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Analytic gaps

by

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Abstract. We investigate when two orthogonal families of sets of integers can be separated if one of them is analytic.

Two sets of integers a and b are *orthogonal* to each other if their intersection is finite. We shall let $a \perp b$ denote this fact. Two families A and B of sets of integers are orthogonal to each other if $a \perp b$ for every $a \in A$ and $b \in B$. Let $A \perp B$ denote this fact. One trivial condition for orthogonality between A and B is the existence of a set of integers c which almost includes every element of A and which is orthogonal to every element of B. In this case we say that c *separates* A and B and that A and B are *separated* from each other. A substantial amount of literature which starts at least with the work of Du Bois-Reymond [4], Hadamard [9] and especially Hausdorff ([10], [11]) is devoted to the converse of this implication. The early work is synthesized in the beautiful paper of Luzin [20] from which we take most of our terminology. In more recent times this type of questions plays a prominent role in a wide range of subjects starting from Banach algebras ([3]) and ending with the most recent "pcf-theory" ([24]). The purpose of this note is to study the questions of Hausdorff and Luzin in the realm of *definable* families of sets of integers. The word definable refers to the classical way of presenting a set of reals as Borel, analytic, coanalytic, etc. (see [13]). Thus, we identify the power-set of the integers with the Cantor set in order to take its topology together with these notions.

Question. When can we separate two orthogonal *definable* families A and B?

To see that this cannot always be done, for an integer i, let a_i be the set of all integers of the form $2^i(2j+1)$. Let $A = \{a_i\}$ and $B = A^\perp$, where $A^\perp$

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is the set of all sets of integers orthogonal to every element of A. That is,

$$B = \{b \subseteq \mathbb{N} : b \perp a_i \text{ for all } i\}.$$

Then A and B are two Borel orthogonal families which cannot be separated. This is our first example of an analytic (in fact, Borel) gap. One thing we learn from this example is that A is rather special, it is *countably generated* in $B^\perp$, i.e., there is a sequence c_n $(= \bigcup_{i\le n} a_i)$ of elements of $B^\perp$ such that every element of A is (almost) included in one of the c_n's. On the other hand, B is rather big—it is *σ-directed* (i.e., for every sequence $\{b_n\}$ of elements of B there is an element of B which almost includes every b_n). Thus, in particular, every countable $B_0 \subseteq B$ can be separated from A. Our first result shows essentially that there are no other kinds of analytic gaps.

Theorem 1. *Suppose A and B are two orthogonal families and that A is analytic. Then A is countably generated in $B^\perp$ iff every countable subset of B can be separated from A.*

Corollary 1. *Suppose A and B are two orthogonal σ-directed families of subsets of $\mathbb{N}$. If one of them is analytic, then they can be separated.*

The assumption that one of the sets A and B is analytic is necessary by the well-known example of Hausdorff ([10], [11]) of an (ω_1, ω_1^*)-gap in the algebra $\mathcal{P}(\mathbb{N})/\mathrm{fin}$. In fact, working in the constructible universe one is able to construct a coanalytic Hausdorff gap so the result cannot be extended to any larger class of definable sets. Note that Theorem 1 is an asymmetrical result which talks about an analytic family A and its separation from an arbitrary family orthogonal to it. To find a symmetrical condition we have to assume at least that both families A and B are analytic and analyze examples of orthogonal pairs (A, B) which are essentially different from the asymmetric pair $A = \{a_i\}$, $B = A^\perp$ discussed above. It turns out that the crucial condition of *nearness* of two orthogonal families used by Hausdorff in constructing his gaps shows also in the definable case. The flexibility of Hausdorff's idea of nearness was fully explored by Luzin ([20]) and especially by Kunen ([17], [18; Ex. II (10), p. 87]). To state the idea in its most general form, suppose that for some single index set I we can write A as a_i $(i \in I)$ and B as b_i $(i \in I)$ in such a way that:

(1) $$a_i \cap b_i = \emptyset \quad \text{for all } i \in I, \text{ and}$$

(2) $$a_i \cap b_j \neq 0 \text{ or } b_i \cap a_j \neq \emptyset \quad \text{for all } i \neq j \text{ in } I.$$

Then for every $c \subseteq \mathbb{N}$ the set of all $i \in I$ such that $a_i \subseteq^* c$ and $c \perp b_i$ must be countable. Thus, if the index set I is uncountable, not only that A and B cannot be separated but no countable set $C \subseteq \mathcal{P}(\mathbb{N})$ separates A from B in the sense that for every $a \in A$ and $b \in B$ we can find a c in C such that $a \subseteq^* c$ and $c \perp b$. We call such a pair (A, B) a *Luzin gap* since Luzin was the first to notice ([20]) that a considerable simplification of Hausdorff's construction gives us an uncountable almost disjoint family E of infinite subsets of $\mathbb{N}$ such that no two uncountable disjoint subsets of E can be separated essentially because they can be refined to two uncountable sets A and B satisfying (1) and (2). If the index set I is a nonempty perfect set of reals and if the mappings $i \mapsto a_i$ and $i \mapsto b_i$ are continuous then we call a pair $\langle \{a_i\}_{i\in I}, \{b_i\}_{i\in I}\rangle$ a *perfect Luzin gap*.

We have seen above that there are no analytic Hausdorff gaps, but it turns out that the weaker form of gaps, Luzin gaps, can be found at this level of complexity. To see this, we identify $\mathbb{N}$ with the set $S = \{0,1\}^{<\omega}$ of all finite sequences of 0's and 1's and concentrate on finding a perfect Luzin gap in $\mathcal{P}(S)$ rather than in $\mathcal{P}(\mathbb{N})$. Let $P = \{0,1\}^{\omega}$ be the Cantor set of all infinite sequences of 0's and 1's viewed as infinite branches of the tree S. For $x \in P$ let a_x be the set of all $\sigma \in S$ such that x end-extends $\sigma 0$, and let b_x be the set of all $\sigma \in S$ such that x end-extends $\sigma 1$. It is clear that $A = \{a_x : x \in P\}$ is orthogonal to $B = \{b_x : x \in P\}$. In fact, $A \cup B$ is an almost disjoint family of sets. It is also clear that $a_x \cap b_x = \emptyset$ for all $x \in P$, so it remains to check the condition (2). For this purpose fix $x \neq y$ in P and let σ be the maximal initial segment of both x and y. Then σ is a member of the intersection $a_x \cap b_y$ or $b_x \cap a_y$ depending on whether the next digit of x after σ is 0 or 1.

Our second result shows that perfect Luzin gaps are essentially the only kind of analytic gaps which cannot be separated by a countable subfamily of $\mathcal{P}(\mathbb{N})$.

THEOREM 2. *If A and B are two orthogonal analytic families then either*

(a) *there is a countable set $C \subseteq \mathcal{P}(\mathbb{N})$ which separates A and B, or*

(b) *the restriction of (A, B) to some end-segment of $\mathbb{N}$ contains a perfect Luzin gap.*

Note that going to some restriction of the form $(\{a \setminus \{1,\ldots,n\} : a \in A\}, \{b \setminus \{1,\ldots,n\} : b \in B\})$ in the alternative (b) is absolutely necessary since one might have two orthogonal families A and B such that some fixed integer is an element of every $a \in A$ and every $b \in B$, so no subgap of (A, B) could ever satisfy (1). It should also be clear that neither separation nor definability properties are changed by the transition from (A, B) to the pair of end-segments of their elements, so it is reasonable to assume that A and B are closed under this operation. Note that if A and B are σ-directed, and if they can be separated by a countable $C \subseteq \mathcal{P}(\mathbb{N})$, then there must be a single element c of C which separates A and B. The phenomenon of Hausdorff's (ω_1, ω_1^*)-gap is really an instance of the general phenomenon that two σ-directed orthogonal families may not be separated. Theorems 1 and 2 tell us that this phenomenon is highly nonanalytic since if, for example, an analytic gap (A, B) contains a Luzin subgap then neither A nor B can be σ-directed.

1. Proof of Theorem 1: Hurewicz's phenomenon. Let $[\mathbb{N}]^{<\omega}$ be the collection of all finite subsets of $\mathbb{N}$ considered as a tree under the relation of end-extension. We identify the set of infinite branches of $[\mathbb{N}]^{<\omega}$ with the set $[\mathbb{N}]^{\omega}$ of infinite subsets of $\mathbb{N}$. Let

$$[\mathbb{N}]^{<\omega} \otimes [\mathbb{N}]^{<\omega} = \{(s,t) \in [\mathbb{N}]^{<\omega} \times [\mathbb{N}]^{<\omega} : |s| = |t|\}.$$

Note that $[\mathbb{N}]^{<\omega} \otimes [\mathbb{N}]^{<\omega}$ with the product ordering is also a tree. For a family B of subsets of $\mathbb{N}$, we say that a subset Σ of $[\mathbb{N}]^{<\omega}$ is a *B-tree* iff:

(3) $\emptyset \in \Sigma$, and

(4) for every $\sigma \in \Sigma$, the set $\{i \in \mathbb{N} : \sigma \cup \{i\} \in \Sigma\}$ is infinite and is included in an element of B.

The following is a more precise version of Theorem 1.

THEOREM 3. *Suppose A and B are two orthogonal families of subsets of $\mathbb{N}$ and that A is analytic and closed under taking subsets of its elements. Then either A is countably generated in $B^{\perp}$ or there is a B-tree all of whose branches are elements of A.*

Proof. Suppose A is not countably generated in $B^{\perp}$ and fix a downwards closed subtree T of $[\mathbb{N}]^{<\omega} \otimes [\mathbb{N}]^{<\omega}$ which codes a closed subset of $([\mathbb{N}]^{\omega})^2$ which projects to $A \cap [\mathbb{N}]^{\omega}$. Thus, an infinite set $a \subseteq \mathbb{N}$ is in A iff there is an infinite branch $f = \{\langle s_n, t_n\rangle\}_{n=0}^{\infty}$ of T such that a is equal to the union of the s_n's. Let f_a denote the leftmost branch of T with this property. For $(s,t) \in T$ set

$$A(s,t) = \{a \in A : f_a \text{ extends } (s,t)\}.$$

Let

$$T_0 = \{(s,t) \in T : A(s,t) \text{ is countably generated in } B^{\perp}\}.$$

Then T_0 is an upward closed subset of T, so its complement $T_1 = T \setminus T_0$ is downwards closed and nonempty by our assumption that A is not countably generated in $B^{\perp}$. In fact, for every a in the nonempty set

$$A_1 = A \setminus \bigcup_{(s,t)\in T_0} A(s,t)$$

the branch f_a is actually a subset of T_1. Moreover, for every $(s,t) \in T_1$, the set

$$A_1(s,t) = \{a \in A_1 : f_a \text{ extends } (s,t)\}$$

is not countably generated inside $B^{\perp}$.

The B-tree $\Sigma \subseteq [\mathbb{N}]^{<\omega}$ satisfying the conclusion of Theorem 3 is built recursively together with a sequence (s_σ, t_σ) $(\sigma \in \Sigma)$ of elements of T_1 and a sequence d_σ $(\sigma \in \Sigma)$ of elements of B such that the following conditions are satisfied:

(5) if τ strictly end-extends σ, then (s_τ, t_τ) strictly extends (s_σ, t_σ),
(6) $\sigma \subseteq s_\sigma$ for all $\sigma \in \Sigma$, and
(7) $b_\sigma = \{i \in d_\sigma : \sigma \cup \{i\} \in \Sigma\}$ is infinite for all $\sigma \in \Sigma$.

We start the recursion by letting $\emptyset \in \Sigma$ and $s_\emptyset = t_\emptyset = \emptyset$. Suppose that we have put some σ in Σ and that we know (s_σ, t_σ) in T_1. Then $A_1(s_\sigma, t_\sigma)$ is not countably generated in $B^\perp$ so its union, call it c_σ, is not orthogonal to B. Fix an element d_σ of B such that $c_\sigma \cap d_\sigma$ is infinite and set

$$b_\sigma = \{i \in c_\sigma \cap d_\sigma : i > \max(s_\sigma)\}.$$

For $i \in b_\sigma$, fix an $a_i \in A_1(s_\sigma, t_\sigma)$ such that $i \in a_i$ and let (s^i, t^i) be the minimal element of the branch f_{a_i} such that $i \in s^i$. Finally, put $\sigma \cup \{i\}$ in Σ whenever $i \in b_\sigma$, and set

$$s_{\sigma\cup\{i\}} = s^i \quad \text{and} \quad t_{\sigma\cup\{i\}} = t^i$$

for every such i. This completes the description of the construction of the B-tree Σ.

To show that it satisfies the conclusion of Theorem 3 let $a = \{i_0, i_1, \ldots,\}$ be a given infinite branch of Σ enumerated increasingly or more precisely an infinite subset of $\mathbb{N}$ which determines (and is determined by) an infinite branch

$$\sigma_0 = \emptyset, \quad \sigma_1 = \{i_0\}, \quad \sigma_2 = \{i_0, i_1\}, \ \ldots$$

of Σ. Then $(s_{\sigma_i}, t_{\sigma_i})$, $i = 1, 2, \ldots$, determines an infinite branch of T whose projection

$$\overline{a} = \bigcup_{i=0}^{\infty} s_{\sigma_i}$$

is a member of A which includes the set a (by (6)). Since A is closed under taking subsets, it follows that a is also a member of A. This finishes the proof.

One may think of Theorem 3 as an instance of a classical phenomenon first touched by Hurewicz [12] and later extended by Kechris, Louveau and Woodin [14]. In fact, it can be shown that the Hurewicz-type result of [14; Theorem 4] is an immediate consequence of Theorem 3. To see this, let us recall that Theorem 4 of [14] says that if E is a compact metric space and if A and B are two disjoint subsets of E such that A is analytic, then either there is an F_σ-set C such that $A \subset C$ and $C \cap B = \emptyset$, or there is a perfect set $P \subset A \cup B$ such that $P \cap B$ is a countable dense subset of P. To see the deduction, note first that E can be assumed to be equal to the Cantor set $\{0,1\}^\omega$ viewed as the branches of the complete binary tree $\{0,1\}^{<\omega}$. For $a \in \{0,1\}^\omega$, let $\widehat{a}$ be the set of all infinite chains of $\{0,1\}^{<\omega}$ whose union is equal to a. Set

$$\widehat{A} = \bigcup_{a\in A} \widehat{a} \quad \text{and} \quad \widehat{B} = \bigcup_{b\in B} \widehat{b}.$$

Then $\widehat{A}$ and $\widehat{B}$ are two orthogonal families of infinite subsets of $\{0,1\}^{<\omega}$ and the two alternatives of Theorem 3 lead to the two alternatives of the Hurewicz-type result.

2. Proof of Theorem 2: An open coloring. We may assume that the given two orthogonal analytic families A and B are closed under finite changes of its elements and we may form the following subset of their product:

$$A \otimes B = \{(a,b) \in A \times B : a \cap b = \emptyset\}.$$

There is a very natural partition of the set $[A \otimes B]^2$ of unordered pairs of elements of $A \otimes B$ that one associates with the problem of separating A and B (see [26; §8]): Let K_0 be the set of all $\{(a,b),(a',b')\}$ from $[A \otimes B]^2$ such that

$$(a \cap b') \cup (b \cap a') \neq \emptyset.$$

It is clear that K_0 is an *open* subset of $[A \otimes B]^2$ in the natural topology induced from the exponential space of $A \otimes B$. So by the Principle of Open Coloring ([26], [5]) we have the following two alternatives:

(i) there is a decomposition $A \otimes B = \bigcup_{n=0}^{\infty} X_n$ such that $[X_n]^2 \cap K_0 = \emptyset$ for all n, or

(ii) there is a nonempty perfect set $P \subseteq A \otimes B$ such that $[P]^2 \subseteq K_0$.

Note that the alternative (ii) implies the alternative (b) of Theorem 2 since, if we write an element x of P as a pair (a_x, b_x), then

$$\{a_x : x \in P\} \subseteq A \quad \text{and} \quad \{b_x : x \in P\} \subseteq B$$

form a perfect Luzin gap by the definition of K_0. To see what the alternative (i) means, let $\pi_0 : A \otimes B \to A$ be the projection, and for $n \in \mathbb{N}$, let c_n be the union of the image of X_n under π_0. Then by the definition of K_0, for all $n \in \mathbb{N}$,

$$a \subseteq c_n \quad \text{and} \quad c_n \cap b = \emptyset \quad \text{for all } (a,b) \in X_n. \tag{8}$$

It is clear that this means that $C = \{c_n\}$ is a countable family which separates A from B.

3. Hausdorff's gaps in the Borel algebra. For $a \in [\mathbb{N}]^{\omega}$, set

$$(\cdot, a]^* = \{b \in [\mathbb{N}]^{\omega} : b \subseteq^* a\}.$$

Then $(\cdot, a]^*$ is an F_σ-subset of $[\mathbb{N}]^{\omega}$ and the operation $a \mapsto (\cdot, a]^*$ is monotonic:

$$a \subseteq^* b \quad \text{implies} \quad (\cdot, a]^* \subseteq (\cdot, b]^*. \tag{9}$$

It follows that (ω_1, ω_1^*)-gaps of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ get transformed into "(ω_1, ω_1^*)-pregaps" of the algebra of Borel subsets of $[\mathbb{N}]^\omega$, i.e., into pairs of families

$$\mathcal{A} = \{(\cdot, a]^* : a \in A\} \quad \text{and} \quad \mathcal{B} = \{(\cdot, b]^* : b \in B\}$$

of F_σ-sets such that $\mathrm{otp}(\mathcal{A}, \subseteq) = \omega_1$, $\mathrm{otp}(\mathcal{B}, \subseteq) = \omega_1^*$, and every element of $\mathcal{B}$ includes every element of $\mathcal{A}$. Is there a Borel set $X \subset [\mathbb{N}]^\omega$ which separates them?

THEOREM 4. *If (A, B) is an (ω_1, ω_1^*)-gap of $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ then there is no analytic set $X \subseteq [\mathbb{N}]^\omega$ such that $(\cdot, a]^* \subseteq X \subseteq (\cdot, b]^*$ for all $a \in A$ and $b \in B$.*

Proof. Suppose such an X exists. Set

$$C = \{\mathbb{N} \setminus b : b \in B\}.$$

Then C is σ-directed and $C \perp X$. By Theorem 1 there is a sequence $\{c_n\}$ of elements of $C^\perp$ which generates X, i.e., which has the property that every element of X is (almost) included in some c_n. Then for some fixed n, the set

$$A_0 = \{a \in A : a \subseteq^* c_n\}$$

is uncountable. But this means that c_n splits the gap (A, B), a contradiction.

Theorem 4 leads to a quite general method of constructing (ω_1, ω_1^*)-gaps in the Borel algebra. Of course, there are other ways for getting such objects but none of them is as canonical as this one, nor produces gaps consisting of sets of such low complexity. For example, one of the more generous sources of Hausdorff's gaps in the Borel algebra is found by considering decompositions of $\mathbb{R}$ into $\aleph_1$ many *disjoint* Borel sets. [It is known that $\mathrm{F}_{\sigma\delta}$ is the smallest possible complexity of sets occurring in such decompositions (see [7]). It is also interesting that the first such decomposition was found by Hausdorff himself using his (ω_1, ω_1^*)-gap (see [11]).] To see the relevance of such decompositions

$$\mathbb{R} = \bigcup_{\xi < \omega_1} X_\xi,$$

note that there must be many subfamilies of X_ξ $(\xi < \omega_1)$ whose unions are not Borel subsets of $\mathbb{R}$. Otherwise, we would be able to find a countable sequence $\{B_n\}$ of such unions with the property that for every $\xi \neq \eta$ there is n such that $B_n \supseteq X_\xi$ and $B_n \cap X_\eta = \emptyset$. So if we define $f : \mathbb{R} \to \{0, 1\}^\omega$ by

$$f(x)(n) = 1 \quad \text{iff} \quad x \in B_n,$$

we get a Borel map whose range (an analytic set) has size exactly $\aleph_1$ no matter what the size of the continuum is. Now given an uncountable co-uncountable set $I \subseteq \omega_1$ such that $\bigcup_{\xi \in I} X_\xi$ is not Borel, the sets $\bigcup_{\xi \in I \cap \alpha} X_\xi$ and $\mathbb{R} \setminus \bigcup_{\xi \in \alpha \setminus I} X_\xi$, for α a countable ordinal, would form an (ω_1, ω_1^*)-gap in the Borel algebra.

4. Analytic ideals. An *ideal* on $\mathbb{N}$ is a family of subsets of $\mathbb{N}$ closed under taking subsets and finite unions. We shall consider only ideals on $\mathbb{N}$ which include the *Fréchet ideal*, the ideal of finite sets. If it is σ-directed under almost inclusion we call it a *P-ideal*. Many of the familiar ideals on $\mathbb{N}$ are analytic and in fact Borel of very low complexity. For example, consider the ideal Z_0 of subsets of $\mathbb{N}$ of asymptotic density 0. It is an example of an analytic (in fact $\mathrm{F}_{\sigma\delta}$) P-ideal which is moreover *dense* in $[\mathbb{N}]^\omega$, i.e., every infinite subset of $\mathbb{N}$ includes an infinite subset which is a member of Z_0. Another interesting example comes when one considers a bounded sequence $\mathbf{f} = \{f_n\}$ of continuous real-valued functions defined on a Polish space X. Assuming that the constantly zero function $\overline{0}$ is a pointwise accumulation point of $\{f_n\}$, let

$$I_{\mathbf{f}} = \{a \subseteq \mathbb{N} : \overline{0} \not\in \overline{\{f_n : n \in a\}}\},$$

where the closure is taken in $\mathbb{R}^X$. Note that the orthogonal $I_{\mathbf{f}}^\perp$ is just the set of all $a \subseteq \mathbb{N}$ which are either finite or have the property that $\{f_n\}_{n\in a}$ pointwise converges to $\overline{0}$. The equality $I_{\mathbf{f}}^{\perp\perp} = I_{\mathbf{f}}$ simply means that every subsequence of $\{f_n\}$ which pointwise accumulates to $\overline{0}$ contains a subsequence which pointwise *converges* to $\overline{0}$. A rather deep result of Bourgain, Fremlin and Talagrand [2] says that this happens whenever the pointwise closure $K = \overline{\{f_n\}} \subseteq \mathbb{R}^X$ is relatively small, e.g. it consists only of Baire class-1 functions. Such compact sets K are in the literature called *separable Rosenthal compacta* ([23]). Many directed sets occurring in Analysis can be represented as analytic ideals on $\mathbb{N}$ ordered by the inclusion. For example, this is true about the lattice ℓ^1 of absolutely converging series, or the lattice $\mathbb{N}^\mathbb{N}$ of integer-valued sequences ($x \le y$ iff $x(n) \le y(n)$ for *all* n). Results that would relate some of these ideals as directed sets are often connected to questions which naturally arise in the context of Real Analysis and Measure Theory (see [6]). For example, a typical question of this sort is the question ([1], [21]) about characters of points of Rosenthal compacta, which reduces to the question about possible cofinalities of ideals of the form $I_{\mathbf{f}}$. Our first result says that the lattice $\mathbb{N}^\mathbb{N}$ has a particular place among analytic ideals on $\mathbb{N}$ and at the same time hints at the relevance of Theorems 1 and 2 in this context.

THEOREM 5. *Suppose A is an analytic ideal whose orthogonal $A^\perp$ is not a P-ideal. Then there is a Borel monotonic map which transfers A to a cofinal subset of* $\mathbb{N}^\mathbb{N}$.

Proof. Choose a sequence $\{c_n\}$ of elements of $A^\perp$ with the property that no element of $A^\perp$ almost includes every member of the sequence. Let $\Phi : A \to \mathbb{N}^\mathbb{N}$ be defined by

$$\Phi(a)(n) = \min\{m \in \mathbb{N} : a \cap c_i \subseteq \{1, \ldots, m\} \text{ for all } i \le n\}.$$

It is clear that Φ is monotonic and Borel and that its range $R = \Phi''A$ consists only of monotonic members of $\mathbb{N}^\mathbb{N}$. Note that R must be unbounded in $\mathbb{N}^{\mathbb{N}}$ even if we take the ordering $<^*$ of eventual dominance. [For $x \in \mathbb{N}^{\mathbb{N}}$, the union of the family $c_n \setminus \{1, \ldots, x(n)\}$ ($n \in \mathbb{N}$) is a subset of $\mathbb{N}$ which almost includes every c_n, and it would be a member of $A^{\perp}$ if x eventually dominates R.] It is well known (see [0]) that an unbounded analytic directed subset of $\mathbb{N}^{\mathbb{N}}$ consisting of monotonic functions must in fact be dominating in the ordering $<^*$. Thus we conclude that for every $x \in \mathbb{N}^{\mathbb{N}}$ there is $r \in R$ such that $x <^* r$. By a Lemma of Kunen [16], this means that there is a $k \in \mathbb{N}$ such that R is dominating on $\mathbb{N} \setminus \{1, \ldots, k\}$ even if we take the ordering of everywhere dominance, that is, for every $x \in \mathbb{N}^{\mathbb{N}}$ there is $r \in R$ such that $x(n) < r(n)$ for *all* $n > k$. Thus, if we define $\Psi : A \to \mathbb{N}^{\mathbb{N}}$ by

$$\Psi(a)(n) = \Phi(a)(n + k)$$

we get the desired mapping. This finishes the proof.

To get a better understanding of this kind of results let us give an illustration by using the separable Rosenthal compactum $K = \overline{\{f_n\}} \subseteq \mathbb{R}^X$ introduced above. Assuming that K contains the constantly zero map $\overline{0}$, let us examine the corresponding analytic ideal $I_{\mathbf{f}}$. We claim that either $\overline{0}$ has countable character in K, or there is a monotonic map from $I_{\mathbf{f}}$ onto a cofinal subset of $\mathbb{N}^{\mathbb{N}}$. To see that this is so, one only needs to compare Theorems 1 and 5 and the result $I_{\mathbf{f}}^{\perp\perp} = I_{\mathbf{f}}$ of [2] mentioned above. It follows that the character of a point in a separable Rosenthal compactum is either countable or at least as big as the cofinality of $\mathbb{N}^{\mathbb{N}}$. This result has already been achieved by Krawczyk [15] by using a quite different argument. In order to state our next result, let us say that an ideal A on $\mathbb{N}$ is *atomic* if it is generated over the Fréchet ideal by a single subset of $\mathbb{N}$.

THEOREM 6. *For every nonatomic analytic P-ideal A on $\mathbb{N}$ there is a Borel monotonic map from A onto a cofinal subset of $\mathbb{N}^{\mathbb{N}}$.*

Proof. It will be convenient to monotonically transfer A to another analytic P-ideal $\overline{A}$ on $\mathbb{N}$, so for this purpose with every subset a of $\mathbb{N}$ we associate the set

$$\overline{a} = \{2^i(2j+1) : i \in \mathbb{N},\ j \leq |a \cap \{1, \ldots, i\}|\}.$$

Let $\overline{A}$ be the ideal generated by $\{\overline{a} : a \in A\}$. It is clear that $a \mapsto \overline{a}$ is a monotonic Borel map from A onto a cofinal subset of $\overline{A}$, so we can from now on concentrate on the ideal $\overline{A}$. Note that $\overline{A}$ is indeed a P-ideal and this is where the assumption that A is nonatomic is used. For suppose we are given a sequence $\{\overline{a}_n\}$ of elements of $\overline{A}$. Applying the fact that A is a nonatomic P-ideal to the corresponding sequence $\{a_n\}$ we find an a in A which not only almost includes all a_n's but it also has the property that $a \setminus a_n$ is infinite for all n. Then $\overline{a}$ will be an element of $\overline{A}$ which almost includes all $\overline{a}_n$'s. For $i \in \mathbb{N}$, set

$$\mathbb{N}_i = \{2^i(2j+1) : j \in \mathbb{N}\},$$

and

$$B = \{b \in \overline{A}^{\perp} : \mathbb{N}_i \setminus b \text{ is finite for all } i\}.$$

By Theorem 5, we may assume that the $\mathbb{N}_i$'s can be separated from $\overline{A}$, or equivalently, that the set B is nonempty. We claim that there is a countable subset of B which cannot be separated from $\overline{A}$, which would give us the conclusion of Theorem 6 similarly to the proof of Theorem 5. If this is not the case, then by Theorem 1 there is a sequence $\{c_n\}$ of elements of $B^{\perp}$ which generates $\overline{A}$, i.e., every element of $\overline{A}$ is almost included in some c_n. Since $\overline{A}$ is a P-ideal, there is a single c_n which almost includes every element of $\overline{A}$. By adding to c_n finitely many elements, we may assume that c_n intersects all $\mathbb{N}_i$'s. For $i \in \mathbb{N}$, set

$$m_i = \max(\mathbb{N}_i \cap c_n) \quad \text{and} \quad b_i = \mathbb{N}_i \setminus \{1, \ldots, m_i - 1\}.$$

Let b be the union of the b_i's. Then $b \cap c_n$ is infinite as it includes the infinite set $\{m_i\}$. Since c_n belongs to $B^{\perp}$ this means that b is not a member of B, so we can find $a \in A$ such that $\overline{a} \cap b$ is infinite. Then $\overline{a}$ intersects b_i for infinitely many i's, so there is an infinite set $E \subseteq \mathbb{N}$ such that $m_i \in \overline{a}$ for all $i \in E$. [This follows from the fact that $\overline{a} \cap \mathbb{N}_i$ is an initial segment of $\mathbb{N}_i$ for all i.] Since A is a proper ideal we can find a $d \in A$ such that $a \subseteq d$ and $d \setminus a \neq \emptyset$. So for almost all $i \in \mathbb{N}$,

$$|d \cap \{1, \ldots, i\}| > |a \cap \{1, \ldots, i\}|.$$

It follows that for almost all $i \in \mathbb{N}$, the set $\overline{d}$ contains at least one integer n_i from $\mathbb{N}_i$ which is above m_i. But this means that $\{n_i : i \in E\}$ is an infinite subset of $\overline{d}$ which is orthogonal to c_n, contradicting the fact that c_n almost includes every element of $\overline{A}$.

In [22], R. Pol showed that if (A, B) is an (ω_1, ω_1^*)-gap in $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ then the set

$$\check{A} = \{c \subseteq \mathbb{N} : a \subseteq^* c \text{ for all } a \in A\}$$

is not analytic. Our Theorem 1 says a bit more, not only that $\check{A}$ is not analytic but no analytic subset of $\check{A}$ contains the other half, B, of the gap. Note that $A^{\perp} = \{\mathbb{N} \setminus c : c \in \check{A}\}$, so this shows that the orthogonal of the lower part of an (ω_1, ω_1^*)-gap is never analytic. There is another result in the literature with a similar conclusion. In [15], Krawczyk showed that if $I_{\mathbf{f}}$ is the analytic ideal associated with the point $\overline{0}$ in a separable Rosenthal compactum $K = \overline{\{f_n\}}$ considered above, then its orthogonal $I_{\mathbf{f}}^{\perp}$ is analytic only in the trivial case when $I_{\mathbf{f}}$ is countably generated. Our next result gives yet another instance of this phenomenon.

Theorem 7. *Let B be a P-ideal on $\mathbb{N}$. Then its orthogonal $B^{\perp}$ is analytic iff it is an F_σ-set iff it is countably generated.*

P r o o f. Assume $A = B^{\perp}$ is analytic. By our assumption that B is a P-ideal every countable subset of B is separated from A. So, applying Theorem 1 to A and B we conclude that A must be countably generated in $B^{\perp} = A$. The rest of the implications are immediate.

Note that if we use the sharper version of Theorem 1, Theorem 3, the results of this section about P-ideals can be extended to the wider class of ideals on $\mathbb{N}$. These are the ideals A with the property that for every sequence $\{a_n\}$ of infinite elements of A there is an element a of A which intersects all a_n's. It turns out that this property has already been considered in the context of analysis of the pointwise convergence of continuous functions, i.e., when A is the ideal of converging subsequences of a fixed bounded sequence $\{f_n\}$ of real functions defined on a Polish space X (see [8], [27], [25], [21]). It simply means that if we are given a sequence $\{g_n^k\}$ of subsequences of $\{f_n\}$ such that $g_n^k \xrightarrow{n} \overline{0}$ for all k, then we can find a diagonal sequence $\{g_{n_k}^k\}$ such that $g_{n_k}^k \xrightarrow{k} \overline{0}$. It is interesting that this property of the orthogonal $A = I_{\mathbf{f}}^{\perp}$ is as restrictive as the assumption that the ideal $I_{\mathbf{f}}$ (of all subsequences of $\{f_n\}$ which do not accumulate to $\overline{0}$) is countably generated. This is the Szlenk Theorem ([25]) as referred to in [21]. It is an immediate corollary of our Theorem 3.

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References

[0] A. Blass, *Near coherence of filters II: Applications to operator ideals, the Stone–Čech remainder of a half-line, order ideal of sequences, and slenderness of groups*, Trans. Amer. Math. Soc. 300 (1987), 557–581.

[1] J. Bourgain, *Some remarks on compact sets of first Baire class*, Bull. Soc. Math. Belg. 30 (1978), 3–10.

[2] J. Bourgain, D. H. Fremlin and M. Talagrand, *Pointwise compact sets of Baire-measurable functions*, Amer. J. Math. 100 (1978), 845–886.

[3] H. G. Dales and W. H. Woodin, *An Introduction to Independence for Analysts*, Cambridge University Press, 1978.

[4] P. du Bois-Reymond, *Eine neue Theorie der Convergenz und Divergenz von Reihen mit positiven Gliedern*, J. Reine Angew. Math. 76 (1873), 61–91.

[5] Q. Feng, *Homogeneity for open partitions of reals*, Trans. Amer. Math. Soc. 339 (1993), 659–684.

[6] D. H. Fremlin, *The partially ordered sets of measure theory and Tukey's ordering*, Note Mat. 11 (1991), 177–214.

[7] D. H. Fremlin and S. Shelah, *On partitions of the real line*, Israel J. Math. 32 (1979), 299–304.
[8] D. C. Gillespie and W. A. Hurwitz, *On sequences of continuous functions having continuous limits*, Trans. Amer. Math. Soc. 32 (1930), 527–543.
[9] J. Hadamard, *Sur les caractères de convergence des séries à termes positifs et sur les fonctions indéfiniment croissantes*, Acta Math. 18 (1894), 319–336.
[10] F. Hausdorff, *Die Graduierung nach dem Endverlauf*, Abh. Königl. Sächs. Gesell. Wiss. Math.-Phys. Kl. 31 (1909), 296–334.
[11] —, *Summen von* $\aleph_1$ *Mengen*, Fund. Math. 26 (1936), 241–255.
[12] W. Hurewicz, *Relativ Perfekte Teile von Punktmengen und Mengen* (A), Fund. Math. 12 (1928), 78–109.
[13] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, 1995.
[14] A. S. Kechris, A. Louveau and W. H. Woodin, *The structure of* σ*-ideals of compact sets*, Trans. Amer. Math. Soc. 301 (1987), 263–288.
[15] A. Krawczyk, *Rosenthal compactum and analytic sets*, Proc. Amer. Math. Soc. 115 (1992), 1095–1100.
[16] K. Kunen, *Some comments on box products*, in: Infinite and Finite Sets, Keszthely 1973, Colloq. Math. Soc. János Bolyai 10, North-Holland, Amsterdam, 1975, 1011–1016.
[17] —, (κ, λ^*) *gaps under MA*, note of August 1976.
[18] —, *An Introduction to Independence Proofs*, North-Holland, 1980.
[19] C. Laflamme, *Bounding and dominating number of families of functions on* ω, Math. Logic Quart. 40 (1994), 207–223.
[20] N. Luzin, *On parts of the natural series*, Izv. Akad. Nauk SSSR Ser. Mat. 11 (1947), 714–722 (in Russian).
[21] R. Pol, *On pointwise and weak topology in function spaces*, preprint, Uniw. Warszawski, 1984.
[22] —, *Note on pointwise convergence of sequences of analytic sets*, Mathematika 36 (1989), 290–300.
[23] H. P. Rosenthal, *Some recent discoveries in the isomorphic theory of Banach spaces*, Bull. Amer. Math. Soc. 84 (1978), 803–831.
[24] S. Shelah, *Cardinal Arithmetic*, Oxford University Press, 1995.
[25] W. Szlenk, *The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces*, Studia Math. 30 (1968), 53–61.
[26] S. Todorčević, *Partition Problems in Topology*, Amer. Math. Soc., Providence, 1989.
[27] Z. Zalcwasser, *Sur une propriété du champ des fonctions continues*, Studia Math. 2 (1930), 63–67.

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