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The Dugundji extension property can fail in ω_μ-metrizable spaces

by

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Abstract. We show that there exist ω_μ-metrizable spaces which do not have the Dugundji extension property (2^{ω_1} with the countable box topology is such a space). This answers a question posed by the second author in 1972, and shows that certain results of van Douwen and Borges are false.

1. Introduction. For a topological space X, let $C(X)$ denote the vector space of continuous real-valued functions on X. If A is a subset of a space X and $\Phi : C(A) \to C(X)$ is a map such that $\Phi(f)$ extends f for each f in $C(A)$, then Φ is called an *extender* (in this setting, the Tietze–Urysohn theorem [9, 2.1.8] says that for every closed set A in a normal space X, there exists an extender $\Phi : C(A) \to C(X)$). The same terminology is used when considering bounded functions and maps $\Phi : C^*(A) \to C^*(X)$. K. Borsuk was the first to construct an extender with interesting properties [3, Theorem 3]. He proved that for every closed, separable subset A of a metric space X there is an extender $\Phi : C^*(A) \to C^*(X)$ which is linear (with respect to the natural vector space structure on $C^*(A)$ and $C^*(X)$) and norm preserving (with respect to the sup norm on $C^*(A)$ and $C^*(X)$). J. Dugundji improved Borsuk's result in three ways [7]. He dropped the hypothesis of separability from the closed subset A, he considered all continuous functions, and his extender preserved convex hulls.

Definition 1.1. A space X is said to have the *Dugundji extension property* if for every closed subspace A of X there is a linear extender $\Phi : C(A) \to C(X)$ such that for each $f \in C(A)$, the range of $\Phi(f)$ is contained in the convex hull of the range of f.

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Thus, we can state succinctly Dugundji's theorem: all metric spaces have the Dugundji extension property. It is known that the Dugundji extension property holds in several classes of generalized metric spaces (see [1], [4], and [12]).

In this paper we give conditions under which certain normal spaces fail to have the Dugundji extension property (Theorem 1.4). In fact, we consider the following weaker extension property.

DEFINITION 1.2. If for each closed subspace A of a space X there is an extender Φ such that $\Phi(f) \leq \Phi(g)$ whenever $f \leq g \in C(A)$, then X is said to satisfy the *unbounded monotone extension property.*

For the definitions of basic terms see [9]. In our theorem, we use the following definition.

DEFINITION 1.3. Let $A \subseteq X$. A family $\mathcal{B}$ of open subsets of X is called a *total-π-base for A in X* provided that

(i) each element of $\mathcal{B}$ has non-empty intersection with A,

(ii) each non-empty open set which contains a point of A contains a member of $\mathcal{B}$ and

(iii) if $B_0 \supseteq B_1 \supseteq B_2 \supseteq \ldots$ is a countable decreasing chain of elements of $\mathcal{B}$ then $\bigcap_{n\in\omega} B_n \neq \emptyset$.

THEOREM 1.4. *If A is a first category, closed subset of a normal space X, and there exists a total-π-base $\mathcal{B}$ for A in X, then there does not exist an extender $\Phi : C(A) \to C(X)$ satisfying the conditions for the unbounded monotone extension property.*

Easy applications of Theorem 1.4 show that certain ω_1-metrizable spaces do not satisfy the Dugundji extension property (see Examples 2.2 and 2.3). These examples have several interesting consequences. They show that a result of Eric van Douwen in his thesis [6, Theorem 7, p. 52; p. 58, line 2] and a result of C. J. R. Borges [2, Theorem 2.1] are false. The present paper resulted from the discovery by Stares that the proof by Borges has a gap [12]. In addition, our examples answer a question raised in 1972 by Vaughan [13, p. 264], and give another interesting difference between the classes of strongly zero-dimensional metric spaces and ω_μ-metrizable spaces. Indeed, R. Engelking [8] proved that in a strongly zero-dimensional metric space, every closed subset is a retract, a stronger property than the Dugundji extension property.

2. Proof and examples

Proof of Theorem 1.4. By way of contradiction, assume such an extender $\Phi : C(A) \to C(X)$ exists. Since A is first category there exist open sets E_i in X for $i \in \omega$ such that $A \cap E_i$ is dense in A and $A \cap (\bigcap_i E_i) = \emptyset$. We construct, by induction, $B_i \in \mathcal{B}$, continuous functions $f_i : X \to [0,\infty)$, and open sets $M_i = \{x \in X : \Phi(f_i\restriction A)(x) > i\}$ satisfying the following properties: B_0 is any member of $\mathcal{B}$ such that $B_0 \subseteq E_0$ and f_0 is the constant function with value 1, and the following hold for $i \geq 1$:

(1) $\mathrm{cl}_X(B_i) \subseteq B_{i-1} \cap M_{i-1} \cap E_i$,
(2) $f_i(\mathrm{cl}_X(B_i)) = i+1$, and $f_i(X \setminus B_{i-1}) = 0$.

We construct B_n and f_n. By (2), $B_{n-1} \cap A \subseteq M_{n-1}$, hence $B_{n-1} \cap M_{n-1} \cap A$ is a non-empty open subset of A, so by denseness $B_{n-1} \cap M_{n-1} \cap E_n$ contains a point of A. By regularity of X, we may pick $B_n \in \mathcal{B}$ such that

$$\mathrm{cl}_X(B_n) \subseteq B_{n-1} \cap M_{n-1} \cap E_n.$$

It is then clear that (1) holds for B_n. By the Tietze–Urysohn theorem there exists a continuous $f_n : X \to [0,\infty)$ such that $f_n(\mathrm{cl}_X(B_n)) = n+1$, and $f_n(X \setminus B_{n-1}) = 0$; so now (2) is also satisfied. This completes the induction.

Define $h : A \to \mathbb{R}$ by $h = \sum_{i=1}^{\infty} f_i\restriction A$. To see that this infinite sum is well-defined and continuous on A, note that for any $a \in A$ there exists n such that $a \notin E_n \supseteq \mathrm{cl}_X(B_n)$; so $(X \setminus \mathrm{cl}_X(B_n))$ is a neighborhood of a on which $h = \sum_{i=1}^{n} f_i\restriction A$, a finite sum of continuous functions. By the hypothesis on $\mathcal{B}$, there exists $y \in \bigcap\{B_i : i \in \omega\}$. Now pick an integer m such that $\Phi(h)(y) < m$. By the monotonicity of Φ, we have $\Phi(f_m\restriction A)(y) \leq \Phi(h)(y)$. By (1), $y \in \bigcap\{M_i : i \in \omega\} \subseteq M_m$, so we have the contradiction

$$m < \Phi(f_m\restriction A)(y) \leq \Phi(h)(y) < m. \ \blacksquare$$

We take the following as the definition of ω_μ-metrizability (see [11] for a list of references concerning ω_μ-metrizability).

DEFINITION 2.1. If ω_μ is a regular, uncountable cardinal then a space X is said to be *ω_μ-metrizable* if there is a collection $\{\mathcal{U}_\alpha : \alpha < \omega_\mu\}$ where each $\mathcal{U}_\alpha$ is a pairwise disjoint open cover of X, $\mathcal{U}_\alpha$ refines $\mathcal{U}_\beta$ if $\alpha > \beta$, and $\bigcup\{\mathcal{U}_\alpha : \alpha < \omega_\mu\}$ is a base for X.

We now present three examples. A simple application of Theorem 1.4 shows that each example fails to have the Dugundji extension property. The first two examples are both ω_1-metrizable and hence give the result stated in the title (among other things). The third example, which is not ω_μ-metrizable, is the well-known Michael line [9, 5.1.32]. That the Michael line does not have the Dugundji extension property was shown in [6] and [10]. Our theorem gives a simpler proof of this, and stems from attempts to generalize van Douwen's proof [6] that the Michael line does not satisfy the unbounded monotone extension property.

EXAMPLE 2.2. *The set 2^{ω_1} with the countable box topology is an ω_1-metrizable topological group without the Dugundji extension property.*

Let X denote the countable box topology on the set 2^{ω_1}. This is the topology having as a base all sets of the form $[x\restriction\alpha]$ for $\alpha < \omega_1$ and $x \in X$, where $[x\restriction\alpha] = \{y \in X : y(\beta) = x(\beta) \text{ for all } \beta < \alpha\}$. Let $\mathcal{U}_\alpha = \{[x\restriction\alpha] : x \in X\}$ for $\alpha < \omega_1$. It is clear that these $\mathcal{U}_\alpha$ satisfy the conditions of Definition 2.1. The subset A is defined to be the set of functions with cofinite support, i.e.,

$$A = \{x \in X : |\{\alpha < \omega_1 : x(\alpha) = 0\}| < \omega\}.$$

Obviously, A is closed. To check that A is a first category set we define open sets E_n for $n \in \omega$ by $E_n = \{x \in X : |\{\alpha < \omega_1 : x(\alpha) = 0\}| > n\}$. Each E_n is open and $A \cap E_n$ is dense in A for all $n \in \omega$. Since any point in $\bigcap_{n\in\omega} E_n$ must have infinitely many coordinates equal to 0, we have $A \cap \bigcap_{n\in\omega} E_n = \emptyset$.

To complete the example we need a total-π-base $\mathcal{B}$ for A in X. Let $\mathcal{B}$ be all the basic open sets in X which have non-empty intersection with A (thus $\mathcal{B}$ is a *clopen base for A in X*). If $B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$ is a decreasing chain of elements of $\mathcal{B}$ then it is clear that $\bigcap_n B_n \neq \emptyset$. All the hypotheses in Theorem 1.4 are satisfied and therefore, X does not satisfy the unbounded monotone extension property and hence, does not have the Dugundji extension property.

The space X in the above example is the special case $\mu = 1$ of the spaces $(2^{\omega_\mu})_{\omega_\mu}$, for ω_μ a regular uncountable cardinal. These spaces are known to be ω_μ-metrizable [5, p. 384]. In the same way as in the above example we can show that $(2^{\omega_\mu})_{\omega_\mu}$ does not have the Dugundji extension property. Thus for *every* regular, uncountable ω_μ, there exists an ω_μ-metrizable space which does not have the unbounded monotone, or Dugundji, extension property.

EXAMPLE 2.3. *The countable product $L(\omega_1)^\omega$ with the box topology does not have the Dugundji extension property. Thus a box product of a countable family of ω_1-metrizable spaces*, *each having the Dugundji extension property*, *need not have the Dugundji extension property.*

By $L(\omega_1)$ we mean the space derived from the space ω_1+1 with the usual order topology by isolating all the points except the point ω_1. By setting $\mathcal{U}_\alpha = \{(\alpha, \omega_1]\} \cup \{\{\beta\} : \beta \leq \alpha\}$ for $\alpha < \omega_1$ we get collections satisfying the conditions of Definition 2.1 and therefore $L(\omega_1)$ is ω_1-metrizable. Our example X will be the product of countably many copies of $L(\omega_1)$ with the box topology. By [14, Theorem 2.9], X is also ω_1-metrizable. The subspace A is defined by $A = \{x \in X : |\{i < \omega : x(i) \neq \omega_1\}| < \omega\}$. As before, A is a closed first category set. The total-π-base $\mathcal{B}$ is, as in the previous example, the collection of all basic open sets in X which have non-empty intersection with A. By Theorem 1.4, X does not have the Dugundji extension property.

It is easy to show that $L(\omega_1)$ has the Dugundji extension property. In fact, if a space has at most one isolated point then a linear extender Φ can be found which is range preserving.

EXAMPLE 2.4 (Heath and Lutzer [10], van Douwen [6]). *The Michael line* $\mathbb{M}$ *does not have the Dugundji extension property.*

Recall that the *Michael line* $\mathbb{M}$ is the set $\mathbb{R}$ of real numbers with the topology obtained by starting with the usual topology and declaring all irrational points to be isolated [9, 5.1.32]. Take the closed set A to be the set of rational numbers in $\mathbb{M}$, and for $\mathcal{B}$ take a base of intervals for the rational numbers with the following properties: $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \omega\}$ such that

(1) each $\mathcal{B}_n$ is a countable family of pairwise disjoint open intervals of length at most $1/(n+1)$ covering the rational numbers,

(2) the closure (in $\mathbb{R}$) of each interval in $\mathcal{B}_{n+1}$ is contained in some interval in $\mathcal{B}_n$.

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