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Transverse Hausdorff dimension of codim-1 C²-foliations

by

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Abstract. The Hausdorff dimension of the holonomy pseudogroup of a codimension-one foliation $\mathcal{F}$ is shown to coincide with the Hausdorff dimension of the space of compact leaves (traced on a complete transversal) when $\mathcal{F}$ is non-minimal, and to be equal to zero when $\mathcal{F}$ is minimal with non-trivial leaf holonomy.

1. Introduction. In [Wa], the second author introduced the notion of a Hausdorff dimension $\dim_{\mathrm{H}} \mathcal{G}$ for finitely generated locally Lipschitz pseudogroups $\mathcal{G}$ acting on compact metric spaces X. Recall that

$$\dim_{\mathrm{H}} \mathcal{G} = \inf\{s > 0 : H^s(\mathcal{G}) = 0\} = \sup\{s > 0 : H^s(\mathcal{G}) = \infty\}, \tag{1}$$

where

$$H^s(\mathcal{G}) = \lim_{\varepsilon\to 0} H^s_\varepsilon(\mathcal{G}), \tag{2}$$

$$H^s_\varepsilon(\mathcal{G}) = \inf\{H_s(A) : A \in \mathcal{A}(\varepsilon)\}, \tag{3}$$

$$H_s(A) = \sum_{g\in A} (\operatorname{diam} D_g)^s, \tag{4}$$

D_g stands for the domain of the map $g \in \mathcal{G}$ and $\mathcal{A}(\varepsilon)$ denotes the family of all finite sets generating $\mathcal{G}$ and consisting of maps with domains of diameter less than ε.

Note that our definitions are analogous to those involved in defining the Hausdorff dimension of metric spaces (see [Ed], for example). In particular,

$$\dim_{\mathrm{H}} X = \dim_{\mathrm{H}} \mathcal{G}(\mathrm{id}_X), \tag{5}$$

where $\mathcal{G}(f_1, \ldots, f_n)$ is the pseudogroup generated by the maps $f_1, \ldots, f_n$.

The dimension $\dim_{\mathrm{H}}$ has the following properties (see [Wa]):

(i) $\dim_{\mathrm{H}} \mathcal{G} \le \dim_{\mathrm{H}} X$,

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(ii) $\dim_{\mathrm{H}} \mathcal{G}_1 = \dim_{\mathrm{H}} \mathcal{G}_2$ when the pseudogroups $\mathcal{G}_1$ and $\mathcal{G}_2$ are Lipschitz equivalent,

(iii) $\dim_{\mathrm{H}} \mathcal{G}' \geq \dim_{\mathrm{H}} \mathcal{G}$ when $\mathcal{G}'$ is a subpseudogroup of $\mathcal{G}$,

(iv) $\dim_{\mathrm{H}}(\mathcal{G}|Y) \leq \dim_{\mathrm{H}} \mathcal{G}$ when Y is a closed $\mathcal{G}$-invariant subset of X and $\mathcal{G}|Y$ denotes the pseudogroup generated by the maps $g|Y$, $g \in \mathcal{G}$,

(v) $\dim_{\mathrm{H}} \mathcal{G} \geq s$ if $\mathcal{G}$ preserves a Borel probability measure μ on X which is s-continuous, i.e. satisfies the condition

$$\mu(Z) \leq c(\operatorname{diam} Z)^s \tag{6}$$

for all Borel subsets Z of X and a positive constant c independent of Z,

(vi) $\dim_{\mathrm{H}} \mathcal{G} = \dim_{\mathrm{H}} X$ when X is a Riemannian manifold and $\mathcal{G}$ a pseudogroup of local isometries of X.

Property (ii) implies that the Hausdorff dimension of the holonomy pseudogroup $\mathcal{H}_T$ of a C^1-foliation $\mathcal{F}$ of a compact manifold M is independent of the choice of a compact complete transversal T. Therefore, the *transverse Hausdorff dimension* $\dim_{\mathrm{H}}^{\pitchfork} \mathcal{F}$ of $\mathcal{F}$ can be defined by

$$\dim_{\mathrm{H}}^{\pitchfork} \mathcal{F} = \dim_{\mathrm{H}} \mathcal{H}_T, \tag{7}$$

T being any transversal as above.

In this article, we compute the transverse Hausdorff dimension for codimension-one C^2-foliations of compact manifolds (Section 2) and collect some examples which show that the assumptions of our Theorem are essential (Section 3).

2. Main results. Let $\mathcal{F}$ be a transversely oriented codimension-1 C^2-foliation of a compact manifold M.

THEOREM. (i) *If $\mathcal{F}$ is not minimal, then* $\dim_{\mathrm{H}}^{\pitchfork} \mathcal{F} = \dim_{\mathrm{H}}(T \cap C(\mathcal{F}))$, *where T is a compact complete transversal and $C(\mathcal{F})$ is the union of all the compact leaves.*

(ii) *If $\mathcal{F}$ is minimal and has non-trivial holonomy, then* $\dim_{\mathrm{H}}^{\pitchfork} \mathcal{F} = 0$.

Let us explain that $\mathcal{F}$ is minimal if its leaves are dense in M; its holonomy is non-trivial when the germ holonomy group of some leaf L is non-trivial.

Proof. (i) The inequality

$$\dim_{\mathrm{H}}^{\pitchfork} \mathcal{F} \geq \dim_{\mathrm{H}}(T \cap C(\mathcal{F}))$$

follows directly from Property (iv) (Section 1) and the fact that the holonomy of $\mathcal{F}|C(\mathcal{F})$ is trivial.

To prove the converse, fix $s > \dim_{\mathrm{H}}(T \cap C(\mathcal{F}))$, $\eta > 0$ and $\varepsilon > 0$. Observe that the center $Z(\mathcal{F}) = C(\mathcal{F}) \cup E_1 \cup \ldots \cup E_m$ is compact and contains finitely many exceptional minimal sets E_i ([La], [HH], etc.). Moreover, every leaf L of $\mathcal{F}$ satisfies $\overline{L} \cap Z(\mathcal{F}) \neq \emptyset$.

First, the Sacksteder Theorem [Sa] allows us to choose points $x_i \in T \cap E_i$ $(i = 1, \ldots, m)$ and holonomy maps h_i contracting some neighbourhoods $J_i \subset T$ of x_i.

Next, since $C(\mathcal{F})$ is compact, we can cover it by a finite number of mutually disjoint foliated I-bundles (possibly reducing to single isolated leaves) $C_1, \ldots, C_n$ bounded by closed semi-isolated leaves L_i^+ and L_i^-. Let $I_j = [x_j, y_j] \subset T$, $j = 1, \ldots, n$, be the fibres of C_j. Extend each of the intervals I_j slightly to get larger intervals $I'_j = [x'_j, y'_j]$ such that the segments $[x'_j, x_j]$ and $[y_j, y'_j]$ are attracted to x_j and y_j by the global holonomy groups Γ_j of the foliated bundles $\mathcal{F}|C_j$. More precisely, for any $j = 1, \ldots, n$ and $\delta > 0$ there should exist $h, h' \in \Gamma_j$ which extend to holonomy maps (denoted by h and h' again) defined on I'_j and bringing x'_j (resp., y'_j) to within distance δ of x_j (resp., y_j).

Set

$$T' = \bigcup_{i=1}^{m} J_i \cup \bigcup_{j=1}^{n} I''_j,$$

where $I''_j = [x''_j, y''_j]$ for some $x''_j \in (x'_j, x_j)$ and $y''_j \in (y_j, y'_j)$ such that the intervals J_i and I''_j remain mutually disjoint. Obviously, T' is a complete transversal for $\mathcal{F}$.

For any j, fix a finite symmetric set Γ_j^0 generating Γ_j. Shrinking the intervals I'_j if necessary we may assume that all the maps of Γ_j^0 extend to holonomy maps defined on I'_j. Let $\mathcal{H}'$ be the subpseudogroup of $\mathcal{H}_{T'}$ generated by the contractions h_i, $i = 1, \ldots, m$, and the (extended to I''_j) holonomy maps of Γ_j^0, $j = 1, \ldots, n$.

Now, cover $T' \cap C(\mathcal{F})$ by intervals $K_1, \ldots, K_N$ with endpoints in $C(\mathcal{F})$ and lengths $l(K_i) < \varepsilon$, and such that

$$\sum_i l(K_i)^s < \eta.$$

(This is possible since $s > \dim_{\mathrm{H}}(T' \cap C(\mathcal{F}))$.)

Put $\delta = \frac{1}{2}\varepsilon \min\{1, l(K_1), \ldots, l(K_N)\}$. For any j choose holonomy maps $f_j, f'_j \in \Gamma_j$ which extend to I'_j and bring x'_j (resp., y'_j) to within distance δ of x_j (resp., y_j). Also, for any gap (i.e., a connected component) $U = (a, b)$ of $I_j \setminus \bigcup_i K_i$ choose points $c, d \in U$, $c < d$, and holonomy maps $f_U, f'_U \in \Gamma_j$ which bring c (resp., d) to within distance δ of b (resp., a).

Now, if $K_i = [\alpha, \beta] \subset (x_j, y_j)$, take the gaps U and U' for which $\alpha \in \overline{U}$ and $\beta \in \overline{U}'$ and let

$$A_i = \{g|K2_i, g \circ f_U^{-1}|V, g \circ (f'_{U'})^{-1}|V' : g \in \Gamma_j^0\},$$

where V and V' are the δ-neighbourhoods of α and β, respectively. If $K_i =$

$[x_j,\beta]\subset I_j$, choose U', $f'_{U'}$ and V' as before, and let

$$A_i=\{g|K2_i, g\circ(f'_{U'})^{-1}|V', g\circ f_j^{-1}|W_j : g\in\Gamma_j^0\},$$

where W_j is the δ-neighbourhood of x_j. Define A_i similarly in the case when $K_i=[\alpha,y_j]$.

Finally, for any $i=1,\dots,m$ choose an exponent $n_i\in\mathbb{N}$ such that the image of J_i under $h_i^{n_i}$ has diameter less than ε and let

$$A=\{h_1^{-n_1},h_1^{-(n_1+1)},\dots,h_m^{-n_m},h_m^{-(n_m+1)}\}\cup A_1\cup\ldots\cup A_N.$$

The set A generates $\mathcal{H}'$, the diameters of the domains of maps in A are bounded by ε and

$$H^s_\varepsilon(\mathcal{H}')\le H_s(A)\le 2m\varepsilon^s+\max_j \#\Gamma_j^0\cdot(1+2\varepsilon^s)\eta.$$

Consequently, $H^s(\mathcal{H}')<\infty$, $s\ge\dim_{\mathrm{H}}\mathcal{H}'$ and $\dim_{\mathrm{H}}\mathcal{H}'\le\dim_{\mathrm{H}}(T\cap C(\mathcal{F}))$. Moreover, by Property (iii), $\dim_{\mathrm{H}}^{\pitchfork}\mathcal{F}=\dim_{\mathrm{H}}\mathcal{H}_{T'}\le\dim_{\mathrm{H}}\mathcal{H}'$.

(ii) The proof is essentially the same. One has to take as T' any segment transverse to $\mathcal{F}$ and short enough to be attracted to a point x_0 by the holonomy of the leaf L_{x_0}. For any symmetric set A_0 generating $\mathcal{H}_{T'}$ and two holonomy maps h_1 and h_2 attracting the endpoints of T' to within distance ε of x_0 and satisfying the condition $\operatorname{diam} R_{h_i}<2\varepsilon$, the set

$$\{g\circ h_1^{-1}, g\circ h_2^{-1} : g\in A_0\}$$

generates $\mathcal{H}_{T'}$ and satisfies the inequality

$$H_s(A)\le\#A_0\cdot(2\varepsilon)^s$$

for any s and $\varepsilon>0$. ∎

3. Examples. First, we will discuss the case of minimal foliations without holonomy.

Example 1. If $\mathcal{F}$ is the suspension of an irrational rotation of S^1, then $\mathcal{F}$ is a Riemannian foliation of the 2-dimensional torus T^2 and therefore (Property (vi) of Section 1) $\dim_{\mathrm{H}}^{\pitchfork}\mathcal{F}=1$. Obviously, $\mathcal{F}$ is minimal and has no holonomy.

Example 2. In [Ar], one can find a construction of a sequence (f_k) of analytic diffeomorphisms of S^1 with the following properties:

(1) $|f_k(z)-f_{k+1}(z)|<\delta_k$,

(2) the rotation numbers $\varrho(f_k)$ are rational, $\varrho(f_k)=p_k/q_k$, and satisfy the inequalities

$$\left|\frac{p_k}{q_k}-\frac{p_{k+1}}{q_{k+1}}\right|<\frac{1}{(k-1)^2(\max_{l<k} q_l)^2},$$

(3) any map f_k is forward semi-stable and has a unique cycle $z_{k,1},\ldots,z_{k,q_k}$ (of length q_k),

(4) there exist exponents $N(k)\in\mathbb{N}$ such that

$$f_k^{N(k)}(S^1\setminus U_k)\subset U_k,$$

where $U_k=U_k^1\cup\ldots\cup U_k^{q_k}$ and U_k^j is the ε_k-neighbourhood of $z_{k,j}$.

In the above, (δ_k) and (ε_k) are sequences of positive numbers converging to 0 sufficiently fast as $k\to\infty$, so we may assume that

$$\varepsilon_k\le q_k^{-1/s_k}, \tag{8}$$

where s_k is an *a priori* given sequence of positive reals converging to 0.

From properties (1)–(4) above it follows that the limit $f=\lim_{k\to\infty}f_k$ exists, is analytic and has an irrational rotation number $\varrho(f)$. Therefore, the suspension $\mathcal{F}$ of f provides us with an analytic minimal foliation without holonomy. Its holonomy pseudogroup is isomorphic to $\mathcal{G}=\mathcal{G}(f)$, the pseudogroup of local diffeomorphisms of S^1 generated by f. Lemma γ of [Ar] shows that if δ_k's are small enough, then

$$f^{N(k)}(S^1\setminus\widetilde{U}_k)\subset\widetilde{U}_k,\qquad k=1,2,\ldots, \tag{9}$$

where $\widetilde{U}_k=\bigcup_j\widetilde{U}_k^j$ and $\widetilde{U}_k^j$ is the ε_k-neighbourhood of U_k^j, $j=1,\ldots,q_k$. From (9) it follows that each of the sets

$$A_k=\{f^{-N(k)}|\widetilde{U}_k^j,f^{-(N(k)+1)}|\widetilde{U}_k^j,f|\widetilde{U}_k^j:j=1,\ldots,q_k\},\qquad k=1,2,\ldots,$$

generates $\mathcal{G}$. Obviously, $A_k\in\mathcal{A}(4\varepsilon_k)$ and $H_{s_k}(A_k)=3q_k(4\varepsilon_k)^{s_k}$. From (8) it follows that for any $k>k_0$ ($k_0\in\mathbb{N}$) we have

$$H_{4\varepsilon_k}^{s_{k_0}}\le 3q_k(4\varepsilon_k)^{s_{k_0}}\le 3\cdot 4^{s_{k_0}}.$$

Consequently, $H^{s_{k_0}}(\mathcal{G})<\infty$ and $\dim_{\mathrm{H}}\mathcal{G}\le s_{k_0}$ for any k_0. Finally,

$$\dim_{\mathrm{H}}^{\pitchfork}\mathcal{F}=\dim_{\mathrm{H}}\mathcal{G}=0.$$

Examples 1 and 2 provide minimal foliations without holonomy with transverse Hausdorff dimension equal to, respectively, 0 and 1. It seems to us that Arnold's construction cannot be modified to get a similar foliation with $\dim_{\mathrm{H}}^{\pitchfork}\in(0,1)$. So, one could search either for other examples of this sort or for the proof of the following: If $\mathcal{F}$ is minimal, C^2-differentiable and has no holonomy, then either $\dim_{\mathrm{H}}^{\pitchfork}\mathcal{F}=0$ or $\dim_{\mathrm{H}}^{\pitchfork}\mathcal{F}=1$.

The following example shows that the assumption of C^2-differentiability in our Theorem is essential.

Example 3. Let $f:S^1\to S^1$ be the classical Denjoy C^1-diffeomorphism constructed as in, for instance, [Ta]. Then S^1 contains a minimal closed invariant set X such that $f|X$ preserves the 1-dimensional Lebesgue measure

λ ($f' = 1$ on X) and $\lambda(X) > 0$. Moreover, λ is 1-continuous, and therefore, by Properties (iv) and (v) of Section 1, we get

$$1 \geq \dim_{\mathrm{H}} \mathcal{G}(f) \geq \dim_{\mathrm{H}} \mathcal{G}(f|X_0) \geq \dim_{\mathrm{H}} X_0 = 1.$$

Suspending f we arrive at a non-minimal C^1-foliation $\mathcal{F}$ which has no compact leaves but has transverse Hausdorff dimension 1.

Remark. The arguments in Example 3 do not work when $\lambda(X) = 0$, X being the minimal set of a Denjoy diffeomorphism f. Examples of such diffeomorphisms are provided in [He], Section X.3. It would be interesting to calculate (or estimate) $\dim_{\mathrm{H}} \mathcal{G}(f)$ for such an f. Is it possible to find a Denjoy diffeomorphism f for which $0 < \dim_{\mathrm{H}} \mathcal{G}(f) < 1$?

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