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On extending automorphisms of models of Peano Arithmetic

by

Roman Kossak (New York, N.Y.)
and **Henryk Kotlarski** (Siedlce)

Abstract. Continuing the earlier research in [10] we give some information on extending automorphisms of models of PA to end extensions and cofinal extensions.

1. Introduction. For any structure, $\mathcal{M}$, we denote by $\mathrm{Aut}(\mathcal{M})$ the group of automorphisms of $\mathcal{M}$. Here we consider only models of PA (Peano Arithmetic); see Kaye [6] for models of PA and Kotlarski [15] for what is known on automorphisms of countable recursively saturated models of PA. Here we consider the question of extendability of automorphisms.

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Observe first that the problem of extending automorphisms divides into three cases, depending on whether we consider end extensions, cofinal extensions, or mixed extensions of models. (An extension is *mixed* if it is neither an end extension nor a cofinal extension. By Gaifman [2], if $\mathcal{K}$ is an extension of $\mathcal{M}$, where $\mathcal{M}, \mathcal{K} \models \mathrm{PA}$, then this extension splits as $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{M}^* \subseteq_{\mathrm{end}} \mathcal{K}$, where

$$\mathcal{M}^* = \{u \in \mathcal{K} : \text{there exists } w \in \mathcal{M} \text{ with } \mathcal{K} \models u < w\}.)$$

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The only known facts about extending automorphisms concern end extensions. Let us review them. There is an obvious necessary condition for extendability of automorphisms to greater models. Namely, let $\mathcal{M}$ and $\mathcal{K}$ be models of PA with $\mathcal{M} \prec_{\text{end}} \mathcal{K}$. If $g \in \text{Aut}(\mathcal{M})$ may be extended to some $\widehat{g} \in \text{Aut}(\mathcal{K})$ then g sends coded subsets of $\mathcal{M}$ onto coded subsets, i.e. for every $a \in \mathcal{K}$ there exists $b \in \mathcal{K}$ so that $g * (a \cap \mathcal{M}) = b \cap \mathcal{M}$.

THEOREM 1.1. *Let $\mathcal{M}$ and $\mathcal{K}$ be countable recursively saturated models of* PA *with $\mathcal{M} \prec_{\text{end}} \mathcal{K}$. If $\mathbb{N}$ does not code $\mathcal{M}$ in $\mathcal{K}$ from above then every $g \in \text{Aut}(\mathcal{M})$ such that g and g^{-1} send coded subsets onto coded subsets, can be extended to some $\widehat{g} \in \text{Aut}(\mathcal{K})$.* ∎

As usual in the theory of models of PA, $\mathbb{N}$ *codes $\mathcal{M}$ in $\mathcal{K}$ from above* if there exists a sequence $u \in \mathcal{K}$ so that $\mathcal{M} = \inf\{u_n : n \in \mathbb{N}\}$.

THEOREM 1.2. *Let $\mathcal{M}$ and $\mathcal{K}$ be countable recursively saturated models of* PA *with $\mathcal{M} \prec_{\text{end}} \mathcal{K}$. Then there exists $g \in \text{Aut}(\mathcal{M})$ which sends a coded subset of $\mathcal{M}$ onto a subset which is not coded* (*hence g cannot extend to an element of* $\text{Aut}(\mathcal{K})$). ∎

THEOREM 1.3. *If $\mathcal{M}$ is a countable recursively saturated model of* PA *then there exists a recursively saturated countable end elementary extension $\mathcal{K}$ of $\mathcal{M}$ such that* id *is the only element of* $\text{Aut}(\mathcal{M})$ *which extends to $\mathcal{K}$.* ∎

Theorems 1.1 and 1.2 are due to Kossak and Kotlarski [10], and Theorem 1.3 is due to Kossak and Schmerl [12].

In [10] we proved Theorem 1.2 by showing that, for $\mathcal{M}$ and $\mathcal{N}$ as above, there is always a subset of $\mathcal{M}$ which is coded in $\mathcal{N}$ and has continuum many automorphic images in $\mathcal{M}$. Then we asked if it is true that every undefinable class of a countable recursively saturated model of PA has continuum many automorphic images (a *class* is a subset of a model whose intersection with every initial segment with a top is definable). Jim Schmerl has provided a short elementary proof showing that this is indeed the case.

A simple observation should be added here. Namely, the assumption that $\mathcal{K}$ is recursively saturated is essential in Theorem 1.2.

THEOREM 1.4. *Let $\mathcal{M} \models$ PA. Let q be a type which is definable in the sense of Gaifman* [3]. *Let $\mathcal{K}$ be the Skolem ultrapower of $\mathcal{M}$ modulo the natural extension of q to $\mathcal{M}$. Then every $g \in \text{Aut}(\mathcal{M})$ extends to $\mathcal{K}$.* ∎

The obvious reason for Theorem 1.4 is that every subset of $\mathcal{M}$ coded in $\mathcal{K}$ is definable in $\mathcal{M}$, exactly as in Theorem 2.1 to be proved in the next section.

2. Nonelementary end extensions. In all the results stated in §1 we required the extension to be elementary. If we do not require elementarity then the situation is different. That is, Theorem 1.2 fails. Namely, we have

THEOREM 2.1. *Let $\mathcal{M}$ be any model of* PA. *Then there exists an end extension $\mathcal{K}$ of $\mathcal{M}$ which is recursively saturated and every $g \in \mathrm{Aut}(\mathcal{M})$ extends to $\mathcal{K}$. In fact, if $r \in \mathbb{N}$ then $\mathcal{K}$ may be chosen to be Σ_r-elementary.*

Proof. Let $\mathcal{M} \models \mathrm{PA}$ and $r \in \mathbb{N}$. Assume first that $\mathcal{M}$ thinks that "$\mathrm{PA} + \mathrm{Tr}_{\Sigma_r}$ is consistent". We use the so-called Arithmetized Completeness Theorem (see e.g. Smoryński [18] or Hájek–Pudlák [4] for more in this direction). That is, we fix a primitive recursive enumeration $\varphi_0, \ldots, \varphi_j, \ldots$ of all sentences and write down the formula $C(\cdot)$ which describes the following procedure: add φ_0 to C if there is no proof of $\neg\varphi_0$ from the axioms of PA and Σ_r sentences which are true, add $\neg\varphi_0$ to C otherwise, continue in the same fashion for all j (but in the axioms there are not only PA and Tr_{Σ_r} but also sentences added in earlier steps). This is done with no troubles, the universal formula Tr_{Σ_r} for Σ_r formulas is used to formalize this inside $\mathcal{M}$.

Now, one constructs a new model $\mathcal{K} = \mathrm{ACT}(\mathcal{M}; C)$. Consider the set of all variable-free terms inside $\mathcal{M}$. Divide this set by the equivalence relation $t_1 \sim t_2$ iff $\mathcal{M} \models C(t_1 = t_2)$. Define addition in the natural manner, i.e. $t_1^{\sim} + t_2^{\sim} = t_3^{\sim}$ iff $\mathcal{M} \models C(t_1 + t_2 = t_3)$. There is no problem in checking that this is well defined. We treat other atomic relations similarly. This completes the definition of $\mathcal{K}$. It requires some minor work to show that $\mathcal{K} \models \mathrm{PA}$. There is a natural embedding of $\mathcal{M}$ onto an initial segment of $\mathcal{M}$ (it is given by sending $b \in \mathcal{M}$ to the equivalence class of the bth numeral, S^b0) and if we identify $\mathcal{M}$ with its image then $\mathcal{K} \succ_{\Sigma_r} \mathcal{M}$. Finally, let us show how to extend automorphisms. So let $g \in \mathrm{Aut}(\mathcal{M})$ be given. For $t^{\sim} \in \mathcal{K}$ we put $\widehat{g}(t^{\sim}) =$ the equivalence class of the term $g(t)$. It is easy to check that $\widehat{g}$ is well defined and is an element of $\mathrm{Aut}(\mathcal{K})$.

If $\mathcal{M}$ thinks that "$\mathrm{PA} + \mathrm{Tr}_{\Sigma_r}$ is inconsistent" then we use the restricted form of the Arithmetized Completeness Theorem as developed in [16]. Let us describe the ideas briefly. First we define a hierarchy Q_n of formulas. We put $Q_0 = \Delta_0$ and $Q_{n+1} =$ the closure of $Q_n \cup \exists Q_n$ under conjunction, negation and bounded quantification. In this hierarchy every subformula of every Q_n-formula is also in Q_n, in contrast to the usual Σ_n-Π_n hierarchy. On the other hand, the universal formula Tr_{Q_n} for Q_n-formulas is itself of class Q_{n+1}. Say that a proof is Q_n iff all formulas which occur in it are of class Q_n. Now one checks that for *standard* n, $\mathcal{M}$ thinks "there exists no Q_n-proof of $0 = 1$ from $\mathrm{PA} + \mathrm{Tr}_{\Sigma_r}$" and hence "the smallest n such that there exists a Q_{n+1}-proof of $0 = 1$ from $\mathrm{PA} + \mathrm{Tr}_{\Sigma_r}$" is nonstandard and definable in $\mathcal{M}$. Call it n_0. Find a completion of $\mathrm{PA} + \mathrm{Tr}_{\Sigma_r}$ exactly as above, but the inductive condition in deciding what to put to C is "there is no Q_{n_0}-proof" rather than "there is no proof" as above. Observe that the use of n_0 in the definition of C is inessential as n_0 is definable in $\mathcal{M}$. Now define an initial segment I_0 of $\mathcal{M}$ by putting $I_0 = \inf\{n_0 - n : n \in \mathbb{N}\}$. Construct the new model $\mathcal{K}$ in the same manner as above, but consider only terms which are terms-minimum for formulas $\varphi \in \mathcal{M}$ with $\mathcal{M} \models \varphi \in Q_j$ for some $j \in I_0$. (For technical reasons it is more convenient to work with formulas in one free variable rather than with variable-free terms.) Then we check that every $g \in \mathrm{Aut}(\mathcal{M})$ fixes I_0 (setwise) and hence we can extend automorphisms exactly as above. This completes our outline of the proof of Theorem 2.1. ■

It should be noticed that the operation $g \to \widehat{g}$ (in both cases considered in the proof sketched above) is canonical enough so that it is, as a matter of fact, an embedding of $\mathrm{Aut}(\mathcal{M})$ in $\mathrm{Aut}(\mathrm{ACT}(\mathcal{M}; C))$. But this embedding is not onto; indeed, $\mathrm{id}_{\mathcal{M}}$ extends to $2^{\aleph_0}$ elements of $\mathrm{Aut}(\mathcal{K})$.

3. Cofinal extensions. By Gaifman's corollary [2] to the Matiyasevich's theorem, we need to consider only elementary cofinal extensions. Moreover, by the result of Smoryński and Stavi [20], recursive saturation is preserved under cofinal extensions.

Observe first that if $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{K}$ then we may speak freely about a subset A of $\mathcal{M}$ being coded in $\mathcal{K}$; this means simply that $A = a \cap \mathcal{M}$ for some $a \in \mathcal{K}$. Also, obviously, we have the analogue of the remark stated before Theorem 1.1, that is, every $g \in \mathrm{Aut}(\mathcal{M})$ extendable to $\mathcal{K}$ must send coded subsets to coded subsets. Let us formulate the concept to be studied in the following manner.

Definition 3.1. The extension $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{K}$ has the *automorphism extension property* (AEC for short) iff for every $g \in \mathrm{Aut}(\mathcal{M})$, if g and g^{-1} send coded subsets onto coded ones, then g is extendable to $\mathcal{K}$.

The goal of this section is the following Theorem 3.2. Its proof will give a sufficient condition for the automorphism extension property. It will be stated in Theorem 3.14.

Theorem 3.2. *Let $\mathcal{M}$ be a countable recursively saturated model of* PA *and let $\mathcal{K} \succ_{\mathrm{cof}} \mathcal{M}$ be countable. Then there exists a countable $\mathcal{R} \succ_{\mathrm{cof}} \mathcal{K}$ such that the extension $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{R}$ has the* AEC.

Before going further let us state a remark which should help the reader's intuition. Let $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{K}$. Let $a \in \mathcal{K}$. Pick any $\alpha \in \mathcal{M}$ so that $\mathcal{K} \models a < \alpha$. Then all properties of a with parameters from $\mathcal{M}$ are determined by

$$\{X \in \mathcal{M} : \mathcal{M} \models X \subseteq (<\alpha) \text{ and } \mathcal{K} \models a \in X\}.$$

Think of this family as an ultrafilter in the Boolean algebra $\{X \in \mathcal{M} : X \subseteq (<\alpha)\}$, i.e. the power set of $<\alpha$ in $\mathcal{M}$.

The argument is as follows. If φ is a formula and $m \in \mathcal{M}$ then we define (inside $\mathcal{M}$) $X = \{x \leq \alpha : \varphi(m, x)\}$, and the fact that $\mathcal{K} \models \varphi(m, a)$ is determined by $\mathcal{K} \models a \in X$. In fact, the idea presented below is just the technique of ensuring that we can speak at least to some extent about the above mentioned ultrafilter inside $\mathcal{K}$.

Let $\mathcal{M} \prec_{\rm cof} \mathcal{K}$, with $\mathcal{M}$ countable. Pick $a \in \mathcal{K} \setminus \mathcal{M}$. Pick $\alpha \in \mathcal{M}$ so that $\mathcal{K} \models a < \alpha$. Enumerate the power set (in the sense of $\mathcal{M}$) of $< \alpha$,

$$\mathrm{P}^{\mathcal{M}}(< \alpha) = \{Z_n : n \in \mathbb{N}\},$$

using the countability assumption. We define a sequence $\langle Y_n : n \in \mathbb{N}\rangle$ by putting $Y_0 = Z_0$ if $a \in Z_0$ and $Y_0 = (< \alpha) \setminus Z_0$ otherwise. If Y_n is defined then we take $Y_{n+1} = Y_n \cap Z_{n+1}$ if $a \in Z_{n+1}$ and $Y_{n+1} = Y_n \cap [(< \alpha) \setminus Z_{n+1}]$ otherwise. Obviously, the sequence $\langle Y_n : n \in \mathbb{N}\rangle$ has the following properties:

1. $Y_n \in \mathcal{M}$ for every $n \in \mathbb{N}$,
2. $Y_n \supseteq Y_{n+1}$ for all $n \in \mathbb{N}$ in both models $\mathcal{K}, \mathcal{M}$,
3. $\mathcal{K} \models a \in Y_n$ for all $n \in \mathbb{N}$,
4. for every set $X \in \mathcal{M}$ with $\mathcal{K} \models a \in X$ there is $n \in \mathbb{N}$ such that $\mathcal{K} \models Y_n \subseteq X$.

It is convenient to think of a sequence $\langle Y_n : n \in \mathbb{N}\rangle$ with these properties as a *describing* sequence; it describes all properties of a over $\mathcal{M}$.

Definition 3.3. The extension $\mathcal{M} \prec_{\rm cof} \mathcal{K}$ has the *description property* (DP for short) iff for every $a \in \mathcal{K} \setminus \mathcal{M}$ there exists a describing sequence $\langle Y_n : n \in \mathbb{N}\rangle$ which is coded in $\mathcal{K}$.

Of course, the describing sequences were constructed above in a completely external manner; the heart of the matter in the above notion is that we require describing sequences to be coded in the greater model. As we shall see below, the description property allows us to perform a single step in the "back and forth" procedure of extending an automorphism. But in order to ensure that the extension also sends coded sets onto coded ones we need an additional idea of covering. But first let us point out that no extension with the description property is finitely generated.

Lemma 3.4. *Let $\mathcal{M} \prec_{\rm cof} \mathcal{R}$ be a proper extension with the description property and let $a \in \mathcal{R}$. Then the Skolem closure of $\mathcal{M} \cup \{a\}$ in R, $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, a)$, is strictly smaller than R.*

Proof. Assume the contrary, $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, a) = \mathcal{R}$ for some $a \in \mathcal{R}$; we derive a contradiction. If $a \in \mathcal{M}$ then we are done as the extension is proper, so assume $a \notin \mathcal{M}$. Pick a sequence $Y \in \mathcal{R}$ describing a. Clearly we may assume that Y is decreasing with respect to inclusion, otherwise we could work with the sequence $Z_n = \bigcap_{j<n} Y_j$ which is coded if Y is. We claim that for every $n \in \mathbb{N}$, $\mathcal{R} \models \mathrm{card}(Y_n) > 1$. For otherwise a would be the only element of Y_n and hence $a \in \mathcal{M}$. By overspill there exists a nonstandard

$r \in \mathcal{R}$ such that
$$\mathcal{R} \models \mathrm{card}(Y_r) > 1 \wedge \forall i < j \leq r\ [Y_i \supseteq Y_j].$$
So fix such an $r \in \mathcal{R}$ and pick $b \neq a$ which is an element of Y_r. This b realizes the same type as a with all parameters from $\mathcal{M}$, by the description property. But also $b = s(m, a)$ for some term $s(\cdot, \cdot)$ and some $m \in \mathcal{M}$. This contradicts the result of Gaifman [3], Theorem 4.1, and Ehrenfeucht [1] (its proof may be also found in Kaye [7], Lemma 4.1) which states that the generator of a simple extension is the only element of the extension realizing its type over the model being extended. ∎

Let us go to the idea of covering.

Definition 3.5. An extension $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{R}$ has the *covering property* (CP for short) iff for every $\gamma \in \mathcal{M}$ there exists a sequence $\langle E_n : n \in \mathbb{N}\rangle$ which is coded in $\mathcal{R}$, is increasing with respect to inclusion and

1. $E_n \in \mathcal{M}$ for all $n \in \mathbb{N}$,
2. $\{x \in \mathcal{M} : \mathcal{M} \models x < \gamma\} = \{x \in \mathcal{R} : \exists n \in \mathbb{N}\ [\mathcal{R} \models x \in E_n]\}$.
3. for every set $e \in \mathcal{R}$ and $n \in \mathbb{N}$, the intersection $e \cap E_n$ is in $\mathcal{M}$.

It is convenient to think of the sequence $\langle E_n\rangle$ as a sequence *covering* $(< \gamma)$ in such a way that for every standard n, $E_n \subseteq \mathcal{M}$, and for nonstandard j, E_j adds no new elements of $\mathcal{M}$ below γ. The last condition may be thought of as some sort of comprehension, also for sets in $\mathcal{R} \setminus \mathcal{M}$. From a more technical point of view the assumption that the extension $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{R}$ has the covering property will play the role of the additional assumption ($\mathbb{N}$ does not $\downarrow$ code $\mathcal{M}$ in $\mathcal{R}$) in Theorem 1.1. To be more exact, it is an analogue of "$\mathbb{N}$ codes $\mathcal{M}$ from below in $\mathcal{R}$".

In order to prove the existence result (Lemma 3.10) we need an auxiliary notion.

Definition 3.6. An extension $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{R}$ has the *strong covering property* iff for every $\gamma \in \mathcal{M}$ there exists a sequence $e \in \mathcal{R}$ such that $\{x \in \mathcal{M} : x < \gamma\} = \{e_n : n \in \mathbb{N}\}$.

Remark 3.7. *Every cofinal extension with strong covering property has the covering property.*

Proof. Indeed, if $e \in \mathcal{R}$ has the property granted by Definition 3.6, then we put $E_n = \{e_0, \ldots, e_{n-1}\}$ and we see that the conditions from Definition 3.5 are satisfied. ∎

The following fact is obvious:

Lemma 3.8. *If the extension $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{K}$ has the strong covering property and $\mathcal{R}$ is a cofinal extension of $\mathcal{K}$ then the extension $\mathcal{M} \prec \mathcal{R}$ has the strong covering property.* ∎

LEMMA 3.9. *If $\mathcal{M} \prec_{\rm cof} \mathcal{K}$ are countable models then there exists a countable $\mathcal{R} \succ_{\rm cof} \mathcal{K}$ such that the extension $\mathcal{M} \prec \mathcal{R}$ has the strong covering property.*

Proof. Let the extension $\mathcal{M} \prec_{\rm cof} \mathcal{K}$ be given. Pick $\gamma \in \mathcal{M}$. Enumerate $(<\gamma)^{\mathcal{M}} = \{u_n : n \in \mathbb{N}\}$, using the countability assumption. Let $e_n = \{u_j : j < n\}$. Extend $\mathcal{K}$ in a cofinal way to obtain $\mathcal{R}_0$ which contains a sequence e with this property. Do this for all γ, i.e. enumerate $\mathcal{M} = \{\gamma_n : n \in \mathbb{N}\}$ and iterate for all n. Clearly we do not loose countability. ∎

COROLLARY 3.10. *If $\mathcal{M} \prec_{\rm cof} \mathcal{K}$ are countable models then there exists a countable $\mathcal{R} \succ_{\rm cof} \mathcal{K}$ such that the extension $\mathcal{M} \prec \mathcal{R}$ has the covering property.* ∎

The following fact was pointed out to us by J. Schmerl; it allowed us to simplify the material of this section considerably.

LEMMA 3.11. *If the extension $\mathcal{M} \prec_{\rm cof} \mathcal{K}$ has the covering property then it has the description property as well.*

Proof. Let $a \in \mathcal{K} \setminus \mathcal{M}$. Pick $u \in \mathcal{M}$ which is greater than a using the cofinality assumption. Let $E \in \mathcal{K}$ witness the covering property of the extension $\mathcal{M} \prec_{\rm cof} \mathcal{K}$ for 2^u. For every n we let $e_n = \{A \in E_n : a \in A\}$; this definition is in $\mathcal{K}$. Then $e_n \in \mathcal{K}$ and hence $E_n \cap e_n \in \mathcal{M}$, so $e_n \in \mathcal{M}$. We let Y_n be the intersection (in the sense of $\mathcal{M}$) of e_n. It is easy to check that Y witnesses the description property. ∎

Let us recall that if $\mathcal{M} \prec \mathcal{R}$ then every element $a \in \mathcal{R} \setminus \mathcal{M}$ determines a new model, the Skolem hull of $\mathcal{M} \cup \{a\}$ in $\mathcal{R}$; we shall denote it $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, a)$.

LEMMA 3.12. *Let $\mathcal{M}$ be recursively saturated, let the extension $\mathcal{M} \prec_{\rm cof} \mathcal{R}$ have the covering property and let $a \in \mathcal{R}$. Then the extension $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, a) \prec \mathcal{R}$ also has the covering property.*

Proof. Let $\gamma \in \mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, a)$. Then γ is of the form $\gamma = t(m, a)$ for some term t and some $m \in \mathcal{M}$. Pick $\delta \in \mathcal{M}$ which is greater than γ by the assumption of cofinality of the extension. Let $\langle E_n \rangle$ be a sequence covering δ in $\mathcal{R}$. Let $\{s_r : r \in \mathbb{N}\}$ be a recursive enumeration of terms in two variables. Consider the type

$$\Delta(\varrho) = \{\forall m\ [s_r(m, a) < \delta \Rightarrow \exists m' < \varrho,\ (s_r(m, a) = s_r(m', a))] : r \in \mathbb{N}\}.$$

This type is clearly consistent. Indeed, for any single s the appropriate sentence is easily provable by induction on δ. For finitely many terms we may take the maximum of the values obtained in this way. So let ϱ realize $\Delta(\cdot)$ in $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, a)$. By recursive saturation of $\mathcal{M}$ and the Smoryński–Stavi theorem, $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, a)$ is also recursively saturated, so let S be a (partial inductive) satisfaction class for $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M},a)$. We let

$$E'_n = \{x < \gamma : \exists j \le n\ \exists m' < \varrho\ [S(s_j(m',a) = x) \wedge m' \in E_n]\}$$

and it is easy to check that the sequence $\langle E'_n : n \in \mathbb{N}\rangle$ covers γ in $\mathcal{R}$ over $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M},a)$. In particular, if $e \in \mathcal{R}$ and $n \in \mathbb{N}$ then $e \cap E'_n \in \mathrm{Hull}^{\mathcal{R}}(\mathcal{M},a)$. Indeed, given a single term s, $\{x < \gamma : \exists m'\ [x = s(m',a) \wedge m' \in e \wedge x \in e \cap E_n]\}$ is in $\mathcal{M} \subseteq \mathrm{Hull}^{\mathcal{R}}(\mathcal{M},a)$ by the third condition from the definition, because $e \cap E_n \in \mathcal{M}$. ∎

By the way, we also have

LEMMA 3.13. *If $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{R}$ has the description property then for every $c \in \mathcal{R}$, the extension $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M},c) \prec_{\mathrm{cof}} \mathcal{R}$ also has the description property.*

Proof. Let $d \in \mathcal{R} \setminus \mathrm{Hull}^{\mathcal{R}}(\mathcal{M},c)$ and let Y be a sequence in $\mathcal{R}$ describing the pair $\langle c,d\rangle$ over M. Pick any $\alpha \in \mathcal{M}$ which is greater than this pair, using the cofinality assumption. Let $Y'_n = \{x < \alpha : \langle c,x\rangle \in Y_n\}$. Obviously all $Y'_n \in \mathrm{Hull}^{\mathcal{R}}(\mathcal{M},c)$. It is equally clear that this new sequence is decreasing and $d \in Y'_n$ for all $n \in \mathbb{N}$. Let $X' \in \mathrm{Hull}^{\mathcal{R}}(\mathcal{M},c)$ be such that $d \in X'$. Then X' is of the form $t(m,c)$ for some Skolem term t and some $m \in \mathcal{M}$. Let $X = \{\langle x,y\rangle : y \in t(m,x) \wedge x,y < \alpha\}$. Then $\langle c,d\rangle \in X \in \mathcal{M}$ and by the properties of this describing sequence, there exists $n \in \mathbb{N}$ so that $Y_n \subseteq X$. For this n we have $d \in Y'_n = \{y < \alpha : \langle c,y\rangle \in Y_n\} \subseteq \{y < \alpha : y \in t(m,c)\} = X$ and we are done. ∎

Theorem 3.2 is a consequence of the above results and

THEOREM 3.14. *Let the extension $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{R}$ have the covering property. Assume that $\mathcal{M}$ is recursively saturated and $\mathcal{R}$ is countable. Then this extension has the automorphism extension property.*

Observe that if the assumptions of Theorem 3.14 are satisfied then both models $\mathcal{M}$ and $\mathcal{R}$ are recursively saturated and countable, by the Smoryński–Stavi result. We shall use this observation without explicit mention.

As a matter of fact we shall have to work with two cofinal submodels, $\mathcal{M}$ and $\mathcal{N}$, of the same model $\mathcal{R}$. Once again, there is no problem in defining the notion of an isomorphism g of $\mathcal{M}$ onto $\mathcal{N}$ sending coded sets onto coded ones. This means simply that for every $c \in \mathcal{R}$ there exists $d \in \mathcal{R}$ so that $g * (c \cap \mathcal{M}) = d \cap \mathcal{N}$.

LEMMA 3.15. *Let $\mathcal{M},\mathcal{N} \prec_{\mathrm{cof}} \mathcal{R}$, and let $g : \mathcal{M} \to \mathcal{N}$ be an isomorphism which sends coded (in $\mathcal{R}$) subsets of $\mathcal{M}$ onto subsets of $\mathcal{N}$ which are also coded. Let $c,d \in \mathcal{R}$ be such that there is an isomorphism $\widehat{g} : \mathrm{Hull}^{\mathcal{R}}(\mathcal{M},c) \to \mathrm{Hull}^{\mathcal{R}}(\mathcal{N},d)$ extending g. Assume moreover that $\mathcal{M}$ is recursively saturated and the extension $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{R}$ has the covering property. Then $\widehat{g}$ also sends coded subsets onto coded ones.*

P r o o f. Of course, $\widehat{g}$ exists iff for every formula φ and every $m \in \mathcal{M}$, $\mathcal{R} \models \varphi(m,c)$ iff $\mathcal{R} \models \varphi(g(m),d)$; and if this condition is satisfied then $\widehat{g}$ must be defined in the "natural" way, that is, for every term t and $m \in \mathcal{M}$, $\widehat{g}(t(m,c)) = t(g(m),d)$.

So assume that the above condition is satisfied. Let $u \in \mathcal{R} \setminus \mathcal{M}$. We want $w \in \mathcal{R}$ so that $\widehat{g} * [u \cap \mathrm{Hull}^{\mathcal{R}}(\mathcal{M},c)] = w \cap \mathrm{Hull}^{\mathcal{R}}(\mathcal{N},d)$. If $u \in \mathrm{Hull}^{\mathcal{R}}(\mathcal{M},c)$ then we let $w = \widehat{g}(u)$, so assume that $u \notin \mathrm{Hull}^{\mathcal{R}}(\mathcal{M},c)$.

Pick $\alpha \in \mathcal{R}$ such that $u < \alpha$. Passing to a greater element if necessary we may assume $\alpha \in \mathcal{M}$ by the assumption of cofinality of the extension. Consider the type

$$\Delta(\gamma) = \{\forall m\ [t(m,c) < \alpha \Rightarrow \exists m' < \gamma\ (t(m,c) = t(m',c))] : t \text{ is a term}\}.$$

This type is consistent. Indeed, for any single (Skolem) term t the appropriate sentence is easily provable in PA (by induction on α). So let $\gamma \in \mathrm{Hull}^{\mathcal{R}}(\mathcal{M},c)$ realize Δ. Once again, we may assume $\gamma \in \mathcal{M}$, otherwise we pass to a greater element.

By the choice of α, for every term t we have

$$\mathcal{R} \models \forall m\ [t(m,c) \in u \Rightarrow \exists m' < \gamma\ (t(m,c) = t(m',c))].$$

Let $t_n(\cdot,\cdot)$ be an enumeration of terms in two variables. We encode in $\mathcal{R}$ the set

$$e = \{\langle m,n\rangle : m < \gamma \wedge t_n(m,c) \in u\}.$$

To be more exact, we realize in $\mathcal{R}$ the type

$$\begin{aligned}\Sigma(e) = \{&\forall x \in e\ [\mathrm{Seq}(e) \wedge \mathrm{lh}(e) = 2]\}\\ &\cup \{\forall m\ [\langle m,n\rangle \in e \equiv m < \gamma \wedge t_n(m,c) \in u] : n \in \mathbb{N}\}.\end{aligned}$$

Obviously this type is consistent; let e realize it in $\mathcal{R}$. By the assumption, g sends coded sets onto coded ones, hence there exists $f \in \mathcal{R}$ such that $g * (e \cap \mathcal{M}) = f \cap \mathcal{M}$. Then for every $n \in \mathbb{N}$ and every $m \in \mathcal{M}$ we have

$$\langle m,n\rangle \in e \equiv \langle g(m),n\rangle \in f.$$

Let us change notation slightly. For every term t we put $e_t = \{m < \gamma : t(m,c) \in u\}$. Then the above fact may be written in the following way:

$$\text{for every term } t, \quad g * [e_t \cap \mathcal{M}] = f_t \cap \mathcal{N}.$$

Consider the type

$$\Xi(w) = \{\forall m''\ [m'' \in f_t \equiv (m'' < g(\gamma) \wedge t(m'',d) \in w)] : t \text{ a term}\}.$$

We claim that there exists $w \in \mathcal{R}$ realizing $\Xi(\cdot)$. Once again, enumerate all terms in two variables as $t_0,\ldots,t_{r-1},\ldots$ By the assumption, for every $r \in \mathbb{N}$ we have

$$\mathcal{R} \models \exists u' \bigwedge_{i<r} \forall m\ [m \in e_i \equiv (m < \gamma \wedge t_i(m,c) \in u')];$$

indeed, $u' = u$ satisfies this. Let $\langle E_n : n \in \mathbb{N}\rangle$ cover γ in $\mathcal{R}$ over $\mathcal{M}$. Thus $E_n \in \mathcal{M}$ for every standard n and

$$\{x \in \mathcal{M} : \mathcal{M} \models x < \gamma\} = \{x \in \mathcal{R} : \exists n \in \mathbb{N}\ [\mathcal{R} \models x \in E_n]\}.$$

For a fixed $n \in \mathbb{N}$ we rewrite the above formula in the following way:

$$\mathcal{R} \models \exists u' \bigwedge_{i<r} \forall m \in E_n\ [m \in e_i \equiv (m < \gamma \wedge t_i(m, c) \in u')].$$

Let us rewrite this formula again:

$$\mathcal{R} \models \exists u' \bigwedge_{i<r} \forall m\ [(m \in E_n \cap e_i) \equiv (m < \gamma \wedge t_i(m, c) \in u')].$$

But g sends coded sets to coded ones, hence there exists $F \in \mathcal{R}$ so that $g(E_n) = F_n$ for all standard n. Fix n. Hence all parameters in this formula are in $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, c)$, by the third condition of Definition 3.5, so this model satisfies this formula. It follows that $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, d)$ satisfies the same formula, but with c replaced by d, E by F, and e_i by f_i, so by elementarity,

$$\mathcal{R} \models \exists w \bigwedge_{i<r} \forall m \in F_n\ [m \in f_i \equiv (m < g(\gamma) \wedge t_i(m, d) \in w)].$$

Let us sum everything up. We have shown that for every $r \in \mathbb{N}$ and every $n \in \mathbb{N}$ the model $\mathcal{R}$ satisfies the above formula. Pick a partial inductive satisfaction class S for $\mathcal{R}$ and define, in $(\mathcal{R}, S)$,

$$n(r) = \max\Big\{n : \exists w \bigwedge_{i<r} \forall m \in F_n\ [m \in f_i \equiv (m < g(\gamma) \wedge t_i(m, d) \in w)]\Big\}.$$

Thus, for every standard r, $n(r)$ is nonstandard, so $r < n(r)$. By undefinability of $\mathbb{N}$ we see that there exists a nonstandard r such that $n(r)$ is nonstandard and the w granted by it realizes $\Xi(\cdot)$. This w codes $\widehat{g}(u)$. ∎

Having proved all the lemmas, we prove Theorem 3.14 by the usual back and forth method. Clearly it suffices to prove the following "back and forth lemma".

LEMMA 3.16. *Let $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{R}$, let finite sequences $\bar{a}$ and $\bar{b}$ of elements of $\mathcal{R}$ be given and let $g : \mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{a}) \to \mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{b})$ be an isomorphism sending sets coded* (*in $\mathcal{R}$*) *onto coded ones. Assume moreover that both extensions $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{a}) \prec \mathcal{R}$ and $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{b}) \prec \mathcal{R}$ have the covering property. Then for every $a \in \mathcal{R}$ there exists $b \in \mathcal{R}$ such that there exists an isomorphism g' : $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{a}, a) \to \mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{b}, b)$ extending g and sending coded sets onto coded ones*, *and both extensions $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{a}, a) \prec \mathcal{R}$ and $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{b}, b) \prec \mathcal{R}$ have the covering property.*

Proof. Observe first that by Lemmas 3.4 and 3.11, $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{a}) \neq \mathcal{R}$ and the same for $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{b})$. Let a be given. If $a \in \mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{a})$ then we change nothing, i.e. we put $b = g(a)$ and $g' = g$. So assume $a \notin \mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{a})$.

Let the sequence $\langle Y_n : n \in \mathbb{N}\rangle$ describe a over $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{a})$ in $\mathcal{R}$. By the assumption and the same reasoning as in the proof of Lemma 3.15 there exists a sequence $Z \in \mathcal{R}$ such that $g(Y_n) = Z_n$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, $\mathcal{R} \models Y_n \neq \emptyset$ (because $a \in Y_n$), hence $Z_n \neq \emptyset$ as well. Moreover, for every n, the sequence $\langle Z_0, \ldots, Z_{n-1}\rangle$ is decreasing because so is the appropriate sequence $\langle Y_0, \ldots, Y_{n-1}\rangle$. By overspill in $\mathcal{R}$, the same two facts hold for some nonstandard n. We take any $w \in Z_n$ for such n and any $b \in w$. By the properties of describing sequences we have

$$\mathcal{R} \models \varphi(m, a) \equiv \varphi(g(m), b)$$

for each formula φ and each $m \in \mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{a})$. Clearly, we can now define g' by $g'(t(m, a)) = t(g(m), b)$. By Lemmas 3.12 and 3.13 both extensions $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{a}, a) \prec \mathcal{R}$ and $\mathrm{Hull}^{\mathcal{R}}(\mathcal{M}, \bar{b}, b) \prec \mathcal{R}$ have the covering property. Finally, by Lemma 3.15, g' sends coded subsets to coded ones. ■

4. Small submodels. Yet another class of elementary extensions of models of PA was used by D. Lascar [17]. His class is defined as follows. Call an elementary submodel $\mathcal{M}$ of $\mathcal{R}$ *small* (in $\mathcal{R}$) if $|\mathcal{M}| = \{c_n : n \in \mathbb{N}\}$ for some sequence $c \in \mathcal{R}$. As Lascar observes himself, a small elementary submodel $\mathcal{M}$ may have $2^{\aleph_0}$ automorphisms (stronger: $\mathcal{M}$ may be recursively saturated), but at most countably many of those automorphisms may be extended to $\mathcal{R}$. Indeed, if $g \in \mathrm{Aut}(\mathcal{M})$ is extendable to $\widehat{g} \in \mathrm{Aut}(\mathcal{R})$, then the restriction of $\widehat{g}$ to $\mathcal{M}$ is entirely determined by $\widehat{g}(c)$.

It is easy to check that if $\mathcal{M}$ is small in $\mathcal{R}$ then $\mathcal{M}$ is neither an initial segment nor cofinal in $\mathcal{R}$. If $|\mathcal{M}| = \{c_n : n \in \mathbb{N}\}$ then $c > c_n$ for all $n \in \mathbb{N}$, so $\mathcal{M}$ is not cofinal in $\mathcal{R}$. In order to check that $\mathcal{M}$ is not a cut of $\mathcal{R}$, it suffices to show that $\mathrm{SSy}(\mathcal{M})$ is strictly smaller than $\mathrm{SSy}(\mathcal{R})$. Indeed, every (notrivial) cut of $\mathcal{R}$ has the same standard system as $\mathcal{R}$. But define (inside $\mathcal{R}$) $E = \{n : n \notin c_n\}$. If $E \cap \mathbb{N} \in \mathrm{SSy}(\mathcal{M})$ then $E \cap \mathbb{N} = \mathbb{N} \cap c_{n_0}$ for some $n_0 \in \mathbb{N}$. Now, $n_0 \in E$ iff $n_0 \in c_{n_0}$ iff $n_0 \notin c_{n_0}$, contradiction, and so $E \cap \mathbb{N} \notin \mathrm{SSy}(\mathcal{M})$.

THEOREM 4.1. *Let $\mathcal{M}$ be a small recursively saturated elementary submodel of a countable recursively saturated $\mathcal{R}$, say $|\mathcal{M}| = \{c_n : n \in \mathbb{N}\}$, where $c \in \mathcal{R}$. Let $\mathcal{K} = \{a \in \mathcal{R} : \exists n \in \mathbb{N}\ (a < c_n)\}$ be the closure of $\mathcal{M}$ in $\mathcal{R}$ under initial segment. Let $g \in \mathrm{Aut}(\mathcal{M})$. Then g is extendable to $\mathcal{K}$ iff g and g^{-1} send subsets of $\mathcal{M}$ which are coded in $\mathcal{K}$ onto subsets with the same property.*

P r o o f. The implication $\Rightarrow$ is obvious. In order to prove the not obvious implication, we simply check that the extension $\mathcal{M} \prec \mathcal{K}$ has the covering property and apply Theorem 3.14; we may apply it as the extension under consideration is cofinal. So let $\gamma \in \mathcal{M}$. Then $\gamma = c_j$ for some $j \in \mathbb{N}$. Enumerate by $\{e_n : n \in \mathbb{N}\}$ all those c_n which are smaller than c_j; clearly this is possible inside $\mathcal{R}$. The same argument as in Lemma 3.7 yields the result. ∎

The following proposition shows that in the case of small submodels the condition of sending coded subsets onto coded ones may be simplified considerably.

PROPOSITION 4.2. *Let $\mathcal{M} \prec \mathcal{R}$ be small, $|\mathcal{M}| = \{c_n : n \in \mathbb{N}\}$, where $c \in \mathcal{R}$. Let $g \in \mathrm{Aut}(\mathcal{M})$. Then g sends coded (in $\mathcal{R}$) subsets onto coded ones iff there exists $d \in \mathcal{R}$ such that $g(c_n) = d_n$ for all $n \in \mathbb{N}$.*

Proof. Assume that g sends coded subsets onto coded ones. Consider the set $U = \{\langle n, c_n\rangle : n \in \mathbb{N}\}$. Here the notion of an ordered pair is in the set-theoretic sense. Obviously, this set is coded in $\mathcal{R}$, so its g-image is coded as well, and we can read off from it the desired d.

For the converse, let $d \in \mathcal{R}$ code $g(c)$. Let $u \in \mathcal{R}$. Put (inside $\mathcal{R}$) $E = \{n < \mathrm{lh}(c) : c_n \in u\}$. For every $m \in \mathbb{N}$,

$$\mathcal{R} \models \exists w\ \forall x\ [x \in w \equiv \exists j < m\ (j \in E \wedge x = d_j)].$$

By overspill this holds for some $m > \mathbb{N}$, and the w granted by this codes $g * (u \cap \mathcal{M})$ in $\mathcal{R}$. ∎

THEOREM 4.3. *Let $\mathcal{R}$ be a countable recursively saturated model of* PA, *let $\mathcal{M} \prec \mathcal{R}$ be small, $|\mathcal{M}| = \{c_n : n \in \mathbb{N}\}$, and let $g \in \mathrm{Aut}(\mathcal{M})$. Then g extends to $\mathcal{R}$ iff there exists $d \in \mathcal{R}$ such that $g(c_n) = d_n$ for all $n \in \mathbb{N}$ and the same holds for g^{-1}.*

Proof. One direction is obvious. For the converse we apply the usual "back and forth" construction. The inductive condition is as follows. We let $\bar{\alpha} = \langle \alpha_0, \ldots, \alpha_{r-1}\rangle$ and similarly for $\bar{\beta}$. Then

$$\widehat{g}(\bar{\alpha}) = \bar{\beta} \Rightarrow \forall n \in \mathbb{N}\ [\mathrm{tp}(c{\restriction}n, \bar{\alpha}) = \mathrm{tp}(d{\restriction}n, \bar{\beta}) \text{ in } \mathcal{R}].$$

So assume that $\bar{\alpha}$ and $\bar{\beta}$ satisfy this condition and let α be given. Consider the type

$$\Gamma(x) = \{\varphi(c{\restriction}n, \bar{\alpha}, \alpha) \equiv \varphi(d{\restriction}n, \bar{\beta}, x) : \varphi, n\}.$$

It suffices to show that this type is consistent; indeed, if it is then we realize it in $\mathcal{R}$. But if it were inconsistent, then for some $m \in \mathbb{N}$, the type

$$\Gamma_m(x) = \{\varphi(c{\restriction}m, \bar{\alpha}, \alpha) \equiv \varphi(d{\restriction}m, \bar{\beta}, x) : \varphi\}$$

would be inconsistent as well (we may add superfluous items of c and d if necessary to get one m). So fix m. We enumerate as $\{\varphi_i : i < k\}$ all formulas φ which occur in (a finite part of) Γ_m. Change them in the following manner. Put $\psi_i = \varphi_i$ if $\mathcal{R} \models \varphi(c{\restriction}m, \bar{\alpha}, \alpha)$ and $\psi_i = \neg\varphi_i$ otherwise. Let ψ be $\bigwedge\!\!\!\bigwedge_{i<k} \psi_i$. Then $\mathcal{R} \models \exists x\ \psi(c{\restriction}m, \bar{\alpha}, x)$; indeed, $x = \alpha$ is good. But this is not true of the sequence $d{\restriction}m, \bar{\beta}$, and this contradicts the inductive assumption. ∎

COROLLARY 4.4. *Under the assumption of Theorem* 4.3, g *extends to* $\mathcal{R}$ *iff the function* $F : \mathbb{N} \to \mathbb{N}$ *defined as* $F(n) = k$ *iff* $f(c_n) = c_k$ *is in* $\mathrm{SSy}(\mathcal{M})$. ■

5. Cuts coded by the standard part from above. In this section we shall show that the annoying extra assumption in Theorem 1.1 cannot be eliminated. That is, we shall prove

THEOREM 5.1. *Let* $\mathcal{M}$ *be a countable recursively saturated model of* PA. *Then there exist* $a \in \mathcal{M}$ *and* $f \in \mathrm{Aut}(\mathcal{M}[a])$ *such that* f *and* f^{-1} *send coded sets onto coded ones*, *but* f *is not extendable to an element of* $\mathrm{Aut}(\mathcal{M})$.

Here, as usual, we denote by $\mathcal{M}(a)$ the smallest elementary cut of $\mathcal{M}$ containing a, i.e.

$$\mathcal{M}(a) = \sup\{t(a) : t \text{ is a Skolem term}\},$$

and

$$\mathcal{M}[a] = \sup\{b : \mathcal{M}(b) < a\}$$

is the greatest elementary cut of $\mathcal{M}$ not containing a. (We treat this last notion as undefined if a is smaller than some definable element of $\mathcal{M}$.) The set-theoretic difference of these, i.e.

$$\mathrm{gap}(a) = \mathcal{M}(a) \setminus \mathcal{M}[a],$$

is called the *gap* about a.

There is an obvious ordering on the family of all gaps of $\mathcal{M}$, i.e.

$$\mathrm{gap}(a) < \mathrm{gap}(b) \quad \text{iff} \quad \mathcal{M}(a) < \mathcal{M}[b].$$

As usual when working with the above notions it is convenient to work with some "fast growing Skolem functions" (cf. [14]). In this paper it will suffice to work with the following sequence:

$$\begin{aligned} F_n(x) = \min\{y : \forall \varphi, u < x\ [\exists v\ \ \mathrm{Tr}_{\Sigma_n}(\varphi, \langle u, v, x\rangle)] \\ \Rightarrow \exists v < y\ \ \mathrm{Tr}_{\Sigma_n}(\varphi, \langle u, v, x\rangle)\}. \end{aligned}$$

The obvious properties of these functions are:

(i) The formula $F_n(x) = y$ is Σ_{n+1}.

(ii) If t is a Σ_n Skolem term and $\mathcal{M} \models \mathrm{PA}$ then for every nonstandard $a \in \mathcal{M}$, $\mathcal{M} \models \forall x < a\ \ [t(x, a) < F_n(a)]$.

We shall use the symbol $\equiv_n$ to denote Σ_n-elementary equivalence.

The proof of Theorem 5.1 will be given in a series of lemmas.

LEMMA 5.2. *If* $a, b \in \mathcal{M}$ *and* $\mathcal{M} \models F_{n+1}(a) < b$ *then there are arbitrarily large* $c \in \mathcal{M}$ *such that* $(\mathcal{M}, a, b) \equiv_n (\mathcal{M}, a, c)$.

Proof. Let $d \in \mathcal{M}$ be given. It suffices to show that the following type is consistent:

$$\{\varphi(a,v) \Leftrightarrow \varphi(a,b) : \varphi\} \cup \{v > d\}.$$

But if this type were inconsistent then for some $\phi \in \Sigma_n$ we would have

$$\mathcal{M} \models \phi(a,b) \wedge \forall v\ [\phi(a,v) \Rightarrow v \le d].$$

Hence, $\mathcal{M} \models t(a) = \max\{v : \phi(a,v)\} \le d$, a contradiction since t is Δ_{n+1}. ∎

LEMMA 5.3. *Let $a_0, \ldots, a_n \in \mathcal{M}$ be such that for all i with $0 < i \le n$, $\mathcal{M} \models F_{2i}(a_i) < a_{i-1}$. Then there are $b_0, \ldots, b_n \in \mathcal{M}$ such that $b_n = a_n$, $\operatorname{gap}(b_n) < \operatorname{gap}(b_{n-1}) < \ldots < \operatorname{gap}(b_0)$ and $(\mathcal{M}, b_0, \ldots, b_n) \equiv_1 (\mathcal{M}, a_0, \ldots, a_n)$.*

Let us note that a slightly strange formulation of Lemma 5.3 is just for technical convenience; the inductive step will be described in a more readable way.

Proof. If $n = 0$ then the conclusion is obvious. Assume that the lemma holds for $n = k$, and let the sequence $a_0, \ldots, a_k, a_{k+1}$ satisfy the assumption. In particular, $\mathcal{M} \models F_{2k+2}(a_{k+1}) < a_k$, hence, by Lemma 5.2, there are $b'_0, \ldots, b'_k$ such that

$$\mathcal{M}(a_{k+1}) < b'_k \quad \text{and} \quad (\mathcal{M}, a_0, \ldots, a_k, a_{k+1}) \equiv_{2k+1} (\mathcal{M}, b'_0, \ldots b'_k, a_{k+1}).$$

Since the formula $F_{2i}(x) < y$ is Σ_{2i+1}, it is easy to verify that the inductive assumptions, for the case $n = k$, are satisfied by the sequence $b'_0, \ldots, b'_{k-1}, \langle b'_n, a_{k+1}\rangle$. This gives $b_0, \ldots, b_k, b_{k+1}$ as required. (Notice that $b_k = b'_k$ and $b_{k+1} = a_{k+1}$.) ∎

It will be convenient to denote by $(a)_n$ the nth term of the sequence coded by a.

LEMMA 5.4. *Let $a \in \mathcal{M}$ be such that for all $n > 0$, $\mathcal{M} \models F_{2n}((a)_n) < (a)_{n-1}$. Then the theory*

$$T(I) = \mathrm{Th}(\mathcal{M}) \cup \{a \in I \subseteq_{\mathrm{end}} \mathcal{M}\} \cup \{I \models \mathrm{PA} + F_n((a)_{k+1}) < (a)_k : k, n \in \mathbb{N}\}$$

is consistent.

Proof. Let Θ be a finite fragment of $T(I)$ and let k be the greatest integer for which $(a)_k$ occurs in Θ. Then by Lemma 5.3, there are $b_0, \ldots, b_k$ such that $(\mathcal{M}, (a)_0, \ldots, (a)_k) \equiv_1 (\mathcal{M}, b_0, \ldots, b_k)$ and $\operatorname{gap}(b_k) < \operatorname{gap}(b_{k-1}) < \ldots < \operatorname{gap}(b_0)$.

Let $J = \mathcal{M}(b_0)$. By Friedman's embeddability criterion (cf. Kaye [6]), there exists $I \subseteq_{\mathrm{end}} \mathcal{M}$ with $(a)_0 \in I$ and $(J, b_0, \ldots, b_k) \simeq (I, (a)_0, \ldots, (a)_k)$. Then $(\mathcal{M}, I, (a)_0, \ldots, (a)_k) \models \Theta$. ∎

COROLLARY 5.5. *Let a be as in Lemma 5.4 and let $\mathcal{K} = \inf\{(a)_n : n \in \mathbb{N}\}$. Then there exists a recursively saturated $\mathcal{N} \subseteq_{\mathrm{end}} \mathcal{M}$ such that $a \in \mathcal{N}$ and*

1. $\mathcal{N}((a)_{n+1}) < (a)_n$ *for all* $n \in \mathbb{N}$,
2. $\mathcal{K} \prec_{\mathrm{end}} \mathcal{N}$ *and the same subsets of* $\mathcal{K}$ *are coded in* $\mathcal{M}$ *and in* $\mathcal{N}$.

Proof. By chronic resplendency of $\mathcal{M}$ (cf. Kaye [6]) and Lemma 5.4 there exists $\mathcal{N} \subseteq \mathcal{M}$ such that $(\mathcal{M}, \mathcal{N}, a)$ is a recursively saturated model of $T(I)$. Now, $a \in \mathcal{N}$ and $\mathcal{N} \subseteq_{\mathrm{end}} \mathcal{M}$, hence $\mathcal{K} \subseteq_{\mathrm{end}} \mathcal{N}$, and, since both extensions are proper, this implies that the same subsets of $\mathcal{K}$ are coded in $\mathcal{N}$ and $\mathcal{M}$. Moreover, the property of a and Tarski's test show that $\mathcal{K} \prec_{\mathrm{end}} \mathcal{N}$, and the result follows. ■

The following fact is known from [11].

LEMMA 5.6. *Let $\mathcal{N}$ be a countable recursively saturated model of* PA *and let $a \in \mathcal{N}$ be such that* $\mathrm{gap}((a)_{n+1}) < \mathrm{gap}((a)_n)$ *for all $n \in \mathbb{N}$. Let $\mathcal{K} = \inf\{(a)_n : n \in \mathbb{N}\}$. Then for all $s, s' \in \mathcal{K}$ with $(\mathcal{K}, s) \equiv (\mathcal{K}, s')$ there exists $f \in \mathrm{Aut}(\mathcal{N})$ such that $f(s) = s'$ and $f * \mathcal{K} = \mathcal{K}$.*

Proof (sketch). Using a standard argument (cf. [11], Lemma 2.2) we may assume that for every $n \in \mathbb{N}$,

$$(\mathcal{N}, s, (a)_0, \ldots, (a)_n) \equiv (\mathcal{N}, s', (a)_0, \ldots, (a)_n).$$

Then, using recursive saturation, we can find $a' \in \mathcal{N}$ such that $\mathcal{K} = \inf\{(a)_n : n \in \mathbb{N}\}$ and $(\mathcal{N}, s, a') \equiv (\mathcal{N}, s', a')$. Hence there exists $f \in \mathrm{Aut}(\mathcal{N})$ such that $f(s, a) = (s', a')$. Clearly for every such f we have $f * \mathcal{K} = \mathcal{K}$. ■

To finish the proof of Theorem 5.1 we need the so-called moving gaps lemma (see [8], Lemma 3.1, or [11], Lemma 5.4).

LEMMA 5.7. *There exists a type $\Gamma(v, w)$ such that for every countable recursively saturated $\mathcal{M} \models$ PA and every $v \in \mathcal{M}$, $\mathcal{M}$ realizes $\Gamma(v, \cdot)$ and whenever $s, a \in \mathcal{M}$ are such that $\Gamma(s, a)$ we have $\mathcal{M}(s) < a$ and for all s, s' and a, a', if $\Gamma(s, a)$, $\Gamma(s', a')$, and $s \neq s'$ then* $\mathrm{gap}(a) \neq \mathrm{gap}(a')$. ■

Proof of Theorem 5.1. Let $a \in \mathcal{M}$ be such that for some $s \in \mathcal{M}$, $\mathcal{M} \models \Gamma(s, a)$. Using recursive saturation it is easy to show that there exists a' such that $\mathrm{gap}(a') = \mathrm{gap}(a)$ and a' satisfies the assumption of Lemma 5.4 and $\mathcal{M}[a] = \mathcal{K} = \inf\{(a')_n : n \in \mathbb{N}\}$. Let $\mathcal{N}$ be the model given by Corollary 5.5. Let $s' \in \mathcal{M}[a]$ be such that $s \neq s'$ and $(\mathcal{M}, s) \equiv (\mathcal{M}, s')$. Then by Lemma 5.6 there exists $h \in \mathrm{Aut}(\mathcal{N})$ with $h(s) = s'$ and $h * \mathcal{K} = \mathcal{K}$. The models $\mathcal{M}, \mathcal{N}$ code the same subsets of $\mathcal{K}$ and h, h^{-1} send subsets coded in $\mathcal{M}$ onto coded ones, so the same happens to subsets coded in $\mathcal{N}$. Finally, let $f = h \restriction \mathcal{M}[a]$. Then f cannot be extended to an element of $\mathrm{Aut}(\mathcal{M})$. Indeed, if $g \in \mathrm{Aut}(\mathcal{M})$ is such that $g * \mathcal{M}[a] = \mathcal{M}[a]$ then $\mathrm{gap}(g(a)) = \mathrm{gap}(a)$, so for each such g, $g(s) = s'$, and g cannot extend f. ■

6. More on the theory of $(\mathcal{M}, \mathcal{M}[a])$**.** Smoryński [19] studied the variety of complete theories and isomorphism types of pairs $(\mathcal{M}, \mathcal{K})$, where $\mathcal{M}$ is a countable recursively saturated model of PA and $\mathcal{K} \prec_{\mathrm{end}} \mathcal{M}$. Among other results, he proved that for every nonstandard a, $\mathrm{gap}(a)$ is definable $(\mathcal{M}, \mathcal{M}(a))$ and, as a consequence, he showed that there are countably many pairwise elementarily inequivalent structures of this form.

Here we will apply the methods of the previous section to prove that for all $a, b \in \mathcal{M} \setminus \mathcal{M}[0]$ we have $(\mathcal{M}, \mathcal{M}[a]) \equiv (\mathcal{M}, \mathcal{M}[b])$. Then, using Smoryński's arguments, we can show that for $a \in \mathcal{M} \setminus \mathcal{M}(0)$, $\mathrm{gap}(a)$ is not definable in $(\mathcal{M}, \mathcal{M}[a])$. This contradicts the claim made about those structures in Smoryński's paper (Theorems 2.11 and 2.12 in [19]).

THEOREM 6.1. *Let $\mathcal{M}$ be a countable recursively saturated model of* PA*. Then for all $a, b \in \mathcal{M} \setminus \mathcal{M}(0)$, the structures $(\mathcal{M}, \mathcal{M}[a])$ and $(\mathcal{M}, \mathcal{M}[b])$ are elementarily equivalent.*

Let $\mathcal{M}$ be a countable recursively saturated model of PA.

LEMMA 6.2. *Let $a_0, \ldots, a_n \in \mathcal{M}$ be such that for some $k \in \mathbb{N}$ with $k > 0$, and for all i with $0 < i \le n$, $\mathcal{M} \models F_{2i+k-1}(a_i) < a_{i-1}$. Then there exist $b_0, \ldots, b_n$ such that $b_n = a_n$, $\mathrm{gap}(b_n) < \mathrm{gap}(b_{n-1}) < \ldots < \mathrm{gap}(b_0)$ and $(\mathcal{M}, b_0, \ldots, b_n) \equiv_k (\mathcal{M}, a_0, \ldots, a_n)$.*

Proof. Analogous to the proof of Lemma 5.3. ∎

LEMMA 6.3. *For all $a, b \in \mathcal{M} \setminus \mathcal{M}(0)$ and every $k \in \mathbb{N}$ there are a', b' such that $\mathrm{gap}(a) = \mathrm{gap}(a')$, $\mathrm{gap}(b) = \mathrm{gap}(b')$, $\mathcal{M}[a] = \inf\{(a')_n : n \in \mathbb{N}\}$, $\mathcal{M}[b] = \inf\{(b')_n : n \in \mathbb{N}\}$, and $(\mathcal{M}, a') \equiv_k (\mathcal{M}, b')$.*

Proof. Let $k \in \mathbb{N}$ be fixed. Using recursive saturation we can assume that $\mathcal{M}[a] = \inf\{(a)_n : n \in \mathbb{N}\}$, $\mathcal{M}[b] = \inf\{(b)_n : n \in \mathbb{N}\}$ and $F_{2n+k-1}((a)_n) < (a)_{n-1}$ for all $n > 0$, and similarly for the sequence (coded by) b.

Let us consider the theory

$$\begin{aligned} T_k(I, x) = \mathrm{Th}(\mathcal{M}) &\cup \{x \in I \prec_{\Sigma_{k+1},\mathrm{end}} \mathcal{M}\} \\ &\cup \{I \models \mathrm{PA} + F_i((x)_{n+1}) < (x)_n : i, n \in \mathbb{N}\}. \end{aligned}$$

Then $T_k(I, a)$ and $T_k(I, b)$ are both consistent. This follows from Lemma 6.2 and the appropriate form of Friedman's embeddability theorem (cf. Kaye [6], Theorem 12.5). Chronic resplendency of $\mathcal{M}$ yields two recursively saturated models $\mathcal{N}_a, \mathcal{N}_b$ such that $\mathcal{M}[a] \prec_{\mathrm{end}} \mathcal{N}_a \prec_{k,\mathrm{end}} \mathcal{M}$ and $\mathcal{M}[b] \prec_{\mathrm{end}} \mathcal{N}_b \prec_{k,\mathrm{end}} \mathcal{M}$. Now, arguing as in the proof of Theorem 5.1, we can replace a, b by a', b' such that $\mathcal{M}[a] = \inf_{n \in \mathbb{N}} (a')_n$, $\mathcal{M}[b] = \inf_{n \in \mathbb{N}} (b')_n$ and$(\mathcal{N}_a, a') \equiv (\mathcal{N}_b, b')$. But $\mathcal{N}_a \prec_k \mathcal{M}$ and $\mathcal{N}_b \prec_k \mathcal{M}$, hence $(\mathcal{M}, a') \equiv_k (\mathcal{M}, b')$. ∎

Proof of Theorem 6.1. Let a' and b' be as in Lemma 6.3. We have $(\mathcal{M}, a') \equiv_k (\mathcal{M}, b')$, hence in the Ehrenfeucht–Fraisse game for Σ_k-elementary equivalence of $(\mathcal{M}, s')$ and $(\mathcal{M}, b')$ the second player has a winning strategy (cf. Hodges [5]). It is not difficult to see that the same strategy can be used by the second player in the game for Σ_k-elementary equivalence of $(\mathcal{M}, \mathcal{M}[a])$ and $(\mathcal{M}, \mathcal{M}[b])$, and, since k was arbitrary, the result follows. ∎

Smoryński [19] shows that in $(\mathcal{M}, \mathcal{M}(a))$ one can define $\mathbb{N}$, satisfaction for $\mathcal{M}(a)$ and, consequently, $\mathrm{gap}(a)$. This in turn allows him to determine whether certain types are realized in $\mathrm{gap}(a)$ just by looking at $\mathrm{Th}((\mathcal{M}, \mathcal{M}(a)))$. The types $\{p_n(v) : n \in \mathbb{N}\}$, used by Smoryński, are such that for $n \neq m$ if a realizes p_m and b realizes p_n then $\mathrm{gap}(a) \neq \mathrm{gap}(b)$. This is the key fact in the proof that for such a and b, $(\mathcal{M}, \mathcal{M}(a)) \not\equiv (\mathcal{M}, \mathcal{M}(b))$. Since there are a and b such that $(\mathcal{M}, \mathcal{M}(a)) \not\equiv (\mathcal{M}, \mathcal{M}(b))$, Theorem 6.1 implies that if $\mathcal{M}$ is a countable and recursively saturated model of PA, then $\mathrm{gap}(a)$ cannot be uniformly definable in $(\mathcal{M}, \mathcal{M}[a])$. We will show that in fact $\mathrm{gap}(a)$ is not definable in $(\mathcal{M}, \mathcal{M}[a])$ for every $a \in \mathcal{M} \setminus \mathcal{M}(0)$. Let us formulate this as a theorem.

THEOREM 6.4. *If $\mathcal{M}$ is a countable recursively saturated model of* PA, *$a \in \mathcal{M} \setminus \mathcal{M}(0)$ and I is an elementary initial segment of $\mathcal{M}$ which contains $\mathcal{M}[a]$ and is definable in $(\mathcal{M}, \mathcal{M}[a])$, then $I = \mathcal{M}[a]$.*

Proof. Suppose that I is an elementary initial segment of $\mathcal{M}$ and that I contains $\mathcal{M}[a]$. If $\mathcal{M}(0) < I$ then it is routine to verify that there is $f \in \mathrm{Aut}(\mathcal{M})$ such that $f * \mathcal{M}[a] = \mathcal{M}[a]$ and $f * I \neq I$. Thus, if I is definable in $(\mathcal{M}, \mathcal{M}[a])$ then $I = \mathcal{M}(a)$ or $I = \mathcal{M}[a]$. But since all structures $(\mathcal{M}, \mathcal{M}[a])$ are elementarily equivalent, the above argument shows that if $\mathcal{M}(a)$ is definable in $(\mathcal{M}, \mathcal{M}[a])$ for some a, then the same would be true for every a, and this would contradict the remark preceding the statement of the theorem. ∎

Let us finish this section with the following problem. The proof of Theorem 6.1 actually shows that if $\mathcal{M}$ and $\mathcal{N}$ are countable recursively saturated models of PA and $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$, then, for all a, b with $a \in \mathcal{M} \setminus \mathcal{M}(0)$ and $b \in \mathcal{N} \setminus \mathcal{N}(0)$, $\mathrm{Th}(\mathcal{M}, \mathcal{M}[a]) = \mathrm{Th}(\mathcal{N}, \mathcal{N}[b])$. The question is what is $\mathrm{Th}(\mathcal{M}, \mathcal{M}[a])$?

7. Some open problems. Let us pose some problems connected with the ideas of this paper.

1. The class of cofinal extensions with the automorphism extension property described above seems to be very narrow. We do not know any wider classes of such cofinal extensions. Perhaps the most important problem in this direction is as follows. Let $\mathcal{R}$ be a countable recursively saturated model

of PA. Let $q(\cdot)$ be a minimal type realized in $\mathcal{R}$. Let A be the set of realizations of q. Which order automorphisms of A extend to $\mathcal{R}$? Equivalently, which automorphisms of $\mathrm{Hull}^{\mathcal{R}}(A)$ extend to $\mathcal{R}$? The moving gaps lemma (3.1 in [8] or 5.4 in [11]) shows that this happens quite rarely; moreover, if the extension exists then it is unique.

2. As pointed out earlier (just after Definition 3.5), the covering property is an analogue of the notion "$\mathbb{N}$ codes $\mathcal{M}$ in $\mathcal{R}$". We do not know how to extend other combinatorial properties of cuts in models of PA, introduced by J. Paris and his school in mid-seventies (see e.g. Kirby's thesis [9]), to the case of cofinal extensions.

3. Let $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{K}$, with both $\mathcal{M}$ and $\mathcal{K}$ countable recursively saturated. Does there exist $g \in \mathrm{Aut}(\mathcal{M})$ which sends a coded subset to a not coded one? Are any extra assumptions on the extension needed?

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Baruch College
City University of New York
Department of Mathematics, Box G0930
17, Lexington Avenue
New York, New York 10010, U.S.A.
E-mail: rkobb@cunyvm.cuny.edu

Institute of Mathematics
Higher Agricultural and
Pedagogical School
Orlicz-Dreszera 19/21
08-110 Siedlce, Poland
E-mail: hkl@impan.impan.gov.pl

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