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Exactly two-to-one maps from continua onto arc-continua

by

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Abstract. Continuing studies on 2-to-1 maps onto indecomposable continua having only arcs as proper non-degenerate subcontinua—called here arc-continua—we drop the hypothesis of tree-likeness, and we get some conditions on the arc-continuum image that force any 2-to-1 map to be a local homeomorphism. We show that any 2-to-1 map from a continuum onto a local Cantor bundle Y is either a local homeomorphism or a retraction if Y is orientable, and that it is a local homeomorphism if Y is not orientable.

Define X to be an *arc-continuum* if X is a (metric) continuum and each proper non-degenerate subcontinuum of X is an arc. In an earlier paper [5] we showed that there is no exactly 2-to-1 continuous map from any continuum onto a tree-like arc-continuum (to partially answer a question raised by Sam Nadler, Jr. and L. E. Ward, Jr. [14]) by first showing that any such map must be a local homeomorphism (i.e. a 2-fold covering map). In this paper we continue our study of exactly 2-to-1 maps from continua onto arc-continua, without the hypothesis of tree-likeness, and we have found some simple conditions on the arc-continuum image (different for orientable and non-orientable cases) that force any 2-to-1 map to be a local homeomorphism. In the case of an indecomposable arc-continuum Y that is a local Cantor bundle, we show that any 2-to-1 map from a continuum onto Y is either a local homeomorphism or a retraction if Y is orientable (both situations can be realized), and any 2-to-1 map from a continuum onto Y is a local homeomorphism if Y is not orientable. Thus more is now known about what kinds of 2-to-1 maps are possible onto these types of spaces, including all solenoids.

A decomposable arc-continuum is the union of two arcs, thus an arc or a simple closed curve. Harrold showed in 1940 [8] that the arc is not the

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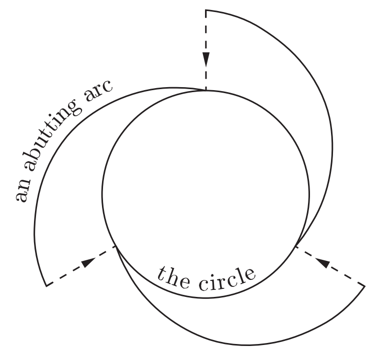

Fig. 1

2-to-1 image of any continuum; see also W. H. Gottschalk (1947) [6] for an independent proof and a generalization. There are 2-to-1 maps onto the simple closed curve. It is easy to construct a 2-fold cover from the circle onto itself; or to retract a circle, plus finitely many abutting arcs (see Fig. 1), 2-to-1 onto the circle; or to construct a 2-to-1 map from the circle onto itself that is neither a local homeomorphism nor a retraction. This paper, however, studies only indecomposable arc-continua.

Now suppose that Y is an indecomposable arc-continuum. Are—as in the case of local Cantor bundles—retractions and 2-fold covers the only 2-to-1 maps from continua onto Y? In general it is not true. In Example 1 below, we define an indecomposable arc-continuum that is the 2-to-1 image of a continuum under a map that is neither a local homeomorphism nor a retraction. The arcs in the continuum Y in Example 1 do not all have a property that we call "approximable", a property that is very useful in our proofs, and is automatically satisfied by any arc-continuum that is a local Cantor bundle. We say that an arc A in an indecomposable arc-continuum Y is *approximable* if every dense half-composant of Y contains a sequence of arcs topologically convergent to A.

Example 1. Let K be an arc-continuum with two endpoints, p and q; let Y be the arc-continuum that results when p and q are identified; and let X be the continuum obtained by gluing together two copies of K, K_1 and K_2, so that p_1 and p_2 are identified and q_1 and q_2 are identified (see Fig. 2). The natural 2-to-1 map from X onto Y obtained by folding K_1 onto K_2 and then identifying the points $\{p_1, p_2\}$ and $\{q_1, q_2\}$ is neither a local homeomorphism nor a retraction. Note that if A is an arc in K with endpoint p and B is an arc in K with endpoint q, then the arc $T = A \cup B$ in Y is not approximable.

In the case when K is an arc, the resulting 2-to-1 map is the map from the circle onto itself (mentioned at the beginning) that is neither a local homeomorphism nor a retraction.

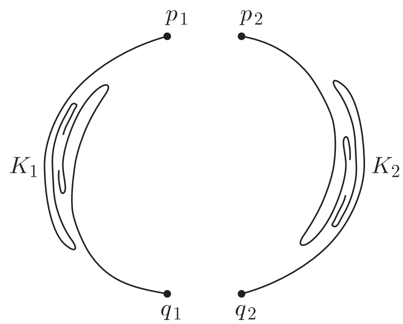

Fig. 2

Another example can be derived from Corollary 3 in [4], where the existence of such maps onto arc-continua from some P-adic solenoids was indicated. This example will be used to show that the approximability properties in the hypotheses of the main theorems are necessary.

1. Basic facts and lemmas concerning 2-to-1 maps onto indecomposable arc-continua. If Y is an indecomposable arc-continuum, then each arc-component of Y is an increasing union of arcs and is either a continuous 1-to-1 image of a ray, or a continuous 1-to-1 image of the line. Suppose that Y is an indecomposable arc-continuum and N is an arc in Y. If we remove the geometric interior of N from the arc component of Y in which N lies, then the remainder splits into two arcwise connected sets, at least one of which, say E, is dense in Y. The set E is a continuous 1-to-1 image of the closed half-line, and such sets will be called *half-composants*. The endpoint common to E and N will be called a *free* endpoint of N. Each arc in Y has at least one free endpoint. The other part of the remainder, if it is not dense, will contain an endpoint of Y. An *endpoint* of Y is any point that is the endpoint (in the usual sense) of each arc in Y that contains it. The unique arc in the arc-continuum joining the points x and y will be denoted by $\langle x, y\rangle$.

One fact that we use often in this paper is that the set of points y in Y at which f is open (i.e. f is open at each of the two points of $f^{-1}(y)$) is open and dense in Y for any 2-to-1 map onto Y (see [13]). We shall call such points in Y *points of openness*.

We will use the following two lemmas from [5]:

LEMMA 1 (Lemma 3 of [5]). *If Y is an indecomposable arc-continuum, f is a* 2*-to-*1 *map from a continuum X onto Y, L is an arc in Y, and C is a component of $f^{-1}(L)$, then $f(C)$ contains a free endpoint of L.*

LEMMA 2 (Lemma 2 of [5]). *Suppose Y is an indecomposable arc-continuum, f is a* 2*-to-*1 *map from a continuum X onto Y, L is an arc in Y from t to x, and C is a component of $f^{-1}(L)$ such that $t \notin f(C)$. Let N be* *an extension of L through x, i.e. an arc with endpoint t containing L. Then there is a component D of $f^{-1}(N)$ for which we still have $t \notin f(D)$.*

LEMMA 3. *Let Y be an indecomposable arc-continuum, and let f be a 2-to-1 map from a continuum onto Y. Let L be an arc contained in Y. Then $f^{-1}(L)$ consists of two components.*

Proof. Suppose that the conclusion does not hold.

By Harrold's result, the inverse image $f^{-1}(L)$ of L is disconnected. Thus, in view of our assumption to the contrary, there are at least three components of $f^{-1}(L)$. This forces the images, under f, of two of them, say C and D, to be disjoint, as otherwise the images of some three components would have a point in common, which contradicts the fact that f is 2-to-1. By Lemma 1, the image of each of them contains a free endpoint of L. Clearly, C and D map to different free endpoints of L, so we see that both endpoints of L are free. It follows that L can be extended to an arc L' whose endpoints are both points of openness.

We shall show that $f^{-1}(L')$ also must have at least three components. Label the endpoints of L by a and b, and the endpoints of L' by a' and b'. Let t denote a point of L that does not belong to either $f(C)$ or $f(D)$. The arc A from t to a' extends the arc from t to a (or b), and so by Lemma 2, some component of $f^{-1}(A)$ maps to a' but not to t. Similarly, some component of the inverse of the arc from t to b' maps to b' but not to t. These two components are also components of $f^{-1}(L')$ and there must be a third component that maps to t.

Hence, without loss of generality, we may assume that the arc L is the extended arc, and that the ends of L are points of openness. Let C, D and t be as before.

Now we will show that in each half-composant determined by L there are arcs that also have at least three components in their inverse. Take disjoint neighborhoods, U_1 of C, U_2 of D and V of $f^{-1}(L) \setminus (C \cup D)$. The neighborhoods U_1 and U_2 can be taken so small that $f(\overline{U}_1) \cap f(\overline{U}_2) = \emptyset$ and $t \notin f(\overline{U}_1) \cup f(\overline{U}_2)$.

For each $\varepsilon > 0$, and each half-composant, there is an arc A in the half-composant within ε of L such that the set $\{a', b'\}$ of endpoints of A is within ε of the set $\{a, b\}$ of endpoints of L and some point t' of A is within ε of t and lies outside of $f(\overline{U}_a \cup \overline{U}_2)$.

To see this, take an ε-neighborhood of L in the form of the union of a closed ε-chain from a to b, $U(a)$ and $U(b)$ denoting the (closed) links to which a and b belong, respectively. Take ε so small that the point t in L is not in $U(a) \cup U(b)$ and the links to which t belongs do not intersect the set $f(\overline{U}_1 \cup \overline{U}_2)$. There exists a sequence of points t_n belonging to the half-composant which converges to t. Let J_n be the component of the point t_n

in the union of the chain from a to b. For sufficiently large n, the endpoints of the arc J_n will lie in $U(a) \cup U(b)$ since otherwise the limit of J_n sequence together with the arc L will be a continuum in Y that is not an arc, contrary to the fact that Y is an arc-continuum. The above mentioned J_n can be taken for the arc A.

If ε is small enough, then (1) $f^{-1}(A) \subset U_1 \cup U_2 \cup V$, (2) some point p of $f^{-1}(A)$ that maps to a' and some point q of $f^{-1}(A)$ that maps to b' lies in $U_1 \cup U_2$ (this is because a and b are points of openness), and (3) both points of $f^{-1}(A)$ that map to t' lie in V. The component of $f^{-1}(A)$ that contains p cannot equal the component of $f^{-1}(A)$ that contains q since these components lie in $U_1 \cup U_2$ and cannot map to t'. Hence $f^{-1}(A)$ has at least three components, one in V that contains a point that maps to t', one containing p and one containing q. So we have three arcs, L_1, L_2, and L_3 in the same composant, each with three or more components in their inverse image, and having the properties of the arcs described above. Label the endpoints of L_i by a_i and b_i, so that their order in the composant is $a_1 < b_1 < a_2 < b_2 < a_3 < b_3$. Let C_i be a component of $f^{-1}(L_i)$ that maps onto a_i but not onto b_i, let D_i be a component of $f^{-1}(L_i)$ that maps onto b_i but not onto a_i, and let t_i denote a point of L_i that is not in the image of $C_i \cup D_i$. Such components C_i and D_i exist, as was mentioned in the introductory fragment of this proof.

Some component of $f^{-1}([a_1, b_3])$ has a point that maps to t_2 and a point that maps to either a_1 or b_3, say a_1, by Lemma 1. By Lemma 0 of [10], this component is arcwise connected (in fact, locally connected), since f is finite-to-1. So, there exists an arc A contained in $f^{-1}([a_1, b_3])$ such that the set of endpoints of A goes onto $\{a_1, t_2\}$ under f and the image of A is the arc $[a_1, t_2]$. But f is 2-to-1, hence f maps A onto $[a_1, t_2]$ homeomorphically. Notice that both C_2 and D_1 are disjoint from A. To see this, suppose there is a point z in both A and D_1, for instance. Then D_1 union the subarc of A from z to the point of A that maps to a_1 maps into L_1 and contains D_1 properly; but this is impossible as D_1 is a component of $f^{-1}(L_1)$. Since f maps A onto $[a_1, t_2]$ homeomorphically, some subarc A' of A maps homeomorphically onto $[b_1, a_2]$. In addition, C_2 has a point x that maps to a_2, and D_1 has a point y that maps to b_1. We showed that x and y do not belong to A', so x and y are the only points outside of A' that map to either b_1 or to a_2. But $f^{-1}([b_1, a_2])$ cannot be connected, so it has a component E different from the component containing A'. Then E must contain one of x or y, by Lemma 1. If E contains x but not y, then $E \cup C_2$ is a component of $f^{-1}([a_1, b_3])$ that fails to map to either a_1 or b_3; if E contains y but not x then $E \cup D_1$ is such a component; and if E contains both y and x, then $E \cup D_1 \cup C_2$ is such a component. In any case, Lemma 1 is violated. This contradiction completes the proof. ■

Suppose ε is a positive number and the arcs B and A lie in the same composant of an indecomposable arc-continuum. Endow the composant with an orientation. We say that the arc B has *reverse orientation within* ε *of* A if B lies in the ε-neighborhood of A and the ε-neighborhood of the last point of A contains the first point of B and the ε-neighborhood of the first point of A contains the last point of B.

LEMMA 4. *Let Y be an indecomposable arc-continuum, let L be an arc in Y containing an endpoint y of Y, and let ε be a positive number. Then there is an arc in the same composant as L with reverse orientation within ε of L.*

Proof. Let L be an arc in Y containing the endpoint y of Y and let ε be a positive number. Denote the other endpoint of L by y', and let U be an ε-neighborhood of L.

Let y_i be a sequence converging to y such that each y_i does not belong to L but does belong to the same composant as y. Assume also that each y_i has a distance less than ε from y. Let L_i be the arc from y' to y_i and let D_i be the closure of the component of $U \cap L_i$ that contains y_i. Assume without loss of generality that the sequence $\{D_i\}$ topologically converges. The topological limit D of $\{D_i\}$ is a continuum, thus an arc, contained in the closure of U, and D has a point on the boundary of U which does not belong to L. Since D contains the point y, the endpoint of the arc component on which L lies, the continuum D must contain the arc L. Thus, if i is sufficiently large, the component D_i contains a point p_i at a distance less than ε from y'. The arc in D_i joining the points p_i and y_i is the desired arc having a reverse orientation within ε of L. ■

2. Two conditions on an indecomposable arc-continuum Y whose arcs are approximable that each imply that any 2-to-1 map from any continuum onto Y must be a local homeomorphism. Suppose f is a 2-to-1 map from a continuum onto a continuum Y. A proper subcontinuum L in Y will be called (following our paper [5]) *trivial* if $f^{-1}(L)$ splits into two disjoint subcontinua, each of which is mapped by f homeomorphically onto L. If f is a 2-to-1 local homeomorphism from a continuum onto an arc-continuum Y, then each arc A in Y is trivial. (This follows from the proof of Lemma 3 in [9] and Harrold's result that the inverse of each arc in Y cannot be connected.) It was also shown in [9] (Theorem 1) that if each proper subcontinuum of Y is trivial, then f is a local homeomorphism.

Now, suppose f is a 2-to-1 map from a continuum onto an indecomposable arc-continuum Y whose arcs are approximable. The two main results of this section (Theorems 1 and 2) are that if either (1) the arcs in any half-composant of Y are trivial, or (2) Y has an endpoint, then f is a local homeomorphism. We also show, in Lemma 5, that, under the hypothesis of approximability of arcs, f is weakly confluent, i.e. at least one component of the inverse of each arc in Y maps onto the arc.

Recall the continuum Y in Example 1 in the introduction. Y is an indecomposable arc continuum that will not satisfy the conclusion of Theorem 1, and some arcs in Y are not approximable (for instance, the arc T, defined in the paragraph preceding the description of Y). Thus it is clear that the hypothesis of approximability will be required in Theorem 1.

THEOREM 1. *Let Y be an indecomposable arc-continuum such that all arcs in Y are approximable and let f be a* 2*-to-*1 *map from a continuum onto Y. If each arc of some half-composant of Y is trivial then all arcs of Y are trivial and hence f is a local homeomorphism.*

Proof. Suppose L is an arc of Y that is not trivial. We can assume without loss of generality that the free ends of L are points of openness of Y and that the end of L which is not free (if there is one) is the endpoint of the composant containing L. This is true because, as before, we can enlarge L to such an arc (since the points of openness form an open set in Y) and since if the larger arc is trivial, so is L. By Lemma 3, $f^{-1}(L)$ has two components and one of them, say C, satisfies $f(C) \neq L$, since L is assumed not to be trivial. By Lemma 1, $f(C)$ contains a free endpoint, say p, of L, which by assumption is a point of openness. Let q be the endpoint of L which does not belong to $f(C)$. Since $f(C)$ does not contain q, choose a neighborhood G of q such that $f(C) \cap \bar{G} = \emptyset$. Let H be a neighborhood of $f^{-1}(L)$ such that $H = H' \cup H''$, where H' and H'' are open and disjoint, $C \subset H'$, and $f(H') \cap G = \emptyset$. Let V be a neighborhood of p so small that $f^{-1}(y) \cap H' \neq \emptyset$ for $y \in V$. (Recall that p is a point where f is open.)

Let S be a half-composant of Y whose arcs are trivial, as assumed by the hypothesis. In view of approximability, take an arc L' in S lying so close to L in the sense of topological limit that (1) $f^{-1}(L') \subset H$ and (2) L' intersects both G and V. Since L' is trivial, $f^{-1}(L')$ splits into two arcs, each of which is mapped homeomorphically under f onto L'. Since L' has points in V, we have $f^{-1}(L') \cap H' \neq \emptyset$. Thus, for one of the two arcs mentioned above, say K, we have $K \cap H' \neq \emptyset$. It follows that $K \subset H'$, and in consequence, $f(K) \cap G = \emptyset$, contrary to the fact that K maps onto L'.

We have shown that all of the arcs in Y are trivial, so from Theorem 1 of [9], f must be a local homeomorphism. ∎

LEMMA 5. *Let Y be an indecomposable arc-continuum such that arcs in Y are approximable*, *and let f be a* 2*-to-*1 *map from a continuum X onto Y. Then f is weakly confluent*, *i.e. if L is an arc contained in Y*, *then there exists a component of $f^{-1}(L)$ that maps onto L under f.*

P r o o f. Let us suppose that the conclusion does not hold. Let A be the union of components of $f^{-1}(L)$ the images of which under f contain one end of L, and let B be the union of components of $f^{-1}(L)$ the images of which contain the other end of L. The sets A and B, being finite unions of components (in fact, exactly one as easily follows from Lemma 3) of $f^{-1}(L)$, are closed. By the assumption, the sets A and B are disjoint and, by Lemma 1, they cover $f^{-1}(L)$. Let U and V be open disjoint sets, one containing A and the other containing B. Since L is approximable, there is an arc L' lying in the same arc component as L, disjoint from L, and so close to L in the sense of Hausdorff distance that (1) $f^{-1}(L') \subset U \cup V$, and (2) the inverse images of the endpoints of L' lie one in U and the other in V. Let M be an arc joining L and L', having only endpoints in common with L and L'. Let C be a component of the inverse image $f^{-1}(L \cup M \cup L')$ of the arc $L \cup M \cup L'$ meeting the set $f^{-1}(M)$. By Lemma 1, $f(C)$ contains an endpoint of $L \cup M \cup L'$. This endpoint lies in L or in L'. Assume that it lies in L'. We have $L' \subset f(C)$. The map f restricted to C is onto $f(C)$ and is weakly confluent, as follows from the result proved by Gryspolakis and Tymchatyn in [6]. This means that there exists a component D of $f^{-1}(L')$ such that $f(D) = L'$. We get a contradiction, as the continuum D is contained in the union of open and disjoint sets U and V, being contained in neither of them. ∎

THEOREM 2. *Suppose f is a continuous* 2*-to-*1 *map from a continuum X onto an indecomposable arc-continuum Y, all arcs in Y are approximable, and Y has an endpoint y. Then f is a local homeomorphism.*

P r o o f. If each arc in Y containing y is trivial, then f is a local homeomorphism by Theorem 1. So assume that L is a non-trivial arc in Y containing y. As in the proof of Theorem 1, we may assume that the other endpoint, x, of L is a point where f is open. Since L is non-trivial there is a component C of $f^{-1}(L)$ that does not map onto L, and by Lemma 1, $f(C)$ contains x and not y. By Lemmas 5 and 3 the other component F of $f^{-1}(L)$ is mapped onto L under f. Note that because of $y \notin f(C)$, F is the only component that maps onto L. It follows that one of the two points that maps to x, say a, belongs to C and the other, say b, belongs to F. Let V and W be open sets such that $C \subset V$, W contains the component F, $\overline{V} \cap \overline{W} = \emptyset$, and $y \notin f(\overline{V})$.

Let U be an open neighborhood of x, an open value of f, such that $f^{-1}(U)$ splits into two open sets each of which is homeomorphically mapped by f onto U. If a point x' lies in U, then $f^{-1}(x')$ has points in both of these open sets. Assume, moreover, that U is so small that the two points of $f^{-1}(x')$ lie in distinct sets V and W. From Lemma 4 we know that there is, for each $\varepsilon > 0$, an arc L', lying in the same arc component of Y as L, with reverse orientation within ε of L. We may choose ε small enough that the first point, x', of L' lies in U (and thus $f^{-1}(x')$ has points in both V and W), that $f^{-1}(L') \subseteq W \cup V$, and that for the other endpoint y' of L', we have $y' \notin f(\overline{V})$. Let C' denote a component of $f^{-1}(L')$ having points in V, and let a' and b' denote the points of $f^{-1}(x')$ in V and W respectively. Note that exactly one component F' of $f^{-1}(L')$ maps onto L', and F' must contain b' and lie in W, since both inverse points of y' lie in W. Let M be the arc from x to x'. Consider the arc $S = L \cup M \cup L'$. By Lemma 5, one of the components of $f^{-1}(S)$, say A, maps onto S.

The following is a list of properties of f restricted to $f^{-1}(S)$:

1. The component A contains components of $f^{-1}(L)$, $f^{-1}(M)$, and $f^{-1}(L')$ that are mapped *onto* the corresponding arcs L, M, and L'. To see this, recall that every map to an arc is weakly confluent [7]. Thus A contains $F \cup F'$ and hence the points b and b' that map to x and x'. Since A is arc-connected [10], some component of $f^{-1}(M)$ contains b and b'.

2. By the theorem of Harrold [8], the counterimage $f^{-1}(M)$ is not connected. Hence there is a component C'' of $f^{-1}(M)$ that does not contain either b or b' and must contain either a or a'.

3. By assumption, the counterimage $f^{-1}(L)$ has a component C containing a such that $y \notin f(C)$ (and there is only one such component).

4. The counterimage $f^{-1}(L')$, of the arc L' close to L, has a component C' containing a' such that $y' \notin f(C')$ (and there is only one such component). Recall that $y' \notin f(\overline{V})$ and $C' \subset V$.

In view of properties 1–4, we infer that there exists a component of $f^{-1}(S)$ whose image contains neither of the points y and y'. This is a contradiction to Lemma 1. To find such a component, consider two cases:

1. C'' contains exactly one of a and a', say a. Then $C'' \cup C$ is such a component.

2. C'' contains both of a and a'. Then $C'' \cup C \cup C'$ is such a component. ∎

Note. Most of the proof of Theorem 2 comes directly from the proof of Lemma 8 of [5], but because the statement of Lemma 8 is quite different from that of Theorem 2, we include the proof for completeness.

Example 2. We construct a continuum Y' as in Example 1 except that we start with an arc continuum K' with three endpoints (and we do not identify this third endpoint with anything in the construction of Y'), then Y' is still an indecomposable arc-continuum, the arc T' is still not approximable, and the conclusion of Theorem 2 does not hold. Furthermore, the map is not weakly confluent at the arc T'. Hence approximability cannot be removed from Lemma 5 either.

Note. An example of an arc-continuum with three endpoints can be found in Hocking and Young [11], p. 142.

3. A study of 2-to-1 maps (from continua) onto indecomposable arc-continua that are local Cantor bundles. A continuum Y is a *local Cantor bundle* if each point of Y has a neighborhood homeomorphic to $C \times (0,1)$, where C denotes the Cantor discontinuum. This is a special case of what Aarts and Martens [2] call a *matchbox manifold*, i.e. a space Z such that if z is a point of Z then there is a zero-dimensional space S_z and a neighborhood of z that is homeomorphic to $S_z \times (0,1)$. Note that S_z need not be compact. Although the Cantor bundle property is a local one, it follows from Aarts and Martens' "Pasting" Lemma [2] that each arc in a local Cantor bundle Y has a neighborhood homeomorphic to $C \times (0,1)$; hence, if A is any arc in Y (where the arc- continuum Y is a local Cantor bundle) then A is approximable. Thus all of our earlier lemmas and theorems apply.

If Y is an orientable (defined below) matchbox manifold, then Aarts and Martens in [2] showed that there is a homeomorphism h from $C \times \{0\}$ onto $C \times \{1\}$ such that Y is homeomorphic to $C \times [0,1]$ with $\langle x, 0\rangle$ identified with $\langle h(x), 1\rangle$. Two special cases of the latter structure theorem were done earlier; by Keynes and Sears [12] if Y is compact, and by Aarts [1] if Y is an arc component of a continuum. These are in fact the two cases we use in this paper.

A technical definition for what it means to say that a matchbox manifold is orientable can be found in [3], but for arc-continua that are local Cantor bundles the definition is equivalent to the following natural one. The arc-continuum Y is *orientable* if each separate arc component can be parameterized (given a direction) so that if A is an arc in Y then there is an $\varepsilon > 0$ such that no arc B of Y has reverse orientation within ε (defined earlier in this paper) of A.

Suppose Y is an indecomposable arc-continuum that is a local Cantor bundle. We will show in Theorem 3 that the only 2-to-1 maps from continua onto Y are retractions or covering maps, and in Theorem 4 we show that if Y is not orientable, then in fact the only 2-to-1 maps from continua onto Y are covering maps. In Theorem 5 we show that if Y is orientable, then there is a 2-to-1 retraction from a continuum onto Y. (Note that Theorem 5 does not require that Y be an indecomposable arc-continuum.) But, as was shown in [4], even for solenoids, in some sense the simplest of orientable indecomposable arc-continua that are local Cantor bundles, 2-fold covers need not exist; for instance the dyadic solenoid does not admit a 2-fold cover. However, as was indicated in [4], 2-folds between some P-adic solenoids can exist.

However, we have no example that demonstrates that the full strength of the hypothesis of local Cantor bundle is necessary in Theorem 3.

QUESTION. *Does there exist a* 2*-to-*1 *map from a continuum onto an indecomposable arc-continuum whose arcs are approximable such that the map is neither a local homeomorphism nor a retraction?*

THEOREM 3. *Let Y be an indecomposable arc-continuum that is a local Cantor bundle. Then every exactly* 2*-to-*1 *map f from a continuum onto Y is a local homeomorphism or a retraction.*

Proof. If we assume that f is not a local homeomorphism, then from Theorem 2 of this paper we know that Y has no endpoint. If each arc of some half-composant of Y is trivial then f is a local homeomorphism, by Theorem 1, so we also may assume that in every half-composant of Y some arc is not trivial. Let A be the set of points a of X with the following property: $(*)$ for each arc L through $f(a)$ there exists an arc M through a such that f restricted to M is a homeomorphism onto L.

We shall show that f is a retraction by showing that (1) $f(A) = Y$, (2) f is 1-to-1 on A, and (3) A is a continuum. To show (3), we need only show that A is closed.

(1) Let $p \in Y$. Let a be such that $f(a) = p$. There is nothing to prove if $a \in A$. So, let us suppose that $a \notin A$. There exists an arc L through $f(a)$ such that a is not in any arc M which is mapped homeomorphically onto L under f. If L' is an arc in Y containing L then by Lemma 5, there exists a component of $f^{-1}(L')$ which is mapped onto L' under f. Since this component is arc-connected, it contains an arc that maps onto L'. But any map from an arc to an arc that sends the endpoints onto both endpoints and is at most 2-to-1, must be 1-to-1; and so there exists an arc M' contained in $f^{-1}(L')$ such that the set of endpoints of M' goes onto the set of endpoints of L' under f and f maps M' onto L' homeomorphically. The arc M' cannot go through a by assumption. Observe that if we diminish L', the above property will be preserved. This means that, if b denotes the point of $f^{-1}(p)$ different from a, then $b \in A$, and so $p \in f(A)$.

(2) Suppose $p = f(a) = f(b)$, where a and b belong to A and $a \neq b$. Since Y has no endpoints, the point p divides the composant on which it lies into two half-composants. In the case under consideration, both of these half-composants contain non-trivial arcs. Let L_1 and L_2 be such arcs. Since arcs that contain non-trivial arcs are themselves non-trivial, we may assume that p is an endpoint of both L_1 and L_2. Since a and b belong to A, there exist arcs, M through a and N through b, which are mapped under f homeomorphically onto $L_1 \cup L_2$. Since the arcs L_1 and L_2 are non-trivial, $M \cap f^{-1}(L_i)$ and $N \cap f^{-1}(L_i)$ intersect for $i = 1, 2$. Hence, for $i = 1, 2$, $(M \cup N) \cap f^{-1}(L_i)$ is connected and is contained in the component of $f^{-1}(L_i)$

that maps onto L_i. Since a and b are in this component, the component of $f^{-1}(L_i)$ that does not map onto L_i maps onto the endpoint of L_i different from p. Hence we have found three components in $f^{-1}(L_1 \cup L_2)$, contrary to Lemma 3.

(3) To finish the proof we shall show that the set A is closed. Let a_n be a sequence of points of A convergent to the point a, and let L be an arc to which $f(a)$ belongs. Because Y is a local Cantor bundle, it follows from the Long and Wide Lemma of Aarts and Martens [2] that there exists a sequence of arcs L_n convergent to L in such a way that $f(a_n) \in L_n$. For each n there exists an arc M_n through a_n such that f maps M_n homeomorphically onto L_n. Without loss of generality we can assume that the sequence of arcs M_n is topologically convergent. Denote by M the topological limit of the sequence M_n.

We shall show that f maps M homeomorphically onto L; this will imply that M is an arc through a and that the point a belongs to A.

Suppose that f does not map M homeomorphically onto L. Then f is not 1-to-1, as $f(M) = L$. Let b and c be different points of M such that $q = f(b) = f(c)$. Let b_n and c_n be different points of M_n such that the sequence $\{b_n\}$ converges to b and the a sequence $\{c_n\}$ converges to c. For each n, let L'_n be a subarc of L_n joining the points $f(b_n)$ and $f(c_n)$. Without loss of generality we can assume that the sequence $\{L'_n\}$ is convergent. The topological limit of the sequence of arcs $\{L'_n\}$ consists of a single point, namely q, since the endpoints converge to q and Y is a local Cantor bundle at the point q. For each n, let M'_n be a subarc of M_n joining the points b_n and c_n. We have $f(M_n) = L_n$ and $f(M'_n) = L'_n$. Without loss of generality we can assume that the sequence of arcs $\{M'_n\}$ is topologically convergent to the arc T from b to c. Then T is contained in the two-point set $f^{-1}(q)$, and in consequence, since T is connected, T must be equal to a one-point set. A contradiction. ■

THEOREM 4. *Suppose Y is an indecomposable arc-continuum that is a non-orientable Cantor bundle. Then the only exactly* 2*-to-*1 *maps from any continuum onto Y are* 2*-fold covers.*

P r o o f. Firstly, we show that if Y is not orientable, then composants of Y are not orientable, i.e. there is an arc A in Y such that if ε is a positive number then there is an arc B *in the same composant as* A that has reverse orientation within ε of A.

So suppose that D is a composant in Y that is itself orientable. We can use Aarts' structure theorem [1] for arc components to construct a totally disconnected (non- compact) space C' and a homeomorphism h from $C' \times \{0\}$ onto $C' \times \{1\}$ such that D is homeomorphic to $C' \times [0,1]$ with $\langle x, 0\rangle$ identified with $\langle h(x), 1\rangle$. This solenoidal structure on D makes it clear that there is an

$\varepsilon > 0$ such that no two arcs in D have opposite orientation within ε of each other. But the orientation on D can be used to construct an orientation on all of Y. For each arc B in Y, there is a sequence of arcs from D converging to B, and all but finitely many are within $\varepsilon/2$ of B and hence must be going in the same direction. Give B the same direction. No arc in Y can have arcs arbitrarily close with opposite direction since this would generate two arcs from D with reverse orientation within ε of each other. Thus we see that if Y is non-orientable, then each composant of Y is also non-orientable.

Now suppose that f is a 2-to-1 map from a continuum X onto Y that is not a 2-fold cover. Then some arc of Y is not trivial and we know that if it is enlarged to an arc L whose endpoints are points of openness then L is also not trivial, since the property of being trivial is hereditary. As in the proof of Theorem 2 we infer that there is a positive number ε such that no arc in one of the half-composants D of L has reverse orientation within ε of L.

This is enough to imply that the composant D itself is orientable. For, suppose D is given a direction and suppose some other arc A in D is a limit of arcs from D whose direction opposes that of A. Using Aarts' "Pasting" Lemma [1], there is, for some arc B in D containing both L and A, a neighborhood of B in D that is a product of a totally disconnected space, C' and $(0,1)$. This is a contradiction for arcs arbitrarily close to A to go in the opposite direction to the long arc B whereas no arc sufficiently close to L goes in the opposite direction. ■

In the orientable case there are examples for both possibilities (see comments in the introductory section), i.e. 2-fold covers and retractions. However, concerning retractions we can state the following stronger result.

THEOREM 5. *If Y is an orientable local Cantor bundle, then there is a continuum Z such that Z admits an exactly 2-to-1 retraction onto Y.*

Proof. We will use the orientable structure described in the introduction of this section.

Let h be a homeomorphism from $C \times \{0\}$ onto $C \times \{1\}$ such that Y is homeomorphic to $C \times [0,1]$ with $\langle x, 0\rangle$ identified with $\langle h(x), 1\rangle$. Let X be the continuum obtained when spikes are added to Y as follows: At each point $(c, 1/4)$ of Y, add an interval $I(c, 1/4)$ so that the collection is homeomorphic to $C \times [0,1]$, and each $I(c, 1/4)$ intersects Y exactly at $(c, 1/4)$. Do the same at the points $(c, 3/4)$ of Y. This describes X. For the retraction, uniformly fold each $I(c, 1/4)$ onto the subarc of Y from $(c, 1/4)$ to $(c, 3/4)$ and fold each $I(c, 3/4)$ onto the subarc of Y with beginning point $(c, 3/4)$ and endpoint $(d, 1/4)$, where $(d, 0)$ is the point that is identified with $(c, 1)$ under the sewing h. ■

References

[1] J. M. Aarts, *The structure of orbits in dynamical systems*, Fund. Math. 129 (1988), 39–58.
[2] J. M. Aarts and M. Martens, *Flows on one-dimensional spaces*, ibid. 131 (1988), 53–67.
[3] J. M. Aarts and L. G. Oversteegen, *Flowbox manifolds*, Trans. Amer. Math. Soc. 327 (1991), 449–463.
[4] W. Dębski, *Two-to-one maps on solenoids and Knaster continua*, Fund. Math. 141 (1992), 277–285.
[5] W. Dębski, J. Heath and J. Mioduszewski, *Exactly two-to-one maps from continua onto some tree-like continua*, ibid., 269–276.
[6] W. H. Gottschalk, *On k-to-1 transformations*, Bull. Amer. Math. Soc. 53 (1947), 168–169.
[7] J. Grispolakis and E. D. Tymchatyn, *Continua which are images of weakly confluent mappings only*, (I), Houston J. Math. 5 (1979), 483–501.
[8] O. G. Harrold, *Exactly $(k,1)$ transformations on connected linear graphs*, Amer. J. Math. 62 (1940), 823–834.
[9] J. Heath, 2-*to*-1 *maps with hereditarily indecomposable images*, Proc. Amer. Math. Soc. 113 (1991), 839–846.
[10] —, *There is no exactly k-to-1 function from any continuum onto* $[0,1]$*, or any dendrite, with only finitely many discontinuities*, Trans. Amer. Math. Soc. 306 (1988), 293–305.
[11] J. Hocking and G. Young, *Topology*, Addison-Wesley, 1961.
[12] H. B. Keynes and M. Sears, *Modeling expansion in real flows*, Pacific J. Math. 85 (1979), 111–124.
[13] J. Mioduszewski, *On two-to-one continuous functions*, Dissertationes Math. (Rozprawy Mat.) 24 (1961).
[14] S. B. Nadler, Jr. and L. E. Ward, Jr., *Concerning exactly $(n,1)$ images of continua*, Proc. Amer. Math. Soc. 87 (1983), 351–354.

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