gehören sie zu verschiedenen Komponenten von $M^{(n)}$, woraus hervorgeht, dass die letzteren mit den Teilstrecken $S^{(n)}_{y,z}$ identisch sind. Nun stimmt aber bekanntlich die Dimension einer kompakten Menge mit der grössten unter den Dimensionen ihrer Komponenten überein [7]), und daraus folgt: dim $M^{(n)} = 1$ $(n = 1, 2, \ldots)$.

Es gilt offenbar

$$M = N^{(1)} + N^{(2)} + \sum_{n=1}^{\infty} M^{(n)},$$

wo die Menge $N^{(1)}$ (bzw. $N^{(2)}$) aus den Punkten $(1, y, 0)$ $(y \in N)$ (bzw. $(0, 0, z)$ $(z \in N)$) gebildet ist. In dieser Zerlegung von M sind alle Summanden abgeschlossen; ferner sind die ersten zwei von ihnen nulldimensional, alle übrigen eindimensional, also ist nach dem „Summensatz" der Dimensionstheorie auch M eindimensional.

Wir behaupten jetzt: *Zu jedem kompakten Raum R gibt es eine Teilmenge M_R der Kurve M und eine oberhalb stetige Zerlegung von M_R in Kontinua, welche einen mit R homöomorphen Hyperraum liefert.*

Bekanntlich ist es bei jedem vorgegebenen kompakten Raum R möglich die Cantorsche Menge N eindeutig und stetig auf R abzubilden [8]). Wir denken uns eine derartige Abbildung f zwischen N und R hergestellt. Für einen Punkt p von R sei N_p die Menge der Punkte von N, die ihn als Bild haben. Für jedes $p \in R$ setzen wir:

$$M_p = \Sigma\, S_{y,z} \quad (y, z \in N_p)$$

M_p ist ein *Teilkontinuum* der Kurve M, und für $p \neq q$ ist $M_p . M_q = 0$. Die Menge $M_R = \Sigma\, M_p$ (Summierung erstreckt über alle $p \in R$) besteht aus allen Strecken $S_{y,z}$ mit $f(y) = f(z)$ und ist offenbar abgeschlossen. Die Kontinua M_p ergeben eine oberhalb stetige Zerlegung von M_R mit zu R homöomorphem Hyperraum womit die eben ausgesprochene Behauptung bewiesen ist. Bemerken wir noch, dass für einen zusammenhängenden Raum R M_R notwendig ein *Kontinuum* ist wie aus einem allgemeinen Satze von Kuratowski hervorgeht [9]) [10]).

[7]) Vgl. Tumarkin, Amst. Proc. 28 (1925), S. 1000, wo der Satz zum ersten Mal ausgesprochen wurde. Er ist auch in dem sub [6]) zitierten Theorem enthalten.

[8]) Vgl. Hausdorff, Mengenlehre (1927), S. 197.

[9]) Kuratowski, Fund. Math. 11, S. 182 (Corollaire 1).

[10]) (Zusatz während der Korrektur). Dadurch ist u. a. gezeigt, dass jedes Kontinuum *stetiges Bild eines eindimensionalen Kontinuums* ist. Dasselbe Resultat hat durch eine ähnliche Konstruktion Mazurkiewicz erhalten und es auf dem im Semptember 1929 in Warschau abgehaltenen I Kongress der Mathematiker der slawischen Länder mitgeteilt.

On Functions Possessing Differentials.

By

Grace Chisholm Young (La Conversion, Switzerland).

Introduction.

The present paper is the outcome of a renewed effort to complete the solution of the problem, attacked by my husband in 1912 [1]), a subject to which we have returned from time to time, but without obtaining any result we thought worthy of publication, — and re-considered by Pollard in 1921 [2]), the problem namely of proving by the methods proper to the Theory of Functions of Two Real Variables, the fundamental theorem in the Theory of Functions of a Complex Variable (Goursat's Theorem), which asserts that the necessary and sufficient condition that w should be a analytic function of z is that $\frac{dw}{dz}$ should exist.

There is no difficulty in expressing this enunciation in terms of two real variables, and indeed, though first clearly brought out by Goursat, this was certainly in the mind of Riemann; the theorem is that

$$w = u(x, y) + i\, v(x, y)$$

involving $i = \sqrt{-1}$, is expressible in the neighbourhood of a point (X, Y) in the form of a power series in $(x - X) + i(y - Y)$, if, and only if, the ratio

$$\Delta w / \Delta z = (u(X + h,\ Y + k) - u(X, Y) + \\ + i\, v(X + h,\ Y + k) - i\, v(X, Y)) / (h + ik)$$

[1]) W. H. Young. „On the Fundamental Theorem in the Theory of Functions of a Complex Variable", (1912), Proc. London M. S., Ser. 2, Vol. 10, pp. 1—6.

[2]) S. Pollard. „On the Conditions for Cauchy's Theorem", (1921), ibid. Vol. 21, pp. 456—482.

has, as $(h, k) \to (0, 0)$, an unique and finite limit $A + iB$, independent of the mode of approach of h and k to zero.

The enunciation at once suggests a query: — What are the necessary and sufficient conditions in order that the limit in question may be unique? This leads to another fundamental theorem, forming with the other the requisite links between the Theory of Functions of a Complex Variable and that of Two Real Variables. This theorem states that the limit is unique, if, and only if, $u(x, y)$ and $v(x, y)$ possess first differentials at (X, Y), and their partial differential coefficients satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The proof of the former theorem (Goursat's theorem) supplied by W. H. Young, and reproduced below in § 17, requires the assumption that the partial differential coefficients of $u(x, y)$ and $v(x, y)$ should be known, *a priori*, to be bounded functions of (x, y), a piece of information which ought only to come, *a posteriori*, as a consequence of the theorem. What I have been able to supply is the proof that in the case of a pair of functions $u(x, y)$ and $v(x, y)$ about which we *do not* know this, we may reduce the problem to that of functions $U(x, y)$ and $V(x, y)$, about which we *do* know it; functions indeed of the typically primeval type, those only considered by Cauchy and Riemann and by the physicists and other scientists, economists and statisticians, of today, innocent functions in fact with continuous partial differential coefficients.

It had seemed to me that our progress was being stopped by a want of clear knowledge of differentials, and I determined to go back to first principles, and patiently investigate the obscure corners in the theory of differentials. The result appears to justify the doubts I felt as to the adequacy of our grasp of this subject. By obtaining *the necessary and sufficient conditions that $f(x, y)$ should possess a first differential*, we render the definition more strictly mathematical, and therefore more useful; but, more than this, we also obtain new and powerful consequences of the possession of a differential. In particular we have the theorem that, *if $f(x, y)$ has a first differential at every point (x, Y) of a linear neighbourhood containing both (X, Y) and (a, Y), $\frac{\partial f}{\partial y}$ is a summable function of x, and differentiation under the integral sign is allowable, that is, writing*

$$F(x, y) = \int_a^x f(x, y)\, dx,$$

we have

$$\frac{\partial F}{\partial y} = \int_a^x \frac{\partial f}{\partial y}\, dx;$$

also $F(x, y)$, like $f(x, y)$, has a first differential at each (x, Y). Finally the double incrementary ratio of $F(x, y)$, namely

$$(F(x+h,\ Y+k) - F(x,\ Y+k) - F(x+h,\ Y) + F(x,\ Y))/hk$$

has $\frac{\partial f}{\partial y}$ for unique limit when $(h, k) \to (0, 0)$, in any manner, except when k/h has zero for a limit; and, in all manners, k times the double incrementary ratio tends to zero.

This result should be compared with the so-called Fundamental Theorem of Differentials [1]) in which no assumption is made as to the existence of a differential except at the point (X, Y) itself, and which states that, *if $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ both have differentials at (X, Y) the double incrementary ratio of $F(X, Y)$ has an unique and finite limit independent of the mode of approach of (h, k) to $(0, 0)$, and that, consequently, at (X, Y)*,

$$\frac{\partial}{\partial x}\cdot\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\cdot\frac{\partial F}{\partial x}.$$

PART I.

Necessary and Sufficient Conditions for the Existence of a Differential.

1. The definition of *the differential coefficient* of a function $f(x)$ of a single variable contains in itself a perfectly satisfactory statement of the necessary and sufficient condition for the existence of

[1]) W. H. Young, „On Differentials", (1908), Proc. L. M. S., Ser. 2, Vol. 7. p. 162.

that differential coefficient at a particular point x, namely the condition that the incrementary ratio

$$\{f(x+h)-f(x)\}/h$$

should tend to an unique limit as $h \to 0$ in any manner whatever. If we admit only finite differential coefficients, we may express this condition by saying that an equation holds of the form

$$f(x+h)-f(x)=h(A+e),$$

where A is a constant, determined by x alone, and e is a function of h, determined by x, which tends to zero as $h \to 0$.

In the case of a function $f(x, y)$ of two variables the *differential* (first differential) plays in many ways the rôle of the differential coefficient in one dimension. But the definition of the differential does not yield at once convenient necessary and sufficient conditions for the existence of the differential at a particular point (x, y). The *necessary* condition that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ should exist has been, of course, pointed out; and conditions which are sufficient, but not necessary, have been formulated and used. The necessary and sufficient conditions given below, (Theorem § 5), reduce the question to one of the uniform convergence of certain functions of a single variable with a single parameter. In this form the conditions are convenient of application, the subject of such uniform convergence having been thoroughly cleared up.

We may, if we prefer, state the necessary and sufficient conditions for the existence of a differential at a point (x, y), where $f(x, y)$ has finite partial differential coefficients $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, in an alternative form more closely analogous to the condition for the existence of a finite differential coefficient in one dimension. The conditions are that *the incrementary ratios obtained by dividing the increment*

$$f(x+h, y+k)-f(x+h, y)-f(x, y+k)+f(x, y)$$

by h and k respectively, both tend to zero as h and k each tend to zero, provided in the former case we omit all modes of approach to zero which make h/k tend to zero, and in the latter case those which make k/h tend to zero.

2. The definition of a first differential is usually given in the folowing form:

A function $f(x, y)$ of two real variables x and y is said to possess, or have, a first differential, or shortly a differential, at the point (x, y), if

(1) $$f(x+h, y+k)-f(x, y)=h(A+e')+k(B+e^*),$$

provided A and B are constants, determined by (x, y), and e' and e^* both tend to zero, when (h, k) approaches $(0, 0)$ in any manner whatever.

When this is verified, the differential is said to „exist“, and its value is then $hA+kB$.

This definition, though logically satisfactory, leaves, by reason of its terseness, something to be desired in respect of mathematical lucidity and applicability. The symbols e' and e^*, introduced without further elucidation, what kind of mathematical entities are they? It is not immediately evident what is their scope, nor whether their definition, when supplied, will be such as all mathematicians would accept as adequate, or whether it would require an infinite number of specifications, that is, of so-called acts of choice.

In practice it is all the more necessary to clear up these points, because we commonly require to make, not only h and k, but also x and y, variable, and to apply processes, such as integration, for which the information conveyed by the definition is inadequate.

3. That e' and e^*, unlike A and B, depend on h, k, as well as on x, y, is evident. It is equally evident that, for all the uniformation conveyed by the definition, they are not determined by (h, k), when (x, y) is fixed, as we shall, for the present, assume to be the case. Indeed, when h and k denote fixed values, either of the pair (e', e^*), say e', may be given any finite value at will, the other, e^*, is then uniquely determined by (1), provided k is not zero, and, if k is zero, may itself be assigned any finite value whatever.

When, however, h and k denote variables, though either e' or e^* may be chosen arbitrarily at any one point, when we try to choose the values of one of them, say e', arbitrarily *in the lump*, we find this is not possible, because of the restriction, imposed by the definition, that e' tends to zero. It is not asserted, or required, by the definition, that the mode in which e' tends to zero should be definable in terms of (h, k), without involving the chosen mode in which (h, k) tends to $(0, 0)$: it is only required that, *for each chosen mode of approach of* (h, k) *to* $(0, 0)$, there is, at least theoretically, a mode of approach to zero which may be taken for that of e', and the equation (1) will then define e^* uniquely in terms of (h, k) and the corresponding e', in such a way that e^* also tends to zero. But even now it is not asserted that e' is *uniquely* determined by the particular mode of approach to $(0, 0)$ assigned to (h, k). We cannot therefore regard e' and e^* as „functions" of (h, k), even it we do not inflict on the term „function" the fetters of single-valuedness, since the values of a function of (h, k) at a set of points are determinate, even when multiple, at each of those points, and do not depend on the choice of that set. Thus, if it were conceived possible to give a many-valued function of (h, k) capable of serving all our purposes, we should have to add a *law* corresponding to each mode of approach of (h, k) to $(0, 0)$, so as to specify how e' was to be chosen from among its many values, for each particular (h, k) involved in that mode of approach; moreover, the function would have to be such that, at any particular (h, k), it had every conceivable value!

Thus e' and e^* in the definition cannot as such be said to be functions, even multi-valued functions, of (h, k).

The dependence of e' and e^* on the mode of approach of (h, k) to $(0, 0)$ requires of itself some device in order to keep it before our eyes. Regarding h and k as the rectangular coordinates of a point P in a plane, any mode of approach of (h, k) to $(0, 0)$ may be considered as implying that the point P *moves*, not necessarily continuously, so as ultimately to *come to rest* at the origin 0. If the motion is discontinuous, the simplest will be that by which P hops from point to point of an ordered sequence of points $P_1, P_2, \ldots$ Sometimes P will approach 0, sometimes recede, and sometimes remain at a constant distance from 0. All we can assert is that, sooner or later, it will get inside, and remain inside, any circle with 0 as centre, however small. This is equally true if the motion is continuous. The simplest case is now that of motion along a straight line, or monotone curve, through the origin, always in the same sense. But P might move, sometimes in one sense, sometimes in the other; and the path might be excessively complicated and curly. A very simple case would be motion along an Archimedean spiral

with the origin as pole. But the path might pass more than once through the origin before the point comes to rest, as for instance the pen passes twice over the same point in forming the sign &.

We will therefore, as a reminder of the possibilities, adopt the written symbol, &, to express the notion conveyed by the phrase „any mode of approach of (h, k) to $(0, 0)$". The generic point which moves in the determinate mode &, we shall then, when desirable, denote by $(h, k, \&)$.

5. Returning now to our definition, we see that it implies that for each $(h, k; \&)$ there exists a pair of theoretically determinate sets of values which we shall denote by $E'(h, k, \&)$ and $E^*(h, k, \&)$, consisting of values $e'(h, k; \&)$ and $e^*(h, k; \&)$ respectively, connected linearly by the relation (1), and that neither of these sets is empty, while they both shrink up to the value zero, as the point $(h, k, \&)$ progresses towards the origin $(0, 0)$.

Geometrically, if at each point $(h, k; \&)$ we erect an ordinate perpendicular to the plane of (h, k), and on it mark the sets E' and E^*, placing the zero point at $(h, k; \&)$, this ordinate, cut off at the two determinate points rendering it as short as possible, consistent with just containing both the sets, shrinks up to the origin $(0, 0)$, as the point $(h, k; \&)$ moves up to that origin.

At any point (h, k) the different ordinates, corresponding to *all* possible &'s, and *all* possible theoretical specifications of the corresponding sets $E'(h, k; \&)$, $E^*(h, k; \&)$ will completely cover the whole ordinate line, infinite in both directions.

6. These considerations shew that, for all that the definition logically requires, the symbols e' and e^* are no elementary mathematical functional symbols. It will be now our task to show that it is unnecessary to retain the definition in this general form, and that, on the contrary, *it is allowable to regard e' and e^* in the definition as single-valued functions of (h, k), mathematically definable.*

The justification of this statement lies in the following theorem:

Theorem. *The necessary and sufficient conditions that a function $f(x, y)$ of two real variables, x and y, should have a first differential at the point (x, y) may be given in the following form:*

(*i*) (*Primary, or Obvious, Condition*).

The partial differential coefficients $\frac{\partial f}{\partial x}$ *and* $\frac{\partial f}{\partial y}$ *exist and are finite at the point* (x, y);

(*ii*) (*Secondary Condition*).

The functions $\varphi_h(t)$ *and* $\psi_k(\tau)$, *each with a single real variable and a single real continuous parameter, being defined by the equations:*

$$\varphi_h(t) = (f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y))/h$$

$$\varphi_h(0) = 0; \quad \varphi_h(\infty) = 0; \quad \varphi_0(t) = 0;$$

where

$$k = ht, \quad (0 < |h| < \infty; \quad 0 < |t| < \infty),$$

and $\psi_k(\tau)$, *got from* $\varphi_h(t)$ *by changing* t *into* τ, *and interchanging* h *and* k, (*except in* $f(x+h, y+k)$, $f(x+h, y)$ *and* $f(x, y+k)$), *these functions converge uniformly to zero in every finite interval of the variable, as the parameter tends to zero.*

To prove *the necessity of the conditions*, we start with the defining equation (1), in which A and B are given finite constants, and $e' = e'(h, k; \&)$ and $e^* = e^*(h, k; \&)$ depend on (h, k) and also on the particular mode of approach selected and denoted by &, and are such that, whatever mode of approach be selected, $e' \to 0$, $e^* \to 0$, when $(h, k; \&) \to (0, 0)$.

First to prove *the necessity of the Primary Condition*, we take the symbol & to denote any mode of approach for which $h \gtrless 0$, $k = 0$, and we get, dividing (1), as we then may, by h,

$$e' = (f(x+h, y) - f(x, y))/h - A = e(h, 0), \tag{2}$$

say, this being independent of the mode of approach. Since, by hypothesis $e' \to 0$, as $h \to 0$, and A is a given constant, this proves that $\frac{\partial f}{\partial x}$ exists and is finite, that, in fact,

$$\frac{\partial f}{\partial x} = A. \tag{3}$$

Similarly, taking the symbol & to denote any mode of approach for which $h = 0$, $k \gtrless 0$, we have,

$$e^* = (f(x, y+k) - f(x, y))/k - B = e^*(0, k), \tag{5}$$

say, and therefore

$$\frac{\partial f}{\partial y} = B. \tag{5}$$

This proves *the necessity of the Primary Condition*.

Next to prove *the necessity of the Secondary Condition*, we define a variable t in terms of the pair of variables h and k by

$$t = k/h, \tag{6}$$

for all finite values of h and k, except when h and k are simultaneously zero. Then, if $t \to \bar{t}$, where $\bar{t}$ is finite, $h \to 0$, $k = ht$ will $\to 0$.

We may therefore denote by the symbol & any mode of approach of (h, k) to $(0, 0)$ by which $k/h \to \bar{t}$, where $\bar{t}$ is any selected finite value, and the mode of approach to $\bar{t}$ is any prescribed mode.

We then write,

$$\varphi_h(t) = \{f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y)\}/h, \quad (h \gtrless 0, \ k \gtrless 0),$$

$$\varphi_h(0) = 0, \quad \varphi_h(\infty) = 0, \quad \varphi_0(t) = 0.$$

These equations show at once that the appearance of zero or infinite values for t during the approach of (h, k) to $(0, 0)$ will not disturb the existence of the unique limit zero for $\varphi_h(t)$; we need therefore only consider the first expression, which, since h is finite and not zero, may be written as follows:

$$\begin{aligned}\varphi_h(t) &= \{f(x+h, y+k) - f(x, y)\}/h - \{f(x+h, y) - f(x, y)\}/h \\ &\qquad - k/h.\{f(x, y+k) - f(x, y)\}/k \\ &= \{A + e^* + (B + e^*)t\} - \{A + e'(h, 0)\} - \{B + e^*(0, k)\}t.\end{aligned}$$

Since, by hypothesis, e', e^*, $e'(h, 0)$ and $e^*(0, k)$ all tend to zero, as $(h, k; \&) \to (0, 0)$, and since $t \to \bar{t}$, which is finite, this shows that $\varphi_h(t) \to 0$. In other words, since the approach of t to $\bar{t}$ was any prescribed mode, it shown that $\varphi_h(t)$ tends uniformly to zero as $h \to 0$, at every finite point $\bar{t}$, and therefore in every finite interval of values of t.

Interchanging the rôles of h and k, and writing

$$\psi_k(\tau) = \{f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y)\}k$$

$$\psi_k(0) = 0; \quad \psi_k(\infty) = 0; \quad \psi_0(\tau) = 0;$$

where

$$\tau = h/k,$$

the same reasoning shows that $\psi_k(\tau)$ converges uniformly to zero as $k \to 0$, in every finite interval of values of τ.

The necessity of both the conditions, primary and secondary, has thus been completely proved.

Next to prove *the sufficiency of the conditions*, we start by hypothecating the validity of the primary and secondary conditions of our theorem, and we now have to deduce as a consequence an equation of the form (1), with a proper definition of the constants A and B, and of the symbols e' and e^*.

Since the primary condition holds, we may assign to A and B definite and finite values as follows:

(9) $$A = \frac{\partial f}{\partial x}; \quad B = \frac{\partial f}{\partial y}.$$

Let now the symbol & denote any chosen mode of approach of a pair of variables (h, k) to $(0, 0)$. Then the generic point $(h, k; \&)$ may assume the position $(0, 0)$; but in this case the equation (1) is identically satisfied, so that, when

(X) $$h = 0, \quad k = 0,$$

we take

(10) $$e'(h, k) = e'(0, 0) = 0; \quad e^*(h, k) = e^*(0, 0) = 0,$$

and the formal equation (1) is satisfiied, with

$$e' = e'(h, k), \quad e^* = e^*(h, k).$$

The generic point $(h, k, \&)$ may assume a position $(h, 0)$, other than $(0, 0)$. In this case we may, by (9), write

$$f(x+h, y+k) - f(x, y) = h\left[A - \frac{\partial f}{\partial x} + \{f(x+h, y) - f(x, y)\}/h\right],$$

so that, when

(XI) $$h \gtrless 0, \quad k = 0,$$

we take

(11) $$\begin{cases} e'(h, k) = e'(h, 0) = -\dfrac{\partial f}{\partial x} + \{f(x+h, y) - f(x, y)\}/h, \\ e^*(h, k) = e^*(h, 0) = 0. \end{cases}$$

Similarly, when

(XII) $$h = 0, \quad k \gtrless 0,$$

we take

(12) $$\begin{cases} e'(h, k) = e'(0, k) = 0, \\ e^*(h, k) = e^*(0, k) = -\dfrac{\partial f}{\partial y} + \{f(x, y+k) - f(x, y)\}/k. \end{cases}$$

Finally, $(h, k, \&)$ having any position other than those considered, we use the identity

(13) $$\begin{cases} f(x+h, y+k) - f(x, y) = \\ \quad = f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y) \\ \quad + \{f(x+h, y) - f(x, y)\} + \{f(x, y+k) - f(x, y)\} \end{cases}$$

which, writing

14) $$t = k/h = 1/\tau$$

and taking $\varphi_h(t)$ and $\psi_k(\tau)$ to be defined as in the enunciation of our theorem, may be written in either of the following forms, since neither h nor k is now zero:

(15) $$\begin{cases} f(x+h, y+k) - f(x, y) = h\varphi_h(t) + \\ \quad + h\left[A - \dfrac{\partial f}{\partial x} + \{f(x+h, y) - f(x, y)\}/h\right] \\ \quad + k\left[B - \dfrac{\partial f}{\partial y} + \{f(x, y+k) - f(x, y)\}/k\right] \end{cases}$$

or

(16) $$\begin{cases} f(x+h, y+k) - f(x, y) = k\psi_k(\tau) + \\ \quad + h\left[A - \dfrac{\partial f}{\partial x} + \{f(x+h, y) - f(x, y)\}/h\right] \\ \quad + k\left[B - \dfrac{\partial f}{\partial y} + \{f(x+h, y) - f(x, y)\}/k\right]. \end{cases}$$

If $(h, k; \&)$ is such that

(XVII) $$0 < |k| \leqq |h|,$$

we take

(17) $$\begin{cases} e'(h, k) = \varphi_h(t) - \dfrac{\partial f}{\partial x} + \{f(x+h, y) - f(x, y)\}/h, \\ e^*(h, k) = -\dfrac{\partial f}{\partial y} + \{f(x, y+k) - f(x, y)\}/k, \end{cases}$$

using (15); while, if

(XVIII) $$0 < |h| < |k|,$$

we take

$$(18)\quad \begin{cases} e'(h,k) = -\dfrac{\partial f}{\partial x} + \{f(x+h,y) - f(x,y)\}/h, \\[2ex] e^*(h,k) = \psi_k(\tau) - \dfrac{\partial f}{\partial y} + \{f(x,y+k) - f(x,y)\}/k, \end{cases}$$

using (16). In both cases we see that the formal equation (1) is satisfied, with $e' = e'(h,k)$, $e^* = e^*(h,k)$.

We have thus proved that the functions $e'(h,k)$ and $e^*(h,k)$, defined by the equations (10), (11), (12), (17) and (18) according as (h,k) satisfies the appropriate relation (X), (XI), (XII), (XVII) or (XVIII), when substituted respectively for e' and e^* in the formal equation (1), render it valid, whatever mode of approach, denoted by the symbol &, be adopted. But we then have $e' \to 0$, $e^* \to 0$, as $(h, k, \&) \to (0,0)$; for the expressions on the right of the formulae (11) and (12) tend to zero by the definitions of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$, and these expressions appear again in (17) and (18); the remaining terms in (17) and (18) also tend to zero, by the secondary condition, since, in (17) $|t| \leqslant 1$, and, in (18), $|\tau| \leqslant 1$. Thus, however $(h, k; \&) \to (0,0)$, all the expressions which from point to point define $e'(h,k)$ and $e^*(h,k)$ tend to zero. We have thus proved the *sufficiency* of our conditions.

The whole theorem is therefore proved.

Corollary. *If $f(x,y)$ is a continuous function of (x,y), we may, instead of introducing the function $\psi_k(\tau)$, say that $\varphi_h(t)/t$ converges uniformly to zero in any interval of values of t, not including zero, but including, if desired, one or both of the infinite values.*

Indeed, by their definitions

$$\tau = 1/t,$$

$$\psi_k(\tau) = \varphi_h(t)/t.$$

Also when $t \to \bar{t}$, $\tau \to 1/\bar{t}$, and *vice versa*, when $\bar{t}$ is finite and not zero; and then $(h,k) \to (0,0)$, provided *either* $k \to 0$, *or* $h \to 0$. Thus the uniform convergence of $\varphi_h(t)/t$ is the same as the uniform convergence of $\psi_k(\tau)$. But, if $\bar{t}$ denotes an infinite value, and $1/\bar{t}$ the value zero, the uniform convergence of $\psi_k(\tau)$ to zero is insufficient to prove the uniform convergence of $\varphi_h(t)/t$ to zero, since there are modes of approach of (t,h) to $(\bar{t}, 0) = (\pm\infty, 0)$ which, since $k = ht$, do not imply that $k \to 0$. For such modes of approach however the numerator in the expression for $\psi_k(\tau) = \varphi_h(t)/t$ tends to zero in virtue of the continuity of $f(x,y)$, while the denominator k has no zero limit, thus these modes of approach also lead to zero as the unique limit of $\varphi_h(t)/t$.

This proves the Corollary. At the same time we see that *the continuity of $f(x,y)$ which we require is continuity with respect to (x,y) at all points $(x+\bar{h}, y)$ of some linear neighbourhood of the point (x,y).*

Similarly, if $f(x,y)$ is continuous with respect to (x,y) at all points $(x, y+\bar{k})$ of a certain linear neighbourhood of the point (x,y), we could replace the function $\varphi_h(t)$ in the secondary condition by $\psi_k(\tau)/\tau$.

In particular, since the equation (1) shows at once that the possessing of a differential implies that $f(x,y)$ is continuous with respect to (x,y) at the point (x,y) itself, we see that *the necessary and sufficient conditions that $f(x,y)$ should possess a first differential at every point (x,y) of a certain interval $\bar{b} < y < \bar{d}$, x constant, are that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ should exist and be finite at each such (x,y) and that $\varphi_h(t)$ and $\varphi_h(t)/t$ should converge uniformly to zero as $h \to 0$, in every interval of values of t which, in the case of $\varphi_h(t)$ does not exceed the finite, and in the case of $\varphi_h(t)/t$ does not contain the origin.*

7. It may be remarked, as not devoid of interest, that the particular functions $e'(h,k)$ and $e^*(h,k)$, defined by the formulae (10), (11), (12), (17) and (18), are not merely single-valued, but of almost elementary form. Each of them can be defined as the unique limit of a function

$$(20)\quad \begin{cases} e'(h,k) = \operatorname*{Lt}_{r\to 0} . q'_r(h,k), \\[1ex] e^*(h,k) = \operatorname*{Lt}_{r\to 0} . q^*_r(h,k), \end{cases}$$

which is continuous with respect to (h,k), except along the pair of straight lines

$$h = \pm k,$$

while on either of these lines, except at the origin, $(0,0)$, it is continuous on one side of the line, and has on the other side an unique limit.

We may, for instance, as is easily verified, replace the formulae defining $e'(h, k)$ and $e^*(h, k)$ by the above (20) and the following; when

(XXI) $$0 < |k| \leqslant |h|,$$

(21) $$\begin{cases} q'_r(h, k) = \{f(x+h, y+k) - f(x, y+k)\}/h \\ \qquad - \{f(x+rh, y) - f(x, y)\}/rh, \\ q^*_r(h, k) = \{f(x, y+k) - f(x, y)\}/k - \\ \qquad - \{f(x, y+rk) - f(x, y)\}/rk, \end{cases}$$

the former of these formulae still holding when $0 \leqslant k < |h|$, while, when

(XXII) $$0 < |h| < |k|,$$

(22) $$\begin{cases} q'_r(h, k) = \{f(x+h, y) - f(x, y)\}/h - \\ \qquad - \{f(x+rh, y) - f(x, y)\}/rh, \\ q^*_r(h, k) = \{f(x+h, y+k) - f(x+h, y)\}/k - \\ \qquad - \{f(x, y+rk) - f(x, y)\}/rk, \end{cases}$$

the latter of these formulae still holding when $0 \leqslant h < |k|$, and, finally

(23) $$q'_r(0, k) = 0, \quad q^*_r(h, = 0.$$

8. The following corollaries to Theorem 1 constitute tests for the existence of a differential of a convenient nature.

Corollary 2. *If the double incrementary ratio of $f(x, y)$ has an unique and finite mixed double limit, i. e. if*

$$\operatorname*{Lt}_{(h,k)\to(0,0)} \{f(x+h, y+k) - f(x+h, y) - f(x, y+k) + f(x, y)\}/hk = C,$$

where C is a finite quantity, then $f(x, y)$ has a first differential at (x, y), provided $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are finite; and then the Secondary Condition is satisfied without any restriction on the uniform convergence of $\varphi_k(t)$, which is true in the whole infinite interval $0 \leqslant |t| \leqslant \infty$.

The same is true if, without tending to an unique limit, the double incrementary ratio is bounded above and below.

Corollary 3. *If $f(x, y)$ has at (x, y) a first differential, and $p(x, y)$ is a bounded function of (x, y) possessing at (x, y) finite partial differential coefficients $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$, then $p(x, y) f(x, y)$ has at (x, y) a first differential.*

For the Primary Condition is satisfied, the partial differential coefficients being given by the usual formulae, and the Secondary Condition is clearly unaffected by multiplying $f(x, y)$ by a bounded function.

9. The special case of Theorem 1 in which $\varphi_h(t)$ converges uniformly in the whole closed infinite interval includes, not only the above Corollary 1, but also all the *sufficient tests* for the existence of a first differential which have hitherto been given. The most general of these, as formulated in Hobson's Treatise on the Theory of Functions of a Real Variable [1]), requires, in addition to the Obvious Condition, that

$$\{f(x+h, y+k) - f(x+h, y)\}/h$$

should be continuous with respect to (h, k) at $(h, k) = (0, 0)$. This is exactly equivalent to the special case of our conditions as given in Theorem 1.

In the special case, but not in the general case, the partial differential coefficient of $f(x, y)$ with respect to x, or y, (according as it is $\varphi_h(t)$ or $\psi_k(\tau)$ which converges uniformly everywhere, including infinity), has the property given in the following theorem.

Theorem 2. *If the conditions of Theorem 1 are satisfied without the restriction on the uniform convergence of $\varphi_h(t)$, the partial derivates of $f(x, y)$ with respect to x are continuous with respect to y at the point (x, y) under consideration.*

For the double limits of $\varphi_h(t)$ include every repeated limit, which in the special case is accordingly unique, and has the value zero. Let us form such a repeated limit by keeping k constant during the first passage to the limit, and making t move up towards $+\infty$, or $-\infty$, by non-zero values, in such a way that h, which is equal to k/t, moves up to zero, and describes such a sequence of

[1]) First Ed., (1907), pp. 312—313; Third Ed., (1927), pp. 418—419.

values as to give for the expression

$$\{f(x+h, y+k) - f(x, y+k)\}/h$$

as unique limit the value of any chosen derivate at $(x, y+k)$; this is possible by the meaning of the term „derivate". At the same time the expression

$$\{f(x+h, y) - f(x, y)\}/h$$

tends to $\frac{\partial f}{\partial x}$. The differen e of these two expressions being $\varphi_h(t)$ tends to zero during the second passage to the limit $k \to 0$. The derivate tends therefore to its own value at (x, y) as limit, in other words it is continuous with respect to y.

This proves the theorem. At the same time we see why the same cannot be asserted in the general, as here in the special, case, since in the general case the final result need not be zero. The following example shows a function $f(x, y)$ having a differential at the origin, and whose derivates, here differential coefficients, with respect to x, are not continuous with respect to y at the origin.

Ex. 1. Let us take a function which is zero on the axes and possesses a first differential at the origin. The primary condition is then satisfied, the partial differential coefficients being both zero at the origin; and we have, in order to satisfy the secondary condition, to make

$$f(h, k)/h = \varphi_h(t), \quad (k = ht, \ 0 < |h|),$$

and

$$f(h, k)/ = \psi_k(\tau), \quad (h = k\tau, \ 0 < |k|),$$

such functions of the single variable t, or τ, that they tend uniformly to zero for every finite value of the variable, as the parameter tends to zero. Taking a stock example, therefore, we write

$$\varphi_h(t) = (1 + akt)/(kt + 1/kt), \quad (k = ht),$$

and therefore

$$\psi_k(t) = (1 + ak/\tau)/(k/\tau + \tau/k).$$

Both these functions satisfy our requirements. Also

$$f(h, k) = h\varphi_h(t) = h(1 + ak^2/h)/(k^2/h + h/k^2) = \\ = hk^2(h + ak^2)/(h^2 + k^4),$$

so that we may put

$$f(x, y) = xy^2(x + ay^2)/(x^2 + x^4), \quad (xy \gtrless 0)$$

$$f(x, 0) = f(0, y) = f(0, 0) = 0.$$

Hence

$$\frac{\partial f}{\partial x}(0, 0) = \underset{h \to 0}{\mathrm{Lt}}.\, f(h, 0)/h = 0,$$

and

$$\frac{\partial f}{\partial y}(0, k) = \underset{h \to 0}{\mathrm{Lt}}.\, f(h, k)/h = a, \quad (k \gtrless 0)$$

shewing that $\frac{\partial f}{\partial x}$ is not continuous with respect to y at the origin.

10. The most familiar test for the existence of a first diffrential is that known as *Thomae's Test*. As it has been stated, this asserts that if $\frac{\partial f}{\partial x}\left(\text{or } \frac{\partial f}{\partial y}\right)$ exists throughout some plane neighbourhood of a point (x, y), and is continuous at (x, y) with respect to (x, y), while the other partial differential coefficient $\frac{\partial f}{\partial y}\left(\text{or } \frac{\partial f}{\partial x}\right)$ exists and is finite at the point (x, y) itself, then $f(x, y)$ has a first differential at (x, y).

This in an immediate corollary from our Theorem, using the Theorem of the Mean; for in this case we have

$$\varphi_h(t) = \frac{\partial f}{\partial x}(x + \theta h, \ y + k) - \frac{\partial f}{\partial x}(x, y), \quad (1 < \theta < 1).$$

But, as a matter of fact, it is unnecessary to hypothecate the existence of $\frac{\partial f}{\partial x}\left(\text{or } \frac{\partial f}{\partial y}\right)$ in any neighbourhood of (x, y); the statement of this simple test is then as follows:

If at a point (x, y) at which both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are finite, and the partial derivates of $f(x, y)$ with respect to x (or y) are known to be continuous with respect to (x, y), then $f(x, y)$ has a first differential at the point (x, y).

For the Theorem of the Mean for Derivates [1]) asserts that in

[1]) „Derivates and the Theorem of the Mean". Quarterly Journ. of Pure and Applied Math. XL, p. 10.

any given interval (a, b) there is a certain point, not one of the end-points, at which the upper derivate of $f(x)$ on one side, (say on the left), is less than or equal to the incrementary ratio

$$\{f(b)-f(a)\}/(b-a),$$

while this incrementary ratio is less than or equal to the lower derivate of $f(x, y)$ on the other side, (the right). Using f^+ and f_- to denote these derivates, this gives in our case,

$$f^+(x+\theta h, y+k)-\{f(x+h, y)-f(x, y)\}/h \leqslant \varphi_h(t)$$
$$\leqslant f_-(x+\theta h, y+k)-\{f(x+h, y)-f(x, y)\}/h.$$

Since, by hypothesis, $\dfrac{\partial f}{\partial x}$ exists at (x, y), $\{f(x+h, y)-f(x, y)\}/h$ tends to this as unique limit when $h \to 0$. Also since, by hypothesis, the upper and lower derivates of $f(x, y)$ are all continuous at (x, y), and $\theta h \to 0$, when $h \to 0$, the derivates which occur in our inequality also tend to $\dfrac{\partial f}{\partial x}$ as $h \to 0$, $k \to 0$. Therefore the extremes in our inequality tend to zero, and therefore also the middle term tends to zero uniformly. Hence, by our Theorem 1, the required result follows.

We may indeed, if we please, go still further, and only hypothecate, for instance, that the derivates with respect to x, instead of being continuous with respect to (x, y), should merely be bounded, and continuous for all modes of approach to the origin other than such as are tangential to the axis of y. The proof is identical with the preceding, but involves the general case of our Theorem 1, and not, as in the preceding, the special case when the convergence is uniform throughout.

11. In conclusion we will cite an example which does not satisfy the conditions of our Theorem 1.

Ex. 2. Let

$$f(x, y) = x \sin 1/y, \quad (y \gtrless 0),$$
$$f(x, 0) = 0.$$

Here

$$\varphi_h(t) = \sin 1/(y+k) - \sin 1/y, \quad (y \gtrless 0),$$

and

$$\varphi_h(t) = \sin 1/k, \quad (y = 0).$$

Thus for points (x, y) not lying on the axis of x, $\varphi_h(t)$ tends uniformly to zero without exception; therefore $f(x, y)$ has a first differential at each such point, since the partial differential coefficients both exists. But at a point other than the origin on the axis of x, the partial differential coefficient does exist, since

$$\{f(x, k)-f(x, 0)\}/k = \frac{x}{k} \sin 1/k,$$

has, as $k \to 0$, the two infinite limits $\pm\infty$. Thus the primary condition of our Theorem 1 is violated, and therefore $f(x, y)$ has not got a first differential.

Finally at the origin, the primary condition is verified, but the secondary condition is violated, since now $\varphi_h(t) = \sin 1/k$ does not converge to zero at any point t. Thus $f(x, y)$ has not a first differential at the origin.

PART II.

On the Integrals of Functions Possessing Differentials.

12. In the case of a function $f(x, y)$ possessing a first differential at the point (X, Y), there is no difficulty in integrating provided f is a measurable function, as of course in Mathematics proper is always the case. We have the following simple theorem:

Theorem 3. *If $f(x, y)$ is a measurable function of x for a certain value $y = Y$, or certain values, $y = Y$, [or is a measurable function of (x, y)], aud possesses at (X, Y) a first differential, then in a certain rectangle $(\bar{a}, \bar{b}; \bar{c}, \bar{d})$ with such a selected point (X, Y) as centre, $f(x, Y)$ is a summable function of x for that value of Y, and for all the other values of Y for which (x, Y) lies inside this rectangle [or is a summable function of (x, y)].*

For since $f(x, y)$ has a first differential at the selected point (X, Y), it is continuous at that point with respect to (x, y) and is therefore bounded in some neighbourhood of that point, and therefore summable with respect to x for that Y, and for any Y for which $f(x, y)$ is measurable, provided all the points (x, Y) considered lie in that same neighbourhood, [or is summable with respect to (x, y), if $f(x, y)$ is a measurable function of (x, y)]. Taking this neighbourhood to be a rectangle with the selected point (X, Y) as centre, this proves the theorem.

13. We shall now denote the integral with respect to x of any function denoted by a small letter, such as f, by the corresponding capital, in this case F, writing

$$F(X, Y) = \int_a^X f(x, Y)\,dx; \tag{24}$$

and we shall discuss the properties of F from the point of view of differentials, assuming that f has a first differential at a point (X, Y), or at certain points (X, Y). It will be assumed, without always specifying it, that f is a measurable fonction of (x, y), — or, at least, is measurable in as far as it is required by Theorem 3, — and is therefore summable in a certain fundamental rectangle $(\bar{a}, \bar{b}; \bar{c}, \bar{d})$, containing all the points involved in the integration, or in other processes occurring in the course of our manipulations. We then have the following preliminary theorem, involving only the differential of $f(x, y)$ at a single point (X, Y).

Theorem. *If $f(x, y)$ possess a first differential at (X, Y), its integral $F(x, y)$ has a double incrementary ratio which tends to an unique and finite mixed double limit at (X, Y), namely*

$$\operatorname*{Lt}_{(H,K)\to(0,0)} \{F(X+H, Y+K) - F(X+H, Y) - F(X, Y+K) + + F(X, Y)\}/HK = \frac{\partial f}{\partial x}(X, Y),$$

provided K/H has not zero for a limit; and, if $K/H \to 0$,

$$\{F(X+H, Y+K) - F(X+H, Y) - F(X, Y+K) + F(X, Y)\}/H \to 0.$$

By (24), the double incrementary ratio in question is expressible as follows:

$$\int_X^{X+H} \{f(x, Y+K) - f(x, Y)\}\,dx/HK =$$

$$= \int_0^H \{f(X+h, Y+K) - f(X+h, Y)\}dh/HK = \{f(X, Y+K) - f(X, Y)\}/K +$$

$$+ \int_0^H \{f(X+h, Y+K) - f(X+h, Y) - f(X, Y+K) + f(X, Y)\}dh/HK =$$

$$= \{f(X, Y+K) - f(X, Y)\}/K + \int_0^{H/K} \psi_K(\tau)\,d\tau \,.\, K/H,$$

where we have put

$$\tau = h/K.$$

If now K/H has not zero for a limit, H/K remains bounded, say numerically $\leqslant N$. Therefore

$$-N \leqslant \tau \leqslant N,$$

so that, since, by the Secondary Condition $\psi_K(\tau)$ converges uniformly to zero in the interval $(-N, N)$ as $K \to 0$, there is a value K_e, corresponding to any assigned positive quantity e, such that for all values of K numerically $\leqslant K_e$, $\psi_K(\tau)$ is numerically $\leqslant e$, for all values of τ numerically $\leqslant N$. Hence

$$\left| \int_0^{H/K} \psi_K(\tau)\,d\tau \,.\, K/H \right| \leqslant \int_0^{|H/K|} e\,d\tau\,|K/H| \leqslant e.$$

Since e is as small as we please, this integral has the unique limit zero when $(H, K) \to (0, 0)$. Hence, since, by the Primary Condition $\dfrac{\partial f}{\partial y}$ exists, our double increment has the unique limit $\dfrac{\partial f}{\partial y}$. This proves the first part of our Theorem.

If, however $K/H \to 0$, and therefore H/K tends to infinity, we may only consider values of H and K so small that H/K is numerically greater than unity, and divide the interval of integration $(0, H/K)$ into two parts $(0, a)$ and $(a, H/K)$ where a denotes unity with the same sign as H/K. By the preceding argument

$$\left| \int_0^a \psi_K(\tau)\,d\tau \,.\, K/H \right| \leqslant e\,|K/H|$$

which is as small as we please. Also, changing the variable from τ to $t = 1/\tau$, and using the function $\varphi_h(t) = \psi_k(\tau)\,K/h$,

$$\left|\int_a^{H/K} \psi_k(\tau)\, d\tau \,.\, K/H\right| = \left|\int_{K/H}^{a} \varphi_h(t)\, h/H \,.\, dt/t^2\right|$$

$$\leqslant \int_{K/H}^{1} e\, dt/t^2 = e(-1 + |H/K|) \leqslant e\,|H/K|,$$

since, by the Secondary Condition $\varphi_h(t)$ converges uniformly to zero in the interval $0 \leqslant |t| \leqslant 1$.

Hence the double incrementary ratio, minus $(f(X, Y+K) - f(X, Y))\, K$, is numerically

$$\leqslant e\,|K/H| + e\,|H/K|,$$

so that, when multiplied by K, it tends to zero, since H, K, K/H and $F(X, Y+K) - F(X, Y)$ all tend to zero, and e is as small as we please.

This proves the theorem.

14. The preceding theorem does not suggest that $F(x, y)$ has a first differential at (X, Y), since, in general, we cannot assert that $\frac{\partial F}{\partial y}$ exists. If $\frac{\partial F}{\partial y}$ does exist, it is, of course, evident, by Thomae's Test, as extended in § 10, without using the preceding theorem at all, that F has a first differential at (X, Y), since $\frac{\partial F}{\partial x}$ exists and is continuous at (X, Y). We have however the following theorem, in which we assume the existence of the differential of $f(x, y)$ for all points (x, Y) of a linear interval.

Theorem 5. *If, not only at the point (X, Y), but also at every point of a linear neighbourhood of that point, with $y = Y$, $f(x, y)$ has a first differential, then $\frac{\partial f}{\partial y}$ is a summable function of x, and $\frac{\partial F}{\partial y}$ exists at every point of that linear neighbourhood, and is given by differentiation under the integral sign, viz.*

$$\frac{\partial F}{\partial y} = \int_a^x \frac{\partial f}{\partial y}\, dx,$$

where $y = Y$, and both a and x belong to the interval in which the integrand is summable.

For, by the preceding, the double incrementary ratio of F at each of the points in question is equal to

$$= \frac{\partial f}{\partial y} + \theta_1 e$$

where e is an arbitrary positive quantity, $|\theta_1| \leqslant 1$, $|H| < |K|$, and K is restricted to be less in absolute value than a certain quantity depending on the choice of e. Therefore

$$\frac{1}{H}\int_x^{X+H} \{f(x, Y+K) - f(x, Y)\}\, dx = K\frac{\partial f}{\partial y}(X, Y) + \theta_1\, e\, K.$$

Letting $H \to 0$,

$$f(X, Y+K) - f(X, Y) = K\frac{\partial f}{\partial y}(X, Y) - \theta_2\, e\, K,$$

where $|\theta_2|$ is numerically $\leqslant 1$, and is a measurable function of in virtue of this equation, and therefore summable. Hence also by this equation $\frac{\partial f}{\partial y}(X, Y)$ is a summable function of X, X denoting any value in the linear interval of values of x mentioned in the enunciation. Therefore, writing x in place of X and integrating from a to X,

$$\{F(X, Y+K) - F(X, Y)\}/K = \int_a^X \frac{\partial f}{\partial y}(x, Y)\, dx + e\int_a^X \theta_2\, dx.$$

Letting $K \to 0$, we get finally, since e is as small as we please,

$$\frac{\partial F}{\partial y} = \int \frac{\partial f}{\partial y}(x, Y)\, dx.$$

This proves the theorem

Corollary. *Under the conditions of the theorem, $F(x, y)$ has a first differential at each of the points (x, Y).*

15. We now have the following

Theorem 6. *If at every point (x, Y) of a linear neighbourhood of (X, Y), containing (a, X), $f(x, y)$ has a first differential, and at each such point $\frac{\partial f}{\partial y}$ is a differential coefficient with respect to x, i. e.*

(A) $$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x},$$

then at each point (x, Y), $F(x, y)$ *has a first differential, and*

$$\frac{\partial F}{\partial y} = g(x, Y) - g(a, Y).$$

For, by the preceding theorem and corollary, $F(x, y)$ has a first differential, $\frac{\partial f}{\partial y}$ is summable with respect to x, and therefore the same is true of $\frac{\partial g}{\partial x}$, and

$$\frac{\partial F}{\partial y} = \int_a^x \frac{\partial f}{\partial y}\, dx.$$

But, since $\frac{\partial g}{\partial x}$ is, by the above, finite and summable, its integral is, by Lebesgue's theorem, the primitive function; therefore

$$\frac{\partial F}{\partial y} = \int_a^x \frac{\partial g}{\partial x}\, dx = g(x, Y) - g(a, Y).$$

This proves the theorem.

Corollary. *If, in addition, we hypothecate that* $g(a, y)$ *is continuous with respect to* y, *and we define* $\overline{F}(x, y)$ *and* $\overline{G}(x, y)$ *as follows*:

$$\overline{F}(x, y) = \int_a^x f(x, y)\, dx + \int_b^y g(a, y)\, dy,$$

$$\overline{G}(x, y) = \int_a^x g(x, y)\, dx + C(y),$$

then $\overline{F}(x, y)$ *possesses a first differential at each* (x, Y), *and we have*

$$\frac{\partial \overline{F}}{\partial y} = \frac{\partial \overline{G}}{\partial x}.$$

16. The following theorem follows immediately:

Theorem 7. *If at every point of a rectangle* $(\overline{a}, \overline{b}; \overline{c}, \overline{d})$ *each of the pair of functions* $u(x, y)$, $v(x, y)$, *possesses a first differential, and these two functions* $u(x, y)$, $a(x, y)$ *satisfy the Cauchy-Riemann equations, that is*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

then the pair of functions $\overline{U}(x, y)$ *and* $\overline{V}(x, y)$, *obtained from* $u(x, y)$ *and* $v(x, y)$ *by integration with respect to* x *as follows*:

$$\overline{U}(x, y) = \int_a^x u(x, y)\, dx - \int_b^y v(a, y)\, dy$$

$$\overline{V}(x, y) = \int_a^x v(x, y)\, dx + \int_b^y u(a, y)\, dy,$$

also satisfy the Cauchy-Riemann equations, and possess differentials.

This follows from the preceding Corollary, putting first

$$f = v,\ g = u,\ C(y) = -\int_b^y v(a, y)\, dy;$$

and then

$$f = u,\ g = -v,\ C(y) = -\int_b^y u(a. g)\, dy.$$

17. The partial differential coefficients of the functions $\overline{U}(x, y$ and $\overline{V}(x, y)$ defined in the preceding paragraph, being the functions $u(x, y)$ and $\pm v(x, y)$, are known to possess first differentials, which was not the case with $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. It will however be seen that these latter partial differential coefficients do in point of fact possess first differentials, so that the conditions imposed on $u(x, y)$ and $v(x, y)$ in Theorem 7 repeat themselves with regard to the pairs of functions obtained by any sort of differentiation as well as by integration with suitable choice of the constants of integration. This will be seen to follow from the following auxiliary theorem which has been proved as here given by W. H. Young[1]).

[1]) W. H. Young, „The Fundamental Theorem in the Theory of Functions of a Complex Variable", (1911), Proc. L. M. S., Ser. 2, Vol. 10. pp. 1—6, where the assumptions made are more general than the above, U_x & U_y being merely bounded functions, while $U(x, y)$ and $V(x, y)$ have differentials and satisfy the Cauchy-Riemann equations.

Theorem. (*Auxiliary Theorem*). *If throughout a circle C, with centre the origin and radius c, the pair of functions $U(x, y)$ and $V(x, y)$ satisfy the Cauchy-Riemann equations*:

$$U_x = V_y,$$
$$U_y = -V_x,$$

while these partial differential coefficients are themselves functions possessing differentials throughout the circle C, then $U(x, y)$ and $V(x, y)$ are expressible in the form of power series, convergent throughout the circle C.

Moreover, when we use polar coordinates, these series are power series in r, and allied Fourier series in θ.

We may evidently assume, without loss of generality that $U(x, y)$ and $V(x, y)$ are both zero at the origin.

Since U_x and V_x possess differentials throughout the circle C, the same is true of U and V, (Theorem 5, Cor.); therefore we may transform to polar coordinates by the usual formulae:

$$U_r = U_x \cos\theta + U_y \sin\theta;$$
$$U_\theta/r = -U_x \sin\theta + U_y \cos\theta, \quad (r \neq 0);$$

whence, using the Cauchy-Riemann equations,

$$\begin{matrix} U_r = V_\theta/r, \\ U_\theta/r = -V_r, \end{matrix} \quad (r \neq 0)$$

where U_r and U_θ/r are continuous with respect to (r, θ) except at the origin, and, as $r \to 0$, they remain bounded. Thus U_r and U_θ/r are summable functions, and we may write

$$U(x, y) = \int_0^r U_r\,dr, \qquad U(x, y) - U(r, 0) = \int_0^\theta U_\theta\,d\theta$$

and therefore expand in a Fourier series, and write

$$U(x, y) = a_0/2 - \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

Also, U_θ/r being summable, its repeated integrals are equal, and therefore

$$\int_0^r dr \begin{Bmatrix} U(x, y) \\ -U(r, 0) \end{Bmatrix} /r = \int_0^r dr . \int_0^\theta d\theta . U_\theta/r = \int_0^\theta d\theta \int_0^r dr . U_\theta/r$$
$$= -\int_0^\theta d\theta \int_0^r dr . V_r = -\int_0^\theta d\theta . V(x, y)$$

since what was pointed out for U is true *mutatis mutandis* for V.

The first of these equal integrals retains its value when r is kept constant and θ is increased by 2π; therefore the same is true of the last of the integrals. Therefore the expansion of $V(x, y)$ as a Fourier series, corresponding to that explicitly given for $U(x, y)$, must contain no first term. This must also be true for $U(x, y)$, since we may always reverse the rôles of U and V, if we pay attention to the difference of signs caused by the Cauchy-Riemann equations. Thus

$$a_0 = 0,$$

and

$$V(x, y) = \sum_{n=1}^{\infty} (a'_n \cos n\theta + b'_n \sin n\theta).$$

Integrating with respect to θ, we get

$$\int_0^r dr \{U(x, y) - U(r, 0)\}/r = -\sum_{n=1}^{\infty} (a'_n \sin n\theta + b'_n \cos n\theta)/n.$$

Therefore, using the usual formulae for the coefficients in a Fourier series,

$$b'/n = \frac{1}{\pi} \int_{-\pi}^{\pi} d\theta . \cos n\theta \int_0^r dr \{U(x, y) - U(r, 0)\}/r.$$

Here we may change the order of integration, since the integrand is a bounded function, as we see using the Theorem of the Mean, by which

$$U(x, y)/r - U(r, 0)/r = \frac{\partial U}{\partial r}(\varrho \cos\theta, \varrho \sin\theta) - \frac{\partial U}{\partial x}(\zeta, 0),$$

for some values ϱ, ζ, where $0 < \varrho < r$, $0 < \zeta < r$.

Hence

$$b'/n = \frac{1}{\pi}\int_0^r dr \int_{-\pi}^{\pi} d\theta . \cos n\theta \, (U(x,y) - U(r,0))/r = \int_0^r dr . a_n/r.$$

Similarly

$$-a_n'/n = \int_0^r dr\, b_n/r; \quad -b_n/n = \int_0^r dr . a_n'/r; \quad a_n/n = \int_0^r dr . b_n'/r.$$

From these equations we see that a_n is a continuo us functio of r, and therefore remains so when divided by r, except at the origin, remaining, however, bounded up to the origin, since, as we saw, a_n/r is the result of a single integration with respect to θ of a bounded function $\cos n\theta (U(x,y) - U(r,0))/r$.

Thus a_n/r is the differential coefficient of its integral

$$a_n = r\frac{\partial}{\partial r}(b_n'/n), \quad (r \neq 0).$$

Similarly

$$b_n' = r\frac{r}{\partial r}(a_n/n), \quad (r \neq 0).$$

Hence

$$m^2 a_n = r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}a_n\right) = d^2 a_n/dt^2,$$

where

$$t = \log r.$$

This gives

$$a_n = K r^n + K' r^{-n} = K r^n,$$

since a_n/r has to be bounded.

Hence also, by the above,

$$b_n' = a_n.$$

Similarly

$$b_n = - a_n' = L r^n.$$

Thus

$$U(x,y) = \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) = \sum_{n=1}^{\infty} r^n (K_n \cos n\theta + L_n \sin n\theta)$$

$$V(x,y) = \sum_{n=1}^{\infty}(a_n' \cos n\theta + b_n' \sin n\theta) = \sum_{n=1}^{\infty} r^n (-L_n \cos n\theta + K_n \sin n\theta).$$

These series are power series in r and allied Fourier series in θ.

Expanding $\cos n\theta$ and $\sin n\theta$, and replacing $r\cos\theta$ by x, and $r\sin\theta$ by y, we thus have $U(x,y)$ and $V(x,y)$ expressed as power series in x and y, convergent throughout the circle C, including the origin, since, by hypothesis $U(x,y)$ and $V(x,y)$ are zero there, and the right-hand sides of the two formulae vanish with r.

This proves the theorem.

Corollary. *Under the conditions of the theorem, $U(x,y)$ and $V(x,y)$ possess all their partial differential coefficients of higher order; and each pair of these, formed from U and V by the same process of differentiation, satisfy the same conditions as U and V, that is they satisfy the Cauchy-Riemann equations, and their partial differential coefficients possess differentials.*

This follows from the fact that a power series may be differentiated term by term, and that, in consequence, the Cauchy-Riemann equations may be differentiated, while the mixed differential coefficients that occur may, in virtue of the Corollary to Theorem 5 have the order of differentiation re-arranged.

Thus, in the case of the pair $\frac{\partial U}{\partial x}, \frac{\partial V}{\partial x}$, or, say $u(x,y)$, $v(x,y)$, we have from the Cauchy-Riemann equations,

$$u_x = U_{xx} = V_{xy} = V_{yx} = v_y;$$

$$u_y = U_{yx} = U_{xy} = -V_{xx} = -v_x.$$

Also u_x and u_y possess differentials, since u_{xx}, u_{yx}, u_{yy} exist and are continuous functions of (x,y). Thus $u(x,y)$, $v(x,y)$, satisfy the same conditions as $U(x,y)$, $V(x,y)$.

Similarly $\frac{\partial U}{\partial y}, \frac{\partial V}{\partial y}$ satisfy the same conditions, and hence, by induction, the same is true in general of any pair $\frac{\partial^{m+n} U}{\partial x^m \partial y^n}, \frac{\partial^{m+n} V}{\partial x^m \partial y^n}$.

18. Combining Theorem 7 with the results of the preceding article, we get the following.

Theorem 8. *If at every point of a rectangle $(\bar{a}, \bar{b}; \bar{c}, \bar{d})$ each of the pair of functions $u(x,y)$, $v(x,y)$, possesses a first differential and the pair satisfy the Cauchy-Riemann equations,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

then $u(x, y)$ and $v(x, y)$ can be expressed in the neighbourhood of any point (x_0, y_0) inside the rectangle by power series in $(x-x_0)$ and $(y-y_0)$, convergent within the largest circle with (x_0, y_0) as centre, and lying entirely in, or on the boundary of the rectangle.

Moreover these series (Taylor series), when expressed in terms of polar coordinates with (x_0, y_0) as origin, and $y = y_0$ for initial line, become series of positive powers of r, and allied Fourier series of θ,

$$u(x, y) = u(x_0, y_0) + \sum_{n=1}^{\infty} r^n \{K_n \cos\theta + L_n \sin n\theta\}$$

$$v(x . y) = v(x_0\, y_0) + \sum_{n=1}^{\infty} r^n \{-L_n \cos n\theta + K_n \sin n\theta\}.$$

We only need, of course to prove the theorem for the case $x_0 = 0$, $y_0 = 0$, since this only implies working with the function $\bar{u}(\bar{x}, \bar{y})$ in place of $u(x, y)$, where $\bar{x} = x - x_0$, $\bar{y} = y - y_0$, and $\bar{u}(\bar{x}, \bar{y}) = u(x, y)$.

We can then use the Auxiliary Theorem and its Corollary, the functions $U(x, y)$ and $V(x, y)$ being taken to be

$$U(x, y) = \int_a^X u(x, y)\, dx - \int_b^Y v(a, y)\, dy,$$

$$V(x, y) = \int_a^X v(x, y)\, dx + \int_b^Y u(x, y)\, dy,$$

which, by Theorem 7, satisfy the conditions of the Auxiliary Theorem. Hence, by the Corollary to the Auxiliary Theorem, $u(x, y) = \frac{\partial U}{\partial x}$ and $v(x, y) = \frac{\partial V}{\partial x}$ satisfy the conditions of the Auxiliary Theorem and therefore have the required properties. This proves the theorem.

Corollary. *If $u(x, y)$ and $v(x, y)$ possess differentials, and satisfy the Cauchy-Riemann equations, then the same is true of the pair of functions equal to $\frac{\partial^{m+n} u}{\partial x^m \partial y^n}$ and $\frac{\partial^{m+n} v}{\partial x^m \partial y^n}$, these partial differential coefficients necessarily existing.*

PART III.

On the Incorporation of the Theory of Functions of a Complex Variable in the Theory of the Functions of Two Real Variables.

19. We are now able to incorporate the Theory of Functions of a Complex Variable in the Theory of Functions of Two Real Variables. This could not hitherto be said to have been satisfactorily done, since one of the two fundamental theorems of the former theory, namely that known as Goursat's Theorem had not yet been proved in its complete generality by the methods of the Real Variable.

Passing over the familiar definitions and conventions, and writing, as usual,

$$z \equiv x + iy,$$

$$w \equiv u + iv \equiv u(x, y) + v(x, y),$$

we merely remark that, from the point of view of the real variable, i is a short-hand symbol (and not a quantity), enabling us to treat the functions $u(x, y)$ and $v(x, y)$ simultaneously; the rules governing the algebraic manipulation of this symbol, discussed in works on algebra, being such that we treat it like an ordinary constant during such processes, putting,

$$i^2 = -1,$$

so that the equations we obtain are all of the form

$$A + iB = C + iD,$$

and express the „real" results that A and C are equal, and B and D also, that is, as we usually say, that the „real parts" (viz. A and C) may be equated, and also the „imaginary parts" (viz. iB and iD).

20. The first of the two fundamental theorems in the Theory of Functions of a Complex Variable is that which deals with the conditions of existence of the differential coefficient $\frac{dw}{dz}$; the proof given below is based on our conditions for the existence of a differential.

Denoting an increment of z by Δz, — or $h+ik$, when we introduce explicitly the real quantities tacitly understood, — the corresponding increment of w is denoted by Δw, or $\Delta u + i\Delta v$, where Δu is the increment of u, and Δv that of v.

The definition of the differential coefficient is then as follows:

Definition. *If the ratio $\Delta w/\Delta z$ tends to an unique and finite limit $A+iB$, as $z\to 0$, then $A+iB$ is called the differential coefficient of w for that value of z, and denoted by $\frac{dw}{dz}$, or, more fully, by $\frac{dw}{dz}(z)$. Under the same conditions we say that the differential coefficient exists for the value of z considered.*

The First Fundamental Theorem is neither more nor less than the exact formulation in the language of the real variable of the conditions here given, (in shorthand), in a form exactly the same as that of the definition of the differential coefficient of a function of a single real variable.

Theorem 9. (*The First Fundamental Theorem in the Theory of Functions of a Complex Variable*).

If, and only if, $u(x,y)$ and $v(x,y)$ possess first differentials, and obey the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},$$

has $\Delta w/\Delta z$ an unique and finite limit, and this limit is then

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}.$$

We introduce the following symbols:

$$\Delta_1 u = u(x+h, y)-u(x,y),\quad \Delta_1 z = h,$$

$$\Delta_2 u = u(x, y+k)-u(x,y),\quad \Delta_2 z = k,$$

$$\varepsilon_1(u)=\Delta u/\Delta z \quad \Delta_1 u/\Delta_1 z;$$

$$\varepsilon_2 u = \Delta u/\Delta z - \Delta_2 u/\Delta_2 z.$$

$$\Delta u = u(x+h, y+k)-u(x+h, y)-u(x, y+k)+u(x,y).$$

The corresponding symbols $\Delta_1 v$ etc. are obtained by changing u into v; and the symbols $\Delta_1 w$ etc by adding to the u-symbol i times the corresponding v-symbol, e. g.

$$\Delta_1 w = \Delta_1 u + i\Delta_1 v.$$

If $\Delta w/\Delta z$ has the unique and finite limit $A+iB$, $\Delta_1 w/\Delta_1 z$ and $\Delta_2 w/\Delta_2 z$ must have the same unique limit, since we have then approach of $\Delta w/\Delta z$ to its limit along $k=0$ and $h=0$ respectively. Therefore, by the definition of a partial differential coefficient, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ all exist, and

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{1}{i}\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)=A+iB.$$

Hence, equating real and imaginary parts, the Cauchy-Riemann equations appear. Also we see that the limit has the value given in the enunciation.

Also, if $hk\neq 0$,

$$\begin{aligned}\Lambda w = \Lambda u + i\Lambda v &= \Delta w - \Delta_1 w - \Delta_2 w\\ &=(h+ik)\,\Delta w/\Delta z - h\Delta_1 w/\Delta_1 z - ik\,\Delta_2 w/\Delta_2 z\\ &= h\,\varepsilon_1(w)+ik\,\varepsilon_2(w),\end{aligned}$$

where $\varepsilon_1(w)$ and $\varepsilon_2(w)$ tend to zero as $z\to 0$, that is as $(h,k)\to(0,0)$. Hence, if h/k does not tend to zero, $\Lambda w/h\to 0$, that is

$$\Lambda u/h\to 0,\quad \Lambda v/h\to 0;$$

and, if h/k does tend to zero, $\Lambda w/k\to 0$, that is

$$\Lambda u/k\to 0,\quad \Lambda v/k\to 0.$$

But this expresses the fact that $u(x,y)$ and $v(x,y)$ satisfy the Secondary Conditions for possessing a first differential, and therefore possess differentials, since the Primary Condition has already been shown to hold.

This proves one half of the Theorem.

Conversely, if the conditions of our theorem hold, the Primary and Secondary Conditions of Theorem 1 hold. By the Primary

Condition $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ all exist, so that, by the definition of a partial differential coefficient,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} = \mathrm{Lt}\,.\,\Delta_1 w/\Delta_1 z,$$

$$\frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x}\right) = \mathrm{Lt}\,.\,\Delta_2 w/\Delta_2 z,$$

which, using the Cauchy-Riemann equations, gives the equality of the unique limits of $\Delta_1 w/\Delta_1 z$ and $\Delta_2 w/\Delta_2 z$, and the value of this limit given in the enunciation.

Again, as before, if $hk \neq 0$,

$$\Delta w = h\,\varepsilon_1(w) + ik\,\varepsilon_2(w),$$

which shews, since the Secondary Condition is satisfied by u and v, — so that, if h/k does not tend to zero, $\Delta w/h \to 0$, and, if $h/k \to 0$, $\Delta w/k \to 0$, — that, in either case, $\varepsilon_1(w)$ and $\varepsilon_2(w)$ both $\to 0$; therefore $\Delta w/\Delta z$ has the same unique limit as $\Delta_1 w/\Delta_2 z$ and $\Delta_2 w/\Delta_2 z$.

This proves the remaining half of the theorem.

Une remarque sur les types de dimensions.

Par

S. Ruziewicz (Lwów).

D'après M. Sierpiński[1]) le type de dimensions (au sens de M. Fréchet) dH est dit *le suivant* pour le type dE (et dE le précédent pour dH), s'il n'existe aucun ensemble Q, tel que

$$dE < dQ < dH.$$

Pour abréger, nous écrirons dans ce cas

$$dE \,\mathrm{p}\, dH.$$

Le but de cette Note est de donner *un exemple de cinq ensembles linéaires* P_1, P_2, P_3, Q_1 *et* Q_2, *tels que*

$$dP_1 \,\mathrm{p}\, dP_2 \,\mathrm{p}\, dP_3$$

et

$$dP_1 \,\mathrm{p}\, dQ_1 \,\mathrm{p}\, dQ_2 \,\mathrm{p}\, dP_3.$$

Cela prouvera que si le type de dimensions dP_2 est le suivant pour dP_1 et le précédent pour dP_3, dP_2 n'est pas nécessairement le type unique intermédiaire entre dP_1 et dP_3, et même qu'il peut exister deux types intermédiaires entre dP_1 et dP_3.

Les ensembles P_1, P_2, P_3, Q_1 et Q_2 sont définis comme il suit.

P_1 est l'ensemble formé des nombres $0, \frac{1}{n}$ et $2+\frac{1}{n}$ $(n=1,2,3,...)$.

P_2 est formé des nombres 0 et $\frac{1}{2^m}+\frac{1}{2^{m+n}}$ $(m=1,2,...;n=1,2,...)$

P_3 est l'ensemble qu'on obtient en adjoignant à P_2 le point $\frac{1}{2}$.

[1]) *Fund. Math.* t. XIV, p. 122.