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A remark on R. G. Woods' paper "The minimum uniform compactification of a metric space"

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by

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Abstract. A question raised in R. G. Woods' paper has a simple solution.

The *minimum uniform compactification* uX of a metric space X is the smallest compactification of X such that every bounded real-valued uniformly continuous function on X has a continuous extension to uX. Two subsets of X are *distant* iff they have disjoint closures in uX. Woods proves that $u\mathbb{R}$ is a perfect compactification of $\mathbb{R}$ and leaves the case of $u\mathbb{R}^n$ open.

THEOREM. *Let X be a convex subset of a normed linear space. Then uX is a perfect compactification of X.*

Proof. Let $f : \beta X \to uX$ be the Stone–Čech extension of the inclusion $X \to uX$. In what follows the bar $\overline{}$ will denote closure in uX. uX is a perfect compactification of X iff f has connected fibers. Suppose that uX is not perfect. Then there is a point p of $uX - X$ such that the closed subspace $f^{-1}(p)$ of βX is not connected. Consequently, $f^{-1}(p)$ is the union of non-empty disjoint closed subsets E, F of βX. As βX is normal, there are disjoint open subsets G, H of βX such that $E \subset G$ and $F \subset H$. Let $A = X - G \cup H$. Now the image under f of the compact space $\beta X - G \cup H$ is a closed subset of the Hausdorff space uX containing A but not p. Hence $p \notin \overline{A}$. Let B be an open neighbourhood of p in the regular space uX such that $\overline{A} \cap \overline{B} = \emptyset$. Then $B \cap X = B_1 \cup B_2$, where $B_1 = B \cap X \cap G$ and $B_2 = B \cap X \cap H$.

As B is open and X is dense in uX, we have $\overline{B} = \overline{B \cap X} = \overline{B}_1 \cup \overline{B}_2$. Thus, without loss of generality, we may assume that $p \in \overline{B}_1$. Note that p also belongs to $f(F)$ and hence to the bigger set $\overline{H \cap X}$. Consequently, $d(B_1, H \cap X) = 0$, where d is the metric induced by a norm $|\cdot|$ on X.

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Let $\varepsilon > 0$. Then there are b in B_1 and c in $H \cap X$ such that $d(b, c) < \varepsilon$. Consider next the line segment $L = \{(1-t)b + tc : 0 \le t \le 1\}$ joining b to c in the convex set X. As L is a connected subspace of βX and G, H are disjoint open sets of it containing b, c, respectively, $A = X - G \cup H$ contains at least one point $a = (1-t)b + tc$ of L. But then $d(a, b) = |a - b| = |-tb + tc| = td(b, c) < \varepsilon$. This implies $d(A, B) = 0$ and hence $\bar{A} \cap \bar{B} \neq \emptyset$. This contradiction establishes the result.

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