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# The Arkhangel'skiĭ–Tall problem under Martin's Axiom

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**Abstract.** We show that $\mathrm{MA}_{\sigma\text{-centered}}(\omega_1)$ implies that normal locally compact metacompact spaces are paracompact, and that $\mathrm{MA}(\omega_1)$ implies normal locally compact metalindelöf spaces are paracompact. The latter result answers a question of S. Watson. The first result implies that there is a model of set theory in which all normal locally compact metacompact spaces are paracompact, yet there is a normal locally compact metalindelöf space which is not paracompact.

**0. Introduction.** In 1971, A. V. Arkhangel'skiĭ [A] proved that every perfectly normal, locally compact, metacompact space is paracompact. This suggests the question, stated in print three years later by Arkhangel'skiĭ [AP] and Tall [T], whether "perfectly normal" can be reduced to "normal":

PROBLEM. *Is every normal locally compact metacompact space paracompact?*

The first positive consistency result on this problem is due to S. Watson [$W_1$] who showed that the answer is "yes" if one assumes Gödel's axiom of constructibility $V = L$. The answer is also positive in a model obtained by adding supercompact many Cohen or random reals, because there normal locally compact spaces are collectionwise normal [$B_1$], and it is well known that metacompact collectionwise normal spaces are paracompact [E].

In [GK] we showed that the answer is not simply positive in ZFC by constructing a consistent example of a normal locally compact metacompact non-paracompact space. Earlier, Watson [$W_2$] had constructed consistent examples of normal locally compact metalindelöf spaces. In particular, his examples followed from "$\mathrm{MA}_{\sigma\text{-centered}}(\omega_1)+\exists$ Suslin line", which is known to be relatively consistent with ZFC. In that paper and subsequently in [$W_3$], Watson asked if $\mathrm{MA}(\omega_1)$ were enough to kill all examples of normal locally

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compact metalindelöf non-paracompact spaces. In this paper we answer his question affirmatively and also show that $\mathrm{MA}_{\sigma\text{-centered}}(\omega_1)$ is enough to kill all such metacompact examples. It follows that in any model of ZFC satisfying "$\mathrm{MA}_{\sigma\text{-centered}}(\omega_1) + \exists$ Suslin line", every normal locally compact metacompact space is paracompact, but there is a normal locally compact metalindelöf space which is not paracompact.

In the course of proving the MA results, we obtain the ZFC result that normal locally compact metalindelöf spaces which are $\omega_1$-collectionwise Hausdorff are paracompact. This implies that, in any model, if there is a normal locally compact metalindelöf space which is non-paracompact, then there is one of Lindelöf degree $\omega_1$, and that is also what enables us to get by with MA for $\omega_1$-many dense sets.

Main Result. (a) *Normal locally compact metalindelöf spaces are paracompact if they are $\omega_1$-collectionwise Hausdorff.*

(b) *If there is a normal locally compact metalindelöf space which is not paracompact, then there is one which is the union of $\omega_1$-many compact sets.*

(c) $\mathrm{MA}(\omega_1)$ *implies normal locally compact metalindelöf spaces are paracompact.*

(d) $\mathrm{MA}_{\sigma\text{-centered}}(\omega_1)$ (*i.e.*, $p > \omega_1$) *implies that normal locally compact metacompact spaces are paracompact.*

**1. Destroying examples with MA.** We prove here the main result given in the introduction. Earlier partial positive solutions to the Arkhangel'skiĭ–Tall problem exploited the fact that closed discrete subsets of the space are *normalized*, i.e., any subset $A$ of a closed discrete set $D$ is contained in some open set whose closure is disjoint from $D \setminus A$. The key new idea of our result is a way to exploit normality with respect to a closed discrete set $D$ and closed sets disjoint from $D$. The proof uses several ideas from Balogh's proof [B$_2$] that normal, locally compact, metalindelöf spaces are paracompact if they are collectionwise Hausdorff. (Note that part (a) of the Main Result is a direct improvement of this.) His proof is by induction on the Lindelöf degree. Recall that the *Lindelöf degree* $L(X)$ of a space $X$ is the least cardinal $\kappa$ such that every open cover of $X$ has a subcoveer of cardinality $\leq\kappa$.

The following is the key new combinatorial tool.

Lemma 1. *Let* $\kappa$ *be a cardinal*, *and assume* $\mathrm{MA}(\kappa)$. *Let* $\{B(\alpha) : \alpha < \kappa\}$ *be a collection of sets such that*, *whenever* $\{F_\alpha : \alpha < \omega_1\}$ *is a disjoint collection of finite subsets of* $\kappa$, *then* $\{\bigcup_{\beta\in F_\alpha} B(\beta) : \alpha < \omega_1\}$ *is not centered.* (*Note that this condition implies that the* $B(\alpha)$*'s are point-countable*, *and is satisfied*, *e.g.*, *if* $\{B(\alpha) : \alpha < \kappa\}$ *is a point-countable collection of compact sets.*) *Let* $\{Y_\alpha : \alpha < \kappa\}$ *be a collection of countable sets such that* $|Y_\alpha \setminus \bigcup_{\beta\in F} B(\beta)| = \omega$

*for every finite $F \subset \kappa \setminus \{\alpha\}$. Then $\kappa = \bigcup_{n<\omega} A_n$, where, for each $n \in \omega$ and $\alpha \in A_n$,*

$$\Big|Y_\alpha \setminus \bigcup_{\beta \in A_n \setminus \{\alpha\}} B(\beta)\Big| = \omega.$$

Proof. We first define a partial order $P$ which will produce one subset of $\kappa$ of the required kind.

Let $P$ be all sequences $p = \langle f^p_\alpha \rangle_{\alpha \in F^p}$ satisfying:

(i) $F^p \in [\kappa]^{<\omega}$.
(ii) $f^p_\alpha$ is a one-to-one function from some $n^p_\alpha \in \omega$ to $Y_\alpha$.
(iii) $\operatorname{ran}(f^p_\alpha) \cap \bigcup_{\beta \in F^p \setminus \{\alpha\}} B(\beta) = \emptyset$.

Define $q \le p$ iff $F^p \subset F^q$ and $f^p_\alpha \subset f^q_\alpha$ for each $\alpha \in F^p$.

First let us suppose that $P$ is CCC, and show that the desired kind of set is produced. The sets

$$D_{p,n} = \{q \in P : q \perp p, \text{ or } q \le p \text{ and } n^q_\alpha \ge n \text{ for each } \alpha \in F^p\}$$

are easily seen to be dense in $P$ for each $p \in P$ and $n \in \omega$. Since $|P| = \kappa$, by $\mathrm{MA}(\kappa)$ there is a filter $G$ meeting them. Let $A = \bigcup_{p \in G} F^p$, and for each $\alpha \in A$, let $f_\alpha = \bigcup_{p \in G} f^p_\alpha$. Then for each $\alpha \in A$, $\operatorname{ran}(f_\alpha)$ is an infinite subset of $Y_\alpha$ missing $\bigcup_{\beta \in A \setminus \{\alpha\}} B(\beta)$ as required.

We now prove that $P$ is CCC. Suppose $\{p_\alpha : \alpha < \omega_1\}$ is an antichain. Without loss of generality, the $F^{p_\alpha}$'s form a $\Delta$-system with root $\Delta$, and for some $k \in \omega$, $|F^{p_\alpha} \setminus \Delta| = k$ for every $\alpha < \omega_1$.

Since for each $\gamma \in \Delta$ there are only countably many possible range values for any $f^p_\gamma$, we may also assume that $f^{p_\alpha}_\gamma = f^{p_\beta}_\gamma$ for each $\gamma \in \Delta$ and $\alpha,\beta \in \omega_1$.

Let $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ list $F^{p_\alpha} \setminus \Delta$ in increasing order. We may assume that there is a sequence $n_0, n_1, \ldots, n_{k-1}$ of natural numbers such that $\operatorname{dom}(f^{p_\alpha}_{\alpha_i}) = n_i$ for each $\alpha < \omega_1$ and $i < k$. For $\alpha < \omega_1$, $i < k$, and $j < n_i$, let $y(\alpha, i, j) = f^{p_\alpha}_{\alpha_i}(j)$. Since $\{B(\alpha) : \alpha < \omega_1\}$ is point-countable, we may, by passing to an uncountable subset if necessary, assume that $y(\beta, i, j) \notin \bigcup_{i<k} B(\alpha_i)$ if $\beta < \alpha$. So if $\beta < \alpha$, since $p_\alpha$ and $p_\beta$ are incompatible, it must be the case that $\bigcup_{i<k} B(\beta_i)$ contains $y(\alpha, i(\alpha,\beta), j(\alpha,\beta))$ for some $i(\alpha,\beta) < k$ and $j(\alpha,\beta) < n_{i(\alpha,\beta)}$. Let $\mathcal{E}$ be a uniform ultrafilter on $\omega_1$. For each $\beta < \omega_1$, there are some $i(\beta), j(\beta) \in \omega$ such that the set $E_\beta = \{\alpha > \beta : i(\alpha,\beta) = i(\beta),\ j(\alpha,\beta) = j(\beta)\}$ is in $\mathcal{E}$. Finally, fix $i, j \in \omega$ such that the set $A(i,j) = \{\beta < \omega_1 : i(\beta) = i,\ j(\beta) = j\}$ is uncountable.

Let $L_\beta = \bigcup_{i<k} B(\beta_i)$, and consider the collection $\mathcal{L} = \{L_\beta : \beta \in A(i,j)\}$. We will show that $\mathcal{L}$ is centered, which will be a contradiction and complete the proof. So suppose $H$ is a finite subset of $A(i,j)$. Choose $\alpha \in \bigcap_{\beta \in H} E_\beta$ with $\alpha > \gamma$ for every $\gamma \in H$. Then $y(\alpha, i, j) \in L_\beta$ for every $\beta \in H$, and the proof that $P$ is CCC is finished.

Now let $P^\omega$ be the finite-support countable power of $P$; i.e., $p \in P^\omega$ iff $p = \langle p_0, p_1, p_2, \ldots\rangle$, where $p_n \in P$ for each $n \in \omega$ and $p_n = \emptyset$ for all but finitely many $n \in \omega$. We may of course assume $\kappa > \omega$, so $\mathrm{MA}(\omega_1)$ holds and this implies $P^\omega$ is CCC (see, e.g., [K]). For each $\alpha \in \kappa$, let $D_\alpha = \{p \in P^\omega : \exists n \in \omega\ (\alpha \in F^{p_n})\}$. Also, for each $p \in P$ and $n, m \in \omega$, let

$$D_{p,n,m} = \{q \in P : q_n \perp p_n, \text{ or } q_n \le p_n \text{ and } n_\alpha^{q_n} \ge m \text{ for each } \alpha \in F_n^p\}.$$

Let $G$ be a filter meeting these dense sets, and let $A_n = \bigcup\{F^{p_n} : p \in G\}$. Then the $A_n$'s are as required. ∎

In the metacompact case (i.e., to prove part (d) of the Main Result), we can use a slightly different version of Lemma 1:

LEMMA 2. *Assume* $\mathrm{MA}_{\sigma\text{-centered}}(\omega_1)$ (*i.e.*, $p > \omega_1$). *Let* $\{B(\alpha) : \alpha < \omega_1\}$ *be a collection of sets*, *and* $\{Y_\alpha : \alpha < \omega_1\}$ *a collection of countable sets such that*

$$y \in Y_\alpha \Rightarrow \{\beta : y \in B(\beta)\} \in [\alpha]^{<\omega}.$$

*Then* $\omega_1 = \bigcup_{n<\omega} A_n$ *such that*, *for each* $n \in \omega$ *and* $\alpha \in A_n$,

$$\Big|Y_\alpha \setminus \bigcup_{\beta \in A_n \setminus \{\alpha\}} B(\beta)\Big| = \omega.$$

The same partial order as in the proof of Lemma 1 is used for Lemma 2. Essentially we just need to show that in this case the partial order is $\sigma$-centered. The next two lemmas will be useful for this. If $F$ and $G$ are sets, then $F \vartriangle G$ denotes the symmetric difference $(F \setminus G) \cup (G \setminus F)$, and if they are sets of ordinals then $F < G$ denotes $\forall \alpha \in F\ \forall \beta \in G\ (\alpha < \beta)$.

LEMMA 3. *There is a partial function* $\psi : [\omega_1]^{<\omega} \to \omega$ *satisfying*:

(a) $\mathrm{dom}(\psi)$ *is cofinal in* $[\omega_1]^{<\omega}$, *i.e.*, *for each* $A \in [\omega_1]^{<\omega}$, *there is* $F \in \mathrm{dom}(\psi)$ *with* $A \subset F$.

(b) *If* $F, G \in \psi^{-1}(n)$, *then* $F \cap G < F \vartriangle G$.

Proof. We inductively define $\psi\restriction[\alpha]^{<\omega}$ for $\alpha \le \omega_1$. Let $[\omega]^{<\omega} \cap \mathrm{dom}(\psi) = \omega$, and let $\psi(n) = n$. Now suppose $\alpha > \omega$ and $\psi\restriction[\beta]^{<\omega}$ has been defined for all $\beta < \alpha$ satisfying the following conditions:

(i) For every $A \in [\beta]^{<\omega}$ there exists $F \in [\beta]^{<\omega} \cap \mathrm{dom}(\psi)$ with $A \subset F$.

(ii) $F, G \in \psi^{-1}(n) \cap [\beta]^{<\omega} \Rightarrow F \cap G < F \vartriangle G$.

(iii) There is $\{F_n(\beta)\}_{n\in\omega} \subset \mathrm{dom}(\psi)$ which is cofinal in $[\beta]^{<\omega}$, $\psi(F_n(\beta)) \ne \psi(F_m(\beta))$ if $n \ne m$, and $F_0(\beta) \subset F_1(\beta) \subset F_2(\beta) \subset \ldots$

If $\alpha = \beta + 1$, extend $\psi\restriction[\beta]^{<\omega}$ by defining $\psi(\{\beta\} \cup F_n) = \psi(F_n)$, where the $F_n$'s are as in (iii). It is easy to check that (i)–(iii) are now satisfied with $\beta = \alpha$.

If $\alpha$ is a limit ordinal, then $\psi\restriction[\alpha]^{<\omega}$ has been defined by virtue of having been defined for each $\beta < \alpha$. Furthermore, it is clear that (i) and (ii) hold.

We need to show (iii) if $\alpha < \omega_1$. Let $\alpha_0, \alpha_1, \ldots$ be an enumeration of $\alpha$. Let $\beta_0, \beta_1, \ldots$ be an increasing sequence of ordinals with supremum $\alpha$, and for each $n < \omega$ let $\{F_{n,m} : m < \omega\}$ witness (iii) for $\beta = \beta_n$. We inductively define $m(n)$ for $n = 0, 1, \ldots$ such that $\{F_{n,m(n)} : n < \omega\}$ satisfies (iii) with $\beta = \alpha$. Given $F_{n,m(n)}$, it suffices to choose $m(n+1)$ such that:

(a) $F_{n+1,m(n+1)} \supset F_{n,m(n)} \cup \{\alpha_k\}$, where $k$ is least such that $\alpha_k \in \beta_{n+1} - F_{n,m(n)}$.

(b) $\psi(F_{n+1,m(n+1)}) \neq \psi(F_{i,m(i)})$ for all $i \leq n$.

It is clear that (iii) for $\beta_{n+1}$ implies that this can be done. ∎

LEMMA 4. *Suppose that $e : [\omega_1]^2 \to \omega$ is such that for every $\alpha \in \omega_1$ the function $e(\cdot, \alpha) : \alpha \to \omega$ is finite-to-one.* (*For $\beta \neq \alpha$ we write $e(\{\beta, \alpha\}) = e(\beta, \alpha) = e(\alpha, \beta)$.*) *Then for every $m, k \in \omega$ there is a partition $\{A_n^{m,k} : n < \omega\}$ of $[\omega_1]^m$ such that*:

(a) $\bigcup_{n<\omega} A_n^{m,k} = [\omega_1]^m$.

(b) *For every $n < \omega$, if $a, b \in A_n^{m,k}$, then $a \cap b < a \triangle b$ and*

$$\forall \alpha \in a - b\ \forall \beta \in b - a\ (e(\beta, \alpha) > k).$$

Proof. Fix $m, k \in \omega$. For every $a \in [\omega_1]^m$ define $E_i(a)$ as follows: $E_0(a) = a$ and

$$E_{i+1}(a) = \{\beta : \exists \alpha \in E_i(a)\ (\beta < \alpha \text{ and } e(\beta, \alpha) \leq k)\},$$

and put $E(a) = \bigcup_{i<\omega} E_i(a)$. Note that since $\max(E_{i+1}) < \max(E_i)$, only finitely many $E_i(a)$'s are non-empty. Hence $E(a)$ is finite because $e(\cdot, \alpha)$ is finite-to-one.

Let $\psi$ be a partial function from $[\omega_1]^{<\omega}$ to $\omega$ satisfying the conditions of Lemma 3. For each $a \in [\omega_1]^m$, choose $F(a) \in \mathrm{dom}(\psi)$ with $E(a) \subseteq F(a)$. Then there is a partition $\{A_n^{m,k} : n < \omega\}$ of $[\omega_1]^m$ such that $a, b \in A_n^{m,k}$ implies:

(i) $\psi(F(a)) = \psi(F(b))$ and $|F(a)| = |F(b)|$.

(ii) The unique order preserving function $h : F(a) \to F(b)$ has the property that $h"(a) = b$.

Suppose $a, b \in A_n^{m,k}$. We need to verify that 4(b) holds. Since $\psi(F(a)) = \psi(F(b))$, we have $F(a) \cap F(b) < F(a) \triangle F(b)$. From this and (ii) it easily follows that $a - b \subset F(a) - F(b)$, $b - a \subset F(b) - F(a)$, and $a \cap b < a \triangle b$. Now suppose $\alpha \in a - b$ and $\beta \in b - a$, and say $\beta < \alpha$. If $e(\beta, \alpha) \leq k$, then $\beta \in E(a) \subset F(a)$, but this contradicts $\beta \in F(b) - F(a)$. ∎

Proof of Lemma 2. Let $\{B(\alpha) : \alpha < \omega_1\}$ be a collection of sets and $\{Y_\alpha : \alpha < \omega_1\}$ a collection of countable sets such that

$$y \in Y_\alpha \Rightarrow \{\beta : y \in B(\beta)\} \in [\alpha]^{<\omega}.$$

Let $P$ be the same poset as in the proof of Lemma 1 (but applied to the above sets, of course). It suffices to prove $P$ is $\sigma$-centered, for then the finite support countable power would be too.

Let $Y_\alpha = \{y_{\alpha,n} : n < \omega\}$. If $\beta < \alpha$ and $B_\beta \cap Y_\alpha \neq \emptyset$, define $e(\beta, \alpha)$ to be the minimal $n$ such that $y_{\alpha,n} \in B_\beta$. Since each $y \in Y_\alpha$ is in at most finitely many $B_\beta$'s, it follows that $e(\cdot, \alpha)$ is finite-to-one. Then $e$ can be extended so that $e(\cdot, \alpha)$ has domain $\alpha$ and still is finite-to-one. This completes the definition of $e : [\omega_1]^2 \to \omega$ (i.e., $e(\{\beta, \alpha\}) = e(\beta, \alpha)$ if $\beta < \alpha$).

Let $\{A_n^{m,k} : n < \omega\}$ satisfy the conditions of Lemma 4. If $p \in P$, put $p \in P_n^{m,k}$ if:

(i) $|F^p| = m$.
(ii) For each $\alpha \in F^p$, $\mathrm{ran}(f_\alpha^p) \subset \{y_{\alpha,i} : i < k\}$.
(iii) $F^p \in A_n^{m,k}$.

To prove that $P_n^{m,k}$ is centered, it suffices to show that whenever $p, q \in P_n^{m,k}$, $\alpha \in F^p$, $\beta \in F^q$, and $\beta \neq \alpha$, then $\mathrm{ran}(f_\alpha^p) \cap B(\beta) = \emptyset$. If both $\alpha$ and $\beta$ are in $F^p$, or both in $F^q$, this follows from the definition of $P$. So we may assume $\alpha \in F^p - F^q$ and $\beta \in F^q - F^p$. Now suppose $\mathrm{ran}(f_\alpha^p) \cap B(\beta) \neq \emptyset$. By the hypothesis of Lemma 2, we have $\beta < \alpha$. Suppose $y_{\alpha,j} \in \mathrm{ran}(f_\alpha^p) \cap B(\beta)$. By condition (ii) in the definition of $P_n^{m,k}$, $j < k$. By definition of $e$, $e(\beta, \alpha) \leq j$. But by Lemma 4(b), $e(\beta, \alpha) > k$. This contradiction completes the proof. ∎

Lemma 5. *Let $\{B(\alpha) : \alpha \in \kappa\}$ be a point-countable collection of sets, and let $\{Y_\alpha : \alpha \in \kappa\}$ be a collection of countable sets. Then $\kappa = \bigcup_{\gamma < \omega_1} A_\gamma$ such that $\beta \neq \alpha \in A_\gamma$ implies $B(\beta) \cap Y_\alpha = \emptyset$.*

Proof.

Claim 1. *Without loss of generality, $\beta < \alpha \Rightarrow Y_\beta \cap B(\alpha) = \emptyset$.*

Note that, by point-countability and an easy closing up argument, each $\gamma \in \kappa$ is in a countable set $M$ such that $\beta \in M$ and $Y_\beta \cap B(\alpha) \neq \emptyset$ implies $\alpha \in M$. Thus $\kappa$ can be written as the union of countable sets $M_\gamma$, $\gamma < \kappa$, having the above property. Let $M_\gamma \setminus \bigcup_{\beta < \gamma} M_\beta = \{x(\gamma, n) : n \in \omega\}$. Let $E_n = \{x(\gamma, n) : \gamma < \kappa\}$. Note that $\beta < \alpha \Rightarrow Y_{x(\beta,n)} \cap B(x(\alpha, n)) = \emptyset$. Thus each $E_n$ satisfies the condition of Claim 1. If the lemma holds for each $E_n$, it holds for $\kappa$, and so Claim 1 follows.

For each $\alpha < \kappa$, let $F(\alpha) = \{\beta \neq \alpha : B(\beta) \cap Y_\alpha \neq \emptyset\} = \{\beta < \alpha : B(\beta) \cap Y_\alpha \neq \emptyset\}$. The following claim completes the proof of the lemma.

Claim 2. *There exists $\theta : \kappa \to \omega_1$ such that $\theta(\beta) = \theta(\alpha) \Rightarrow \beta \notin F(\alpha)$ (and hence $B(\beta) \cap Y_\alpha = \emptyset$).*

To see this, simply define $\theta$ inductively by letting $\theta(\alpha) = \sup\{\theta(\beta) + 1 : \beta \in F(\alpha)\}$. ∎

If $Y$ and $H$ are subsets of a space $X$, let us say $Y$ *converges to* $H$, and write $Y \to H$, if every neighborhood of $H$ contains all but finitely many elements of $Y$.

Lemma 6. *Let $\mathcal{U}$ be a point-countable cover of a space $X$ by open $\sigma$-compact sets with compact closures. Let $O \in \mathcal{U}$, and suppose that $H = O \setminus \bigcup(\mathcal{U} \setminus \{O\}) \neq \emptyset$. Let $Z \subset X$ such that $\overline{Z} \cap H \neq \emptyset$. Then there is a countable subset $Y$ of $Z$ such that $Y \to H$.*

Proof. For each $U \in \mathcal{U}$, let $U = \bigcup_{n\in\omega} U(n) = \bigcup_{n\in\omega} U(n)^\circ$, where each $U(n)$ is compact. For every $y \in X \setminus H$, let $\{U^y_n : n < \omega\}$ enumerate $\{U \in \mathcal{U} \setminus \{O\} : y \in U\}$. Inductively choose points $y_n$, $n < \omega$, such that

$$y_n \in Z \cap \overline{O} \setminus \bigcup\{U^{y_i}_j(k) : i, j, k < n\}.$$

It is easy to check that $Y = \{y_n : n < \omega\}$ has no limit point outside of $H$. Since $Y \subset \overline{O}$ and $\overline{O}$ is compact, it follows that $Y \to H$. ∎

Lemma 7. *Every open cover of a metalindelöf locally compact space has a point-countable open refinement by $\sigma$-compact open sets.*

Proof. Note that a locally compact Hausdorff space has a base of $\sigma$-compact open sets (use complete regularity). So this is a corollary of [GM; Cor. 4.1], which states that every base for a locally Lindelöf, metalindelöf space contains a point-countable subcover. ∎

Lemma 8. *The following are equivalent*:

(a) *There is a normal locally compact metalindelöf space that is not $\kappa$-CWH.*

(b) *There is a normal locally compact metalindelöf space of Lindelöf degree $\leq \kappa$ which is not paracompact.*

Proof. (a)⇒(b). Suppose $D$ is a closed discrete unseparated subset of cardinality $\kappa$ in a normal locally compact metalindelöf space $X$. For each $d \in D$, let $U_d$ be an open $\sigma$-compact subset of $X$ containing $d$. By normality, there is a closed neighborhood $N$ of $D$ contained in $\bigcup_{d\in D} U_d$. Then $L(N) \leq \kappa$ and $D$ cannot be separated in $N$.

(b)⇒(a). Suppose $X$ satisfies the hypotheses of (b). By Balogh's theorem [$B_2$], there is a closed discrete subset $D$ of $X$ which cannot be separated. Since $L(X) \leq \kappa$, $X$ is the union of $\leq\kappa$-many compact sets, so $|D| \leq \kappa$. Hence $X$ is not $\kappa$-CWH. ∎

Lemma 9. *Let $\kappa$ be the least cardinal such that there is a normal locally compact metalindelöf non-paracompact space $X$ with $L(X) = \kappa$. Then $\kappa$ is regular.*

Proof. Let $\kappa$ and $X$ satisfy the hypotheses. Note that by the minimality of $\kappa$ and Lemma 8, $X$ is $<\kappa$-CWH. Write $X = \bigcup\{U_\alpha : \alpha < \kappa\}$, where each $U_\alpha$ is a $\sigma$-compact open set. For $\alpha < \kappa$, let $V_\alpha = \bigcup\{U_\beta : \beta < \alpha\}$.

First suppose some $\overline{V}_\alpha$ is not paracompact. Since $X$ is $<\kappa$-CWH, there is a closed discrete $D \subset \overline{V}_\alpha$ with $|D| = \kappa$. Suppose $|\alpha|^+ < \kappa$. Then any subset of $D$ of cardinality $|\alpha|^+$ has a discrete open expansion. But this is impossible, since $\overline{V}_\alpha$ has a dense subset which is the union of $|\alpha|$-many compact sets. So $\kappa = \alpha^+$, and the lemma is proved in this case.

Now suppose each $\overline{V}_\alpha$ is paracompact. Then there is a $\sigma$-discrete cover $\mathcal{W}'_\alpha$ of $\overline{V}_\alpha$ by relatively open sets with compact closures (e.g., take any $\sigma$-discrete open (in $\overline{V}_\alpha$) refinement of any cover of $\overline{V}_\alpha$ by open sets with compact closures). Let $\mathcal{W}_\alpha = \{W \cap V_\alpha : W \in \mathcal{W}'_\alpha\}$. Then $\mathcal{W}_\alpha$ is a $\sigma$-discrete (in $X$) cover of $V_\alpha$ by open (in $X$) sets with compact closures. Let $A$ be a cofinal subset of $\kappa$ of cardinality $\mathrm{cf}(\kappa)$. Let $\mathcal{W} = \bigcup_{\alpha \in A} \mathcal{W}_\alpha$.

Then $\mathcal{W}$ is a cover of $X$ by open sets with compact closures, and each member of $\mathcal{W}$ meets at most $\mathrm{cf}(\kappa)$-many others. Thus by a standard chaining argument, $X$ is the union of disjoint clopen subspaces of Lindelöf degree $\le \mathrm{cf}(\kappa)$. Since $X$ is not paracompact, one of these subspaces cannot be paracompact. Then by the minimality of $\kappa$, $\mathrm{cf}(\kappa) = \kappa$. ∎

Proof of Main Result. Part (b) follows from (a), Lemma 8, and local compactness. We prove (a), (c), and (d) simultaneously by induction on the Lindelöf degree. So suppose $\kappa$ is the least cardinality of a counterexample $X$ with $L(X) = \kappa$. By Lemma 9, $\kappa$ is regular, and by Lemma 8, $X$ is $<\kappa$-CWH. By Lemma 7, $X$ has a point-countable cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ by $\sigma$-compact open sets.

We first take care of part (a) when $\kappa = \omega_1$. In this case, $X$ has no closed discrete subsets of cardinality greater than $\omega_1$, so $X$ being $\omega_1$-CWH implies $X$ is CWH, hence paracompact by Balogh's theorem. Thus we may assume from now on that $\kappa > \omega_1$ when dealing with part (a).

Let $V_\alpha = \bigcup_{\gamma < \alpha} U_\gamma$.

Case 1. *For some $\delta < \kappa$, $\overline{V}_\delta$ is not paracompact.*

If $\overline{V}_\delta$ is not paracompact, it is not $\kappa$-CWH but is $<\kappa$-CWH (by choice of $\kappa$). Thus there is a closed discrete set $D$ of $\overline{V}_\delta$ of cardinality $\kappa$. Since $V_\delta$ is the union of less than $\kappa$-many compact sets, we may assume $D$ is a subset of the boundary $\partial V_\delta$ of $V_\delta$. By metalindelöf, there is a point-countable cover $\mathcal{W}$ of $\overline{V}_\delta$ by open $\sigma$-compact sets with compact closures such that each member of $\mathcal{W}$ contains at most one member of $D$, and each point of $D$ is in only one member of $\mathcal{W}$. (To see this, apply Lemma 7 to any such open cover to get a point-countable cover $\mathcal{W}'$ by $\sigma$-compact open sets with compact closures, and for each $d \in D$, if more than one member of $\mathcal{W}'$ contains $d$, replace them with their union. Let $\mathcal{W}$ be the result of modifying $\mathcal{W}'$ in this way.)

Now let $\mathcal{O} = \mathcal{W} \cup \{U_\gamma : \gamma < \delta\}$, let $D = \{x_\alpha : \alpha < \kappa\}$, and let $O_\alpha$ be the unique member of $\mathcal{O}$ which contains $x_\alpha$. Let $H_\alpha = O_\alpha \setminus \bigcup(\mathcal{O} \setminus \{O_\alpha\})$. Note that $x_\alpha \in H_\alpha \subset \partial V_\delta$, and that $H_\alpha$ is a closed (in $X$) subset of $O_\alpha$, so it is compact. By Lemma 6, there is a countable subset $Y_\alpha$ of $V_\delta$ such that $Y_\alpha \to H_\alpha$.

Let $B(\alpha)$ be a compact neighborhood of $H_\alpha$ with $B(\alpha) \subset O_\alpha$. Then $\{B(\alpha) : \alpha < \kappa\}$ and $\{Y_\alpha : \alpha < \kappa\}$ satisfy the hypotheses of both Lemma 1 and Lemma 5. If $\kappa > \omega_1$, apply Lemma 5 and the fact that $\kappa$ is regular to obtain a subset $A$ of $\kappa$ of cardinality $\kappa$ satisfying the conclusion of Lemma 5 (i.e., $\beta \neq \alpha \in A$ implies $Y_\alpha \cap B(\beta) = \emptyset$). If $\kappa = \omega_1$, we know we are considering part (c) or (d). If (c), by $\mathrm{MA}(\omega_1)$ and Lemma 1, there is a subset $A$ of $\kappa$ of cardinality $\kappa$ satisfying the conclusion of Lemma 1 (one of the $A_n$'s given by Lemma 1 must have cardinality $\kappa$; take $A$ to be such an $A_n$). Suppose we are in case (d). Since $\{H_\alpha : \alpha < \omega_1\}$ is a closed discrete collection of closed sets in a metacompact space, it has a point-finite open expansion, and so we may assume that $\{B(\alpha) : \alpha \in \omega_1\}$ is point-finite. Each $Y_\alpha$ meets at most countably many $B(\beta)$'s. So it is not difficult to see that $\omega_1 = \bigcup_n W_n$, where $\alpha < \beta \in W_n$ implies $B(\beta) \cap Y_\alpha = \emptyset$ (see, e.g., the proof of Claim 1 in the proof of Lemma 5). Choose $n$ so that $W_n$ is uncountable. By re-indexing via the unique order preserving map from $W_n$ onto $\omega_1$, the sets $B(\alpha)$ and $Y_\alpha$ for $\alpha \in W_n$ satisfy the hypotheses of Lemma 2. So again, but now by $\mathrm{MA}_{\sigma\text{-centered}}(\omega_1)$, there is a set $A$ as in cases (a) and (b). (In any case, we only need an $A$ which satisfies the conclusion of Lemmas 1 or 2, which is of course weaker than the conclusion of Lemma 5.)

Let $H = \bigcup_{\alpha\in A} H_\alpha$ and $K = X \setminus \bigcup_{\alpha \in A} B(\alpha)^\circ$. We aim for a contradiction by showing that $H$ and $K$ cannot be separated in $X$. To this end, suppose $G$ is an open set containing $H$, and let $G_\alpha = G \cap B(\alpha)^\circ$. By the property of the set $A$, $Y_\alpha \setminus \bigcup_{\beta \in A\setminus\{\alpha\}} B(\beta)$ is infinite. Since $Y_\alpha \to H_\alpha$, we can choose a point $y_\alpha \in G_\alpha \cap [Y_\alpha \setminus \bigcup_{\beta\in A\setminus\{\alpha\}} B(\beta)]$. Since $V_\delta$ is the union of less than $\kappa$-many compact sets and each $y_\alpha \in V_\delta$, some compact subset of $V_\delta$ contains $\kappa$-many $y_\alpha$'s. Thus there is a point $y \in V_\delta$ every neighborhood of which contains $\kappa$-many $y_\alpha$'s. But the $y_\alpha$'s are relatively discrete in $X \setminus K = \bigcup_{\alpha\in A} B(\alpha)^\circ$, so $y \in K \cap \overline{G}$. Thus $H$ and $K$ cannot be separated, a contradiction which completes the proof of Case 1.

Case 2. *Each $\overline{V}_\alpha$, $\alpha < \kappa$, is paracompact.*

Let $S = \{\alpha < \kappa : \overline{V}_\alpha \neq V_\alpha\}$. We first show that $S$ is stationary. Suppose $C \subset \kappa$ is a club missing $S$. Given $\alpha \in C$, let $\alpha'$ be the least element of $C$ greater than $\alpha$. We may assume $0 \in C$. Then $\{V_{\alpha'} \setminus V_\alpha : \alpha \in C\}$ is a partition of $X$ into clopen paracompact pieces, whence $X$ is paracompact, a contradiction.

Since $V_\alpha$ is a dense subset of $\overline{V}_\alpha$ and is the union of $<\kappa$-many compact sets, any $\sigma$-discrete cover of $\overline{V}_\alpha$ by open sets with compact closures has cardinality less than $\kappa$. Since $\overline{V}_\alpha$ is paracompact, it follows that $L(\overline{V}_\alpha) < \kappa$. Thus there is $\gamma(\alpha) < \kappa$ such that $\overline{V}_\alpha \subset V_{\gamma(\alpha)}$. Let $C \subset \kappa$ be a club such that $\delta \in C$ and $\alpha < \delta$ implies $\gamma(\alpha) < \delta$. Let $S' = S \cap C$. Then $S'$ is stationary and $\{\partial V_\alpha : \alpha \in S'\}$ is a closed discrete collection in $X$ (since each $U_\beta$ meets at most one member of the collection).

For each $\alpha \in S'$, choose $\mu(\alpha) \in \kappa$ such that $U_{\mu(\alpha)} \cap \partial V_\alpha \neq \emptyset$. Note that $\mu(\alpha) \neq \mu(\alpha')$ for distinct $\alpha, \alpha' \in S'$. Let $O_\alpha$ denote $U_{\mu(\alpha)}$. By complete regularity, we can find a compact $G_\delta$-set $K_\alpha \subset O_\alpha$ with $K_\alpha \cap \partial V_\alpha \neq \emptyset$. Let $\mathcal{U}(\alpha)$ be the modification of the open cover $\mathcal{U}$ obtained by removing $K_\alpha$ from each member of $\{U_\gamma : \gamma \geq \alpha,\ \gamma \neq \mu(\alpha)\}$. This modification is still a cover of $X$ by open $\sigma$-compact sets. Let $H'_\alpha = O_\alpha \setminus \bigcup(\mathcal{U}(\alpha) \setminus \{O_\alpha\})$ and let $H_\alpha = H'_\alpha \cap \partial V_\alpha$. Note that $K_\alpha \cap \partial V_\alpha \subset H_\alpha$; so $\emptyset \neq H_\alpha \subset \partial V_\alpha \cap O_\alpha$ and $H'_\alpha \cap V_\alpha = \emptyset$. By Lemma 6, there is a countable subset $Y_\alpha$ of $V_\alpha$ such that $Y_\alpha \to H'_\alpha$; note that in fact $Y_\alpha \to H_\alpha$. We finish the proof as in Case 1. Let $B(\alpha)$ be a compact neighborhood of $H_\alpha$ with $B(\alpha) \subset O_\alpha$. Then $\{B(\alpha) : \alpha \in S'\}$ and $\{Y_\alpha : \alpha \in S'\}$ satisfy the hypotheses of both Lemma 1 and Lemma 5.

If $\kappa > \omega_1$, apply Lemma 5 and the fact that $\kappa$ is regular to obtain a stationary subset $A$ of $S'$ satisfying the conlusion of Lemma 5 (i.e., $\beta \neq \alpha \in A$ implies $Y_\alpha \cap B(\beta) = \emptyset$). If $\kappa = \omega_1$, we know we are considering part (c) or (d). If (c), by $\mathrm{MA}(\omega_1)$, there is a stationary subset $A$ of $S'$ satisfying the conclusion of Lemma 1. If (d), follow the proof as in Case 1 but choose $n$ such that $W_n \cap S'$ is stationary, and then use Lemma 2 to conclude the existence of a stationary $A$ as in the other cases. (Again, in any case, we only need a stationary $A$ which satisfies the conclusion of Lemma 1 or 2.)

Let $H = \bigcup_{\alpha \in A} H_\alpha$ and $K = X \setminus \bigcup_{\alpha \in A} B(\alpha)^\circ$. We aim for a contradiction by showing that $H$ and $K$ cannot be separated in $X$. To this end, suppose $G$ is an open set containing $H$, and let $G_\alpha = G \cap B(\alpha)^\circ$. By the property of $A$, $Y_\alpha \setminus \bigcup_{\beta \in A \setminus \{\alpha\}} B(\beta)$ is infinite. Since $Y_\alpha \to H_\alpha$, we can choose a point $y_\alpha \in G_\alpha \cap [Y_\alpha \setminus \bigcup_{\beta \in A \setminus \{\alpha\}} B(\beta)]$. Now $y_\alpha \in U_{\beta(\alpha)}$ for some $\beta(\alpha) < \alpha$ (since $y_\alpha \in V_\alpha$), so by the pressing-down lemma, the set $E(\beta) = \{y_\alpha : \alpha \in A$ and $\beta(\alpha) = \beta\}$ is uncountable for some $\beta < \kappa$. Since $U_\beta$ is $\sigma$-compact, such an $E(\beta)$ must have a limit point $y$ in $U_\beta$. But $E(\beta)$ is relatively discrete in $X \setminus K$, so $y \in \overline{G} \cap K$. That completes the proof. ∎

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