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A dimension raising hereditary shape equivalence

by

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Abstract. We construct a hereditary shape equivalence that raises transfinite inductive dimension from ω to $\omega + 1$. This shows that ind and Ind do not admit a geometric characterisation in the spirit of Alexandroff's Essential Mapping Theorem, answering a question asked by R. Pol.

1. Introduction. Every space in this paper is assumed to be separable and metric. A continuous function is called a *map*. Compactness is assumed for the results quoted in the introduction.

Consider the following beautiful (and powerful) geometric characterisation of topological dimension (Alexandroff [1]):

THEOREM 1.1. *A compact space has dimension at least n if and only if the space has an essential mapping into the n-cell I^n.*

A map $f : X \to I^n$ is *essential* if every continuous extension $g : X \to I^n$ of $f|f^{-1}(\partial I^n)$ is surjective. D. W. Henderson [13] has attempted to extend this result to transfinite dimension. He constructed a transfinite sequence of compact ARs J^α with closed subsets ∂J^α and he proved that if a space has an essential map onto J^α then $\operatorname{Ind} X \geq \alpha$. Unfortunately, the converse of this statement turned out to be false (Pol [17, 18], Borst and Dijkstra [5]).

However, as Pol observed in [17, 18], these results leave open the question whether there exists a characterisation of transfinite inductive dimension in the spirit of Alexandroff's Theorem (by making a different choice for the J^α's). Our aim is to show that such a geometric characterisation for ind or Ind cannot exist, solving a problem formulated by Pol in [18, Remark 2.4.b]. A pair (M, S) consisting of an ANR M and a closed subset S is called an *AH-pair*. A map $f : X \to M$ is called *essential* if every continuous extension $g : X \to M$ of $f|f^{-1}(S)$ is surjective. We prove

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Theorem 1.2. *There exist compact ARs X and Y with the properties*: $\operatorname{ind} X = \operatorname{Ind} X = \omega$, $\operatorname{ind} Y = \operatorname{Ind} Y = \omega + 1$, *and for every AH-pair* (M, S) *the space X has an essential map onto M if and only if Y has such a map.*

Returning to the J^α's, they do correspond to a dimension function. Borst [4, Theorem 4.2.1] constructed a transfinite extension of the covering dimension and showed that $\dim X \geq \alpha$ precisely if $X \times C$ has an essential map onto J^α, where C is the Cantor set. Since Pol's index equals $\omega^{\dim}$ ([4, Theorem 3.3.8]) Borst's result produces also an essential mapping characterisation for index.

The means by which we reach Theorem 1.2 is through the construction of a dimension raising cell-like map, which is of interest in its own right. It was shown by Dranishnikov [9] that there exist cell-like images of finite-dimensional spaces that fail to be countably dimensional. We, however, are interested in the more restrictive concept of a hereditary shape equivalence.

It is a well-known corollary of Theorem 1.1 that hereditary shape equivalences do not raise finite dimension (cf. Lemma 2.1). For infinite-dimensional spaces the following is known: hereditary shape equivalences preserve weak infinite dimensionality (because a strongly infinite-dimensional space is a space with an essential map onto the Hilbert cube) and the property C (van Mill and Mogilski [16], Ancel [3]), but they do not preserve strong countable dimensionality (Dijkstra, van Mill, and Mogilski [6]). In addition, it follows from the aforementioned characterisations of index and dim in terms of essential maps that the values of these dimension functions cannot be raised by hereditary shape equivalences (cf. Lemma 2.1).

We present the following result that shows that hereditary shape equivalences (or fine homotopy equivalences) can raise both small and large inductive dimension.

Theorem 1.3. *There exists an AR-map $H : X \to Y$, where X and Y are compact ARs with* $\operatorname{ind} X = \operatorname{Ind} X = \omega$ *and* $\operatorname{ind} Y = \operatorname{Ind} Y = \omega + 1$.

The main unsolved problem in this area remains whether countable dimensionality is preserved by hereditary shape equivalences. Theorem 1.3 is connected to this problem in the sense that the existence of ind raising hereditary shape equivalences is a necessary condition for the failure of countable dimensionality to be preserved under such maps (Dijkstra, van Mill and Mogilski [6]).

The results in this paper were announced in [7].

2. Definitions and preliminaries. A space is called *countably dimensional* if it is a countable union of finite-dimensional spaces. If α is a countable ordinal ($\alpha < \omega_1$), then $\operatorname{ind} X \leq \alpha$ if every point in X has arbitrarily small neighbourhoods U such that $\operatorname{ind}\operatorname{Bd}(U) < \alpha$. $\operatorname{Ind} X \le \alpha$ if every pair of disjoint closed subsets of X has a closed separator S with $\operatorname{Ind} S < \alpha$.

We know that ind and Ind coincide for finite-dimensional spaces and that $\operatorname{ind} X \le \operatorname{Ind} X$ in general. It was proved by Hurewicz [14] and Smirnov [19] that if X is a compactum then X is countably dimensional if and only if $\operatorname{ind} X < \omega_1$ if and only if $\operatorname{Ind} X < \omega_1$.

A proper surjection $f : X \to Y$ is called a *shape equivalence* if for every ANR Z the map f produces a one-to-one correspondence f^* between the homotopy classes of $C(Y, Z)$ and $C(X, Z)$. Here $C(A, B)$ stands for the space of continuous mappings from A into B. A proper surjection $f : X \to Y$ is called a *hereditary shape equivalence* if for every subset A of Y the restriction $f|f^{-1}(A)$ is a shape equivalence between $f^{-1}(A)$ and A. Obviously, every hereditary shape equivalence is *cell-like*, i.e. is a proper map whose fibres have trivial shape. Under certain conditions the converse is valid: a cell-like map between ANRs is a fine homotopy equivalence and hence a hereditary shape equivalence (Haver [12], Kozlowski [15]) and a cell-like map whose range is countably dimensional is a hereditary shape equivalence (Ancel [2]). Another useful property is that the image of an ANR under a hereditary shape equivalence is also an ANR (Kozlowski [15]). An AR-map is a proper surjection whose fibres are absolute retracts.

The following two lemmas are not new. The first supplies the link between Theorem 1.2 and Theorem 1.3.

LEMMA 2.1. *If $h : X \to Y$ is a hereditary shape equivalence and f is an essential map from Y into an AH-pair (M, S) then $f \circ h$ is also essential.*

Proof. Put $S' = f^{-1}(S)$ and $S'' = h^{-1}(S')$. We assume that $f \circ h$ is not essential. Then there is a point $p \in M$ and a map $g'' : X \to M \setminus \{p\}$ with $g''|S'' = f \circ h|S''$. We may assume that $p \notin S$ because otherwise f would not be onto and hence would be inessential. Since h is a shape equivalence and $M \setminus \{p\}$ is an ANR there is a map $g' : Y \to M \setminus \{p\}$ such that $g' \circ h$ and g'' are homotopic. So $g' \circ h|S''$ is homotopic to $g''|S'' = f \circ h|S''$. Since $h|S''$ is a shape equivalence we see that $g'|S'$ and $f|S'$ are homotopic as maps into $M \setminus \{p\}$. Now the homotopy extension theorem guarantees that $f|S'$ can be extended to a map $g : Y \to M \setminus \{p\}$. This proves that f is inessential.

For any space X let $\operatorname{AE}(X)$ stand for the collection of all ANRs that are absolute extensors for X.

LEMMA 2.2. *If $h : X \to Y$ is a hereditary shape equivalence then* $\operatorname{AE}(X) \subset \operatorname{AE}(Y)$.

Proof. Let A be closed in Y and let $f \in C(A, S)$. Put $g = f \circ h|h^{-1}(A) \in C(h^{-1}(A), S)$. If $S \in \operatorname{AE}(X)$ then we can extend g to a $\widetilde{g} \in C(X, S)$. The map h is a shape equivalence so there is an $\widetilde{f} \in C(Y, S)$ such that $\widetilde{g}$ and

$\widetilde{f} \circ h$ are homotopic. The restriction $\widetilde{f} \circ h|h^{-1}(A)$ is then homotopic to $\widetilde{g}|h^{-1}(A) = f \circ h|h^{-1}(A)$. Since $h|h^{-1}(A)$ is a shape equivalence we find that f is homotopic to the extendable map $\widetilde{f}|A$ and is therefore extendable.

3. Peano maps and Cantor manifolds. We first have a look at a standard example of a space filling curve. Define the homeomorphism h between the Cantor sets $3^{\mathbb{N}} = \prod_{i=1}^{\infty}\{0,1,2\}$ and $3^{\mathbb{N}} \times 3^{\mathbb{N}}$ by $h(\varepsilon) = (\alpha, \beta)$, where

$$\alpha_i = \begin{cases} \varepsilon_{2i-1} & \text{for } \sum_{j<i} \beta_i \text{ even}, \\ 2 - \varepsilon_{2i-1} & \text{for } \sum_{j<i} \beta_i \text{ odd}, \end{cases}$$

and

$$\beta_i = \begin{cases} \varepsilon_{2i} & \text{for } \sum_{j\le i} \alpha_i \text{ even}, \\ 2 - \varepsilon_{2i} & \text{for } \sum_{j\le i} \alpha_i \text{ odd}. \end{cases}$$

Let $q : 3^{\mathbb{N}} \to I$ stand for the usual quotient map $q(\varepsilon) = \sum_{i=1}^{\infty} \varepsilon_i 3^{-i}$. Since the fibres of q contain at most two points, the fibres of $(q \times q) \circ h$ contain at most four points. One readily verifies that $q(\varepsilon) = q(\varepsilon')$ implies $(q \times q) \circ h(\varepsilon) = (q \times q) \circ h(\varepsilon')$. So there exists a continous map $p : I \to I \times I$ with $(q \times q) \circ h = p \circ q$. Then p is obviously surjective and its fibres consist of at most four points. For $n \ge 2$ we define functions $p^n : I \to I^n$ by iterating p as follows. Let p_1 and p_2 stand for the two components of p. Define $p^2 = p$, $p_i^n(t) = p_i^{n-1}(p_1(t))$, and $p_n^n(t) = p_2(t)$ for $i = 1, \ldots, n-1$ and $t \in I$. It is then easily seen that we have:

Claim 3.1. *The map p^n is surjective and each of its fibres consists of at most 4^{n-1} points.*

For $f : X \to Y$, $\operatorname{sing}(f)$ consists of all points in X such that $f^{-1}(\{f(x)\}) \neq \{x\}$. If f is a closed map then $\operatorname{sing}(f)$ is necessarily an F_σ-set and hence for compact X, $\operatorname{sing}(f)$ is σ-compact.

Claim 3.2. $\operatorname{sing}(p^n)$ *is a* 0*-dimensional σ-compact subset of* $(0,1)$.

P r o o f. Observe that $p^{-1}(0,0) = \{0\}$ and $p^{-1}(1,1) = \{1\}$ and hence that $(p^n)^{-1}(0,\ldots,0) = \{0\}$ and $(p^n)^{-1}(1,\ldots,1) = \{1\}$. So $\operatorname{sing}(p^n) \subset (0,1)$.

Consider the following countable dense subset of $(0,1)$:

$$Q = \{q(\varepsilon) \mid \varepsilon_i = 1 \text{ from some index on}\}.$$

It follows from the definition of h that $p(Q) = Q \times Q$ and hence $p^n(Q) = Q^n$. Note that every element of Q has a unique ternary representation, i.e. $q^{-1}(Q) \cap \operatorname{sing}(q) = \emptyset$. This implies that Q and $\operatorname{sing}(p^n)$ are disjoint, proving Claim 3.2.

Claim 3.2 means that we may assume that $\operatorname{sing}(p^n)$ is contained in the 0-dimensional absorber of $(0,1)$, implying that the set is thin (Geoghegan and Summerhill [11]). This means that there is an autohomeomorphism ξ_n of

$(0,1)$, arbitrarily close to the identity, such that $\xi_n(\operatorname{sing}(p^n)) \cap \operatorname{sing}(p^n) = \emptyset$. Extend ξ_n over I by putting $\xi_n(0) = 0$ and $\xi_n(1) = 1$. We define $\widetilde{p}^n = p^n \circ \xi_n$ and note that $\operatorname{sing}(\widetilde{p}^n)$ and $\operatorname{sing}(p^n)$ are disjoint.

We need some notation for decompositions. Let $\mathcal{A}$ and $\mathcal{B}$ be two decompositions of the same space X. If for each $A \in \mathcal{A}$ there is a $B \in \mathcal{B}$ with $A \subset B$ then we denote this by $\mathcal{A} < \mathcal{B}$. We say that $\mathcal{A}$ and $\mathcal{B}$ are *compatible* if for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $A \subset B$, $B \subset A$, or $A \cap B = \emptyset$. If $\mathcal{A}$ and $\mathcal{B}$ are compatible then we can define a decomposition $\mathcal{A} \vee \mathcal{B} = \{A \cup B \mid A \in \mathcal{A},\ B \in \mathcal{B},\ A \cap B \neq \emptyset\}$. If $\mathcal{A}$ and $\mathcal{B}$ are upper semicontinuous then so is $\mathcal{A} \vee \mathcal{B}$.

We now start the construction of the centrepiece of the counterexample. Let $n \geq 2$ be fixed. Let $J = [-1,1]$ and consider $D = J \times I$. Let $\pi_1 : D \to J$ and $\pi_2 : D \to I$ be the projections. Let I_{L} and I_{R} stand for the intervals $\{-1\} \times I$ and $\{1\} \times I$, respectively. $\mathcal{A}_{\mathrm{L}}$ (resp. $\mathcal{A}_{\mathrm{R}}$) stands for the decomposition of D that p^n (resp. $\widetilde{p}^n$) generates on I_{L} (resp. I_{R}). $\mathcal{B}_{\mathrm{L}}$ (resp. $\mathcal{B}_{\mathrm{R}}$) stands for the decomposition of D consisting of the fibres of $p^n \circ \pi_2$ (resp. $\widetilde{p}^n \circ \pi_2$). We have $\mathcal{A}_{\mathrm{L}} < \mathcal{B}_{\mathrm{L}}$, $\mathcal{A}_{\mathrm{R}} < \mathcal{B}_{\mathrm{R}}$, $\mathcal{A}_{\mathrm{L}}$ is compatible with $\mathcal{A}_{\mathrm{R}}$, and since $\operatorname{sing}(p^n)$ and $\operatorname{sing}(\widetilde{p}^n)$ are disjoint $\mathcal{B}_{\mathrm{L}}$ is compatible with $\mathcal{B}_{\mathrm{R}}$. So $\mathcal{A}_n = \mathcal{A}_{\mathrm{L}} \vee \mathcal{A}_{\mathrm{R}}$ and $\mathcal{B}_n = \mathcal{B}_{\mathrm{L}} \vee \mathcal{B}_{\mathrm{R}}$ are upper semicontinuous and $\mathcal{A}_n < \mathcal{B}_n$. Consider the quotient maps

$$\alpha_n : D \to D/\mathcal{A}_n \quad \text{and} \quad \beta_n : D/\mathcal{A}_n \to D/\mathcal{B}_n.$$

Observe that $I^n_{\mathrm{L}} = \alpha_n(I_{\mathrm{L}})$ and $I^n_{\mathrm{R}} = \alpha_n(I_{\mathrm{R}})$ are copies of I^n and that $D/\mathcal{A}_n$ consists of the mapping cylinders of p^n and $\widetilde{p}^n$ attached at $\{0\} \times I$. So $D/\mathcal{A}_n$ is an absolute retract. Let A_n stand for the compact AR we obtain from $D/\mathcal{A}_n$ if we attach an $(n+1)$-cube I^{n+1}_{L} with one of its n-faces to I^n_{L} and an $(n+1)$-cube I^{n+1}_{R} to I^n_{R}. Let B_n be the space we obtain if we divide A_n by the decomposition determined by β_n and let $h_n : A_n \to B_n$ be the quotient map that extends β_n. Figure 1 shows the map h_n with sample fibres.

Observe that the fibres of β_n are simply cones over the fibres of p^n and $\widetilde{p}^n$ which are finite. So both β_n and h_n are AR-maps. Note that $D/\mathcal{B}_n = \beta_n \circ \alpha_n(I_{\mathrm{L}})$ and that the fibres of $\beta_n \circ \alpha_n | I_{\mathrm{L}}$ are essentially the fibres of p^n and $\widetilde{p}^n$ and hence consist of at most 4^{n-1} points. According to [10, Theorem 4.3.3] this implies that $\dim(D/\mathcal{B}_n) \leq 4^{n-1}$. A cell-like map with finite-dimensional range is a hereditary shape equivalence and hence β_n and h_n are hereditary shape equivalences and B_n is an $(n+1)$-dimensional AR. We have

Claim 3.3. *$h_n : A_n \to B_n$ is an AR-map between ARs.*

A *Cantor manifold* X is a finite-dimensional continuum such that every separator S of X has the property $\dim S \geq \dim X - 1$. The standard exam-

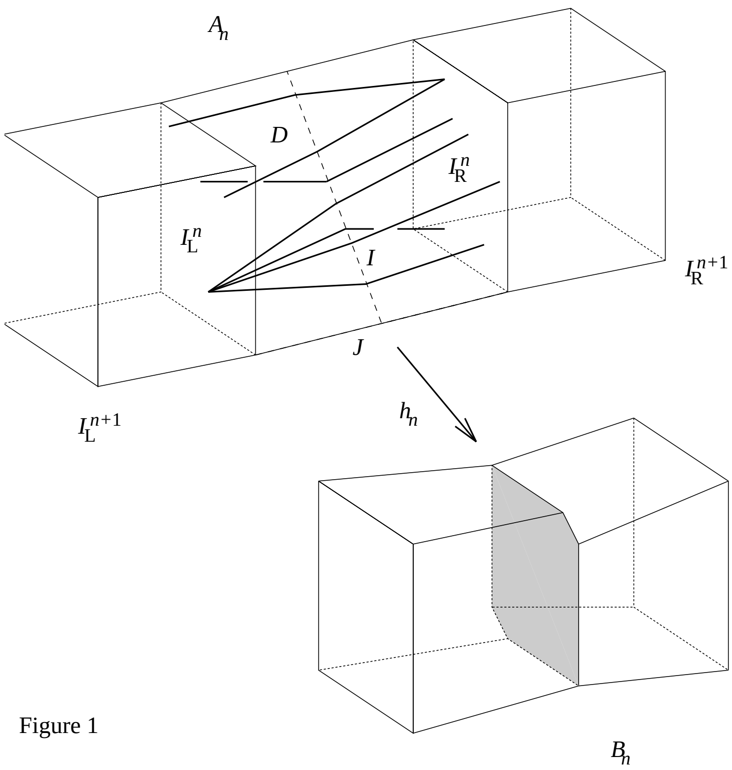

Figure 1

ples of Cantor manifolds are the cells I^n. The following lemma is trivial but useful.

LEMMA 3.4. *If X is the union of two n-dimensional Cantor manifolds whose intersection is at least $(n-1)$-dimensional then X is a Cantor manifold.*

CLAIM 3.5. *B_n is an $(n+1)$-dimensional Cantor manifold.*

Proof. It is obvious that $h_n(I_L^{n+1})$ and $h_n(I_R^{n+1})$ are Cantor manifolds so it suffices to prove that their intersection $D/\mathcal{B}_n$ is n-dimensional. Note that $\beta_n|I_L^n$ has finite fibres. A map with 0-dimensional fibres cannot lower dimension ([10, Theorem 4.3.6]) and hence $\beta_n(I_L^n) = D/\mathcal{B}_n$ is at least n-dimensional.

We now consider the face of I_R^{n+1} that is opposite to I_R^n and attach a third $(n+1)$-cube to that face in the manner descibed above, i.e. by using a double mapping cylinder like $D/\mathcal{A}_n$. We repeat this process until we have a sequence of n copies of I^{n+1} connected by $n-1$ copies of $D/\mathcal{A}_n$ and call this space $\bigvee A_n$. Let $\bigvee B_n$ be the quotient space we obtain if we replace every $D/\mathcal{A}_n$ by $D/\mathcal{B}_n$, and let $\bigvee h_n : \bigvee A_n \to \bigvee B_n$ be the quotient map.

The following statement is obvious.

CLAIM 3.6. $\bigvee h_n : \bigvee A_n \to \bigvee B_n$ *is an AR-map between compact ARs and* $\bigvee B_n$ *is an* $(n+1)$*-dimensional Cantor manifold.*

4. Proof of Theorem 1.3. Consider the compact AR K which consists of a copy of I with at each point $1/n$ $(n \geq 2)$ a cube I^n attached with the point $(0,\ldots,0)$. The topology is such that $\lim_{n\to\infty} I^n = 0 \in I$. Let $C \subset I$ be the "middle third" Cantor set and let $(s_i, r_i)_{i\geq 2}$ be an enumeration of the gaps of C. Consider the following compact subset of $I \times K$:

$$\widetilde{K} = (I \times I) \cup (C \times K) \cup \bigcup_{i=3}^{\infty} \bigcup_{n=2}^{i-1} (s_i, r_i) \times I^n.$$

Let $\mathcal{D}$ be a decomposition of $\widetilde{K}$ whose nontrivial elements are all intervals of the form $[s_i, r_i] \times \{x\}$ that are contained in $\widetilde{K}$.

The space $\widetilde{K}$ displays a number of "gaps" which we shall fill with double mapping cylinders to obtain the space X of the theorem. If $n \geq i \geq 2$ then

$$(\{s_i\} \times I^n) \cup ([s_i, r_i] \times \{1/n\}) \cup (\{r_i\} \times I^n)$$

is the boundary of such a gap. We fill this gap with a copy of $D/\mathcal{A}_n$ by identifying I^n_{L} with $\{s_i\} \times I^n$, I^n_{R} with $\{r_i\} \times I^n$, and $J \times \{0\}$ with $[s_i, r_i] \times \{1/n\}$. Note that the resulting space X consists of $I \times I$ with for each $n \geq 2$ a copy of the space $\bigvee A_n$ attached to $I \times \{1/n\}$ and that X has a natural retraction R onto $I \times I$ which maps every $\bigvee A_n$ onto $I \times \{1/n\}$. The topology is such that basic neighbourhoods in X of points of $I \times \{0\}$ have the form $R^{-1}(U)$, where U is a neighbourhood of the point in $I \times I$. The decomposition $\mathcal{F}$ of X is obtained by adding to $\mathcal{D}$ the decompositions that come with the double mapping cylinders $D/\mathcal{A}_n$. We leave it to the reader to verify that X is a compact AR and that $\mathcal{F}$ is an upper semicontinuous decomposition whose nontrivial elements are cones of finite sets. Put $Y = X/\mathcal{F}$ and let $H : X \to Y$ be the quotient map (see Figure 2).

We identify C with the subset $C \times \{0\} \subset X$. Note that $H(C)$ is an arc and that $H(I \times I) = H(C) \times I$ is a 2-cell. The space Y consists of $H(C) \times I$ with for each $n \geq 2$ a copy of the space $\bigvee B_n$ attached to $H(C) \times \{1/n\}$. So Y is countably dimensional and H is a hereditary shape equivalence. We may conclude:

CLAIM 4.1. $H : X \to Y$ *is an AR-map between compact ARs.*

It remains to verify the dimensions.

CLAIM 4.2. $\operatorname{ind} Y \geq \omega + 1$.

Proof. Let S be a closed subset of Y that separates the endpoints $(0,0)$ and $(1,0)$ of $H(C)$ in Y. Then S separates the left-hand endface of

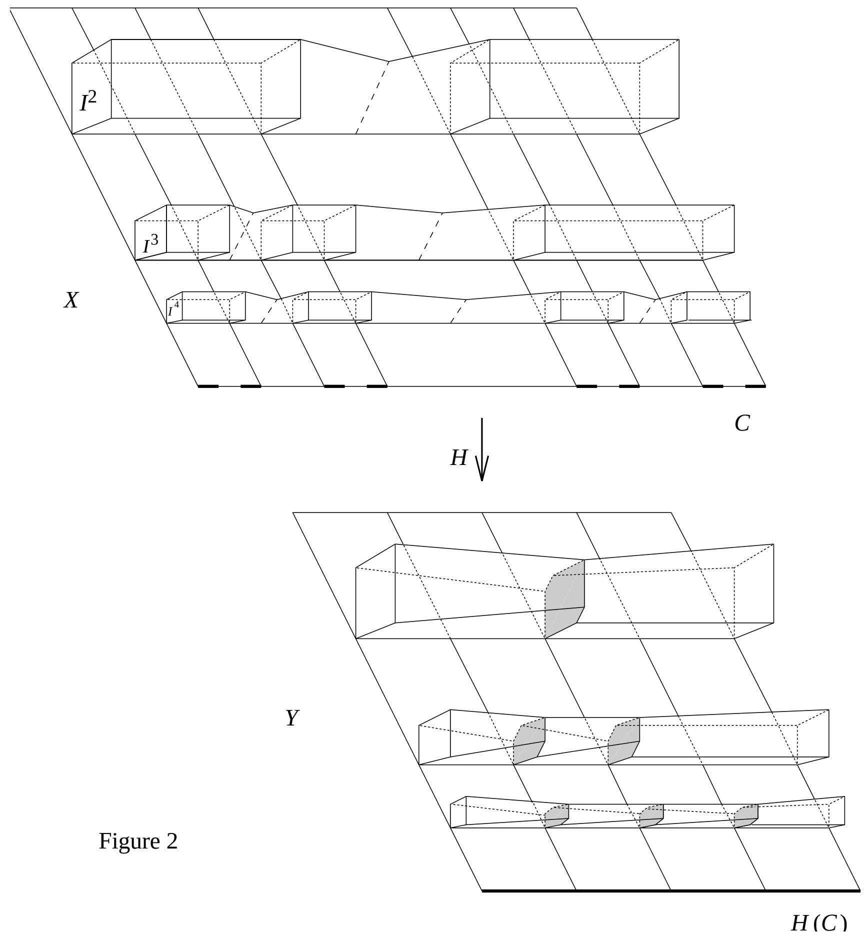

Figure 2

$\bigvee B_n$ from the right-hand endface for all n greater than some N. In view of Claim 3.6 we have $\dim(S \cap \bigvee B_n) \geq n$ and hence S is infinite-dimensional.

We shall use the following result from [5].

Proposition 4.3. *If* $\alpha < \omega_1$ *and* A *is a subset of a space* Z *such that every closed set* F *that is disjoint from* A *has* $\operatorname{Ind} F < \alpha$ *then* $\operatorname{Ind} Z \leq \alpha + \operatorname{Ind} A$.

Claim 4.4. $\operatorname{Ind} Y \leq \omega + 1$.

Proof. We apply the proposition to Y with $\alpha = \omega$ and $A = H(C)$. The complement of every neighbourhood of $H(C)$ is obviously finite-dimensional, so $\operatorname{Ind} Y \leq \omega + \operatorname{Ind} H(C) = \omega + 1$.

Claim 4.5. $\operatorname{Ind} X \leq \omega$.

P r o o f. We apply Proposition 4.3 to X with $\alpha = \omega$ and $A = C$. Observe that any closed subset F of X that is disjoint from C is contained in a set of the form

$$R^{-1}(I \times [1/N, 1]) \cup \bigcup_{i=2}^{N} R^{-1}((s_i, r_i) \times [0, 1/N))$$

for some N. The set $R^{-1}(I \times [1/N, 1])$ has dimension $N+1$ and the sets $R^{-1}((s_i, r_i) \times [0, 1/N))$ are 2-dimensional. Consequently, $\operatorname{Ind} X \le \omega + \operatorname{Ind} C = \omega$.

The proof of Theorem 1.3 is complete.

5. Conclusion. We now verify Theorem 1.2. Fix a point p in the space X of Theorem 1.3 and let X_n be a sequence of copies of X. Consider the compact AR $\widetilde{X}$ that is obtained by attaching every X_n with the point p to the point n in the interval $[1, \infty]$ and topologise the result in such a way that $\lim_{n\to\infty} X_n = \infty$. If we replace in $\widetilde{X}$ the first copy X_1 by Y then we obtain the space $\widetilde{Y}$. Let $\widetilde{H} : \widetilde{X} \to \widetilde{Y}$ be the extension with the identity of $H : X_1 \to Y$. Obviously, $\widetilde{H}$ is a hereditary shape equivalence, $\widetilde{Y}$ is a compact AR, $\operatorname{ind} \widetilde{X} = \operatorname{Ind} \widetilde{X} = \omega$, and $\operatorname{ind} \widetilde{Y} = \operatorname{Ind} \widetilde{Y} = \omega + 1$. If we identify the AR $Y \cup [1, 2]$ in $\widetilde{Y}$ to a point then we get a topological copy of $\widetilde{X}$, so there is also a hereditary shape equivalence $\widetilde{G} : \widetilde{Y} \to \widetilde{X}$. Applying Lemma 2.1 we find that $\widetilde{X}$ and $\widetilde{Y}$ are indistinguishable as far as the existence of essential maps into AH-pairs is concerned, proving Theorem 1.2.

Observe that we are dealing here with a strong counterexample, in the sense that $\widetilde{X}$ and $\widetilde{Y}$ are independent of the choice of M and S. This implies that the example also excludes the use of certain variations of the definition of essential map to characterise inductive dimension, in particular those involving (transfinite) sequences of essential maps into different spaces.

A somewhat different approach to geometric characterisation of dimension involves absolute extensors. For instance, $\dim X \le n$ if and only if $S^n \in \mathrm{AE}(X)$. One may define different dimension functions in this manner by providing a suitable sequence of ANRs (cf. Dobrowolski and Rubin [8]). Lemma 2.2 implies $\mathrm{AE}(\widetilde{X}) = \mathrm{AE}(\widetilde{Y})$ and hence we cannot characterise transfinite inductive dimension in this way.

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