FUNDAMENTA
MATHEMATICAE
150 (1996)

The dimension of X^n where X is a separable metric space

by

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Abstract. For a separable metric space X, we consider possibilities for the sequence $S(X) = \{d_n : n \in \mathbb{N}\}$ where $d_n = \dim X^n$. In Section 1, a general method for producing examples is given which can be used to realize many of the possible sequences. For example, there is X_n such that $S(X_n) = \{n, n+1, n+2, \ldots\}$, Y_n, for $n > 1$, such that $S(Y_n) = \{n, n+1, n+2, n+2, n+2, \ldots\}$, and Z such that $S(Z) = \{4, 4, 6, 6, 7, 8, 9, \ldots\}$.

In Section 2, a subset X of $\mathbb{R}^2$ is shown to exist which satisfies $1 = \dim X = \dim X^2$ and $\dim X^3 = 2$.

0. Introduction and preliminaries. In this paper, we are concerned with problems related to the following question:

Question. *Suppose $D = \{d_n : n \in \mathbb{N}\}$ is a sequence of positive integers. Under what conditions is there a separable metric space X_D such that, for each $n \in \mathbb{N}$, $\dim X_D^n = d_n$?*

In case a sequence D has an X_D, we say D is an *allowable* sequence and that X_D *realizes* D. The sequence $\{kn : n \in \mathbb{N}\}$ is realized by $X = I^k$, but there are other allowable sequences. The well-known example of Erdős (see [E]) shows that the sequence $\{d_n : n \in \mathbb{N}\}$ where each d_n is 1 is allowable; Anderson and Keisler [AK] improved this, showing that each $d_n = k$ is allowable. In [Ku1], it is shown that, given m and k with $k \geq m$, there is a sequence D where $d_1 = m$ and for all large enough n, $d_n = k$.

Obviously, if D is an allowable sequence, then D is nondecreasing, and for each n, $d_{n+1} - d_n \leq d_1$, but not all sequences with these properties are allowable. For example, no sequence starting out as $1, 1, 2, 3$ is allowable since if $\dim X^2 = 1$, then $\dim X^4 = \dim(X^2)^2 \leq 2$.

We say a sequence $D = \{d_n : n \in \mathbb{N}\}$ of positive integers is *subadditive* provided that, whenever $s, t \in \mathbb{N}$, $d_{s+t} \leq d_s + d_t$. It is not hard to see that

1991 *Mathematics Subject Classification*: Primary 54F45, 54G20.

an allowable sequence is both increasing and subadditive. The following conjecture says that all such sequences are allowable.

CONJECTURE. *The allowable sequences are precisely those which are increasing and subadditive.*

In Section 1 we give a unified method for generating all allowable sequences noted above as well as several new examples of allowable sequences. Two new types of examples give, for $n \geq 2$, the sequences $n, n+1, n+2, n+3, \ldots$ and $n, n+1, n+2, n+2, n+2, \ldots$ It is also shown that, given $n \in \mathbb{N}$, there exists an allowable sequence D which, for n distinct values k, satisfies $d_k = d_{k+1} < d_{k+2}$; in other words, there is a space X so that for n distinct values k, $\dim X^k = \dim X^{k+1} < \dim X^{k+2}$.

This method has limitations. One is that if D is a sequence realized by an example X_D from Section 1, $s < t$ and $d_s < d_t$, then there is a subspace S of X_D such that $\dim S^s < \dim S^{s+1}$; so while $\dim X^s$ might be the same as $\dim X^{s+1}$ there is a subspace of X whose dimension is growing, and we have only hidden this growth. This precludes, for example, the method giving an example whose sequence starts out as $1, 1$ and is not constant.

In Section 2 we give an example X which satisfies $\dim X = \dim X^2 = 1$, while $\dim X^3 = 2$. In addition to giving an example not available in Section 1, it has the property that if $Y \subseteq X$, then $\dim Y^2 \leq \dim Y$ but for $Y = X$, $\dim Y^3 > \dim Y$.

In Section 3, we pose several questions.

Throughout this paper, we make extensive use of the technique of Anderson and Keisler [AK], which appears to be almost essential, in some form, for constructions of this sort.

We use $\mathbb{R}$ to denote the real numbers and $\mathfrak{c}$ to denote the cardinality of $\mathbb{R}$; $\mathbb{N}$ denotes the natural numbers. By a *hyperplane* in $\mathbb{R}^n$, we mean a translate of a linear subspace of $\mathbb{R}^n$; the *dimension* of the hyperplane is its algebraic dimension, which is the same as its topological dimension. Hyperplanes P and Q of dimension p and q in $\mathbb{R}^n$, where $p + q \geq n$, are *in general position* provided the hyperplane $P \cap Q$ has dimension $= \max\{0, p+q-n\}$.

For most of this paper we will use subsets of $\mathbb{R}^{2k}$, for $k \geq 1$; it is convenient to view $\mathbb{R}^{2k}$ as $(\mathbb{R}^2)^k = \{(\alpha(1), \ldots, \alpha(k)) : \alpha(i) \in \mathbb{R}^2\}$. For S a subset of $\{1, \ldots, k\}$, let $H(S,k) = \{(\alpha(1), \ldots, \alpha(k)) : \alpha(i) = \alpha(j)$ if $i, j \in S$, and $\alpha(i) = (0,0)$ if $i \notin S\}$. Then there are finitely many hyperplanes, for fixed k, of the form $H(S,k)$, and each is two-dimensional. By a *standard translate* of $H(S,k)$ we mean a hyperplane of the form $H(S,k) + \sigma$, where $\sigma(i) = (0,0)$ if $i \in S$.

For information on dimension theory, the reader is referred to [E].

1. A construction and several examples. In this section we construct infinitely many spaces with several properties interrelating them. Theorem 1.1 describes these spaces. Then we show how to produce several examples of products related to the conjecture.

THEOREM 1.1. *There is a collection* $\{A(i,j) : i,j \in \mathbb{N}\}$ *of subsets of* $\mathbb{R}^2$ *such that*:

(1) *For all* $i,j \in \mathbb{N}$, $1 = \dim A(i,j) = \dim A(i,j)^\omega$.
(2) *For each* $j \in \mathbb{N}$, *and* $K \subseteq \mathbb{N}$, $\dim_i \prod_{i\in K} A(i,j) = |K|$.
(3) *If* $k,n \in \mathbb{N}$, *and for each* $j \le k$, $i_j \in \mathbb{N}$, *then*

$$\dim \prod_{1\le j\le k} A(i_j,j)^n = 1.$$

Remark. Condition (1) follows from conditions (2) and (3) together, condition (2) giving $\dim A(i,j)^\omega \ge 1$, and condition (3) giving $\dim A(i,j)^\omega \le 1$. It is listed here because it is so fundamental to our applications.

In order to prove Theorem 1.1, we need the following lemmas. Lemma 1.1 can be found in [Ku1; Lemma] in a more geometric form, and is due to Anderson and Keisler [AK].

LEMMA 1.1. *There is a subset* T_{2k} *of* $\mathbb{R}^{2k}$ *such that* $\dim \mathbb{R}^{2k}\setminus T_{2k} = 1$ *and if* $H(S,k)+\sigma$ *is a standard translate of* $H(S,k)$, *then*

$$|(H(S,k)+\sigma)\cap T_{2k}| \le \omega.$$

Let $\mathbb{Q} = [0,1]^{\mathbb{N}}$, and let $\{(A_i,B_i) : i \in \mathbb{N}\}$ denote its standard essential family, that is, $A_i = \{x \in \mathbb{Q} : x(i) = 0\}$ and $B_i = \{x \in \mathbb{Q} : x(i) = 1\}$. Let $\mathcal{S} = \{\bigcap_{i\in\mathbb{N}} S_{2i} : S_{2i}$ is a separator between A_{2i} and $B_{2i}\}$. Then $|\mathcal{S}| = \mathfrak{c}$, so we can write $\mathcal{S} = \{S_\alpha : \alpha < \mathfrak{c}\}$.

The following lemmas are not difficult, and are essentially proved in [Ku1; p. 559, step 1].

LEMMA 1.2. *Suppose* $x \in \mathbb{Q}$ *and* $S_\alpha \in \mathcal{S}$. *There is* $s \in S_\alpha$ *such that*, *for all* $i \in \mathbb{N}$, $s(2i-1) = x(2i-1)$. (*In other words*, *the projection of* S_α *onto the infinite-dimensional cube determined by the odd coordinates is onto.*)

LEMMA 1.3. *If* $X \subseteq \mathbb{Q}$ *intersects each element of* $\mathcal{S}$, *then*:

(i) X *is strongly infinite-dimensional.*
(ii) *Viewing* $\mathbb{Q}$ *as* $([0,1]^2)^{\mathbb{N}}$, *the projection of* X *to the* $2k$*-cube determined by any* k *coordinates is* $\ge k$*-dimensional.*

Proof of Theorem 1.1. We use transfinite induction. For each $\alpha < \mathfrak{c}$, we find sets $\{A(i,j)_\alpha : i,j \in \mathbb{N}\}$ such that, starting with $A(i,j)_0 = \emptyset$:

(i) If α is a limit ordinal, then $A(i,j)_\alpha = \bigcup\{A(i,j)_\beta : \beta < \alpha\}$, and otherwise $A(i,j)_\alpha = A(i,j)_{\alpha-1} \cup \{a(i,j)_{\alpha-1}\}$.

(ii) $A(i,j)_\alpha \subseteq \mathbb{R}^2$.

(iii) Given $j \in \mathbb{N}$ and $\beta < \alpha$, $(a(1,j)_\beta, a(2,j)_\beta, \ldots) \in S_\beta$.

(iv) For each choice of $k, n \in \mathbb{N}$, and $i_j \in \mathbb{N}$ for each $j \le k$, $(A(i_1,1)_\alpha)^n \times (A(i_2,2)_\alpha)^n \times \ldots \times (A(i_k,k)_\alpha)^n \cap T_{2nk} = \emptyset$.

We need only consider the case when α is not a limit ordinal. Assume we already have $\{A(i,j)_\alpha : i,j \in \mathbb{N}\}$ satisfying (i)–(iv) above. We will get $\{A(i,j)_{\alpha+1} : i,j \in \mathbb{N}\}$ by induction on j.

Suppose that, for all $r < j$ and $i_r \in \mathbb{N}$, $A(i_r,r)_{\alpha+1}$ satisfies (i)–(iii) for $\alpha+1$ and that (iv) is satisfied with $A(i_r,r)_{\alpha+1}$ replacing $A(i_r,r)_\alpha$ whenever $r < j$.

We must find, for each $i \in \mathbb{N}$, $a(i,j)_\alpha \in \mathbb{R}^2$ so that letting $A(i,j)_{\alpha+1} = A(i,j)_\alpha \cup \{a(i,j)_\alpha\}$, the previous sentence holds with $r \le j$ instead of $r < j$.

To achieve (iv), we use Lemma 1.1. We assume k, n are fixed, as well as $j \le k$ and $i_1, \ldots, i_k$, with $i_j = i$. Then we assume

$$(A(i_1,1)_{\alpha+1})^n \times \ldots \times (A(i_{j-1},j-1)_{\alpha+1})^n \times (A(i,j)_\alpha)^n \times \ldots \times (A(i_k,k)_\alpha)^n \cap T_{2nk} = \emptyset.$$

We want to choose $a(i,j)_\alpha$, for each i, so that $A(i,j)_{\alpha+1} = A(i,j)_\alpha \cup \{a(i,j)_\alpha\}$ can be substituted for $A(i,j)_\alpha$ in the above statement. If $\mathbf{a}$ is a candidate for $a(i,j)_\alpha$, then $\mathbf{a}$ is possibly a bad choice if placing $\mathbf{a}$ in some of the n $\mathbb{R}^2$ coordinates, the coordinates in $W = \{n(j-1)+1, n(j-1)+2, \ldots, n(j-1)+n\}$, reserved for $A(i,j)_{\alpha+1}$ in $\mathbb{R}^{2nk}$, and filling in the other coordinates with points from

$$Z = \bigcup\{A(i_t,t)_{\alpha+1} : t < j\} \cup \bigcup\{A(i_t,t)_\alpha : j \le t \le k\}$$

causes an intersection with T_{2nk}. Thus possible bad choices for $a(i,j)_\alpha$ are contained in intersections of T_{2nk} with standard translates of $H(S,nk)$ by elements of $\mathbb{R}^{2nk}$ whose nonorigin coordinates are in Z, where $S \subseteq W$. By Lemma 1.1, each such intersection is countable, and there are clearly fewer than $\mathfrak{c}$ translates involved. Thus there are fewer than $\mathfrak{c}$ bad choices for $a(i,j)_\alpha$ with respect to property (iv) and this choice of k, n, and $i_1, \ldots, i_{j-1}, i, i_{j+1}, \ldots, i_k$. Over all k's, n's and i_j's there are then fewer than $\mathfrak{c}$ bad choices. Thus there is a set $Y \subseteq \mathbb{R}$ with fewer than $\mathfrak{c}$ points in it such that, for any $i \in \mathbb{N}$, if $a(i,j)_\alpha = (u,t)$ where $u \notin Y$, then (iv) will be satisfied. Fix $b \in [0,1]\backslash Y$. By Lemma 1.2, there is $s \in S_\alpha$ such that $s(2i-1) = b$ for each $i \in \mathbb{N}$. Let $a(i,j)_\alpha = (b, s(2i))$. Then (i)–(iv) are satisfied.

Finally, let $A(i,j) = \bigcup\{A(i,j)_\alpha : \alpha < \mathfrak{c}\}$. Condition (3), with $\le$ instead of $=$, follows from (iv); in particular, $\dim A(i,j) \le 1$. By (iii) and Lemma 1.3(ii), condition (2) is satisfied with $\ge$ used in place of $=$; taken together these give the desired equalities for conditions (2) and (3). ∎

Now assume that $\{A(i,j) : i,j \in \mathbb{N}\}$ satisfy Theorem 1.1. For each $j \in \mathbb{N}$, let $A(j) = \{A(i,j) : i \in \mathbb{N}\}$, and let $A = \bigcup\{A(j) : j \in \mathbb{N}\}$. We will use the symbol $\mathbb{U}$ to denote free union; if C is a collection of topological spaces, then $\mathbb{U}C$ is the free union of the spaces in C. We call a space X a *term* if X is a product of distinct elements of A, and call a space Y a *base space* if Y is a free union of at most ω terms; then all base spaces are separable and metrizable. All of our examples are base spaces. For a base space Y and $j \in \mathbb{N}$, let $\deg(Y,j)$ denote $|\{i : A(i,j) \text{ is a factor of a term of } Y\}|$, and let $\deg(Y) = \sup_j(\deg(Y,j))$ (so $\deg(Y) \in \mathbb{N}$ or $\deg(Y) = \omega$).

Suppose $Y = \mathbb{U}X$, where $X = \{X_i : i \in \{1,\ldots,m\}\}$ or $X = \{X_i : i \in \mathbb{N}\}$, and each X_i is a term. The following facts are immediate from Theorem 1.1.

FACT 1. $\dim Y = \sup_j(\deg(X_i))$ (*if the supremum is* ω, *then* $\dim Y = \infty$).

FACT 2. $\dim Y^k = \sup_K \deg(\mathbb{U}K)$, *where* $K \subseteq X$ *and* $|K| \le k$.

These facts make the computation of dimension easy in powers of base spaces. We are now in a position to give several examples.

EXAMPLE 1. There is a space Y_1 which realizes the constant sequence $\{n\}$, the Anderson–Keisler sequence. Just let Y_1 be the term $\prod_{1\le i\le n} A(i,1)$.

EXAMPLE 2. There is an example Y_2 which realizes the sequence n, $n+1, n+2, n+3, \ldots$ Let $Y_2 = Y_1 \cup \mathbb{U}\{A(i,1) : i > n\}$. The question of the existence of such a space was posed to the author by Gary Gruenhage.

EXAMPLE 3. There is an example Y_3 which realizes the sequence n, $n+1, n+2, \ldots, k, k, k, \ldots$, where k is any integer $> n$. This is just like Y_2, but with fewer terms. Set $Y_3 = Y_1 \cup \mathbb{U}\{A(i,1) : n+1 \le i \le k\}$. This gives the examples in [Ku1] with knowledge of intermediate values.

EXAMPLE 4. There is an example Y_4 which realizes the sequence $4, 4, 6, 6, 7, 8, 9, 10, \ldots$ Let $T_1 = A(1,1) \times A(2,1) \times A(3,1) \times A(4,1)$, let $T_2 = A(1,2) \times A(2,2) \cup A(3,2) \times A(4,2) \cup A(5,2) \times A(6,2)$, and let $T_3 = \mathbb{U}\{A(1,3) \times A(2,3), A(3,3) \times A(4,3), A(5,3), A(6,3), A(7,3), \ldots\}$. Then let $Y_4 = T_1 \cup T_2 \cup T_3$.

The space Y_4 is an example which, for 2 distinct values of k (1 and 5), satisfies $\dim Y^k = \dim Y^{k+1} < \dim Y^{k+2}$. Using this idea it is easy to see that the following can be achieved.

EXAMPLE 5. Given $n \in \mathbb{N}$, there is an example $Y_5(n)$ which satisfies, for some $k_1 < \ldots < k_n$, $\dim Y^{k_i} = \dim Y^{k_i+1} < \dim Y^{k_i+2}$.

It is possible to vary the specifics of the sequences in the types of examples given above, and clearly there are many other possibilities but there are limitations. One was alluded to in the introduction. Another related limitation is that the n in Example 5 cannot be replaced by an infinite set. In the next section, we give a partial solution to the problem from the introduction. We pose the other problem as a question in Section 3.

2. An example X **where** $\dim X = \dim X^2 = 1$ **but** $\dim X^3 = 2$**.** The main idea here is to produce three subsets A, B, and C of $\mathbb{R}^2$, each of which is one-dimensional, so that:

(i) the square of each is one-dimensional,
(ii) the product of any two is also one-dimensional, and
(iii) the product of all three is two-dimensional.

Getting properties (i) and (ii) together was accomplished in Section 1, as well as getting properties (i) and (iii) together, with "at least two" in place of "two" for property (iii). The main difficulty is in getting properties (ii) and (iii) simultaneously; this imposes constraints on our use of geometry, which the following addresses.

Let

$$P^{\#} = \{(x, t, y, t) : x, y, t \in \mathbb{R}\};$$

then $P^{\#}$ is a three-dimensional hyperplane in $\mathbb{R}^4$. For $k \in \mathbb{R}$, let $P^{\#}(k) = \{(x, k, y, k) : x, y \in \mathbb{R}\}$; then $P^{\#}(k)$ is a two-dimensional hyperplane and a translate of $P^{\#}(0)$. Let $\mathbf{0}$ denote $(0, 0)$, and let $P_{\mathbf{a}} = \{(a, \mathbf{0}) : a \in \mathbb{R}^2\}, P_{\mathbf{b}} = \{(\mathbf{0}, b) : b \in \mathbb{R}^2\}$, and $P_{\mathbf{aa}} = \{(a, a) : a \in \mathbb{R}^2\}$. Then $P_{\mathbf{a}}$, $P_{\mathbf{b}}$, and $P_{\mathbf{aa}}$ are all two-dimensional hyperplanes in $\mathbb{R}^4$. Let

$$F = \{P^{\#}, P^{\#}(\mathbf{0}), P_{\mathbf{a}}, P_{\mathbf{b}}, P_{\mathbf{aa}}\}.$$

We need the following technical lemma, which is crucial for reconciling the opposing demands that $\dim X^2 = 1$ and $\dim X^3 = 2$ impose. It can be viewed as a generalization of Lemma 1.1 in the particular case $k = 2$.

LEMMA 2.1. *In* $\mathbb{R}^4$ *there are* 3 *collections* $\mathcal{H}(i) = \{H_j(i) : j \in \mathbb{N}\}$ *for* $i \in \{1, 2, 3\}$ *of three-dimensional hyperplanes and corresponding* $\mathcal{S}(i) = \{S_j(i) : j \in \mathbb{N}\}$ *of two-spheres such that*:

(i) *For* $i \in \{1, 2, 3\}$ *and* $j \in \mathbb{N}$, $S_j(i) \subseteq H_j(i)$.

(ii) *For* $i \in \{1, 2, 3\}$, $\dim \mathbb{R}^4 \setminus \bigcup \mathcal{S}(i) = 1$.

(iii) $H_j(i)$ *is in general position with respect to each translate of each element of* F.

(iv) *If* $u \in \{1, 2, 3\}$ *and* $j \in \mathbb{N}$, *then* $\Pi_{1,2} S_j(u) \cap P^{\#}$ *and* $\Pi_{3,4} S_j(u) \cap P^{\#}$ *are either one-point sets or nondegenerate ellipses.*

(v) *If* $u, v \in \{1, 2, 3\}$ *with* $u \neq v$, *and* $j, k \in \mathbb{N}$, *then the ellipses in* $\{\Pi_{1,2} S_j(u) \cap P^{\#}, \Pi_{3,4} S_j(u) \cap P^{\#}\}$ *are distinct from the ellipses in* $\{\Pi_{1,2} S_k(v) \cap P^{\#}, \Pi_{3,4} S_k(v) \cap P^{\#}\}$ (*assuming they are not one-point sets*).

In order to prove Lemma 2.1, we need the following.

LEMMA 2.2. *Let $\mathcal{H} = \{H : H$ is a three-dimensional hyperplane in $\mathbb{R}^4$ whose equation can be written as $ax+by+z+dw = e$, where $a \neq 0\}$. Then*:

(1) *If $H \in \mathcal{H}$ and $S \subseteq H$ is a two-sphere, then*

  (i) *$H \cap P^{\#}$ is a two-dimensional hyperplane.*

  (ii) *If $S \cap P^{\#} \neq \emptyset$, then $S \cap P^{\#}$ is a circle or a point. If $S \cap P^{\#}$ is a circle, then $\Pi_{1,2}S \cap P^{\#}$ and $\Pi_{3,4}S \cap P^{\#}$ are nondegenerate ellipses.*

(2) *If $H_1, H_2 \in \mathcal{H}$ with $S_1 \subseteq H_1$ and $S_2 \subseteq H_2$, where S_1 and S_2 are two-spheres with $S_1 \cap P^{\#}$ and $S_2 \cap P^{\#}$ both circles, and H_i is given by the equation $a_ix + b_iy + z + d_iw = e_i$, then the ellipses in $\{\Pi_{1,2}S_1 \cap P^{\#}, \Pi_{3,4}S_1 \cap P^{\#}\}$ are distinct from the ellipses in $\{\Pi_{1,2}S_2 \cap P^{\#}, \Pi_{3,4}S_2 \cap P^{\#}\}$ provided that for $i \in \{1,2\}$*:

  (a) *$a_i(b_i+d_i)/(a_i^2+1)$ are distinct (this gives $\Pi_{1,2}S_1 \cap P^{\#} \neq \Pi_{1,2}S_2 \cap P^{\#}$);*

  (b) *$a_1(b_1+d_1)/(a_1^2+1) \neq (b_2+d_2)/(a_2^2+1)$ (this gives $\Pi_{1,2}S_1 \cap P^{\#} \neq \Pi_{3,4}S_2 \cap P^{\#}$);*

  (c) *$(b_1+d_1)/(a_1^2+1) \neq a_2(b_2+d_2)/(a_2^2+1)$ (this gives $\Pi_{3,4}S_1 \cap P^{\#} \neq \Pi_{1,2}S_2 \cap P^{\#}$);*

  (d) *$(b_i+d_i)/(a_i^2+1)$ are distinct (this gives $\Pi_{3,4}S_1 \cap P^{\#} \neq \Pi_{3,4}S_2 \cap P^{\#}$).*

Proof of (1)(i). $H \cap P^{\#}$ can easily be seen to be two-dimensional.

Proof of (1)(ii). $S \cap P^{\#}$ is contained in $H \cap P^{\#}$, and thus it is the intersection of a two-sphere with a two-plane. This is either empty, a point, or a circle. Suppose it is a circle C; then, by translating, we may assume C is centered at the origin. Since translating does not affect the x^2, z^2, t^2, xt, or tz terms of the ellipses, in what follows this causes no problems, and simplifies matters. Thus C is the set of points simultaneously satisfying (substituting t for y and w)

$$ax + (b+d)t + z = 0, \tag{I}$$

$$x^2 + 2t^2 + z^2 = r^2. \tag{II}$$

Solving for z in (I), we have $z = -(ax + (b+d)t)$, and substituting in (II), we obtain

$$(a^2+1)x^2 + (2 + (b+d)^2)t^2 + 2a(b+d)xt = r^2. \tag{$*$}$$

Solving for x in (I), we have $x = -a^{-1}(z + (b+d)t)$, and substituting in (II), we obtain

$$(1 + a^{-2})z^2 + 2(b+d)a^{-2}zt + ((b+d)^2a^{-2} + 2)t^2 = r^2. \tag{$**$}$$

Now $(*)$ is the equation for $\Pi_{1,2}S \cap P^{\#}$ and $(**)$ is the equation for $\Pi_{3,4}S \cap P^{\#}$. Each of these is easily seen to be the equation of a nondegenerate ellipse.

Proof of (2). This follows from examining $(*)$ and $(**)$ with the H_1 and H_2 given in the statement of the lemma and observing that in any equivalent formulation of the conic section $ax^2+bxy+cy^2=d$, the ratio of the coefficient of xy to the coefficient of x^2 will always be constant at b/a. The statements (a)–(d) just say that for the appropriate ellipses these ratios are different. ∎

COROLLARY 2.1. *Let $\mathcal{H}$ be as in Lemma 2.2, and let K be a countable subset of $\mathcal{H}$. Let $H \in \mathcal{H}$, where H has equation $ax+by+z+dw=e$. Then given $\varepsilon > 0$, there is b' within ε of b such that if H' is the plane whose equation is $ax+b'y+z+dw=e$, and C is a circle in $H' \cap P^{\#}$, then for any $H^* \in K$, and any circle C^* in H^*, the ellipses in $\{\Pi_{1,2}C, \Pi_{3,4}C\}$ are distinct from the ellipses in $\{\Pi_{1,2}C^*, \Pi_{3,4}C^*\}$.*

Proof. By (2) of Lemma 2.2, there are only 4 possible choices for b' for each element of H^* which could be bad, that is, giving equality in one of the conditions in (2) of Lemma 2.2, if a, d, and e are left fixed. Thus there are only countably many bad choices altogether. ∎

Proof of Lemma 2.1. Let $B = \{B_i : i \in \mathbb{N}\}$ be a base of balls for $\mathbb{R}^4$; letting W_i be the boundary of B_i, $W = \{W_i : i \in \mathbb{N}\}$ is a collection of three-spheres in $\mathbb{R}^4$ such that $\dim \mathbb{R}^4 \setminus \bigcup W = 0$. If Z is a subset of $\mathbb{R}^4$ whose intersection with each W_i is zero-dimensional, then $\dim(\mathbb{R}^4 \setminus \bigcup W \cup Z) \le 1$. This is because it has a base of open sets whose boundaries are zero-dimensional.

For each $i \in \mathbb{N}$, we can find a collection H_i of countably many hyperplanes of dimension three so that W_i intersects each hyperplane in a two-sphere, and so that $\dim W_i \setminus \bigcup H_i = 0$. Then $\bigcup_{i\in\mathbb{N}} H_i$ almost serves as an $\mathcal{H}(j)$ required by this lemma, with, for $h \in \mathcal{H}(j)$, $h \in H_i$ for some i, so the sphere corresponding to h is $h \cap W_i$. The problem is now only to satisfy conditions (iii), (iv), and (v).

For each W_i let D_i be a countable dense subset, and let $\mathcal{T} = \{(W_i, d, n) : d \in D_i,\ n \in \mathbb{N}\}$. Let $\mathcal{T} = \{T_i : i \in \mathbb{N}\}$ enumerate the elements of $\mathcal{T}$ in such a way that for each $i \in \mathbb{N}$, T_{3i-2}, T_{3i-1}, T_{3i} are the same element of $\mathcal{T}$. Inductively, we choose a hyperplane V_i in $\mathcal{H}$ of Lemma 2.2 such that:

(i) If $T_i = (W_t, d, n)$, then $N_i = V_i \cap W_t$ separates the $(1/n)$-neighborhood of d in W_t from the $(2/n)$-neighborhood of d in W_t.

(ii) V_i is in general position with respect to each translate of each member of F.

(iii) For any circle C in $V_i \cap P^{\#}$, $\{\mathit{\Pi}_{1,2}C, \mathit{\Pi}_{3,4}C\}$ are distinct ellipses from those in $\{\mathit{\Pi}_{1,2}C^*, \mathit{\Pi}_{3,4}C^*\}$, where C^* is a circle in $V_t \cap P^{\#}$ for some $t < i$.

This can be accomplished as (i) is easy to get, and if V_i' satisfies (i) then so will any small alteration of it (that is, of the coefficients in the equation defining V_i'). With an appropriate small alteration V_i'' of V_i', we can get (ii); now any small enough alteration of V_i'' will satisfy (i) and (ii). By applying Corollary 2.1, a small enough alteration V_i of V_i'' additionally gives (iii).

Now let $H_i(j) = V_{3i-j+1}$, $S_i(j) = N_{3i-j+1}$; then $\mathcal{H}(j) = \{H_i(j) : i \in \mathbb{N}\}$ and $\mathcal{S}(j) = \{S_i(j) : i \in \mathbb{N}\}$ will satisfy the conditions of the lemma. ∎

For the next lemma, we assume, for $j \in \{1, 2, 3\}$, the collections $\mathcal{H}(j)$ and $\mathcal{S}(j)$ that Lemma 2.1 gives. For fixed $t \in \mathbb{R}$, let $P_t = \{(x, t, y, t, z, t) : x, y, z \in \mathbb{R}\}$. Then P_t is a three-dimensional hyperplane in $\mathbb{R}^6$ (as opposed to $P^{\#}(t)$ which is a two-dimensional hyperplane in $\mathbb{R}^4$). Also let $B_t = \{(x, t, y, t, z, t) \in P_t : (x, t, y, t) \in \bigcup \mathcal{S}(1),\ (y, t, z, t) \in \bigcup \mathcal{S}(2)$, and $(x, t, z, t) \in \bigcup \mathcal{S}(3)\}$.

Lemma 2.3. *Let* $\mathbf{B} = \{t \in \mathbb{R} : \dim P_t \backslash B_t < 2\}$. *Then* $\mathbf{B}$ *is countable.*

Proof. From (v) of Lemma 2.1, since 5 points determine an ellipse, for $r, s \in \{1, 2, 3\}$ with $r \neq s$ and $i, j \in \mathbb{N}$, we have:

(i) $|\mathit{\Pi}_{1,2}S_i(r) \cap P^{\#} \cap \mathit{\Pi}_{1,2}S_j(s) \cap P^{\#}| \le 4$,
(ii) $|\mathit{\Pi}_{1,2}S_i(r) \cap P^{\#} \cap \mathit{\Pi}_{3,4}S_j(s) \cap P^{\#}| \le 4$, and
(iii) $|\mathit{\Pi}_{3,4}S_i(r) \cap P^{\#} \cap \mathit{\Pi}_{3,4}S_j(s) \cap P^{\#}| \le 4$.

Now let $K = \{(a, t) : (a, t)$ is in one of the intersections from (i)–(iii) above$\}$. Then K is countable, and so is $T = \{t : (a, t) \in K$ for some $a \in \mathbb{R}\}$.

Fix $t \notin T$; we claim that $\dim P_t \backslash B_t \ge 2$.

If $x \in \mathbb{R}$, there is at most one $r \in \{1, 2, 3\}$ such that for some $i \in \mathbb{N}$, $(x, t) \in \mathit{\Pi}_{1,2}S_i(r) \cap P^{\#}$ or $(x, t) \in \mathit{\Pi}_{3,4}S_i(r) \cap P^{\#}$. So if $(x, t, y, t) \in S_i(r)$ and $(a, t, b, t) \in S_j(s)$ where $r \neq s$, then $x \neq a$, $x \neq b$, $y \neq a$, and $y \neq b$. Also, since $H_i(r)$ is in general position with respect to $P^{\#}(t)$, the dimension of $P^{\#}(t) \cap H_i(r)$ is 1, so $|S_i(r) \cap P^{\#}(t)| \le 2$, since $S_i(r) \cap P^{\#}(t)$ is the intersection of a sphere with a line.

Now let, for $i \in \{1, 2, 3\}$, $M_i = \bigcup \mathcal{S}(i) \cap P^{\#}(t) = \{(x, t, y, t) : (x, t, y, t) \in \bigcup \mathcal{S}(i)\}$. Then for $i \neq j$, $\mathit{\Pi}_1 M_i \cap \mathit{\Pi}_1 M_j = \emptyset$, $\mathit{\Pi}_3 M_i \cap \mathit{\Pi}_1 M_j = \emptyset$, and $\mathit{\Pi}_3 M_i \cap \mathit{\Pi}_3 M_j = \emptyset$, and M_i is countable.

Thus B_t is a union of lines in P_t of 3 types, countably many of each type. Namely:

$$
\begin{aligned}
L_1 &= \{(x, t, y, t, z, t) : (x, t, y, t) \in M_1,\ z \in \mathbb{R}\},\\
L_2 &= \{(z, t, x, t, y, t) : (x, t, y, t) \in M_2,\ z \in \mathbb{R}\},\\
L_3 &= \{(x, t, z, t, y, t) : (x, t, y, t) \in M_3,\ z \in \mathbb{R}\}.
\end{aligned}
$$

Now we claim that no two lines in $L_1 \cup L_2 \cup L_3$ intersect. Clearly this is true for lines in a fixed L_i. Suppose without loss of generality that $l \in L_1$ and $m \in L_2$, and $(x,t,y,t,z,t) \in l \cap m$. Then $y \in \Pi_3 M_1 \cap \Pi_1 M_2$, which is impossible.

Thus B_t is a union of countably many pairwise disjoint lines in P_t. By the argument of Sitnikov (see [E]), $\dim P_t \backslash B_t \geq 2$. ∎

The Example. We view $\mathbb{R}^6$ as $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$, and the example X will be the disjoint union of subsets A, B, and C of $\mathbb{R}^2$. The requirements are that:

(1) $\dim A^2 = \dim B^2 = \dim C^2 = 1$,
(2) $\dim A \times B = \dim A \times C = \dim B \times C = 1$,
(3) $\dim A \times B \times C = 2$.

We assume $\mathcal{H}(j)$ and $\mathcal{S}(j)$ for $j \in \{1,2,3\}$ as in Lemma 2.1, and also the corresponding $\mathbf{B}$ as in Lemma 2.3. Let $G = \{g : g \text{ is a } G_\delta \text{ subset of } \mathbb{R}^6 \text{ such that } \dim g = 1\}$. Then we can write $G = \{g_\alpha : \alpha < \mathfrak{c}\}$. If $Y \subseteq \mathbb{R}^6$ intersects the complement of each element of G, then $\dim Y \geq 2$.

The sets A, B, C are produced by transfinite induction so that $A = \{a_\alpha : \alpha < \mathfrak{c}\}$, $B = \{b_\alpha : \alpha < \mathfrak{c}\}$, and $C = \{c_\alpha : \alpha < \mathfrak{c}\}$. At stage α, we start with approximations $A_\alpha = \{a_\beta : \beta < \alpha\}$, $B_\alpha = \{b_\beta : \beta < \alpha\}$, and $C_\alpha = \{c_\beta : \beta < \alpha\}$ such that:

(i) None of $A_\alpha^2, B_\alpha^2, C_\alpha^2$ intersect $\bigcup \mathcal{S}(1)$.
(ii) $A_\alpha \times B_\alpha \cap \bigcup \mathcal{S}(1) = \emptyset$, $A_\alpha \times C_\alpha \cap \bigcup \mathcal{S}(3) = \emptyset$, $B_\alpha \times C_\alpha \cap \bigcup \mathcal{S}(2) = \emptyset$.
(iii) For each $\beta < \alpha$, $(a_\beta, b_\beta, c_\beta) \in \mathbb{R}^6 \backslash g_\beta$.

We must produce, simultaneously, a_α, b_α, c_α so that on setting $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$, $B_{\alpha+1} = B_\alpha \cup \{b_\alpha\}$, $C_{\alpha+1} = C_\alpha \cup \{c_\alpha\}$, the conditions (i)–(iii) are satisfied with $\alpha + 1$ in place of α.

With A, B, and C so constructed, condition (1) is met due to (i), condition (2) is met due to (ii), and condition (3) is met due to (iii).

Remembering condition (iii) of Lemma 2.1, each element of $\mathcal{H}(j)$ is in general position with respect to each translate of the planes $P_{\mathbf{a}}$, $P_{\mathbf{b}}$, and $P_{\mathbf{aa}}$. Thus, by using an argument similar to that used in the proof of Theorem 1.1, the set $K = \{u \in \mathbb{R} : \text{there is } v \in \mathbb{R} \text{ such that choosing } a_\alpha, b_\alpha, \text{ or } c_\alpha \text{ to be } (v,u) \text{ would cause (i) to fail}\}$ has fewer than $\mathfrak{c}$ points.

We know, for example, that $A_\alpha \times B_\alpha \cap \bigcup \mathcal{S}(1) = \emptyset$; similarly to the above there are fewer than $\mathfrak{c}$ choices for a_α so that $\{a_\alpha\} \times B_\alpha \cap \bigcup \mathcal{S}(1) \neq \emptyset$. More generally, the sets $L_a, L_b, L_c \subseteq \mathbb{R}$ given by $L_a = \{u \in \mathbb{R} : \text{there is } v \in \mathbb{R} \text{ such that if } a_\alpha = (v,u) \text{ then } \{a_\alpha\} \times B_\alpha \cap \bigcup \mathcal{S}(1) \neq \emptyset \text{ or } \{a_\alpha\} \times C_\alpha \cap \mathcal{S}(3) \neq \emptyset\}$, $L_b = \{u \in \mathbb{R} : \text{there is } v \in \mathbb{R} \text{ such that if } b_\alpha = (v,u) \text{ then } A_\alpha \times \{b_\alpha\} \cap \mathcal{S}(1) \neq \emptyset \text{ or } \{b_\alpha\} \times C_\alpha \cap \bigcup \mathcal{S}(2) \neq \emptyset\}$, and $L_c = \{u \in \mathbb{R} : \text{there is } v \in \mathbb{R} \text{ such that}$

if $c_\alpha = (v,u)$ then $A_\alpha \times \{c_\alpha\} \cap \mathcal{S}(3) \neq \emptyset$ or $B_\alpha \times \{c_\alpha\} \cap \mathcal{S}(2) \neq \emptyset\}$ all have fewer than $\mathfrak{c}$ points.

Let $L = L_a \cup L_b \cup L_c$; then L has fewer than $\mathfrak{c}$ points. Now choose $t \in \mathbb{R}\backslash(K \cup L \cup \mathbf{B})$. Since $t \notin \mathbf{B}$, $\dim P_t\backslash B_t \geq 2$; in particular, $(P_t\backslash B_t)\backslash g_\alpha \neq \emptyset$. Choose $(a,t,b,t,c,t) \in (P_t\backslash B_t)\backslash g_\alpha$, and let $a_\alpha = (a,t)$, $b_\alpha = (b,t)$, and $c_\alpha = (c,t)$. Then (i) is satisfied since $t \notin K$; (iii) is obviously satisfied. We check (ii) is satisfied in the case of $A_{\alpha+1} \times B_{\alpha+1}$. We have, by assumption, $A_\alpha \times B_\alpha \cap \bigcup \mathcal{S}(1) = \emptyset$. Since $t \notin L_a \cup L_b$, it follows that $\{a_\alpha\} \times B_\alpha \cap \bigcup \mathcal{S}(1) = \emptyset = A_\alpha \times \{b_\alpha\}$. We need only check that $(a_\alpha, b_\alpha) \notin \bigcup \mathcal{S}(1)$; if $(a,t,b,t) \in \bigcup \mathcal{S}(1)$ we would have $(a,t,b,t,c,t) \in B_t$, which is not possible. The other cases are checked similarly. ∎

3. Questions. The first question is the ultimate goal.

QUESTION 1. Is the conjecture true or false? If it is false, what is the correct conjecture?

Question 1 may be very difficult. There are other interesting parts of the Conjecture. A couple of them are contained in:

QUESTION 2. Given k and $n \gg k$, is there X such that $\dim X^r = k$ if $r < n$, and $\dim X^n > k$? More specifically, is there a space X_n, for each $n > 3$, such that $\dim X_n^r = 1$ if $r < n$ and $\dim X_n^n = 2$? Our example in Section 2 is X_3.

QUESTION 3. Is there a space X such that, for infinitely many k, $\dim X^k = \dim X^{k+1} < \dim X^{k+2}$? Note that if $1 = \dim X = \dim X^2$, and $2 = \dim X^3$, and $\dim X^\omega = \infty$, then X is such an example, since $\dim X^{2m} \leq \dim(X^2)^m \leq m$, so half the time $\dim X^m = \dim X^{m+1}$.

QUESTION 4. Suppose $D = \{d_n : n \in \mathbb{N}\}$ and $E = \{e_n : n \in \mathbb{N}\}$ are allowable sequences. Is $D + E = \{d_n + e_n : n \in \mathbb{N}\}$ allowable? This question is due to Jim Lawrence. Any sequences built in a nontrivial way from old sequences would be interesting.

For a finite-dimensional separable metric space X, let $e(X) = \min\{n : X \text{ embeds in } \mathbb{R}^n\}$. We will call $e(X)$ the *embedding number* for X.

QUESTION 5. What are the possibilities for the sequence $e(X), e(X^2), e(X^3), \ldots$? The embedding number sequences are related to allowable sequences since $\dim X \leq e(X) \leq 2\dim X + 1$. The results in [DRS], [Sp] and [K], along with that in [L], suggest that, at least for locally compact spaces, the connection may be stronger, while the result in [Ku2] shows it is not as strong in the general case.

Hattori [H] showed that examples with the properties of those found in [Ku1] can be chosen to be topological groups.

QUESTION 6. Can any allowable sequence be realized by a topological group? If so, a precompact topological group? Less generally, can any of the allowable sequences realized by base spaces be realized by topological groups (precompact topological groups)?

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Received 4 September 1995;
in revised form 17 November 1995