \ldots + e_n$. (n is the same for all elements of x_ξ^ζ by ($\star\star$).) Atoms of A_α correspond to isolated points of X_α. We delete the isolated points $e_i < a$ for all $a \in x_\xi^\zeta$, and put into their places copies of a one-point compactification of the discrete ω. Let $X_{\alpha+1}$ denote the topological space thus obtained. We put $A_{\alpha+1} = \mathcal{B}(X_{\alpha+1})$. Since X_α is a continuous image of $X_{\alpha+1}$, we have $A_\alpha \subseteq A_{\alpha+1}$. Obviously, each automorphism of A_α can be extended to an automorphism of $A_{\alpha+1}$. This finishes the proof. ∎

Now we consider the case of ¬CH. It is known that there exists a model of ZFC+MA+¬CH in which the algebra $P(\omega_1)$ is not embeddable in $P(\omega)/\text{fin}$ ([2]). On the other hand, it is consistent with ZFC and MA(σ-linked) that the cardinality of the continuum is arbitrarily large and each Boolean algebra of cardinality $\leq 2^\omega$ can be embedded in $P(\omega)/\text{fin}$ ([1]). Thus the existence of such embeddings does not imply CH. The assertion of Proposition 1 is stronger and we may ask if the converse holds. The answer is negative. We prove that:

THEOREM 1. *It is consistent with* ZFC + MA(σ*-linked*) *that the cardinality of the continuum is arbitrarily large and for each Boolean algebra B of cardinality $\leq 2^\omega$, there is an embedding $i : B \to P(\omega)/\text{fin}$ such that each automorphism of $i(B)$ can be extended to an automorphism of $P(\omega)/\text{fin}$.*

Proof. Let $\mathbf{V}$ be a ground model satisfying the generalized continuum hypothesis (GCH). Thus there exists a regular cardinal κ in $\mathbf{V}$ such that $\kappa > \omega_1$ and if $\kappa = \lambda^+$, then $\text{cf}(\lambda) > \omega$, moreover $\diamondsuit_\kappa$ (the diamond principle) holds in the form:

There is a sequence $(T_\alpha : \alpha < \kappa,\ \text{cf}(\alpha) = \omega_1)$ such that for every set $X \subseteq H(\kappa)$ the set $\{\alpha < \kappa : \text{cf}(\alpha) = \omega_1,\ X \cap H_\alpha = T_\alpha\}$ is stationary in κ.

$H(\kappa)$ denotes as usual the family of all sets of hereditary power $< \kappa$, $H(\kappa) = \bigcup_{\alpha<\kappa} H_\alpha$, and $(H_\alpha : \alpha < \kappa)$ is a continuously increasing sequence of sets of cardinality $< \kappa$.

We will define a finite support iteration $(\mathsf{P}_\alpha : \alpha < \kappa)$ having the c.c.c. (countable chain condition) such that, in the corresponding generic extension $\mathbf{V}[G]$, the conclusion of Theorem 1 will be satisfied. The model $\mathbf{V}$ will be extended in such a way that given a system of generators for a certain Boolean algebra B ($\text{card}\, B < 2^\omega$) there will be an embedding sending the generators to generic sets added at some steps $\alpha < \kappa$. The embedding will be defined by induction: If a certain monomorphism embeds the subalgebra B_0 of B generated by the initial α ($\alpha < 2^\omega$) generators and if the image (under the monomorphism) of each of them is a generic subset of $P(\omega)/\text{fin}$ then the next generator determines in B_0 two sets (which form a gap): one consists of elements less than the generator (called the "lower class"), and the elements of the other (called the "upper class") are disjoint from the generator. An image of the gap is a gap in the subalgebra of $P(\omega)/\mathrm{fin}$ which is generated by some generic sets. The monomorphism can be extended if there is an element of $P(\omega)/\mathrm{fin}$ which fills this gap. Thus, to ensure embeddability of Boolean algebras, we will add generic sets $X_\alpha \subseteq \omega$ filling gaps in subalgebras of $P(\omega)/\mathrm{fin}$ generated by some previously added $X_\beta, \beta < \alpha$. Simultaneously, in a similar way, we will extend automorphisms of the subalgebras. In constructing embeddings of Boolean algebras and extensions of their automorphisms, we have to avoid the following problem: It is well known that there are gaps in $P(\omega)/\mathrm{fin}$ which are unfillable by c.c.c. forcing. It could happen, unless steps are taken to prevent it, that an image (under an extension of an automorphism of one of the embedded algebras) of some gap filled in a later step is an unfillable gap.

To ensure that every automorphism of an embedded Boolean algebra can be extended we use the $\diamondsuit$ principle. It guarantees that each such automorphism is "approximated" by an increasing sequence of automorphisms which belong to models $\mathbf{V}[G|\alpha]$. To be more precise: if F is a canonical P_κ-name for some automorphism f of a given algebra B (from the model $\mathbf{V}[G]$) then there is a subset A of κ such that

$$\operatorname{card} A = \kappa \quad \text{and} \quad \bigcup_{\alpha\in A}(F \cap H_\alpha) = \bigcup_{\alpha\in A} T_\alpha$$

and T_α is a P_α-name for an automorphism from $\mathbf{V}[G|\alpha]$. We will extend automorphisms using those of the T_α's which are their names. To obtain MA(σ-linked) we will enumerate at some stages (with repetition) all σ-linked forcings R with $\operatorname{card}\mathsf{R} < \kappa$ (cf. [1], [4]).

Assume the following notation:

Let X be a set and let T_ξ denote a homomorphism. Then for $\varepsilon \in \{-1, 1\}$, εX denotes X, if $\varepsilon = 1$, or $\setminus X$, if $\varepsilon = -1$. Moreover, T_ξ^ε is T_ξ, if $\varepsilon = 1$, or T_ξ^{-1}, if $\varepsilon = -1$. (We abbreviate $(T_\xi^\varepsilon)^n$ to $T_\xi^{\varepsilon n}$.)

For $\varphi : \alpha \to \{0, 1\}$ let $B(\varphi)$ be the subalgebra generated by $\{X_\beta : \varphi(\beta) = 1\}$, where X_β is a generic subset of ω added at stage β. If s is a finite sequence with $\operatorname{dom}(s) \subseteq \{\beta : \varphi(\beta) = 1\}$ and $\operatorname{rg}(s) \subseteq \{-1, 1\}$ then

$$X(s) = \bigcap_{s(\xi)=1} X_\xi \cap \bigcap_{s(\zeta)=-1} (\omega \setminus X_\zeta).$$

Thus $B(\varphi)$ consists of finite unions of sets of the form $X(s)$. A *gap* in $B(\varphi)$ is a system of the form

$$\mathcal{L} = (\{X(s) : s \in S\}, \{X(t) : t \in T\}),$$

where $X(s) \cap X(t) =_\star \emptyset$ for all $s \in S$ and $t \in T$.

An *increasingly ordered gap* $\mathcal{L}$ *of type* (λ,γ) is a gap as above such that there are enumerations $S=\{s_\alpha:\alpha<\lambda\}$ and $T=\{t_\beta:\beta<\gamma\}$ satisfying

$$\alpha_1<\alpha_2<\lambda \ \Rightarrow\ X(s_{\alpha_1})\subseteq_\star X(s_{\alpha_2}),$$
$$\beta_1<\beta_2<\gamma \ \Rightarrow\ X(t_{\beta_1})\subseteq_\star X(t_{\beta_2}).$$

We assume that each gap except the increasingly ordered ones satisfies the condition: if $s_1,\ldots,s_n\in S$ and $X(s)\subseteq_\star X(s_1)\cup\ldots\cup X(s_n)$ then $s\in S$ (and similarly for T).

We will use two notions of forcing: Kunen's forcing filling a gap, and the other, which adds an uncountable antichain to Kunen's forcing of type (ω_1,ω_1).

Now we describe the two forcings:

Let $\mathcal{L}=(\{X(s):s\in S\},\{X(t):t\in T\})$ be a gap. *Kunen's forcing* $\mathsf{Q}(\mathcal{L})$ consists of elements of the form $(u_\mathsf{q},x_\mathsf{q},w_\mathsf{q})$, where u_q and w_q are finite subsets of S and T (respectively) and x_q is a finite zero-one sequence. Moreover,

$$\bigcup_{s\in u_\mathsf{q}} X(s)\cap\bigcup_{t\in w_\mathsf{q}} X(t)\subseteq \mathrm{dom}(x_\mathsf{q}).$$

Let $\mathsf{p}=(u_\mathsf{p},x_\mathsf{p},w_\mathsf{p})$ and $\mathsf{q}=(u_\mathsf{q},x_\mathsf{q},w_\mathsf{q})$; then p is an extension of q (written $\mathsf{p}\le\mathsf{q}$) iff $u_\mathsf{q}\subseteq u_\mathsf{p}$, $w_\mathsf{q}\subseteq w_\mathsf{p}$, $x_\mathsf{q}\subseteq x_\mathsf{p}$ and for each i with $\mathrm{dom}(x_\mathsf{q})\le i<\mathrm{dom}(x_\mathsf{p})$,

$$\text{if } i\in\bigcup_{s\in u_\mathsf{q}} X(s) \text{ then } x_\mathsf{p}(i)=1 \quad\text{and}\quad \text{if } i\in\bigcup_{t\in w_\mathsf{q}} X(t) \text{ then } x_\mathsf{p}(i)=0.$$

It is known that if $\mathcal{L}$ is separated, then $\mathsf{Q}(\mathcal{L})$ has the c.c.c.

Now let $\mathcal{L}=(\{X(s_\alpha):\alpha<\omega_1\},\{X(t_\beta):\beta<\omega_1\})$ be an increasingly ordered gap. A condition of forcing $\mathsf{E}(\mathcal{L})$ is a finite set e consisting of sequences of the type $(\alpha,s_\alpha,t_\alpha)$ such that if $(\alpha,s_\alpha,t_\alpha),(\beta,s_\beta,t_\beta)\in\mathsf{e}$ and $\alpha\ne\beta$ then either $X(s_\alpha)\cap X(t_\beta)\ne\emptyset$ or $X(s_\beta)\cap X(t_\alpha)\ne\emptyset$. $\mathsf{E}(\mathcal{L})$ is ordered by inverse inclusion. It is well known that if $\mathcal{L}$ is an unfilled gap then $\mathsf{E}(\mathcal{L})$ has the c.c.c. and

$$\mathsf{E}(\mathcal{L})\Vdash \text{“}\mathsf{Q}(\mathcal{L}) \text{ has an uncountable antichain”}.$$

The definition of the iteration is inductive and uses a “bookkeeping” technique. At each inductive step $\alpha<\kappa$ we enumerate some objects in $\mathbf{V}^{(\mathsf{P}_\alpha)}$, and at higher stages we add some generic sets to them. The objects occur in an order determined by a function Nb. To be more precise, we divide κ into five unbounded sets:

$$A=\{\alpha<\kappa:\mathrm{cf}(\alpha)=\omega_1\},\qquad M=\{\alpha\in\kappa\setminus A:\alpha \text{ is odd}\},$$
$$E=k(A),\quad Q_1=k(M),\quad Q_2=k(\kappa\setminus(A\cup M)),$$

where $k:\kappa\to\kappa\setminus(A\cup M)$ is an increasing bijection.

Let $\{\nu_\alpha : \alpha < \kappa\}$ be an increasing enumeration of the set $\{\beta < \kappa : \beta \geq \lambda\}$, if $\kappa = \lambda^+$, or of the set $\{\beta < \kappa : \beta \text{ is a cardinal and } \operatorname{cf}(\beta) > \omega\}$, if κ is a limit cardinal.

Let $n : \kappa \times \kappa \to \kappa$ be a pairing function satisfying:

$$\begin{array}{ll} \xi, \zeta < n(\xi, \zeta) & \text{for all } \xi, \zeta < \kappa, \\ n(\alpha, \beta) \in M & \text{for all } \alpha \in M,\ \beta < \kappa, \\ n(\alpha, \beta) \in Q_1 & \text{for all } \alpha \in Q_1,\ \beta < \kappa, \\ n(\alpha, \beta_1) < n(\alpha, \beta_2) & \text{for all } \alpha \in A,\ \beta_1 < \beta_2 < \kappa, \\ n(\alpha, \beta) \in Q_2 & \text{for all } \alpha \in A,\ \beta < \nu_\alpha, \\ n(\alpha, \beta) \in E & \text{for all } \alpha \in A,\ \beta > \nu_\alpha, \\ n(\alpha_1, \beta_1) < n(\alpha_2, \beta) & \text{for } \beta_1 < \nu_{\alpha_1},\ \alpha_1 < \alpha_2, \beta < \kappa. \end{array}$$

Using this function we will define (by induction) a function Nb. At stages $\xi \in M$, we will add generic filters to σ-linked forcings. In steps $\xi \in Q_1$, we add (by Kunen's forcing) the generic set X_ξ which fills a gap consisting of some sets previously added (in Q_1 steps). In the model $\mathbf{V}[G]$, each Boolean algebra will be embedded in a certain algebra generated by sets obtained in these steps. At stage $\xi \in Q_2$ we also add (by the same forcing) the generic set X_ξ which separates a gap, but this gap is generated by sets previously added both in Q_1 and Q_2 steps. In the model $\mathbf{V}[G]$ each of these sets X_ξ, $\xi \in Q_2$, will be an image (under one of the extended automorphisms) of some element of $P(\omega)/\text{fin}$ which appeared in some model $\mathbf{V}[G|\delta]$, $\delta < \xi$. In steps $\xi \in E$ we will add uncountable antichains to Kunen's forcing to keep gaps in the ranges (of the extended automorphisms) unfilled.

The sequence in which new elements of $P(\omega)/\text{fin}$ appear is important in our construction. It will be described by the function Ind from $P(\omega)/\text{fin}$ into κ, defined inductively simultaneously with iteration. We begin with the condition: if $x \in P(\omega)/\text{fin} \cap \mathbf{V}$ then $\operatorname{Ind}(x) = 0$. At each higher stage we extend the function Ind according to the rule:

$$\text{If} \quad \mathsf{P}_{\alpha+1} \Vdash \text{“}x \notin \operatorname{dom}(\operatorname{Ind}) \text{ and } x \in P(\omega)/\text{fin”} \quad \text{then} \quad \operatorname{Ind}(x) = \alpha + 1.$$

If $\xi < \kappa$, $\operatorname{cf}(\xi) = \omega_1$ and $\mathsf{P}_\xi \Vdash$ “T_ξ is an automorphism of $B(\varphi)$” (T_ξ is an element of the $\diamondsuit$-sequence), then we begin to define (inductively) families of monomorphisms according to the following conditions:

(a) $T^\xi_\xi = T_\xi$.

(b) For $\gamma \geq \xi$,

$$\mathsf{P}_\gamma \Vdash \text{“}T^\gamma_\xi \text{ is a monomorphism from a subalgebra of } P(\omega)/\text{fin into } P(\omega)/\text{fin”}.$$

(c) If $\gamma_1 < \gamma_2$ then $T^{\gamma_2}_\xi$ is an extension of $T^{\gamma_1}_\xi$.

(d) If $\gamma_1 \leq \gamma_2$, $0 < \xi_i \leq \gamma_i$, $\mathsf{P}_{\xi_i} \Vdash$ “T_{ξ_i} is an automorphism of $B(\varphi_i)$”

$(i = 1, 2)$ and

$$\mathsf{P}_{\max(\xi_1,\xi_2)} \Vdash \text{“For some ordinal } \varrho,\ \varphi_1 \restriction \varrho = \varphi_2 \restriction \varrho \text{ and } T_{\xi_1} \text{ and } T_{\xi_2} \text{ agree on } B(\varphi_1 \restriction \varrho)\text{”},$$

then

$$T^{\gamma_1}_{\xi_1} \restriction \{X \in P(\omega)/\mathrm{fin} : \mathrm{Ind}(X) < \varrho\} \cap \mathrm{dom}(T^{\gamma_1}_{\xi_1}) = T^{\gamma_2}_{\xi_2} \restriction \{X \in P(\omega)/\mathrm{fin} : \mathrm{Ind}(X) < \varrho\} \cap \mathrm{dom}(T^{\gamma_2}_{\xi_2}).$$

(e) If $\lambda \le \alpha$ is a limit ordinal then $T^{\lambda}_{\xi} = \bigcup_{\gamma<\lambda} T^{\gamma}_{\xi}$.

At each stage we will compute $\mathrm{card}\, P(\omega)/\mathrm{fin}$ using the following two (well known) theorems (see e.g. [5], [6]):

THEOREM 2. *Assume that* P *has the c.c.c. in* $\mathbf{V}$ *and let* ν *be a cardinal in* $\mathbf{V}$ *such that* $\mathbf{V} \Vdash$ “$\mathrm{card}\, \mathsf{P} \le \nu$, $\nu^\omega = \nu$”. *Let* Q *be such that* $\mathsf{P} \Vdash$ “$\mathrm{card}\, \mathsf{Q} \le \nu$”. *Then* $\mathrm{card}\, \mathsf{P} \star \mathsf{Q} \le \nu$ *in* $\mathbf{V}$.

THEOREM 3. *Assume that* P *has the c.c.c. in* $\mathbf{V}$ *and* $\lambda, \nu \ge \omega$ *are cardinals in* $\mathbf{V}$ *such that* $\mathbf{V} \Vdash$“$\mathrm{card}\, \mathsf{P} \le \nu$ *and* $\lambda = \nu^\omega$”. *Let* G *be* P*-generic over* $\mathbf{V}$. *Then* $2^\omega \le \lambda$ *in* $\mathbf{V}[G]$.

Thus we have to show that for each α, P_α has the c.c.c. We will do that in the second part of the proof; now we assume that it is true.

We describe the inductive step $\alpha \Rightarrow \alpha + 1$. Assume that the forcing P_α and families of monomorphisms T^{γ}_{ξ} $(\xi \le \gamma \le \alpha)$ satisfying the above conditions (a)–(e) are already defined. Assume also that $\mathrm{card}\, \mathsf{P}_\alpha \le \nu_\alpha$ and $\mathsf{P}_\alpha \Vdash$ “$2^\omega \le \nu_\alpha$”. Since the cardinality of each of the forcings occurring in Cases 1 to 5 below is $\le \nu_\alpha$, by Theorems 1 and 2 we have $\mathrm{card}\, \mathsf{P}_{\alpha+1} \le \nu_{\alpha+1}$ and $\mathsf{P}_\alpha \Vdash$ “$2^\omega \le \nu_{\alpha+1}$”.

We distinguish five cases. In Cases 1, 2 and 5 we set $T^{\alpha+1}_{\xi} = T^{\alpha}_{\xi}$.

C a s e 1: $\alpha \in M$. We enumerate all P_α-names of σ-linked forcings of cardinality $< \kappa$ so that each forcing occurs κ times in the enumeration and the following holds:

If R is ξth element of the enumeration then $\mathsf{P}_\alpha \Vdash$ “$\mathrm{card}\, \mathsf{R} \le \xi$”.

We extend the function Nb: if

$$\mathsf{P}_\alpha \Vdash \text{“}\mathsf{R} \text{ is } \sigma\text{-linked and } \mathrm{card}\, \mathsf{R} < \beta\text{”}$$

and R is the βth element of the above enumeration then $\mathrm{Nb}(\mathsf{R}) = n(\alpha, \beta)$.

If there are $\gamma < \alpha$ and $\beta < \kappa$ such that $\alpha = n(\gamma, \beta)$, then we put

$$\mathsf{P}_{\alpha+1} = \mathsf{P}_\alpha \star \mathsf{R},$$

where $\mathsf{P}_\gamma \Vdash$ “R is σ-linked and $\mathrm{card}\, \mathsf{R} < \kappa$” and $\mathrm{Nb}(\mathsf{R}) = \alpha$.

C a s e 2: $\alpha \in Q_1$. In this case we enumerate all pairs $(\mathcal{L}, \varphi)$ of P_α-names such that P_α forces the following properties:

(a) $\varphi \in \mathcal{D}_\alpha$, where $\mathcal{D}_\alpha$ consists of all $\psi \in \mathbf{V}^{\mathsf{P}_\alpha}$ with $\mathrm{dom}(\psi) \le \alpha$, $\mathrm{rg}(\psi) \subseteq \{0,1\}$ and $\Gamma = \{\gamma : \psi(\gamma) = 1\} \subseteq Q_1$, and such that if $\{\gamma_\xi : \xi < \delta\}$ is an increasing enumeration of Γ, then for each $\xi < \delta$ there is a gap $\mathcal{L}_\xi$ in $B(\psi \restriction \gamma_\xi + 1)$ satisfying $\gamma_{\xi+1} = \mathrm{Nb}(\mathcal{L}_\xi, \psi \restriction \gamma_\xi + 1)$.

(b) $\mathcal{L}$ is a gap in $B(\varphi)$.

Each of these pairs occurs κ times in the enumeration.

If there are $\gamma < \alpha$ and $\beta < \kappa$ such that $n(\gamma, \beta) = \alpha$ then we put

$$\mathsf{P}_{\alpha+1} = \mathsf{P}_\alpha \star \mathsf{Q}(\mathcal{L}),$$

where $\mathsf{P}_\gamma \Vdash$ "$\mathcal{L}$ is a gap in $B(\varphi)$" for some $\varphi \in \mathcal{D}_\gamma$.

C a s e 3: $\mathrm{cf}(\alpha) = \omega_1$. If

$$\mathsf{P}_\alpha \Vdash \text{ "For some } \gamma < \alpha \text{ and } \varphi \in \mathcal{D}_\gamma,\ T_\alpha \text{ is an automorphism of } B(\varphi)\text{"}$$

then two cases are possible:

($\star$) $\mathsf{P}_\alpha \Vdash$ "There is no $\xi < \alpha$ such that T_ξ is an automorphism of $B(\psi)$ and T_α and T_ξ agree on $B(\psi_\xi)$, where $\psi_\xi = \varphi \restriction \varrho_\xi = \psi \restriction \varrho_\xi$ for some ordinal $\varrho_\xi \le \xi$", and

($\star\star$) $\mathsf{P}_\alpha \Vdash$ "There are ordinals ξ, ϱ_ξ and a function $\psi \in \mathcal{D}_\xi$ such that $\varrho_\xi \le \xi < \alpha$, $\mathrm{cf}(\xi) = \omega_1$, $\psi_\xi = \varphi \restriction \varrho_\xi = \psi \restriction \varrho_\xi$, T_ξ is an automorphism of $B(\psi)$ and T_α is an extension of $T_\xi \restriction B(\psi_\xi)$".

Let Υ denote the set of all pairs (ϱ_ξ, ξ) such that P_α forces that T_ξ is an automorphism of $B(\psi)$, $\psi_\xi = \varphi \restriction \varrho_\xi = \psi \restriction \varrho_\xi$ and T_α is an extension of $T_\xi \restriction B(\psi_\xi)$. Let $\zeta = \sup\{\varrho_\xi : (\varrho_\xi, \xi) \in \Upsilon\}$. We enumerate all triples $(X, T_\alpha, \varepsilon n)$ of P_α-names such that

$$\mathsf{P}_\alpha \Vdash \text{ "} X \in P(\omega)/\mathrm{fin}\text{"},$$

and $\mathrm{Ind}(X) < \mathrm{dom}(\varphi)$ (case ($\star$)), or $\zeta \le \mathrm{Ind}(X) < \mathrm{dom}(\varphi)$ (case ($\star\star$)), $\varepsilon \in \{-1, 1\}$, $n \in \omega$. We fix a function j from the set of these triples into κ with the following properties:

(a) If $X \in B_{\mathrm{dom}(\varphi)} = \{X_\gamma : \gamma \in Q_1,\ \gamma < \mathrm{dom}(\varphi)\}$ and $Y \notin B_{\mathrm{dom}(\varphi)}$ then

$$j((X, T_\alpha, \varepsilon n)) < j((Y, T_\alpha, \varepsilon m)) \quad \text{for all } n, m < \omega.$$

(b) If $X_1, X_2 \in B_{\mathrm{dom}(\varphi)}$ [resp. $Y_1, Y_2 \notin B_{\mathrm{dom}(\varphi)}$] and $\mathrm{Ind}(X_1) < \mathrm{Ind}(X_2)$ [resp. $\mathrm{Ind}(Y_1) < \mathrm{Ind}(Y_2)$] then

$$j((X_1, T_\alpha, \varepsilon n)) < j((X_2, T_\alpha, \varepsilon m)) \quad [\text{resp. } j((Y_1, T_\alpha, \varepsilon n)) < j((Y_2, T_\alpha, \varepsilon m))].$$

(c) For all X with $\mathrm{Ind}(X) < \mathrm{dom}(\varphi)$ and all $n \in \omega$,

$$j((X, T_\alpha, -n)) = j((X, T_\alpha, n)) + 1,$$

$$j((X,T_\alpha,n+1))=j((X,T_\alpha,-n))+1.$$

Since $\mathsf{P}_\alpha \Vdash$ "card $P(\omega)/\mathrm{fin} \le \nu_\alpha$", the domain of the sequence of the triples is $\le \nu_\alpha$.

Using ordinals $> \nu_\alpha$ we also enumerate all triples $(\mathcal{L},T_\alpha,\varepsilon n)$ of P_α-names such that P_α forces: $\mathcal{L} = (\{X(s_\gamma) : \gamma < \omega_1\},\{X(t_\beta) : \beta < \omega_1\})$ is an increasingly ordered gap of the type (ω_1,ω_1), $\mathrm{Ind}(X(s_\gamma)) \le \mathrm{dom}(\varphi)$ and $\mathrm{Ind}(Y(t_\beta)) \le \mathrm{dom}(\varphi)$ for all $\gamma<\omega_1$ and $\beta<\omega_1$, and $\mathsf{Q}(\mathcal{L})$ does not have the c.c.c. We can assume that $(\mathcal{L},T_\alpha,-n)$ follows $(\mathcal{L},T_\alpha,n)$ and precedes $(\mathcal{L},T_\alpha,n+1)$.

We extend the function Nb to the set of objects described above in the following way:

$$\mathrm{Nb}((X,T_\alpha,\varepsilon n))=n(\alpha,j(X,T_\alpha,\varepsilon n))$$

and if $(\mathcal{L},T_\alpha,n)$ is the βth element of the (second) enumeration then

$$\mathrm{Nb}((\mathcal{L},T_\alpha,n))=n(\alpha,\beta).$$

We set

$$\mathsf{P}_{\alpha+1}=\mathsf{P}_\alpha.$$

We also define $T^\alpha_\alpha = T^{\alpha+1}_\alpha = T_\alpha$ in case $(\star)$ and $T^\alpha_\alpha = T^{\alpha+1}_\alpha =$ monomorphism generated by T_α and $\bigcup_{(\varrho_\xi,\xi)\in\Upsilon} T^\alpha_\xi \restriction \{X \in P(\omega)/\mathrm{fin} : \mathrm{Ind}(X) < \varrho_\xi\}$ in case $(\star\star)$. The families T^α_γ defined at earlier stages are not changed: $T^{\alpha+1}_\gamma = T^\alpha_\gamma$.

It is easy to check by using Sikorski's theorem and the following lemma that the above definitions are correct.

Lemma 1. *Let X be an element of $P(\omega)/\mathrm{fin}$ in* $\mathbf{V}$, *let* $\mathsf{p}=(u_\mathsf{p},x_\mathsf{p},w_\mathsf{p}) \in \mathsf{Q}(\mathcal{L})$ *and let X_γ stand for a generic subset added by* Q. *Then we have*:

$$\text{if } \mathsf{p} \Vdash \text{“}X \subseteq_\star X_\gamma\text{” then } X \subseteq_\star \bigcup_{s\in u_\mathsf{p}} X(s),$$

$$\text{if } \mathsf{p} \Vdash \text{“}X \cap X_\gamma =_\star \emptyset\text{” then } X \subseteq_\star \bigcup_{t\in w_\mathsf{p}} X(t).$$

Case 4: $\alpha \in Q_2$. Suppose that $\alpha = \mathrm{Nb}((X,T_\gamma,\varepsilon n))$, where

$$\mathsf{P}_\gamma \Vdash \text{“}T_\gamma \text{ is an automorphism of } B(\varphi),\ X \notin B(\varphi)''.$$

If $\varepsilon = 1$ and $X \notin \mathrm{dom}(T^\alpha_\xi)$ or $\varepsilon=-1$ and $X \notin \mathrm{rg}(T^\alpha_\xi)$ then we extend the monomorphism T^α_ξ.

Suppose that $\varepsilon = 1$. Let $\mathcal{L}$ be a gap in $\mathrm{rg}(T^\alpha_\xi)$ defined by X:

$$\mathcal{L}=(\{(T^\alpha_\xi)^{\varepsilon n}(Z) : Z \subseteq_\star X\},\{(T^\alpha_\gamma)^{\varepsilon n}(Y) : X \subseteq_\star Y\})$$

(P_α forces all the properties). All elements of the gap have been defined at the previous stages, because of the definition of j (Case 3). We set

$$\mathsf{P}_{\alpha+1}=\mathsf{P}_\alpha \star \mathsf{Q}(\mathcal{L}).$$

We extend T^α_ξ setting

$$T^{\alpha+1}_\gamma = \text{homomorphism generated by } T^\alpha_\xi \cup \{((T^\alpha_\xi)^{\varepsilon n}(X), X_{\alpha+1})\},$$

where

$$X_{\alpha+1} = \{i \in \omega : \exists \mathsf{p} \in G\ [x_{\mathsf{p}}(i) = 1]\},$$

and $G \subseteq \mathsf{Q}(\mathcal{L})$ is a generic filter. If T_ξ is a P_α-name such that

$\mathsf{P}_\alpha \Vdash$ "T_ξ is an automorphism of $B(\psi)$ and
for some ordinal ϱ, $\varphi \restriction \varrho = \psi \restriction \varrho$ and T_ξ and T_γ agree on $B(\varphi \restriction \varrho)$",

and $\mathrm{Ind}(X) < \mathrm{dom}(B(\psi))$, then

$$T^{\alpha+1}_\xi = \text{homomorphism generated by } T^\alpha_\xi \cup \{((T^\alpha_\xi)^{\varepsilon n}(X), X_{\alpha+1})\},$$

and $T^{\alpha+1}_\zeta = T^\alpha_\zeta$ in the remaining cases. If $\varepsilon = -1$ we proceed with the construction in a similar way: we add a generic set to the domain of T^α_ξ and to the domains of each of the T^α_γ's which agree with T^α_ξ on an "initial segment" of their domains.

(It is easy to prove, by using Lemma 1 and Sikorski's theorem, that the definitions of the monomorphism T^α_γ are correct.)

Case 5: $\alpha \in E$. Assume that $\alpha = \mathrm{Nb}((\mathcal{L}, T_\gamma, \varepsilon n))$, where $\mathsf{P}_\gamma \Vdash$ "$\mathcal{L}$ is an increasingly ordered gap in $P(\omega)/\mathrm{fin}$ and $\mathsf{Q}(\mathcal{L})$ does not have the c.c.c." Suppose that $\mathcal{L} = (\{X(s_\zeta) : \zeta < \omega_1\}, \{X(t_\beta) : \beta < \omega_1\})$ and let $\mathcal{L}^\star$ denote the gap

$$(\{(T^\alpha_\gamma)^{\varepsilon n}(X(s_\zeta)) : \zeta < \omega_1\}, \{(T^\alpha_\gamma)^{\varepsilon n}(X(t_\beta)) : \beta < \omega_1\}).$$

We set

$$\mathsf{P}_{\alpha+1} = \mathsf{P}_\alpha \star \mathsf{E}(\mathcal{L}^\star) \quad \text{and} \quad T^{\alpha+1}_\gamma = T^\alpha_\gamma.$$

For limit ordinals $\lambda < \kappa$ we define P_λ as a direct limit of $\{\mathsf{P}_\alpha : \alpha < \lambda\}$. We also assume $\mathsf{P}_{\alpha+1} = \mathsf{P}_\alpha$ in all cases not mentioned above. This completes the definition of the iteration.

We conclude this part of the proof by checking that the above construction is correct, i.e. that there is no gap $\mathcal{L}$ of the type (ω_1, ω_1), consisting of generic subsets of ω, which is an image (under one of the extending monomorphisms) of some gap $\mathcal{L}'$ such that $\mathsf{Q}(\mathcal{L}')$ does not have the c.c.c. and which is filled by the generic set X_γ at some stage $\gamma < \kappa$.

Claim 1. *Let B_i $(i = 1, 2)$ denote one of the following subalgebras of $P(\omega)/\mathrm{fin}$: $B(\varphi_i)$ (where $\varphi_i \in \mathcal{D}_{\eta_i}$); the domain of T^α_γ; the range of T^α_ξ. Assume that $\bigcup_{i=1}^n X(s_i) \in B_1$, $\bigcup_{j=1}^m X(t_j) \in B_2$ and $\bigcup_{i=1}^n X(s_i) \subseteq_\star \bigcup_{j=1}^m X(t_j)$. Then there are finite functions $r_1, \ldots, r_k$ such that* $\mathrm{rg}(r_l) \subseteq \{-1, 1\}$ $(l = 1, \ldots, k)$ *and for each $\xi \in \bigcup_{l=1}^k \mathrm{dom}(r_l)$ we have $X_\xi \in B_1 \cap B_2$*

and

$$\bigcup_{i=1}^{n} X(s_i) \subseteq_\star \bigcup_{l=1}^{k} X(r_l) \subseteq_\star \bigcup_{j=1}^{m} X(t_j).$$

This is proved by using Lemma 1.

LEMMA 2. *Let $\alpha < \kappa$. Assume that P_α has the c.c.c. Suppose that P_α forces the following*:

(1) *For each $\beta < \alpha$ and each $\varphi \in \mathcal{D}_\beta$, if $\mathcal{L}_\varphi$ is a gap in $B(\varphi)$, then $\mathsf{Q}(\mathcal{L}_\varphi)$ has the c.c.c.*

(2) *If $\mathcal{L}_\zeta = (\mathcal{S}_\zeta, \mathcal{U}_\zeta)$ is a gap in the domain or range of $(T^\alpha_\zeta)^{\varepsilon k}$ such that $\mathsf{P}_\alpha \Vdash$ "$\mathsf{Q}(\mathcal{L})$ has the c.c.c." and for all $X(s) \in \mathcal{S}_\zeta$ and $X(t) \in \mathcal{U}_\zeta$ there are $\bigcup_{i=1}^n X(s_i) \in \mathcal{S}_\zeta$ and $\bigcup_{j=1}^m X(t_j) \in \mathcal{U}_\zeta$ such that*

$$X(s) = \bigcup_{i=1}^{n} X(s_i) \wedge X(t) = \bigcup_{j=1}^{m} X(t_j)$$
$$\wedge \operatorname{Ind}\Big(\bigcup_{i=1}^{n} X(s_i)\Big) < \operatorname{Ind}\Big((T^\alpha_\zeta)^{\varepsilon k}\Big(\bigcup_{i=1}^{n} X(s_i)\Big)\Big)$$
$$\wedge \operatorname{Ind}\Big(\bigcup_{j=1}^{m} X(t_j)\Big) < \operatorname{Ind}\Big((T^\alpha_\zeta)^{\varepsilon k}\Big(\bigcup_{j=1}^{m} X(t_j)\Big)\Big)$$

then $\mathsf{P}_\alpha \Vdash$ "$\mathsf{Q}((T^\alpha_\zeta)^{\varepsilon k}(\mathcal{L}))$ has the c.c.c."

(3) *$\mathcal{L}_\xi = (\{(T^\alpha_\xi)^{\varepsilon_1 m}(X(s_\eta)) : X(s_\eta) \in \mathcal{S}\}, \{(T^\alpha_\xi)^{\varepsilon_1 m}(X(t_\eta)) : X(t_\eta) \in \mathcal{U}\})$ is an increasingly ordered gap such that $\mathsf{P}_\xi \Vdash$ "T_ξ is an automorphism of $B(\psi)$, $\mathcal{L} = (\mathcal{S}, \mathcal{U})$ is an increasingly ordered gap of the type (ω_1, ω_1) in $P(\omega)/\mathrm{fin}$ and $\mathsf{Q}(\mathcal{L})$ does not have the c.c.c."*

Under the above assumptions, there is an $\alpha_0 < \omega_1$ such that for each $\gamma > \alpha_0$ and any finite subsets $\{X(s^1), \ldots, X(s^n)\}$, $\{X(t^1), \ldots, X(t^m)\}$ of the lower and upper classes (respectively) of the gap $\mathcal{L}_\varphi$ or $\mathcal{L}_\zeta$ the following holds:

$$X(s_\gamma) \not\subseteq_\star \bigcup_{i=1}^{n} X(s^i) \quad \text{or} \quad X(t_\gamma) \not\subseteq_\star \bigcup_{j=1}^{m} X(t^j)$$

where $X(s_{\alpha_0})$, $X(s_\gamma)$ and $X(t_{\alpha_0})$, $X(t_\gamma)$ are elements of the lower and upper classes of $\mathcal{L}_\xi$ (respectively).

Proof. Assume to the contrary that for every $\gamma \in \omega_1$ there are $s_1^\gamma, \ldots, s_n^\gamma, t_1^\gamma, \ldots, t_m^\gamma$ such that

$$\Big[X(s_\gamma) \subseteq_\star \bigcup_{i=1}^{n} X(s_i^\gamma)\Big] \wedge \Big[X(t_\gamma) \subseteq_\star \bigcup_{j=1}^{m} X(t_j^\gamma)\Big].$$

Applying Claim 1 to $X(s_\gamma) \subseteq_\star \bigcup_{i=1}^{n} X(s_i^\gamma)$ and $X(t_\gamma) \subseteq_\star \bigcup_{j=1}^{m} X(t_j^\gamma)$ (for each $\gamma < \omega_1$) we obtain elements $\bigcup_{i=1}^{n_\gamma} X(r_i^\gamma)$, $\bigcup_{j=1}^{m_\gamma} X(p_j^\gamma)$ such that $X(s_\gamma) \subseteq_\star \bigcup_{i=1}^{n_\gamma} X(r_i^\gamma) \subseteq_\star \bigcup_{i=1}^{n} X(s_i^\gamma)$, $X(t_\gamma) \subseteq_\star \bigcup_{j=1}^{m_\gamma} X(p_j^\gamma) \subseteq_\star \bigcup_{j=1}^{m} X(t_j^\gamma)$ and $X_\eta \in B \cap \mathrm{rg}((T_\xi^\alpha)^{\varepsilon_1 m})$ for each $\eta \in \bigcup_{i=1}^{n_\gamma} \mathrm{dom}(r_i) \cup \bigcup_{j=1}^{m_\gamma} \mathrm{dom}(p_j)$. (Here B denotes $B(\varphi)$ or the domain or range of T_ζ^α.) Thus

$$\mathcal{L}' = \Big(\Big\{\bigcup_{i=1}^{n_\gamma} X(r_i^\gamma) : \gamma < \omega_1\Big\}, \Big\{\bigcup_{j=1}^{m_\gamma} X(p_j^\gamma) : \gamma < \omega_1\Big\}\Big)$$

is a gap in $B \cap \mathrm{rg}(T_\xi^\alpha)$.

If $B = B(\varphi)$ then $\mathcal{L}'$ is a gap in $B(\varphi \cap \psi)$. Thus $\mathcal{L}'_\xi$, the image of $\mathcal{L}'$ under $(T_\xi^\alpha)^{-\varepsilon_1 m}$, is a gap in $B(\psi)$ and by (1), $\mathsf{Q}(\mathcal{L}'_\xi)$ has the c.c.c., but by (3) it does not have the c.c.c., a contradiction.

If $B = \mathrm{dom}(T_\gamma^\alpha)$ then $(T_\xi^\alpha)^{-\varepsilon_1 m}(\mathcal{L}') = \mathcal{L}''$ is a gap in $\mathrm{dom}((T_\xi^\alpha)^{\varepsilon_1 m})$. By (2), $\mathsf{Q}(\mathcal{L}'')$ has the c.c.c. but by (3) it does not have the c.c.c., a contradiction. ∎

We show that the assertion of Theorem 1 holds in the extension $\mathbf{V}[G]$, where $G \subseteq \mathsf{P}_\kappa$ is a generic filter. It is clear that $\mathbf{V}[G] \Vdash$ "$2^\omega = \kappa$" and (by Theorems 2 and 3), $\mathbf{V}[G|\alpha] \Vdash$ "$2^\omega < \kappa$" for each $\alpha < \kappa$. Let B be a Boolean algebra in $\mathbf{V}[G]$ with $\mathrm{card}\, B = \kappa$. There are elements $b_\gamma \in B$ for $\gamma < \kappa$ such that $B = \bigcup_{\alpha<\kappa} B_\alpha$, where B_α is the subalgebra generated by b_γ, $\gamma \le \alpha$.

Assume inductively that we have an embedding $i : B_\alpha \to P(\omega)/\mathrm{fin}$ such that $i(b_\xi) = X_{\beta_\xi}$ with $\beta_\xi \in Q_1$ for each $\xi < \alpha$. We define a sequence $\varphi_\alpha : \sup\{\beta_\xi : \xi < \alpha\} \to \{0,1\}$ putting $\varphi_\alpha(\beta_\xi) = 1$ for each $\xi < \alpha$, and $\varphi_\alpha(\zeta) = 0$ otherwise. Thus $B(\varphi_\alpha)/\mathrm{fin}$ is an isomorphic image of the algebra B_α. Let

$$b(s) = \Big(\prod_{s(\zeta)=1} b_\zeta\Big) \cdot \Big(\prod_{s(\eta)=-1} -b_\eta\Big),$$

where s is a finite function on α with $\mathrm{rg}(s) \subseteq \{-1,1\}$. The next generator b_α determines a gap

$$\mathcal{L}^{B_\alpha} = (\{b(s) : b(s) \le b_\alpha\}, \{b(t) : b(t) \cdot b_\alpha = 0\})$$

in the algebra B_α. Let $\mathcal{L}$ be the image of $\mathcal{L}^{B_\alpha}$ under i. So $\mathcal{L}$ is a gap in $B(\varphi_\alpha)$ and

$$\mathcal{L} = (\{X(s_i)\}, \{X(t_i)\}),$$

where s_i is defined on $\{\beta_\xi : \xi \in \mathrm{dom}(s)\}$ by the equality $s_i(\beta_\xi) = s(\xi)$ (t_i is defined similarly).

Let $\gamma > \sup(\varphi_\alpha)$, $\gamma \in Q_1$ and $\gamma = \mathrm{Nb}(\mathcal{L}, \varphi_\alpha)$. We define $i(b_\alpha) = X_\gamma$ and $\varphi_{\alpha+1} = \varphi_\alpha \cup \{(\beta, 0) : \mathrm{dom}(\varphi_\alpha) < \beta < \gamma\} \cup \{(\gamma, 1)\}$. This extends i to an embedding from $B_{\alpha+1}$ onto $B(\varphi_{\alpha+1})/\mathrm{fin}$ (we check this using Lemma 1).

Let $\Phi = \bigcup_{\alpha<\kappa} \varphi_\alpha$. It is clear that B is isomorphic to $B(\Phi)$.

Let f be an automorphism of B. Then $i \circ f \circ i^{-1}$ is an automorphism of $B(\Phi)$ and there is a canonical name F for it, $F \subseteq H(\kappa)$, consisting of some pairs $((x,y)^{(\mathsf{P}_\kappa)}, \mathsf{p})$, where x, y are canonical names for the elements of $B(\Phi)$ and the set $F(x,y) = \{\mathsf{p} \in \mathsf{P}_\kappa : ((x,y),\mathsf{p}) \in F\}$ is an antichain. Since P_κ has the c.c.c., the set

$$N_1 = \{\alpha < \kappa : \forall x, y\ [x, y \in \mathbf{V}^{(\mathsf{P}_\alpha)} \to F(x,y) \subseteq \mathsf{P}_\alpha]\}$$

is ω_1-club (closed and unbounded) in κ. For any $\alpha \in N_1$ the restriction

$$F_\alpha = F \cap (\mathbf{V}^{(\mathsf{P}_\alpha)} \times \mathsf{P}_\alpha)$$

is a P_α-name and $F_\alpha[G|\alpha] = i \circ f \circ i^{-1} \cap \mathbf{V}[G|\alpha]$. So, for all $\alpha \in N_1$, the monomorphism $F_\alpha[G|\alpha]$ belongs to $\mathbf{V}[G|\alpha]$. On the other hand, the sets

$$N_2 = \{\alpha < \kappa : \beta < \alpha,\ \mathrm{cf}(\alpha) = \omega_1,\ F \cap H_\alpha = F_\alpha\}$$

are ω_1-club for all $\beta < \kappa$. From the diamond principle it follows that there is an increasing sequence $\{\gamma_\beta \in N_1 \cap N_2 : \beta < \kappa\}$ such that $F_{\gamma_\beta} = T_{\gamma_\beta}$. Let $A(F) = \bigcup_{\beta<\kappa} T_{\gamma_\beta}^{\gamma_{\beta+1}}$ and $\bar{f} = A(F)[G]$. Then $\bar{f}$ is an automorphism of $P(\omega)/\mathrm{fin}$ and $i \circ f \circ i^{-1} \subseteq \bar{f}$.

It remains to show that P_α has the c.c.c. for each $\alpha \le \kappa$. Let P'_α consist of all $\mathsf{p} \in \mathsf{P}_\alpha$ satisfying the following conditions:

1. For each $\gamma \in \mathrm{supp}(\mathsf{p}) \cap (Q_1 \cup Q_2)$ there are $u_\gamma(\mathsf{p})$, $x_\gamma(\mathsf{p})$, $w_\gamma(\mathsf{p})$ such that

$$\mathsf{p}\restriction\gamma \Vdash \text{“}\mathsf{p}(\gamma) = (u_\gamma(\mathsf{p}),\ x_\gamma(\mathsf{p}),\ w_\gamma(\mathsf{p}))\text{”}$$

and $\mathrm{dom}(s) \subseteq \mathrm{supp}(\mathsf{p})$ for each $s \in u_\gamma(\mathsf{p}) \cup w_\gamma(\mathsf{p})$. Moreover, for each $\gamma \in \mathrm{supp}(\mathsf{p}) \cap (Q_1 \cup Q_2)$, the number $\mathrm{dom}(x_\gamma(\mathsf{p}))$ is constant (independent of γ). We write $l(\mathsf{p})$ for this value.

2. For each $\gamma \in \mathrm{supp}(\mathsf{p}) \cap E$ there are $(\alpha_1, s_{\alpha_1}, t_{\alpha_1}), \ldots, (\alpha_n, s_{\alpha_n}, t_{\alpha_n})$ such that

$$\mathsf{p}\restriction\gamma \Vdash \text{“}\mathsf{p}(\gamma) = \{(\alpha_1, s_{\alpha_1}, t_{\alpha_1}), \ldots, (\alpha_n, s_{\alpha_n}, t_{\alpha_n})\}\text{”}$$

and $\mathrm{dom}(s_{\alpha_i}) \cup \mathrm{dom}(t_{\alpha_i}) \subseteq \mathrm{supp}(\mathsf{p})$ for $i \le n$.

Let $\mathsf{P}^\star_\alpha \subseteq \mathsf{P}'_\alpha$ be the set of all $\mathsf{p} \in \mathsf{P}'_\alpha$ with the property:

3. If $\gamma \in M$ then there is an $n \in \omega$ such that

$$\mathsf{p}\restriction\gamma \Vdash \text{“}h_\gamma(\mathsf{p}(\gamma)) = n\text{”},$$

where h_γ is a P_γ-name of a function such that

$$\mathsf{P}_\gamma \Vdash \text{“}h_\gamma : \mathsf{R}_\gamma \to \omega \text{ and } \forall n \in \omega\ [h_\gamma^{-1}(n) \text{ is linked}]\text{”}.$$

(We can choose the h_γ since $\mathsf{P}_\gamma \Vdash$ “R_γ is σ-linked”.)

Lemma 3. *For each $\mathsf{p} \in \mathsf{P}_\alpha$ and $m \in \omega$, there is a $\mathsf{q} \in \mathsf{P}^\star_\alpha$ such that $\mathsf{p} \ge \mathsf{q}$ and $l(\mathsf{q}) \ge m$.*

Proof. The proof (except for the case $\beta \in E$) is similar to the proof of Lemma 4.4 in Chapter 9 of [5].

Assume (inductively) that the lemma holds for α, $\beta = \alpha+1$, $\beta \in \operatorname{supp}(\mathsf{p})$ and $\beta \in E$. There is a $\mathsf{p}_1 \le \mathsf{p}\restriction\beta$ such that

$$\mathsf{p}_1 \Vdash \text{“}\mathsf{p}(\beta) = \{(\alpha_1, s_{\alpha_1}, t_{\alpha_1}), \ldots, (\alpha_n, s_{\alpha_n}, t_{\alpha_n})\}\text{”}$$

for some P_β-names $(\alpha_1, s_{\alpha_1}, t_{\alpha_1}), \ldots, (\alpha_n, s_{\alpha_n}, t_{\alpha_n})$. We may assume that $\operatorname{dom}(s_{\alpha_i}) \cup \operatorname{dom}(t_{\alpha_i}) \subseteq \operatorname{supp}(\mathsf{p})$ for $i \le n$. By the inductive assumption there is a $\mathsf{p}_2 \le \mathsf{p}_1$ such that $\mathsf{p}_2 \in \mathsf{P}^\star_\beta$ and $l(\mathsf{p}_2) \ge m$. Thus, the element $\mathsf{p}_2 \star \mathsf{p}(\beta)$ has all the required properties. ∎

We precede the next two lemmas with the following note: Fix $\alpha < \kappa$ and suppose that P_α has the c.c.c. and the assumptions of Lemma 2 are satisfied. Let P_α force that $\mathcal{L}$ is the image under T^α_γ of an increasingly ordered gap $\mathcal{L}'$ such that

$$P_\gamma \Vdash \text{“}\mathsf{Q}(\mathcal{L}') \text{ does not have the c.c.c.”}$$

Suppose that $\{\mathsf{p}_\xi : \xi < \omega_1\} \subseteq \mathsf{P}^\star_\alpha$ is a set of pairwise compatible conditions and that e_ξ are P_α-names of conditions of the forcing $\mathsf{E}(\mathcal{L})$ such that

$$\forall \xi < \omega_1\ [\mathsf{p}_\xi \Vdash \text{“}\mathsf{e}_\xi = \{(\alpha^\xi_1, s_{\alpha^\xi_1}, t_{\alpha^\xi_1}), \ldots, (\alpha^\xi_{n_\xi}, s_{\alpha^\xi_{n_\xi}}, t_{\alpha^\xi_{n_\xi}})\}\text{”}].$$

Let z_i, $i = 1, \ldots, n$, be finite functions with $\operatorname{dom}(z_i) \subseteq \bigcap_{\xi<\omega_1} \operatorname{supp}(\mathsf{p}_\xi) \cap (Q_1 \cup Q_2)$. From Lemma 2 it follows that there are (at most) two possibilities:

1. There is an uncountable set $B \subseteq \omega_1$ such that

$$\forall \xi_1, \xi_2 \in B\ [\mathsf{r}_{\xi_1,\xi_2} \Vdash \text{“}\mathsf{e}_{\xi_1} = \mathsf{e}_{\xi_2}\text{”}],$$

where $\mathsf{r}_{\xi_1,\xi_2} \le \mathsf{p}_{\xi_1}, \mathsf{p}_{\xi_2}$.

2. Any set $A \subseteq \omega_1$ satisfying the following condition:

If $\xi_1, \xi_2 \in A$ then for some $i_0 \in \{1, \ldots, n_{\xi_1}\}$ and $j_0 \in \{1, \ldots, n_{\xi_2}\}$ we have

$$\mathsf{p}_{\xi_1} \Vdash \text{“}X(s_{\alpha^{\xi_1}_{i_0}}) \subseteq_\star \bigcup_{i=1}^n X(z_i)\text{”} \quad \textit{and} \quad \mathsf{p}_{\xi_2} \Vdash \text{“}X(t_{\alpha^{\xi_2}_{j_0}}) \subseteq_\star \omega \setminus \bigcup_{i=1}^n X(z_i)\text{”}$$

and for all $\mathsf{r}_{\xi_1,\xi_2} \le \mathsf{p}_{\xi_1}, \mathsf{p}_{\xi_2}$,

$$\mathsf{r}_{\xi_1,\xi_2} \Vdash \text{“}\forall k \in \{1, \ldots, n_{\xi_2}\}\ [\alpha^{\xi_1}_{i_0} \ne \alpha^{\xi_2}_k] \text{ and } \forall l \in \{1, \ldots, n_{\xi_1}\}\ [\alpha^{\xi_2}_{j_0} \ne \alpha^{\xi_1}_l]\text{”}$$

is at most countable.

Lemma 4 ([5]). *Let $\mathsf{p} \Vdash$ “$X(s) \in \mathrm{fin}$” and $\gamma = \max\operatorname{dom}(s)$. If $\mathsf{p} \in \mathsf{P}'_\alpha$ then there is an $\mathsf{r} \in \mathsf{P}'_\alpha$ with $\mathsf{r} \le \mathsf{p}$ and $l(\mathsf{p}) = l(\mathsf{r})$ such that if $\mathsf{r}\restriction\gamma \Vdash$ “$\mathsf{r}(\gamma) = (u^{\mathsf{r}}_\gamma, x^{\mathsf{r}}_\gamma, w^{\mathsf{r}}_\gamma)$”, then $\mathsf{r} \Vdash$ “$s\restriction\gamma \in u^{\mathsf{r}}_\gamma$” (if $s(\gamma) = -1$) or $\mathsf{r} \Vdash$ “$s\restriction\gamma \in w^{r}_\gamma$” (if $s(\gamma) = 1$).*

LEMMA 5. *Assume that* $\mathsf{p}, \mathsf{q} \in \mathsf{P}^{\star}_{\alpha+1}$ *satisfy the following conditions*:

1. $\mathsf{p}\restriction\alpha$ *and* $\mathsf{q}\restriction\alpha$ *are compatible.*

2. *If* $\xi \in \operatorname{supp}(\mathsf{p}) \cap \operatorname{supp}(\mathsf{q}) \cap M$ *and*

$$\mathsf{p}\restriction\xi \Vdash \text{“}\mathsf{p}(\xi) \in h_\xi^{-1}(n)\text{”} \quad \textit{and} \quad \mathsf{q}\restriction\xi \Vdash \text{“}\mathsf{q}(\xi) \in h_\xi(m)\text{”}$$

then $n = m$.

3. *If* $\xi \in \operatorname{supp}(\mathsf{p}) \cap \operatorname{supp}(\mathsf{q}) \cap (Q_1 \cup Q_2)$ *and*

$$\mathsf{p}\restriction\xi \Vdash \text{“}\mathsf{p}(\xi) = (u^{\mathsf{p}}_\xi, x^{\mathsf{p}}_\xi, w^{\mathsf{p}}_\xi)\text{”} \quad \textit{and} \quad \mathsf{q}\restriction\xi \Vdash \text{“}\mathsf{q}(\xi) = (u^{\mathsf{q}}_\xi, x^{\mathsf{q}}_\xi, w^{\mathsf{q}}_\xi)\text{”}$$

then $x^{\mathsf{p}}_\xi = x^{\mathsf{q}}_\xi$.

4. *Let* $\xi \in \operatorname{supp}(\mathsf{p}) \cap \operatorname{supp}(\mathsf{q}) \cap E$ *and*

$$\mathsf{p}\restriction\xi \Vdash \text{“}\mathsf{p}(\xi) = \{(\alpha^\xi_1, s_{\alpha^\xi_1}, t_{\alpha^\xi_1}), \ldots, (\alpha^\xi_{n_\xi}, s_{\alpha^\xi_{n_\xi}}, t_{\alpha^\xi_{n_\xi}})\}\text{”},$$

$$\mathsf{q}\restriction\xi \Vdash \text{“}\mathsf{q}(\xi) = \{(\beta^\xi_1, s_{\beta^\xi_1}, t_{\beta^\xi_1}), \ldots, (\beta^\xi_{m_\xi}, s_{\beta^\xi_{m_\xi}}, t_{\beta^\xi_{m_\xi}})\}\text{”}.$$

Define $A_\xi = \{i : (\alpha^\xi_i, s_{\alpha^\xi_i}, t_{\alpha^\xi_i}) \in \mathsf{p}(\xi)$ *and* $\alpha^\xi_i \neq \beta^\xi_j$ *for all* j *such that* $(\beta^\xi_j, s_{\beta^\xi_j}, t_{\beta^\xi_j}) \in \mathsf{q}(\xi)\}$ (B_ξ *is defined in a similar way*). *Assume that for any* $i \in A_\xi$ *and* $j \in B_\xi$ *there is no* s_l *with* $\operatorname{dom}(s_l) \subseteq \operatorname{supp}(\mathsf{p}) \cap \operatorname{supp}(\mathsf{q})$ *such that*

$$\mathsf{p} \Vdash \text{“}X(s_{\alpha^\xi_i}) \subseteq_\star \bigcup X(s_l)\text{”} \quad \textit{and} \quad \mathsf{q} \Vdash \text{“}X(t_{\beta^\xi_j}) \subseteq_\star \omega \setminus \bigcup X(s_l)\text{”}.$$

Then p *and* q *are compatible.*

P r o o f. Denote by Δ the set $\operatorname{supp}(\mathsf{p}) \cap \operatorname{supp}(\mathsf{q}) \cap E$. The required condition will be constructed in the following way: First we define extensions of the conditions p and q by extending zero-one sequences x^{p}_ξ and x^{q}_ξ such that

$$\mathsf{p}\restriction\xi \Vdash \text{“}\mathsf{p}(\xi) = (u^{\mathsf{p}}_\xi, x^{\mathsf{p}}_\xi, w^{\mathsf{p}}_\xi)\text{”} \quad \text{and} \quad \mathsf{q}\restriction\xi \Vdash \text{“}\mathsf{q}(\xi) = (u^{\mathsf{q}}_\xi, x^{\mathsf{q}}_\xi, w^{\mathsf{q}}_\xi)\text{”}.$$

This will be done in such a way that if $\varrho \in \Delta$ and if some extension r of the conditions p and q forces

$$X(s_{\alpha_i}) = \varepsilon^i_1 X_{\gamma^i_1} \cap \ldots \cap \varepsilon^i_{n_i} X_{\gamma^i_{n_i}}, \qquad i \in A_\varrho,$$

and

$$X(t_{\beta_i}) = \varepsilon^j_1 X_{\xi^j_1} \cap \ldots \cap \varepsilon^j_{m_j} X_{\xi^j_{m_j}}, \qquad j \in B_\varrho,$$

then for some $n \geq l(\mathsf{p})$,

$$\overline{x}_{\xi^j_k}(n) = \begin{cases} 0, & \varepsilon_{\xi^j_k} = -1, \\ 1, & \varepsilon_{\xi^j_k} = 1, \end{cases} \qquad \overline{x}_{\gamma^i_l}(n) = \begin{cases} 0, & \varepsilon_{\gamma^i_l} = -1, \\ 1, & \varepsilon_{\gamma^i_l} = 1, \end{cases}$$

$$x_{\xi^j_k} \subset \overline{x}_{\xi^j_k}, \qquad\qquad x_{\gamma^i_l} \subset \overline{x}_{\gamma^i_l}.$$

Thus we obtain extensions p' and q' which force “$n \in X(s_{\alpha_i})$” and “$n \in X(t_{\beta_j})$” respectively. In the next step of the proof we will consider the conditions $(u_{\gamma_l^i}, x_{\gamma_l^i}, w_{\gamma_l^i})$ and $(u_{\xi_k^j}, x_{\xi_k^j}, w_{\xi_k^j})$ and extend some of x_γ, x_ξ for $\gamma \in \mathrm{dom}(s)$, $s \in u_{\gamma_l^i} \cup w_{\gamma_l^i}$, $\xi \in \mathrm{dom}(t)$, $t \in u_{\xi_k^j} \cup w_{\xi_k^j}$. We will repeat this step for all x_γ which have been just extended. Finally, we extend each remaining x_γ with $\gamma \in (\mathrm{supp}(\mathsf{p}) \cup \mathrm{supp}(\mathsf{q})) \cap (Q_1 \cup Q_2)$. The construction should be careful in order to avoid a situation where for some $\gamma \in (\mathrm{supp}(\mathsf{p}) \cap \mathrm{supp}(\mathsf{q})) \cap (Q_1 \cup Q_2)$ there are $s \in u_\gamma^{\mathsf{p}}$, $t \in w_\gamma^{\mathsf{q}}$ and $n \geq l(\mathsf{p})$ such that the extensions p' and q' force that "$n \in X(s)$" and "$n \in X(t)$". (Such conditions p' and q' are incompatible.)

For all $\varrho \in \mathrm{supp}(\mathsf{p}) \cap \mathrm{supp}(\mathsf{q}) \cap E = \Delta$ we define a function

$$\Psi^\varrho : (\alpha + 1) \cap (\mathrm{supp}(\mathsf{p}) \cup \mathrm{supp}(\mathsf{q})) \cap (Q_1 \cup Q_2) \to \{1, 0\}.$$

(At the end of the proof we will extend the sequences x_γ with $\gamma \in (\mathrm{supp}(\mathsf{p}) \cup \mathrm{supp}(\mathsf{q})) \cap (Q_1 \cup Q_2)$ putting $\overline{x}_\gamma(n_\varrho) = \Psi^\varrho(\gamma)$, where $n_\varrho = l(\mathsf{p}) + k(\varrho)$ and k is an increasing enumeration of the set Δ.)

Let $\mathsf{r} < \mathsf{p}\restriction\alpha, \mathsf{q}\restriction\alpha$ force that

$$X(s_{\alpha_i}) = \varepsilon_1^i X_{\gamma_1^i} \cap \ldots \cap \varepsilon_{n_i}^i X_{\gamma_{n_i}^i}, \quad i \in A_\varrho,$$

and

$$X(t_{\beta_i}) = \varepsilon_1^j X_{\xi_1^j} \cap \ldots \cap \varepsilon_{m_j}^j X_{\xi_{m_j}^j}, \quad j \in B_\varrho.$$

(We denote α_i^ϱ, β_j^ϱ by α_i, β_j respectively.) We put

$$\Psi^\varrho(\gamma_l^i) = \begin{cases} 1 & \text{if } \varepsilon_l^i = 1, \\ 0 & \text{if } \varepsilon_l^i = -1,\ i \in A_\varrho,\ l \leq n_i, \end{cases}$$

and

$$\Psi^\varrho(\xi_k^j) = \begin{cases} 1 & \text{if } \varepsilon_k^j = 1, \\ 0 & \text{if } \varepsilon_k^j = -1,\ j \in B_\varrho,\ k \leq m_j. \end{cases}$$

Note that there are no s_f, s_l^{p}, s_k^{q} with $\mathrm{dom}(s_f)$, $\mathrm{dom}(s_l^{\mathsf{p}})$, $\mathrm{dom}(s_k^{\mathsf{q}}) \subseteq \mathrm{supp}(\mathsf{p}) \cap \mathrm{supp}(\mathsf{q})$ such that

$$\begin{aligned} \mathsf{p} \Vdash \text{"} & \bigcap_{i \in A'_\varrho} \bigcap_{l \in N'_i} \varepsilon_l^i X_{\gamma_l^i} \cap \bigcup X(s_k^{\mathsf{q}}) \subseteq_\star \bigcup X(s_f) \text{ and} \\ & \bigcap_{i \in A''_\varrho} \bigcap_{l \in N''_i} \varepsilon_l^i X_{\gamma_l^i} \subseteq_\star \bigcup X(s_l^{\mathsf{p}})\text{"}, \\ \mathsf{q} \Vdash \text{"} & \bigcap_{j \in B'_\varrho} \bigcap_{k \in M'_j} \varepsilon_k^j X_{\xi_k^j} \cap \bigcup X(s_l^{\mathsf{p}}) \subseteq_\star \omega \setminus \bigcup X(s_f) \text{ and} \\ & \bigcap_{j \in B''_\varrho} \bigcap_{k \in M''_j} \varepsilon_k^j X_{\xi_k^j} \subseteq_\star \bigcup X(s_k^{\mathsf{q}})\text{"} \end{aligned}$$

(where $A'_\varrho, A''_\varrho \subseteq A_\varrho$, $B'_\varrho, B''_\varrho \subseteq B_\varrho$, $N'_i, N''_i \subseteq n_i$ and $M'_j, M''_j \subseteq m_j$). Thus, if we put $\overline{x}_{\gamma_i^l}(l(\mathsf{p})) = \Psi^\varrho(\gamma_i^l)$ and $\overline{x}_{\xi_j^k}(l(\mathsf{p})) = \Psi^\varrho(\xi_j^k)$ then the extensions we have obtained will be compatible.

Let $\mathsf{p}\restriction\gamma_i^l \Vdash$ "$\mathsf{p}(\gamma_i^l) = (u_{\gamma_i^l}, x_{\gamma_i^l}, w_{\gamma_i^l})$" and $s \in u_{\gamma_i^l}$, $t \in w_{\gamma_i^l}$. If we put $\overline{x}_{\gamma_i^l}(l(\mathsf{p})) = \Psi^\varrho(\gamma_i^l)$ then the sequences $\overline{x}_\gamma$, $\gamma \in \mathrm{dom}(s) \cup \mathrm{dom}(t)$, have to be defined in such a way that the extension we obtain forces "$l(\mathsf{p}) \notin X(s)$" when $\Psi^\varrho(\gamma_i^l) = 0$, and "$l(\mathsf{p}) \notin X(t)$" when $\Psi^\varrho(\gamma_i^l) = 1$. (Similar conditions should hold for q.)

For each $X_{\gamma_l^i}$, $X_{\xi_k^j}$ we define

$$v_{\gamma_l^i} = \begin{cases} u_{\gamma_l^i} & \text{if } \varepsilon_l^i = -1, \\ w_{\gamma_l^i} & \text{if } \varepsilon_l^i = 1, \end{cases}$$

where

$$\mathsf{p}\restriction\gamma_l^i \Vdash \text{ "}\mathsf{p}(\gamma_l^i) = (u_{\gamma_l^i}, x_{\gamma_l^i}, w_{\gamma_l^i})\text{"}.$$

(The definitions of $v_{\xi_k^j}$ are similar.)

Let $\{s_i^{\mathsf{p}} : i \le k_\varrho\}$ be an enumeration of all $s \in \bigcup_{i\in A_\varrho}\bigcup_{k\le n_i} v_{\gamma_k^i}$. We enumerate also $\overline{s}_i^{\mathsf{p}} = \{-s_i^{\mathsf{p}}(\gamma)X_\gamma : \gamma \in \mathrm{dom}(s_i^{\mathsf{p}})\} = \{\varepsilon_1^i X_1^i, \ldots, \varepsilon_{k_i}^i X_{k_i}^i\}$. Denote by $\mathbf{I}(a)$ the intersection $\varepsilon_{a(1)}^1 X_{a(1)}^1 \cap \ldots \cap \varepsilon_{a(k_\varrho)}^{k_\varrho} X_{a(k_\varrho)}^{k_\varrho}$, where $a : k_\varrho + 1 \ni i \to a(i) \le k_i$, and by $\mathbf{I}$ the set of all the functions a. ($\mathbf{J}(b)$ and $\mathbf{J}$ are defined in a similar way for $v_{\xi_k^j}$.)

Thus

$$X(s_{\alpha_{\min}}) \subseteq_\star \omega \setminus \bigcup_{i\in A_\varrho}\bigcup_{l=1}^{n_i}\bigcup_{s\in v_{\gamma_l^i}} X(s) = \bigcup_{a\in\mathbf{I}} \mathbf{I}(a),$$

where $\alpha_{\min} = \min\{\alpha_1^\varrho, \ldots, \alpha_{n_\varrho}^\varrho\}$. It is easy to check that there exist sequences $a \in \mathbf{I}$ and $b \in \mathbf{J}$ such that

$$\mathsf{p} \Vdash \text{ "}X(s_{\alpha_{\min}}) \cap \mathbf{I}(a) \ne_\star \emptyset\text{"} \quad \text{and} \quad \mathsf{q} \Vdash \text{ "}X(t_{\beta_{\min}}) \cap \mathbf{J}(b) \ne_\star \emptyset\text{"}$$

and the following holds: for any

$$a', a'' \subseteq a, \quad b', b'' \subseteq b, \quad A'_\varrho, A''_\varrho \subseteq A_\varrho,$$
$$N'_i, N''_i \subseteq n_i, \quad M'_j, M''_j \subseteq m_j, \quad B'_\varrho, B''_\varrho \subseteq B_\varrho,$$

there are no s_l^{p}, s_k^{q}, $s_l^{\mathbf{I}}$, $s_k^{\mathbf{J}}$, s_f which satisfy the conditions below:

(1) The domains of the functions are subsets of $\mathrm{supp}(\mathsf{p}) \cap \mathrm{supp}(\mathsf{q})$.

(2) $\mathsf{p} \Vdash$ "$\mathbf{I}(a') \cap \bigcap_{i\in A'_\varrho}\bigcap_{l\in N'_i} \varepsilon_l^i X_{\gamma_l^i} \cap \bigcup X(s_k^{\mathsf{q}}) \cap \bigcup X(s_k^{\mathbf{J}}) \subseteq_\star \bigcup X(s_f)$,

$$\bigcap_{i\in A''_\varrho}\bigcap_{l\in N''_i} \varepsilon_l^i X_{\gamma_l^i} \subseteq_\star \bigcup X(s_l^{\mathsf{p}}) \text{ and } \mathbf{I}(a'') \subseteq_\star \bigcup X(s_l^{\mathbf{I}})\text{"}.$$

(3) $\mathsf{q} \Vdash$ "$\mathbf{J}(b') \cap \bigcap_{j\in B'_\varrho}\bigcap_{k\in M'_j} \varepsilon_k^j X_{\xi_k^j} \cap \bigcup X(s_l^{\mathsf{p}}) \cap \bigcup X(s_l^{\mathbf{I}}) \subseteq_\star \omega \setminus \bigcup X(s_f)$,

$$\bigcap_{i\in B''_\varrho}\bigcap_{k\in M''_j}\varepsilon^j_k X_{\xi^j_k}\subseteq_\star\bigcup X(s^{\mathsf{q}}_k)\text{ and }\mathbf{J}(b'')\subseteq_\star\bigcup X(s^{\mathbf{J}}_k)\text{''}.$$

For δ such that $X_\delta = X^i_{a(i)}$ or $X_\delta = X^j_{b(j)}$ we define

$$\Psi^\varrho(\delta)=\begin{cases}1 & \text{if } \varepsilon^i_{a(i)}=1 \text{ (resp. } \varepsilon^j_{b(j)}=1),\\ 0 & \text{if } \varepsilon^i_{a(i)}=-1 \text{ (resp. } \varepsilon^j_{b(j)}=-1).\end{cases}$$

We proceed with the construction in the following way:

We replace $\{X_{\gamma^i_l} : i\le n,\ l\le n_i\}$ and $\{X_{\xi^j_k} : j\le m,\ k\le m_j\}$ with $\{X^i_{a(i)} : i\in \mathrm{dom}(a)\}$ and $\{X^j_{b(j)} : j\in\mathrm{dom}(b)\}$ and repeat that until each v_δ is empty for each $X_\delta = X^i_{a_k(i)}$ and $X_\delta = X^j_{b_k(j)}$, where a_k and b_k are the sequences obtained in the $(k-1)$th iteration of the construction. Thus

$$\mathsf{p}\Vdash \text{“}X(s_{\alpha_{\min}})\cap \mathbf{I}(a_0)\cap\ldots\cap\mathbf{I}(a_k)\neq_\star\emptyset\text{”},$$
$$\mathsf{q}\Vdash \text{“}X(t_{\beta_{\min}})\cap \mathbf{J}(b_0)\cap\ldots\cap\mathbf{J}(b_k)\neq_\star\emptyset\text{”}$$

and there are no s_l with $\mathrm{dom}(s_l)\subseteq\mathrm{supp}(\mathsf{p})\cap\mathrm{supp}(\mathsf{q})$ such that

$$\mathsf{p}\Vdash \text{“}X(s_{\alpha_{\min}})\cap \mathbf{I}(a_0)\cap\ldots\cap\mathbf{I}(a_k)\subseteq_\star\bigcup X(s_l)\text{”},$$
$$\mathsf{q}\Vdash \text{“}X(t_{\beta_{\min}})\cap \mathbf{J}(b_0)\cap\ldots\cap\mathbf{J}(b_k)\subseteq_\star\omega\setminus\bigcup X(s_l)\text{”}.$$

Let

$$\Gamma=\{\gamma : X_\gamma = X^i_{a_j(i)},\ j\le k \text{ or } \gamma=\gamma^i_l,\ i\in A_\varrho,\ l\le n_i\},$$
$$\Xi=\{\xi : X_\xi = X^j_{b_i(j)},\ i\le k \text{ or } \xi=\xi^j_l,\ j\in B_\varrho,\ l\le m_j\}.$$

We defined $\Psi^\varrho(\beta)$ for $\beta\in\Gamma\cup\Xi$. It remains to define $\Psi^\varrho(\beta)$ for $\beta\notin\Gamma\cup\Xi$. Assume that ϱ is the lth element of Δ and let $c=l(\mathsf{p})+l$. Denote by $\mathbf{P}_\xi$ the formula “$c\in X_\xi$”, and by $\mathbf{P}_s$ the conjunction $\bigwedge_{\xi\in\mathrm{dom}(s)} s(\xi)\mathbf{P}_\xi$, where

$$s(\xi)\mathbf{P}_\xi=\begin{cases}\mathbf{P}_\xi & \text{if } s(\xi)=1,\\ \neg\mathbf{P}_\xi & \text{if } s(\xi)=-1.\end{cases}$$

Consider the following scheme: If $\xi\in(\mathrm{supp}(\mathsf{p})\cup\mathrm{supp}(\mathsf{q}))\cap(Q_1\cup Q_2)=\Omega$ then $\mathbf{R}_\xi$ is the formula

$$\bigvee_{s\in u^{\mathsf{p}}_\xi\cup u^{\mathsf{q}}_\xi}\mathbf{P}_s\wedge\bigvee_{t\in w^{\mathsf{p}}_\xi\cup w^{\mathsf{q}}_\xi}\mathbf{P}_t$$

(we assume that $\mathbf{R}_\xi$ is false if $u^{\mathsf{p}}_\xi\cup u^{\mathsf{q}}_\xi$ or $w^{\mathsf{p}}_\xi\cup w^{\mathsf{q}}_\xi$ is empty), and $\mathbf{K}_\xi$ is the formula

$$\Big(\bigvee_{s\in u^{\mathsf{p}}_\xi\cup u^{\mathsf{q}}_\xi}\mathbf{P}_s\Rightarrow\mathbf{P}_\xi\Big)\wedge\Big(\bigvee_{t\in w^{\mathsf{p}}_\xi\cup w^{\mathsf{q}}_\xi}\mathbf{P}_t\Rightarrow(\neg\mathbf{P}_\xi)\Big).$$

We want to find an assignment such that

$$(\star)\qquad \bigwedge_{\xi\in\Omega}\neg\mathbf{R}_\xi\wedge\bigwedge_{\xi\in\Omega}\mathbf{K}_\xi \text{ is true}$$

and

$$(\star\star)\quad \mathbf{P}_\xi \text{ is true if } \Psi^\varrho(\beta)=1, \text{ and } \mathbf{P}_\xi \text{ is false if } \Psi^\varrho(\beta)=0 \text{ for } \beta\in\Gamma\cup\Xi.$$

(Since p and q are compatible there exists an assignment such that $(\star)$ is true, but $(\star\star)$ need not be satisfied.)

Suppose to the contrary that for all assignments which satisfy $(\star\star)$ the sentence

$$\bigvee_{\xi\in\Omega}\mathbf{R}_\xi\vee\bigvee_{\xi\in\Omega}\Big(\bigvee_{s\in u^{\mathsf{p}}_\xi\cup u^{\mathsf{q}}_\xi}\mathbf{P}_s\wedge\neg\mathbf{P}_\xi\Big)\vee\Big(\bigvee_{t\in w^{\mathsf{p}}_\xi\cup w^{\mathsf{q}}_\xi}\mathbf{P}_t\vee\mathbf{P}_\xi\Big)$$

is true. (Note that this sentence is an alternative of sentences $\mathbf{P}_v$ when $\mathsf{q}\Vdash$ "$X(v)=_\star\emptyset$".) Thus there are $\xi_1,\dots,\xi_l\in\Omega$ and $\zeta_1,\dots,\zeta_d\in\Gamma\cup\Xi$ such that

$$\bigvee_{\varepsilon\in\Theta}\varepsilon(1)\mathbf{P}_{\xi_1}\wedge\dots\wedge\varepsilon(l)\mathbf{P}_{\xi_l}\wedge\mathbf{P}_{\zeta_{i_1(\varepsilon)}}\wedge\dots\wedge\mathbf{P}_{\zeta_{i_{d(\varepsilon)}(\varepsilon)}}$$

is equivalent to

$$\bigvee_{\xi\in\Omega'}\mathbf{R}_\xi\vee\bigvee_{\xi\in\Omega''}\Big(\bigvee_{s\in u^{\mathsf{p}}_\xi\cup u^{\mathsf{q}}_\xi}\mathbf{P}_s\wedge\neg\mathbf{P}_\xi\Big)\vee\Big(\bigvee_{t\in w^{\mathsf{p}}_\xi\cup w^{\mathsf{q}}_\xi}\mathbf{P}_t\vee\mathbf{P}_\xi\Big)$$

($\Omega',\Omega''\subseteq\Omega$ and $\Theta=\{\varepsilon:\varepsilon:l+1\to\{-1,1\}\}$).

Since $\mathsf{p}\restriction\alpha$ and $\mathsf{q}\restriction\alpha$ are compatible, $\{\zeta_1,\dots,\zeta_d\}\neq\emptyset$. It is easy to see that there are $\zeta_1,\dots,\zeta_i\in\Gamma$ and $\zeta_{i+1},\dots,\zeta_d\in\Xi$. We divide the set of ξ_j's into three disjoint sets: $\xi_1,\dots,\xi_{l_1}\in\operatorname{supp}(\mathsf{p})\setminus\operatorname{supp}(\mathsf{q})$, $\xi_{l_1+1},\dots,\xi_{l_2}\in\operatorname{supp}(\mathsf{p})\cap\operatorname{supp}(\mathsf{q})$, $\xi_{l_2+1},\dots,\xi_l\in\operatorname{supp}(\mathsf{q})\setminus\operatorname{supp}(\mathsf{p})$. Thus each $\varepsilon=(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ and $\operatorname{dom}(\varepsilon_i)$ is equal to $\{1,\dots,l_1\}$, $\{l_1+1,\dots,l_2\}$ or $\{l_2+1,\dots,l\}$ respectively. Denote by $\mathbf{I}(a^\varepsilon)$ (resp. $\mathbf{J}(b^\varepsilon)$) the intersection $\bigcap_{\zeta_{i(\varepsilon)}\in\Gamma}X_{\zeta_{i(\varepsilon)}}$ (resp. $\bigcap_{\zeta_{i(\varepsilon)}\in\Xi}X_{\zeta_{i(\varepsilon)}}$). There are two possibilities:

1. $\mathbf{I}(a^{\varepsilon_1,\varepsilon_2,\varepsilon_3})\cap\bigcap_{i\in\operatorname{dom}(\varepsilon_1\varepsilon_2)}(\varepsilon_1\varepsilon_2)(i)X_{\xi_i}=_\star\emptyset$.
2. There is $\varrho_\varepsilon\in\operatorname{supp}(\mathsf{p})\cap\operatorname{supp}(\mathsf{q})$ such that

$$\mathsf{p}\Vdash\text{“}\mathbf{I}(a^{\varepsilon_1,\varepsilon_2,\varepsilon_3})\cap\bigcap_{i\in\operatorname{dom}(\varepsilon_1\varepsilon_2)}(\varepsilon_1\varepsilon_2)(i)X_{\xi_i}\subseteq_\star X_{\varrho_\varepsilon}\cap\bigcap_{i\in\operatorname{dom}(\varepsilon_2)}\varepsilon_2(i)X_{\xi_i}\text{”}$$

and

$$\mathsf{q}\Vdash\text{“}\mathbf{J}(b^{\varepsilon_1,\varepsilon_2,\varepsilon_3})\cap\bigcap_{i\in\operatorname{dom}(\varepsilon_3)}\varepsilon_3(i)X_{\xi_i}\subseteq_\star\omega\setminus\Big(X_{\varrho_\varepsilon}\cap\bigcap_{i\in\operatorname{dom}(\varepsilon_2)}\varepsilon_2(i)X_{\xi_i}\Big)\text{”}.$$

Thus

$$\mathsf{p} \Vdash \text{“}X(s_{\alpha_{\min}}) \cap \mathbf{I}(a_0) \cap \ldots \cap \mathbf{I}(a_k) \subseteq_\star \mathbf{I}\Big(\bigcup_{\varepsilon\in\Theta} a^{\varepsilon_1,\varepsilon_2,\varepsilon_3}\Big)$$
$$\subseteq_\star \bigcup_{\varepsilon_1}\bigcup_{\varepsilon_2}\bigcup_{\varepsilon_3}\Big(X_{\varrho_\varepsilon} \cap \bigcap_{i\in\mathrm{dom}(\varepsilon_2)} \varepsilon_2(i)X_{\xi_i}\Big)\text{”}$$

and

$$\mathsf{q} \Vdash \text{“}X(s_{\beta_{\min}}) \cap \mathbf{J}(b_0) \cap \ldots \cap \mathbf{J}(b_k) \subseteq_\star \mathbf{J}\Big(\bigcup_{\varepsilon\in\Theta} b^{\varepsilon_1,\varepsilon_2,\varepsilon_3}\Big)$$
$$\subseteq_\star \omega \setminus \Big(\bigcup_{\varepsilon_1}\bigcup_{\varepsilon_2}\bigcup_{\varepsilon_3}\Big(X_{\varrho_\varepsilon} \cap \bigcap_{i\in\mathrm{dom}(\varepsilon_2)} \varepsilon_2(i)X_{\xi_i}\Big)\Big)\text{”},$$

a contradiction.

Thus there is an assignment satisfying $(\star\star)$ such that $(\star)$ is true. We define

$$\Psi^{\varrho}(\xi) = \begin{cases} 1 & \text{if } \mathbf{P}_\xi \text{ is true,}\\ 0 & \text{if } \mathbf{P}_\xi \text{ is false.}\end{cases}$$

It is easy to prove (by induction) that there is $\mathsf{r} \in \mathsf{P}'_\alpha$ satisfying the following conditions:

(†) $l(\mathsf{r}) = l(\mathsf{p}) + \operatorname{card}\Delta$.

(††) If $\xi \in \Omega$ and $\mathsf{r}\restriction\xi \Vdash$ “$\mathsf{r}(\xi) = (u^{\mathsf{r}}_\xi, x^{\mathsf{r}}_\xi, w^{\mathsf{r}}_\xi)$” then $x^{\mathsf{r}}_\xi \restriction l(\mathsf{p}) = x^{\mathsf{p}}_\xi$ and $x_\xi(l(\mathsf{p}) + i) = \Psi^{\varrho_i}(\xi)$, where ϱ_i is the ith element of Δ.

The proof (except for the case $\beta \in E$) is identical to the proof of Lemma 4.5 of [5] (Ch. 9).

Assume that $\beta \in E$ and β is the ith element of Δ. Let $\mathsf{r}_1 \le \mathsf{p}\restriction\beta, \mathsf{q}\restriction\beta$ be an element of P'_β satisfying (†) and (††). Thus

$$\mathsf{r}_1 \Vdash \text{“}X(s_{\alpha^\beta_i}) \cap X(t_{\beta^\beta_j}) =_\star \emptyset\text{”},$$

where

$$\mathsf{p}\restriction\beta \Vdash \text{“}\mathsf{p}(\beta) = \{(\alpha^\beta_1, s_{\alpha^\beta_1}, t_{\alpha^\beta_1}), \ldots, (\alpha^\beta_{n_\beta}, s_{\alpha^\beta_{n_\beta}}, t_{\alpha^\beta_{n_\beta}})\}\text{”},$$
$$\mathsf{q}\restriction\beta \Vdash \text{“}\mathsf{q}(\beta) = \{(\xi^\beta_1, s_{\xi^\beta_1}, t_{\xi^\beta_1}), \ldots, (\xi^\beta_{m_\beta}, s_{\xi^\beta_{m_\beta}}, t_{\xi^\beta_{m_\beta}})\}\text{”}.$$

If $\alpha^\beta_i \notin A_\beta$ or $\xi^\beta_j \notin B_\beta$ then

$$\mathsf{r}_1 \Vdash \text{“}X(s_{\alpha^\beta_i}) \cap X(t_{\xi^\beta_j}) \ne \emptyset\text{”}.$$

If $\alpha^\beta_i \in A_\beta$ and $\xi^\beta_j \in B_\beta$ then

$$\mathsf{r}_1 \Vdash \text{“}l(\mathsf{p}) + i \in X(s_{\alpha^\beta_i}) \cap X(t_{\xi^\beta_j})\text{”}.$$

Thus $\mathsf{r}_1 \Vdash$ “$\tau = \mathsf{p}(\beta) \cup \mathsf{q}(\beta) \in \mathsf{E}_\beta$” and $\mathsf{r} = \mathsf{r}_1 \star \tau$ is the required element. ∎

LEMMA 6. *Assume inductively that*:

$(1)_\alpha$ P_α *has the c.c.c.*

$(2)_\alpha$ *If* $\mathcal{L}$ *is a gap in* $B(\varphi)$ *and* $\varphi \in \mathcal{D}_\alpha$ *then* $\mathsf{P}_\alpha \Vdash$ “$\mathsf{Q}(\mathcal{L})$ *has the c.c.c.*”

$(3)_\alpha$ *If* $\zeta < \alpha$ *and* $\mathcal{L}_\zeta = (\mathcal{S}_\zeta, \mathcal{U}_\zeta)$ *is a gap in the domain or range of* $(T^\alpha_\zeta)^{\varepsilon k}$ *such that* $\mathsf{P}_\alpha \Vdash$ "$\mathsf{Q}(\mathcal{L})$ *has the c.c.c.*" *and for all* $X(s) \in \mathcal{S}_\zeta$ *and* $X(t) \in \mathcal{U}_\zeta$ *there are* $\bigcup_{i=1}^n X(s_i) \in \mathcal{S}_\zeta$ *and* $\bigcup_{j=1}^m X(t_j) \in \mathcal{U}_\zeta$ *such that*

$$X(s) = \bigcup_{i=1}^n X(s_i) \wedge X(t) = \bigcup_{j=1}^m X(t_j)$$
$$\wedge \operatorname{Ind}\Big(\bigcup_{i=1}^n X(s_i)\Big) < \operatorname{Ind}\Big((T^\alpha_\zeta)^{\varepsilon k}\Big(\bigcup_{i=1}^n X(s_i)\Big)\Big)$$
$$\wedge \operatorname{Ind}\Big(\bigcup_{j=1}^m X(t_j)\Big) < \operatorname{Ind}\Big((T^\alpha_\zeta)^{\varepsilon k}\Big(\bigcup_{j=1}^m X(t_j)\Big)\Big)$$

then $\mathsf{P}_\alpha \Vdash$ "$\mathsf{Q}((T^\alpha_\zeta)^{\varepsilon k}(\mathcal{L}))$ *has the c.c.c.*"

Then $\mathsf{P}_{\alpha+1}$ *has the c.c.c. and the conditions* $(2)_{\alpha+1}$–$(3)_{\alpha+1}$ *hold.*

Proof. Let $P = \{\mathsf{p}_\xi : \xi \in \omega_1\} \subseteq \mathsf{P}_{\alpha+1}$. Then, by Lemma 3, for each p_ξ there is $\mathsf{p}'_\xi \le \mathsf{p}_\xi$ with $\mathsf{p}'_\xi \in \mathsf{P}^\star_{\alpha+1}$. Applying the Δ-system lemma we find a set $P'_\Delta \subseteq \{\mathsf{p}'_\xi : \xi \in \omega_1\}$ of cardinality ω_1 consisting of conditions whose supports have a common root. By Lemma 2 deleting (at most) countably many conditions we can divide P'_Δ into ω sets P^n_Δ on which the assumptions of Lemma 5 are satisfied. Thus there are no uncountable antichains in $\mathsf{P}_{\alpha+1}$. Conditions $(2)_{\alpha+1}$–$(3)_{\alpha+1}$ are proved in a similar way. ∎

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