The Dugundji extension property can fail in \( \omega_\mu \)-metrizable spaces

by

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Abstract. We show that there exist \( \omega_\mu \)-metrizable spaces which do not have the Dugundji extension property (\( 2^{\omega_1} \) with the countable box topology is such a space). This answers a question posed by the second author in 1972, and shows that certain results of van Douwen and Borges are false.

1. Introduction. For a topological space \( X \), let \( C(X) \) denote the vector space of continuous real-valued functions on \( X \). If \( A \) is a subset of a space \( X \) and \( \Phi : C(A) \to C(X) \) is a map such that \( \Phi(f) \) extends \( f \) for each \( f \) in \( C(A) \), then \( \Phi \) is called an extender (in this setting, the Tietze–Urysohn theorem [9, 2.1.8] says that for every closed set \( A \) in a normal space \( X \), there exists an extender \( \Phi : C(A) \to C(X) \)). The same terminology is used when considering bounded functions and maps \( \Phi : C^*(A) \to C^*(X) \). K. Borsuk was the first to construct an extender with interesting properties [3, Theorem 3]. He proved that for every closed, separable subset \( A \) of a metric space \( X \) there is an extender \( \Phi : C^*(A) \to C^*(X) \) which is linear (with respect to the natural vector space structure on \( C^*(A) \) and \( C^*(X) \)) and norm preserving (with respect to the sup norm on \( C^*(A) \) and \( C^*(X) \)). J. Dugundji improved Borsuk’s result in three ways [7]. He dropped the hypothesis of separability from the closed subset \( A \), he considered all continuous functions, and his extender preserved convex hulls.

Definition 1.1. A space \( X \) is said to have the Dugundji extension property if for every closed subspace \( A \) of \( X \) there is a linear extender \( \Phi : C(A) \to C(X) \) such that for each \( f \in C(A) \), the range of \( \Phi(f) \) is contained in the convex hull of the range of \( f \).
Thus, we can state succinctly Dugundji’s theorem: all metric spaces have the Dugundji extension property. It is known that the Dugundji extension property holds in several classes of generalized metric spaces (see [1], [4], and [12]).

In this paper we give conditions under which certain normal spaces fail to have the Dugundji extension property (Theorem 1.4). In fact, we consider the following weaker extension property.

**Definition 1.2.** If for each closed subspace $A$ of a space $X$ there is an extender $\Phi$ such that $\Phi(f) \leq \Phi(g)$ whenever $f \leq g \in C(A)$, then $X$ is said to satisfy the unbounded monotone extension property.

For the definitions of basic terms see [9]. In our theorem, we use the following definition.

**Definition 1.3.** Let $A \subseteq X$. A family $B$ of open subsets of $X$ is called a total-$\pi$-base for $A$ in $X$ provided that

(i) each element of $B$ has non-empty intersection with $A$,
(ii) each non-empty open set which contains a point of $A$ contains a member of $B$ and
(iii) if $B_0 \supseteq B_1 \supseteq B_2 \supseteq \ldots$ is a countable decreasing chain of elements of $B$ then $\bigcap_{n \in \omega} B_n \neq \emptyset$.

**Theorem 1.4.** If $A$ is a first category, closed subset of a normal space $X$, and there exists a total-$\pi$-base $B$ for $A$ in $X$, then there does not exist an extender $\Phi : C(A) \rightarrow C(X)$ satisfying the conditions for the unbounded monotone extension property.

Easy applications of Theorem 1.4 show that certain $\omega_1$-metrizable spaces do not satisfy the Dugundji extension property (see Examples 2.2 and 2.3). These examples have several interesting consequences. They show that a result of Eric van Douwen in his thesis [6, Theorem 7, p. 52; p. 58, line 2] and a result of C. J. R. Borges [2, Theorem 2.1] are false. The present paper resulted from the discovery by Stares that the proof by Borges has a gap [12]. In addition, our examples answer a question raised in 1972 by Vaughan [13, p. 264], and give another interesting difference between the classes of strongly zero-dimensional metric spaces and $\omega_1$-metrizable spaces. Indeed, R. Engelking [8] proved that in a strongly zero-dimensional metric space, every closed subset is a retract, a stronger property than the Dugundji extension property.

**2. Proof and examples**

**Proof of Theorem 1.4.** By way of contradiction, assume such an extender $\Phi : C(A) \rightarrow C(X)$ exists. Since $A$ is first category there exist open
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sets $E_i$ in $X$ for $i \in \omega$ such that $A \cap E_i$ is dense in $A$ and $A \cap \bigcap_i E_i = \emptyset$. We construct, by induction, $B_i \in \mathcal{B}$, continuous functions $f_i : X \to [0, \infty)$, and open sets $M_i = \{x \in X : \Phi(f_i[A])(x) > i\}$ satisfying the following properties: $B_0$ is any member of $\mathcal{B}$ such that $B_0 \subseteq E_0$ and $f_0$ is the constant function with value 1, and the following hold for $i \geq 1$:

1. $\text{cl}_X(B_i) \subseteq B_{i-1} \cap M_{i-1} \cap E_i$,
2. $f_i(\text{cl}_X(B_i)) = i + 1$, and $f_i(X \setminus B_{i-1}) = 0$.

We construct $B_n$ and $f_n$. By (2), $B_{n-1} \cap A \subseteq M_{n-1}$, hence $B_{n-1} \cap M_{n-1} \cap A$ is a non-empty open subset of $A$, so by denseness $B_{n-1} \cap M_{n-1} \cap E_n$ contains a point of $A$. By regularity of $X$, we may pick $B_n \in \mathcal{B}$ such that

$$\text{cl}_X(B_n) \subseteq B_{n-1} \cap M_{n-1} \cap E_n.$$ 

It is then clear that (1) holds for $B_n$. By the Tietze–Urysohn theorem there exists a continuous $f_n : X \to [0, \infty)$ such that $f_n(\text{cl}_X(B_n)) = n + 1$, and $f_n(X \setminus B_{n-1}) = 0$; so now (2) is also satisfied. This completes the induction.

Define $h : A \to \mathbb{R}$ by $h = \sum_{i=1}^\omega f_i[A]$. To see that this infinite sum is well-defined and continuous on $A$, note that for any $a \in A$ there exists $n$ such that $a \notin E_n \supseteq \text{cl}_X(B_n)$; so $(X \setminus \text{cl}_X(B_n))$ is a neighborhood of $a$ on which $h = \sum_{i=1}^\omega f_i[A]$, a finite sum of continuous functions. By the hypothesis on $\mathcal{B}$, there exists $y \in \bigcap\{B_i : i \in \omega\}$. Now pick an integer $m$ such that $\Phi(h)(y) < m$. By the monotonicity of $\Phi$, we have $\Phi(f_m[A])(y) \leq \Phi(h)(y)$. By (1), $y \in \bigcap\{M_i : i \in \omega\} \subseteq M_m$, so we have the contradiction

$$m < \Phi(f_m[A])(y) \leq \Phi(h)(y) < m.$$ 

We take the following as the definition of $\omega_\mu$-metrizability (see [11] for a list of references concerning $\omega_\mu$-metrizability).

**Definition 2.1.** If $\omega_\mu$ is a regular, uncountable cardinal then a space $X$ is said to be $\omega_\mu$-metrizable if there is a collection $\{\mathcal{U}_\alpha : \alpha < \omega_\mu\}$ where each $\mathcal{U}_\alpha$ is a pairwise disjoint open cover of $X$, $\mathcal{U}_\alpha$ refines $\mathcal{U}_\beta$ if $\alpha > \beta$, and $\bigcup\{\mathcal{U}_\alpha : \alpha < \omega_\mu\}$ is a base for $X$.

We now present three examples. A simple application of Theorem 1.4 shows that each example fails to have the Dugundji extension property. The first two examples are both $\omega_1$-metrizable and hence give the result stated in the title (among other things). The third example, which is not $\omega_\mu$-metrizable, is the well-known Michael line [9, 5.1.32]. That the Michael line does not have the Dugundji extension property was shown in [6] and [10]. Our theorem gives a simpler proof of this, and stems from attempts to generalize van Douwen’s proof [6] that the Michael line does not satisfy the unbounded monotone extension property.

**Example 2.2.** The set $2^{\omega_1}$ with the countable box topology is an $\omega_1$-metrizable topological group without the Dugundji extension property.
Let $X$ denote the countable box topology on the set $2^{\omega_1}$. This is the topology having as a base all sets of the form $[x|\alpha]$ for $\alpha < \omega_1$ and $x \in X$, where $[x|\alpha] = \{y \in X : y(\beta) = x(\beta) \text{ for all } \beta < \alpha\}$. Let $\mathcal{U}_a = \{[x|\alpha] : x \in X\}$ for $\alpha < \omega_1$. It is clear that these $\mathcal{U}_a$ satisfy the conditions of Definition 2.1. The subset $A$ is defined to be the set of functions with cofinite support, i.e.,

$$A = \{x \in X : |\{\alpha < \omega_1 : x(\alpha) = 0\}| < \omega\}.$$ 

Obviously, $A$ is closed. To check that $A$ is a first category set we define open sets $E_n$ for $n \in \omega$ by

$$E_n = \{x \in X : |\{\alpha < \omega_1 : x(\alpha) = 0\}| > n\}.$$ 

Each $E_n$ is open and $A \cap E_n$ is dense in $A$ for all $n \in \omega$. Since any point in $\bigcap_{n \in \omega} E_n$ must have infinitely many coordinates equal to 0, we have $A \cap \bigcap_{n \in \omega} E_n = \emptyset$.

To complete the example we need a total-$\pi$-base $\mathcal{B}$ for $A$ in $X$. Let $\mathcal{B}$ be all the basic open sets in $X$ which have non-empty intersection with $A$ (thus $\mathcal{B}$ is a clopen base for $A$ in $X$). If $B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$ is a decreasing chain of elements of $\mathcal{B}$ then it is clear that $\bigcap_n B_n \neq \emptyset$. All the hypotheses in Theorem 1.4 are satisfied and therefore, $X$ does not satisfy the unbounded monotone extension property and hence, does not have the Dugundji extension property.

The space $X$ in the above example is the special case $\mu = 1$ of the spaces $(2^{\omega_\mu})_\omega$, for $\omega_\mu$ a regular uncountable cardinal. These spaces are known to be $\omega_\mu$-metrizable [5, p. 384]. In the same way as in the above example we can show that $(2^{\omega_\mu})_\omega$ does not have the Dugundji extension property. Thus for every regular, uncountable $\omega_\mu$, there exists an $\omega_\mu$-metrizable space which does not have the unbounded monotone, or Dugundji, extension property.

**Example 2.3.** The countable product $L(\omega_1)^\omega$ with the box topology does not have the Dugundji extension property. Thus a box product of a countable family of $\omega_1$-metrizable spaces, each having the Dugundji extension property, need not have the Dugundji extension property.

By $L(\omega_1)$ we mean the space derived from the space $\omega_1 + 1$ with the usual order topology by isolating all the points except the point $\omega_1$. By setting $\mathcal{U}_a = \{(\alpha, \omega_1) \cup \{\beta : \beta \leq \alpha\} : \alpha < \omega_1\}$ we get collections satisfying the conditions of Definition 2.1 and therefore $L(\omega_1)$ is $\omega_1$-metrizable. Our example $X$ will be the product of countably many copies of $L(\omega_1)$ with the box topology. By [14, Theorem 2.9], $X$ is also $\omega_1$-metrizable. The subspace $A$ is defined by $A = \{x \in X : |\{i < \omega : x(i) \neq \omega_1\}| < \omega\}$. As before, $A$ is a closed first category set. The total-$\pi$-base $\mathcal{B}$ is, as in the previous example, the collection of all basic open sets in $X$ which have non-empty in-
tersection with $A$. By Theorem 1.4, $X$ does not have the Dugundji extension property.

It is easy to show that $L(\omega_1)$ has the Dugundji extension property. In fact, if a space has at most one isolated point then a linear extender $\Phi$ can be found which is range preserving.

**Example 2.4** (Heath and Lutzer [10], van Douwen [6]). *The Michael line $M$ does not have the Dugundji extension property.*

Recall that the *Michael line $M$ is the set $\mathbb{R}$ of real numbers with the topology obtained by starting with the usual topology and declaring all irrational points to be isolated [9, 5.1.32]. Take the closed set $A$ to be the set of rational numbers in $M$, and for $B$ take a base of intervals for the rational numbers with the following properties: $B = \bigcup\{B_n : n \in \omega\}$ such that

1. each $B_n$ is a countable family of pairwise disjoint open intervals of length at most $1/(n + 1)$ covering the rational numbers,
2. the closure (in $\mathbb{R}$) of each interval in $B_{n+1}$ is contained in some interval in $B_n$.

**References**


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