FUNDAMENTA
MATHEMATICAE
149 (1996)

Losing Hausdorff dimension while generating pseudogroups

by

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Abstract. Considering different finite sets of maps generating a pseudogroup $\mathcal{G}$ of locally Lipschitz homeomorphisms between open subsets of a compact metric space X we arrive at a notion of a Hausdorff dimension $\dim_{\mathrm{H}} \mathcal{G}$ of $\mathcal{G}$. Since $\dim_{\mathrm{H}} \mathcal{G} \leq \dim_{\mathrm{H}} X$, the dimension loss $\mathrm{dl}_{\mathrm{H}} \mathcal{G} = \dim_{\mathrm{H}} X - \dim_{\mathrm{H}} \mathcal{G}$ can be considered as a "topological price" one has to pay to generate $\mathcal{G}$. We collect some properties of $\dim_{\mathrm{H}}$ and dl_{H} (for example, both of them are invariant under Lipschitz isomorphisms of pseudogroups) and we either estimate or calculate $\dim_{\mathrm{H}} \mathcal{G}$ for pseudogroups arising from classical dynamical systems, group actions, foliations, etc.

Introduction. In this article, we define the Hausdorff dimension $\dim_{\mathrm{H}} \mathcal{G}$ of a finitely generated pseudogroup $\mathcal{G}$ acting on a compact metric space X. We show that $\dim_{\mathrm{H}} \mathcal{G}$ does not exceed $\dim_{\mathrm{H}} X$, the Hausdorff dimension of X, so one has a kind of dimension loss $\mathrm{dl}_{\mathrm{H}} \mathcal{G} = \dim_{\mathrm{H}} X - \dim_{\mathrm{H}} \mathcal{G} \geq 0$. We show that Lipschitz equivalent pseudogroups have the same Hausdorff dimensions, so—in particular—the (transverse) dimension loss of a foliation $\mathcal{F}$ can be defined as that of its holonomy pseudogroup $\mathcal{H}$ acting on any compact complete transversal T. Several examples provided here show that the dimension loss $\mathrm{dl}_{\mathrm{H}} \mathcal{G}$ is positive when there is enough contraction (or, expansion) by elements of $\mathcal{G}$.

The motivation of this research comes from the following.

First, Hausdorff dimensions (and other related dimensions) turn out to be useful in defining and studying fractals which appear often in the theory of (especially complex) dynamical systems as, for example, minimal invariant sets (see [Ed], [Fa] and the references there). For some classes of sets (like quasi-circles which are defined as subsets of $\mathbb{R}^n$ homeomorphic to S^1 and satisfying some other natural conditions and which appear naturally in the study of Kleinian groups [Bo]), the equality $\dim_{\mathrm{H}} X = \dim_{\mathrm{H}} Y$ implies that

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1991 *Mathematics Subject Classification*: Primary 54H20.

Research supported partially by the grant KBN 0241 P3 93 04.

X and Y are quasi-isometric [FM], so $\dim_{\mathrm{H}}$ becomes a good invariant to study dynamics of some systems.

Second, recent years brought wide interest in general dynamical systems like relations, group actions, pseudogroups, foliations, etc. There is a large variety of problems and results in this area. We list below just some of them (and give only some references):

(1) invariant measures for general dynamics, ergodicity, amenability ([Pl], [Gar], [Zi2], [Zi3], etc.),

(2) entropy for relations, pseudogroups, foliations ([GLW], [LW1], [LW2], [Fr], [Hu1], [Hu2], [Wi], [Bi], etc.),

(3) other invariants measuring dynamics of general systems ([Eg1], [Eg2], [LW3], etc.),

(4) rigidity of group actions ([Gh], [Hu3], [Hu4], [Zi1], etc.),

(5) geometry and dynamics of hyperbolic groups ([Gr], [GH1], [GH2], [CDP], [GHV], [C1], [C2], [Ch], etc.).

Finally, the ultimate impulse came from [Le], where the author defined a measure-theoretic cost $l(\Phi)$ of countably generated measure-preserving relations R (pseudogroups, in particular) given a countable generating set Φ. $l(\Phi)$ and $l(R) = \inf_{\Phi} l(\Phi)$ have the property that they become larger if one wants to create more complicated dynamics. Since non-trivial $\mathcal{G}$-invariant measures do not exist for several pseudogroups $\mathcal{G}$, we realized that it could be useful and interesting to find a kind of topological cost of generating. Thinking about it we arrived at our dimension loss which has (to some extent) a similar property: If the dynamics of $\mathcal{G}$ is complicated enough, then $\mathrm{dl}_{\mathrm{H}}\,\mathcal{G}$ is positive. However, we do not expect any direct relation between $l(\mathcal{G})$ and $\mathrm{dl}_{\mathrm{H}}\,\mathcal{G}$. A remark of Section 3.3 shows a relation between $l(\mathcal{G})$ and $H^s(\mathcal{G})$, a kind of Hausdorff measure defined in Section 2.1 and essentially involved in calculation of $\dim_{\mathrm{H}}\mathcal{G}$, in the case when the invariant measure under consideration is s-continuous in the sense of Section 3.2.

The paper is organized as follows. Section 1 contains basic definitions of pseudogroups, pseudogroup morphisms, holonomy pseudogroups for foliations, etc. In Section 2, we define the Hausdorff dimension and dimension loss for pseudogroups, collect their elementary properties and discuss some simple examples. In Section 3, we estimate $\dim_{\mathrm{H}}$ and dl_{H} assuming the existence of good (called s-continuous there) invariant measures. Section 4 provides more examples like pseudogroups of local isometries, hyperbolic groups acting on the ideal boundary, rational maps, etc. In particular, we show (Prop. 4.4.1) that existence of attractors allows estimating $\dim_{\mathrm{H}}$ from above. In Section 5, we collect some final remarks. Since our examples come from different fields, to make the paper easier to follow by different readers, we decided to include subsections (1.3, 3.1 and a large part of 4.2) containing some information about holonomy of foliations, invariant measures and hyperbolic groups.

1. Pseudogroups

1.1. *Basic definitions.* Throughout this paper X is a compact metric space with the distance function ϱ, and $\mathcal{G}$ is a pseudogroup acting on X, i.e. a set of homeomorphisms $g : D_g \to R_g$ between open subsets D_g and R_g of X which is closed under composition, inversion, restriction to open subdomains and unions. More precisely, $\mathcal{G}$ satisfies the following conditions:

(i) if $g, h \in \mathcal{G}$, then $g \circ h : h^{-1}(D_g) \to g(R_h)$ is in $\mathcal{G}$,
(ii) if $g \in \mathcal{G}$, then $g^{-1} : R_g \to D_g$ is in $\mathcal{G}$,
(iii) if $g \in \mathcal{G}$ and $U \subset D_g$ is open, then $g|U \in \mathcal{G}$,
(iv) if $g : D_g \to R_g$ and $\mathcal{U}$ is an open cover of D_g such that $g|U \in \mathcal{G}$ for any $U \in \mathcal{U}$, then $g \in \mathcal{G}$.

We shall also assume that

(v) $\bigcup\{D_g : g \in \mathcal{G}\} = X$ so that $\mathrm{id}_X \in \mathcal{G}$.

If A is a set of homeomorphisms between open subsets of X and

$$\bigcup_{g\in A} (D_g \cup R_g) = X, \tag{1.1.1}$$

then the smallest pseudogroup of local homeomorphisms of X which contains A exists, is said to be *generated* by A and will be denoted by $\mathcal{G}(A)$. It consists of all the maps $g : D_g \to R_g$ such that for any $x \in D_g$ there exist an open neighbourhood U of x, maps $g_1, \ldots, g_n \in A$ and exponents $e_1, \ldots, e_n \in \{\pm 1\}$ for which $g = g_1^{e_1} \circ \ldots \circ g_n^{e_n}$ on $D_g \cap U$. If A is finite, $A = \{g_1, \ldots, g_n\}$, then we write $\mathcal{G}(g_1, \ldots, g_n)$ instead of $\mathcal{G}(A)$. If $\mathcal{G}$ is a pseudogroup and $A \subset \mathcal{G}$, then $\mathcal{G}(A)$ is a subpseudogroup of $\mathcal{G}$.

A pseudogroup $\mathcal{G}$ is *finitely generated* iff there exists a finite set A such that $\mathcal{G} = \mathcal{G}(A)$. Hereafter, all the pseudogroups are supposed to be finitely generated. A generating set A is *symmetric* iff $\mathrm{id}_X \in A$ and $A = A^{-1}$ (i.e., $g^{-1} \in A$ whenever $g \in A$). If $\mathcal{G}$ is finitely generated, then it admits a finite symmetric generating set.

Let $\mathcal{G}$ and $\mathcal{H}$ be pseudogroups of local homeomorphisms of compact metric spaces X and Y, respectively. A *morphism* $\Phi : \mathcal{G} \to \mathcal{H}$ is a family Φ of homeomorphisms $\phi : D_\phi \to R_\phi$ between open sets $D_\phi \subset X$ and $R_\phi \subset Y$ for which

(vi) $\bigcup\{D_\phi : \phi \in \Phi\} = X$ and
(vii) $\phi_1 \circ g \circ \phi_2^{-1} \in \mathcal{H}$ for all $\phi_1, \phi_2 \in \Phi$ and all $g \in \mathcal{G}$.

A morphism Φ is an *isomorphism* if $\Phi^{-1} = \{\phi^{-1} : \phi \in \Phi\}$ occurs to be a morphism between $\mathcal{H}$ and $\mathcal{G}$. In this case,

(viii) $\bigcup\{R_\phi : \phi \in \Phi\} = Y$.

Enlarging a morphism Φ as defined above we can obtain a larger family $\widetilde{\Phi}$ closed under unions and such that

(ix) $h \circ \phi \circ g \in \widetilde{\Phi}$ whenever $g \in \mathcal{G}$, $h \in \mathcal{H}$ and $\phi \in \widetilde{\Phi}$.

The smallest $\widetilde{\Phi}$ like this will also satisfy conditions (vi) and (vii) and will become a pseudogroup morphism in the sense of Haefliger [Ha]. In this situation, we say that $\widetilde{\Phi}$ is *generated* by Φ. We assume that all our morphisms are finitely generated.

In this paper, we work with Lipschitz and locally Lipschitz maps: We assume that our pseudogroups and morphisms are generated by Lipschitz maps and, therefore, consist of locally Lipschitz ones. Of course, the elements of a pseudogroup (or of a morphism) are not uniformly Lipschitz: the Lipschitz constants of different maps are different.

1.2. *Generating pseudogroups.* Let $A_0 = \{g_1, \ldots, g_N\}$ be any finite set generating a pseudogroup $\mathcal{G}$. Put

$$A = \{g_{i_1,\ldots,i_m}|U : U \in \mathcal{U}_{i_1,\ldots,i_m},\ i_1, \ldots, i_m \le N,\ m \le m_0\}, \tag{1.2.1}$$

where $g_{i_1,\ldots,i_m} : D_{i_1,\ldots,i_m} \to R_{i_1\ldots,i_m}$ denotes the composition $g_{i_1} \circ \ldots \circ g_{i_m}$ and $\mathcal{U}_{i_1,\ldots,i_m}$ is a finite (possibly empty) family of open subsets of $D_{i_1,\ldots i_m}$.

The set A generates $\mathcal{G}$ if for any $i = 1, \ldots, N$ and any $x \in D_i$ there exist a map $g_{i_1,\ldots,i_m}$ and an open neighbourhood V of x such that $g_i = (g_{i,i_1,\ldots,i_m}|U')\circ(g_{i_1,\ldots,i_m}|U)^{-1}$ on V for some $U \in \mathcal{U}_{i_1\ldots,i_m}$ and $U' \in \mathcal{U}_{i,i_1,\ldots,i_m}$. This happens when, for any i,

$$\bigcup_I \bigcup_{U\in\mathcal{U}_I} \bigcup_{U'\in\mathcal{U}_{i,I}} g_I(U \cap U') = D_i, \tag{1.2.2}$$

where I denotes a multiindex $(i_1, \ldots, i_m)$. (The condition (1.2.2) is not necessary for A to generate $\mathcal{G}$ since, in general, the maps g_i can be obtained by composing elements of A_0 in different ways. However, it is useful when calculating (or estimating) the Hausdorff dimension of $\mathcal{G}$.)

In particular, if $f : X \to X$ is a homeomorphism, $\mathcal{G} = \mathcal{G}(f)$ and

$$A = \{f^m|U : U \in \mathcal{U}_m,\ m = 0, \pm1, \ldots, \pm m_0\}, \tag{1.2.3}$$

where $\mathcal{U}_m$ are, as before, finite families of open subsets of X, and

$$\bigcup_m \bigcup_{U\in\mathcal{U}_m} \bigcup_{U'\in\mathcal{U}_{m+1}} f^m(U \cap U') = X, \tag{1.2.4}$$

then A generates $\mathcal{G}(f)$. In fact, in this situation any point $x \in X$ has an open neighbourhood V such that $f|V = (f^{m+1}|U') \circ (f^m|U)^{-1}|V$ for some $m \in \mathbb{Z}$, some $U \in \mathcal{U}_m$ and some $U' \in \mathcal{U}_{m+1}$ such that $x \in f^m(U \cap U')$ and $|m| < m_0$.

Similarly, if $f : X \to X$ is a local homeomorphism, then the pseudogroup $\mathcal{G}(f)$ generated by all the maps of the form $f|U$, where $U \subset X$ is open and $f|U$ is one-to-one, is generated by the set A defined by (1.2.3) and satisfying (1.2.4) provided that the maps $f^m|U$ are one-to-one for all $U \in \mathcal{U}_m$, $m \in \mathbb{Z}$.

1.3. *Holonomy pseudogroups.* The basic example we have in mind while thinking about pseudogroups is the holonomy pseudogroup $\mathcal{H}$ of a foliation $\mathcal{F}$ of a connected compact manifold M. (The reader not familiar with foliations should consult [CL], [Go], [HH] or [Ta].)

To construct $\mathcal{H}$ we cover M with a finite family of charts $U_1, \ldots, U_N$ distinguished by $\mathcal{F}$ and satisfying the following condition: any plaque $P \subset U_i$ intersects at most one plaque $Q \subset U_j$, $i, j = 1, \ldots, N$. (A cover like this is called *nice* [HH]. Nice covers always exist.) Let $\widetilde{T}_i = U_i/(\mathcal{F}|U_i)$ be the space of plaques of U_i. Without loosing generality, we may assume that the plaques are relatively compact in M, $\widetilde{T}_i$ is homeomorphic (diffeomorphic of class C^r if $\mathcal{F}$ is C^r-differentiable, $r \geq 1$) to an open ball $B^q(0, 1+\eta) \subset \mathbb{R}^q$, $q = \operatorname{codim} \mathcal{F}$, $\eta > 0$, and that the plaques corresponding to the points of the closed ball $\overline{B}^q(0,1)$ form compact spaces T_i with the following property: every leaf L of $\mathcal{F}$ intersects the disjoint union $T = \bigsqcup_i T_i$ (i.e., every leaf L contains a plaque $P \in T_i$ for some i). The compact space T (becoming a C^r-manifold with boundary if $\mathcal{F}$ is C^r-differentiable, $r \geq 1$) is called a *complete transversal* of $\mathcal{F}$.

If $U_i \cap U_j \neq \emptyset$, then one has the *holonomy homeomorphism* $h_{ij} : D_{ij} \to R_{ij}$ between open sets $D_{ij} \subset T_i$ and $R_{ij} \subset T_j$ which maps a plaque $P \in T_i$ to the unique plaque $Q \in T_j$ such that $P \cap Q \neq \emptyset$ (if one exists). Then $\mathcal{H} = \mathcal{H}_T$ is the pseudogroup of local homeomorphisms of T generated by the maps h_{ij}, $i, j = 1, \ldots, N$. If $\mathcal{F}$ is C^r-differentiable, $r \geq 1$, then $\mathcal{H}$ consists of local C^r-diffeomorphisms of T.

It is well known that the holonomy pseudogroups $\mathcal{H}_T$ and $\mathcal{H}_{T'}$ of $\mathcal{F}$ corresponding to two different complete transversals T and T' are isomorphic ([Go], p. 76). In fact, if $T \sqcup T'$ is a complete transversal, then the holonomy maps $h_{ij'}$ corresponding to components $T_i \subset T$ and $T_{j'} \subset T'$ (whenever defined) form a morphism $\Phi_{TT'} : \mathcal{H}_T \to \mathcal{H}_{T'}$. It is an isomorphism since $\Phi_{T'T}$ is its inverse. The general case can be reduced to that discussed above by considering another transversal T'' corresponding to a nice covering subordinated to nice coverings defining T and T'.

A complete transversal T has no metric structure *a priori.* However, any Riemannian metric $\langle \cdot, \cdot \rangle$ on M provides M with the Riemannian distance function ϱ. The latter induces the Hausdorff distance function ϱ_i in T_i. Without loosing generality, we may assume that $\operatorname{diam}(T_i, \varrho_i) \leq 1$ for any i. The distance function ϱ_T on T can be defined by

$$\varrho_T(x,y) = \begin{cases} \varrho_i(x,y) & \text{when } x,y \in T_i,\ i=1,\ldots,N, \\ 1 & \text{otherwise.} \end{cases} \tag{1.3.1}$$

If $\mathcal{F}$ is C^r-differentiable and $r \geq 1$, then T can be considered as a C^r-submanifold of M transverse to $\mathcal{F}$. In this case, a Riemannian structure on M induces a Riemannian metric on T. The latter provides the components of T with distance functions, denoted by ϱ_i again, which could be used to define the distance function ϱ_T on T by (1.3.1).

Since M is compact, any two metric spaces (T,ϱ_T) and (T,ϱ'_T) with the distance functions obtained from two Riemannian structures on M are quasi-isometric: there exists a constant $c \geq 1$ such that for all x and y in T,

$$c^{-1}\varrho_T(x,y) \leq \varrho'_T(x,y) \leq c\varrho_T(x,y). \tag{1.3.2}$$

Moreover, if T' is another transversal equipped with a distance function $\varrho'_{T'}$, then all the maps $h_{ij'}$ generating the morphism $\Phi_{TT'}$ are uniformly Lipschitz: there exists a constant $c_1 \geq 1$ such that

$$c_1^{-1}\varrho_T(x,y) \leq \varrho'_{T'}(h_{ij'}(x),h_{ij'}(y)) \leq c_1\varrho_T(x,y) \tag{1.3.3}$$

for all $x,y \in D_{ij'}$ and all i and j.

Similarly, holonomy pseudogroups can be defined for laminations $\mathcal{L}$, i.e. compact (more generally, separable and locally compact) metrizable spaces X equipped with open covers $\mathcal{U}$ and distinguished charts ϕ which map homeomorphically $U \in \mathcal{U}$ onto $D \times T$, D being an open subset of $\mathbb{R}^k$ ($k = \dim\mathcal{L}$), and satisfy the following condition: if $U,U' \in \mathcal{U}$ overlap, and ϕ and ϕ' are the corresponding charts, then

$$\phi' \circ \phi^{-1}(x,t) = (f(x,t),h(t)) \tag{1.3.4}$$

for all $(x,t) \in \phi(U \cap U') \subset D \times T$. Usually some smoothness conditions are required. The typical assumption is that f has all partial x-derivatives and all of them should be continuous on $\phi(U \cap U')$ [Ca].

Closed saturated subsets of foliated manifolds provide a class of examples of laminations. Other examples appear in [Su].

The following shows that there is a large class of pseudogroups which can be realized as holonomy of some foliations or laminations.

Examples 1. If Γ is a finitely generated group of diffeomorphisms of a compact manifold T, N is a compact manifold with the fundamental group $\pi_1(N)$ isomorphic to Γ, then any isomorphism $h : \pi_1(N) \to \Gamma$ provides us with a foliation $\mathcal{F}_h$, the *suspension* of h, of the manifold $M = (\widetilde{N} \times T)/\Gamma$, where $\widetilde{N}$ is the universal covering space of N. This foliation is induced by the canonical projection $\pi : \widetilde{N} \times T \to M$ from $\mathcal{F} = \{\widetilde{N} \times \{t\} : t \in T\}$. The holonomy pseudogroup of $\mathcal{F}_h$ is isomorphic to $\mathcal{G}(A)$, A being a finite set generating Γ.

2. A pseudogroup $\mathcal{G}$ acting on the interval $I = [0, 1]$ is said to be *Markov* [CC] if it admits a finite set A of generators which satisfy the following conditions: for any $g, g' \in A$,

(i) if $R_g \cap R_{g'} \neq \emptyset$, then $g = g'$,
(ii) either $R_g \subset D_{g'}$ or $R_g \cap D_{g'} = \emptyset$.

It is known [In] that any Markov pseudogroup of C^2-diffeomorphisms is isomorphic to the holonomy pseudogroup of a codim-1 foliation restricted to a neighbourhood of an exceptional minimal set.

2. Hausdorff dimension and dimension loss

2.1. *Definitions.* Let $\mathcal{G}$ be a finitely generated pseudogroup acting on a compact metric space X. For any $\varepsilon > 0$ let $\mathcal{A}(\varepsilon) = \mathcal{A}_{\mathcal{G}}(\varepsilon)$ be the family of all finite sets A generating $\mathcal{G}$ such that $\operatorname{diam} D_g \leq \varepsilon$ for all $g \in A$. Since X is compact, $\mathcal{A}(\varepsilon) \neq \emptyset$ for any ε.

Fix $s > 0$ and let

$$H^s_\varepsilon(\mathcal{G}) = \inf\{H_s(A) : A \in \mathcal{A}(\varepsilon)\}, \tag{2.1.1}$$

where

$$H_s(A) = \sum_{g\in A} (\operatorname{diam} D_g)^s. \tag{2.1.2}$$

Obviously, $H^s_\varepsilon(\mathcal{G}) \geq H^s_{\varepsilon'}(\mathcal{G})$ whenever $0 < \varepsilon \leq \varepsilon'$. Therefore, we may put

$$H^s(\mathcal{G}) = \lim_{\varepsilon\to 0} H^s_\varepsilon(\mathcal{G}) = \sup_{\varepsilon>0} H^s_\varepsilon(\mathcal{G}). \tag{2.1.3}$$

From (2.1.1) through (2.1.3) it follows immediately that $H^{s_1}(\mathcal{G}) = \infty$ and $H^{s_3}(\mathcal{G}) = 0$ if $s_1 < s_2 < s_3$ and $0 < H^{s_2}(\mathcal{G}) < \infty$. Therefore, the *Hausdorff dimension* $\dim_{\mathrm{H}} \mathcal{G}$ can be defined by

$$\dim_{\mathrm{H}} \mathcal{G} = \inf\{s > 0 : H^s(\mathcal{G}) = 0\} = \sup\{s > 0 : H^s(\mathcal{G}) = \infty\} \tag{2.1.4}$$

with the obvious convention when $H^s(\mathcal{G}) = 0$ (or ∞) for all $s > 0$.

Note that the Hausdorff dimension $\dim_{\mathrm{H}} X$ of X equals $\dim_{\mathrm{H}}(\mathcal{G}(\mathrm{id}_X))$, so we write $H^s_\varepsilon(X)$ and $H^s(X)$ instead of $H^s_\varepsilon(\mathcal{G}(\mathrm{id}_X))$ and $H^s(\mathcal{G}(\mathrm{id}_X))$, respectively. Also, for any finite open covering $\mathcal{U}$ of X we write $H_s(\mathcal{U})$ in place of $H_s(\{\mathrm{id}_U : U \in \mathcal{U}\})$.

The equality $\dim_{\mathrm{H}} \mathcal{G} = \dim_{\mathrm{H}} X$ does not hold in general but we have the following.

2.1.1. PROPOSITION. $\dim_{\mathrm{H}} \mathcal{G} \leq \dim_{\mathrm{H}} X$ *for any pseudogroup* $\mathcal{G}$ *acting on* X.

Proof. Fix any $s > \dim_{\mathrm{H}} X$, $\varepsilon > 0$ and $\eta > 0$, and take any finite set $A_0 = \{g_1, \ldots, g_N\}$ generating $\mathcal{G}$. For any $i = 1, \ldots, N$ take a finite open covering $\mathcal{U}_i$ of X such that $\operatorname{diam}\mathcal{U}_i \le \varepsilon$ and $H_s(\mathcal{U}_i) < \eta/2^i$. Let $A = \{g_i|U : U \in \mathcal{U}_i, i = 1, 2, \ldots\}$. Obviously, $A \in \mathcal{A}_{\mathcal{G}}(\varepsilon)$ and

$$H_s(A) \le \sum_i H_s(\mathcal{U}_i) \le \sum_i \eta/2^i \le \eta.$$

Therefore, $H^s_\varepsilon(\mathcal{G}) = 0$ for all ε and $H^s(\mathcal{G}) = 0$. ∎

The above proposition shows that

$$\operatorname{dl_H}\mathcal{G} = \dim_{\mathrm{H}} X - \dim_{\mathrm{H}} \mathcal{G} \tag{2.1.5}$$

is always non-negative. This difference will be called the *dimension loss* of the pseudogroup $\mathcal{G}$.

2.2. *First examples.* 1. If $\mathcal{G} = \mathcal{G}(\Gamma)$, where $\Gamma = \{f_1, \ldots, f_m\}$ is a finite group of Lipschitz homeomorphisms of X, then $\operatorname{dl_H}\mathcal{G} = 0$. In fact, if $A = \{f_k|U : U \in \mathcal{U}_k,\ k = 1, \ldots, m\} \in \mathcal{A}_{\mathcal{G}}(\varepsilon)$ (since Γ is finite, we do not lose generality by considering generating sets of this form only!), then the sets $f_j(U)$ $(U \in \mathcal{U}_k,\ j, k = 0, 1, \ldots, m-1)$ cover X and

$$H_s(A) = \sum_{k=1}^m H_s(\mathcal{U}_k) \ge \frac{1}{mK^s} \sum_{j,k=1}^m \sum_{U \in \mathcal{U}_k} (\operatorname{diam} f_j(U))^s \ge \frac{1}{mK^s} H^s_{K\varepsilon}(X),$$

where K is the maximum of Lipschitz constants for $f_1, \ldots, f_m$. Consequently,

$$H^s_\varepsilon(\mathcal{G}) \ge \frac{1}{mK^s} H^s_{K\varepsilon}(X), \qquad H^s(\mathcal{G}) \ge \frac{1}{mK^s} H^s(X)$$

and finally $\dim_{\mathrm{H}} \mathcal{G} \ge \dim_{\mathrm{H}} X$.

2. If $\mathcal{G} = \mathcal{G}(f)$, where $f : S^1 \to S^1$, $f(z) = z^2$, then $\mathcal{A}_{\mathcal{G}}(\varepsilon)$ contains a generating set A consisting of 8 maps. In fact, if $U = \{z \in S^1 : |\arg z| < \varepsilon/2\}$, m is the smallest natural number for which $f^m(U) = S^1$ and ε' is small enough, then the set $A = \{f^{m+1}|U_i, f^m|U_i : i = 1, \ldots, 4\}$ with $U_j = \{z : (j-3)\varepsilon/4 - \varepsilon' < \arg z < (j-2)\varepsilon/4 + \varepsilon'\}$ generates $\mathcal{G}$. Consequently,

$$H^s_\varepsilon(\mathcal{G}) \le 4\varepsilon^s \quad \text{and} \quad H^s(\mathcal{G}) = 0$$

for any $s > 0$. It follows that $\dim_{\mathrm{H}} \mathcal{G} = 0$ and $\operatorname{dl_H}\mathcal{G} = 1$. ($U$ is split into four pieces to have the maps of the family A invertible.)

3. Let $K = \bigcup_j f_j(K)$ be a compact invariant set for a finite system $f = \{f_1, \ldots, f_m\}$ of similarities $f_j : \mathbb{R}^N \to \mathbb{R}^N$ with ratio r_j, $0 < r_j < 1$. For any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\operatorname{diam} f_j^n(K) < \varepsilon$ for $j = 1, \ldots, m$. Let $A = \{(f_j^n|K)^{-1}, (f_j^{n+1}|K)^{-1} : j = 1, \ldots, m\}$. Then A generates the pseudogroup $\mathcal{G} = \mathcal{G}(f_1|K, \ldots, f_m|K)$ (in fact, $f_j|K = (f_j^n|K)^{-1} \circ ((f_j^{n+1}|K)^{-1})^{-1}$) and

$$\sum_{g \in A} (\operatorname{diam} D_g)^s \le 2m\varepsilon^s,$$

so $H^s(\mathcal{G}) = 0$ for all $s > 0$. Therefore, $\dim_{\mathrm{H}} \mathcal{G} = 0$ and $\mathrm{dl}_{\mathrm{H}} \mathcal{G} = \dim_{\mathrm{H}} K$. If K and f satisfy Marion's open set condition [Ma], then $\mathrm{dl}_{\mathrm{H}}\mathcal{G} = \dim_{\mathrm{h}} f$, where $\dim_{\mathrm{h}} f$ denotes the similarity dimension of the system f, i.e. the unique exponent s for which $\sum_j r_j^s = 1$.

In general, calculating Hausdorff dimensions of Cartesian products can be difficult (see [Fa], Chapter 7, for some results and examples concerning subsets of $\mathbb{R}^n$) but some pseudogroups $\mathcal{G}$ acting on spaces Z for which $0 < \mathrm{dl}_{\mathrm{H}} \mathcal{G} < \dim_{\mathrm{H}} Z$ can be produced from the above examples by acting on Cartesian products (or finite quotients of Cartesian products).

4. Let $X \subset \mathbb{R}^n$ be compact, $Y = S^1$, $Z = X \times Y$ and $\mathcal{G} = \mathcal{G}(\mathrm{id}_X \times f)$, where $f(z) = z^2$ as in Example 2. If $A = \{\mathrm{id}_U \times (f|V) : U \in \mathcal{U},\ V \in \mathcal{V}\}$ generates $\mathcal{G}$, then $\mathcal{U}$ covers X and $\mathrm{diam}(U \times V) \ge \mathrm{diam}\, U$. Therefore,

$$H^s_\varepsilon(\mathcal{G}) \ge H^s_\varepsilon(\mathcal{G}(\mathrm{id}_X))$$

for all s and ε. On the other hand, the argument similar to that of Example 2 above shows that

$$H^s_\varepsilon(\mathcal{G}) \le 8c^s H^s_\varepsilon(\mathcal{G}(\mathrm{id}_X)),$$

where c is a constant which depends only on the choice of a metric ϱ on Z ($c = 1$ when $\varrho((x,y),(x',y')) = \max\{\varrho_X(x,x'), \varrho_Y(y,y')\}$, ϱ_X and ϱ_Y being the distance functions on X and Y, respectively). The inequalities above imply that

$$\dim_{\mathrm{H}} \mathcal{G} = \dim_{\mathrm{H}} \mathcal{G}(\mathrm{id}_X) = \dim_{\mathrm{H}} X.$$

Since $Y = S^1$ is sufficiently regular, we have ([Fa], Corollary 7.4)

$$0 < 1 = \mathrm{dl}_{\mathrm{H}} \mathcal{G} < 1 + \dim_{\mathrm{H}} X = \dim_{\mathrm{H}} Z$$

provided $\dim_{\mathrm{H}} X > 0$.

2.3.1. *Morphisms.* In this subsection, we obtain relations between the Hausdorff dimensions of pseudogroups and subpseudogroups, and show how to define the transverse Hausdorff dimension and the dimension loss for foliations.

2.3.1. Proposition. *If $\mathcal{G}'$ is a finitely generated subpseudogroup of a pseudogroup $\mathcal{G}$, then*

$$\dim_{\mathrm{H}} \mathcal{G}' \ge \dim_{\mathrm{H}} \mathcal{G} \quad \textit{and} \quad \mathrm{dl}_{\mathrm{H}} \mathcal{G}' \le \mathrm{dl}_{\mathrm{H}} \mathcal{G}. \tag{2.3.1}$$

Proof. Let $A_0 = \{g_0, g_1, \ldots, g_N\}$, $g_0 = \mathrm{id}_X$, be a symmetric generating set for $\mathcal{G}$. Let $c \ge 1$ be a Lipschitz constant for all g_i's:

$$c^{-1}\varrho(x,y) \le \varrho(g_i(x), g_i(y)) \le c\varrho(x,y)$$

for all $i = 1, \ldots, N$ and $x, y \in D_i = D_{g_i}$.

Take any $\varepsilon > 0$ and a finite generating set $A' \in \mathcal{A}_{\mathcal{G}'}(\varepsilon)$. Let

$$A = \{g_i \circ h \circ g_j : h \in A',\ i, j = 1, \ldots, N\}.$$

Then A generates $\mathcal{G}$. In fact, if $x \in X$, then there exist a neighbourhood V of x and elements $h_1, \ldots, h_m$ of A' such that $\mathrm{id}_V = h_1^{e_1} \circ \ldots \circ h_m^{e_m} | V$ for some $e_1, \ldots, e_m \in \{\pm 1\}$. (Obviously, one can take $m = 2$ and $h_2 = h_1^{-1}$.) Let $x_k = h_1^{e_1} \circ \ldots \circ h_k^{e_k}(x)$ for $k = 1, \ldots, m-1$. For any k find $j_k \le N$ such that $x_k \in D_{j_k}$. For any $j_0 \le N$ the equality

$$g_{j_0} = (g_{j_0} \circ h_1^{e_1} \circ g_{j_1}) \circ \ldots \circ (g_{j_{m-1}}^{-1} \circ h_m^{e_m} \circ g_0)$$

holds on a neighbourhood $V' \subset V$ of x.

Moreover, $D_{g_i \circ h \circ g_j} \subset g_j(D_h)$, so $\operatorname{diam} D_{g_i \circ h \circ g_j} \le \operatorname{diam} g_j(D_h) \le c \operatorname{diam} D_h \le c\varepsilon$, $A \in \mathcal{A}_{\mathcal{G}}(c\varepsilon)$ and

$$H_s(A) \le c^s N^2 H_s(A')$$

for any $s > 0$. It follows that

$$H^s_{c\varepsilon}(\mathcal{G}) \le c^s N^2 H^s_\varepsilon(\mathcal{G}') \quad \text{and} \quad H^s(\mathcal{G}) \le c^s N^2 H^s(\mathcal{G}').$$

This ends the proof. ■

2.3.2. PROPOSITION. *If pseudogroups $\mathcal{G}_i$ acting on X_i, $i = 1, 2$, are isomorphic via $\Phi = \{\phi_1, \ldots, \phi_N\}$, where all the maps ϕ_i are Lipschitz, then*

$$\dim_{\mathrm{H}} \mathcal{G}_1 = \dim_{\mathrm{H}} \mathcal{G}_2 \quad \textit{and} \quad \mathrm{dl}_{\mathrm{H}} \mathcal{G}_1 = \mathrm{dl}_{\mathrm{H}} \mathcal{G}_2. \tag{2.3.2}$$

Proof. If $A_1 \in \mathcal{A}_{\mathcal{G}_1}(\varepsilon)$, then $A_2 = \{\phi_i \circ g \circ \phi_j^{-1} : g \in A_1,\ i, j \le N\} \in \mathcal{A}_{\mathcal{G}_2}(c\varepsilon)$, where c is a Lipschitz constant for all the maps $\phi_i \in \Phi$. Moreover,

$$H_s(A_2) \le c^s N^2 H_s(A_1)$$

so—as in the proof of Proposition 2.3.1—we have $H^s(\mathcal{G}_2) \le c^s N^2 H^s(\mathcal{G}_1)$ and $\dim_{\mathrm{H}} \mathcal{G}_2 \le \dim_{\mathrm{H}} \mathcal{G}_1$. Of course, the converse inequality holds as well.

The second equality in (2.3.2) holds because the spaces X_1 and X_2 have the same Hausdorff dimension: For any $i \le N$, $\dim_{\mathrm{H}} D_{\phi_i} = \dim_{\mathrm{H}} R_{\phi_i}$, $X_1 = \bigcup_i D_{\phi_i}$ and $X_2 = \bigcup_i R_{\phi_i}$, so $\dim_{\mathrm{H}} X_1 = \max_i \dim_{\mathrm{H}} D_{\phi_i} = \max_i \dim_{\mathrm{H}} R_{\phi_i} = \dim_{\mathrm{H}} X_2$. ■

2.3.3. COROLLARY. *If T and T' are complete transversals of a* C^1-*foliation $\mathcal{F}$ of a compact manifold M, then $\dim_{\mathrm{H}} \mathcal{H}_T = \dim_{\mathrm{H}} \mathcal{H}_{T'}$ and $\mathrm{dl}_{\mathrm{H}} \mathcal{H}_T = \mathrm{dl}_{\mathrm{H}} \mathcal{H}_{T'}$.* ■

Therefore, we can define the (transverse) Hausdorff dimension $\dim_{\mathrm{H}}^{\pitchfork} \mathcal{F}$ and the dimension loss $\mathrm{dl}_{\mathrm{H}}^{\pitchfork} \mathcal{F}$ of a C^1-foliation $\mathcal{F}$ as follows:

$$\dim_{\mathrm{H}}^{\pitchfork} \mathcal{F} = \dim_{\mathrm{H}} \mathcal{H}_T \quad \text{and} \quad \mathrm{dl}_{\mathrm{H}}^{\pitchfork} \mathcal{F} = \mathrm{dl}_{\mathrm{H}} \mathcal{H}_T = \operatorname{codim} \mathcal{F} - \dim_{\mathrm{H}}^{\pitchfork} \mathcal{F}, \tag{2.3.3}$$

where T is any complete transversal of $\mathcal{F}$.

2.3.4. COROLLARY. *If M' is a compact manifold and $f : M' \to M$ is a* C^1-*map transverse to a* C^1-*foliation $\mathcal{F}$ of a compact manifold M, then*

$$\dim_{\mathrm{H}}^{\pitchfork} \mathcal{F}' \ge \dim_{\mathrm{H}}^{\pitchfork} \mathcal{F} \quad \textit{and} \quad \mathrm{dl}_{\mathrm{H}}^{\pitchfork} \mathcal{F}' \le \mathrm{dl}_{\mathrm{H}}^{\pitchfork} \mathcal{F}, \tag{2.3.4}$$

where $\mathcal{F}'$ is the pullback of $\mathcal{F}$ via f.

Proof. We have $\operatorname{codim}\mathcal{F}' = \operatorname{codim}\mathcal{F}$ and the holonomy pseudogroup of $\mathcal{F}'$ is isomorphic to a subpseudogroup of the holonomy pseudogroup of $\mathcal{F}$ ([Go], p. 76). ∎

EXAMPLES. 1. For the standard Reeb foliation $\mathcal{F}$ of S^3 ([Go], p. 36) one has $\dim_{\mathrm{H}}^{\pitchfork}\mathcal{F} = 0$ and $\mathrm{dl}_{\mathrm{H}}^{\pitchfork}\mathcal{F} = 1$. In fact, any arbitrarily short closed segment $T = [-\eta, \eta]$ transverse to $\mathcal{F}$ and intersecting the unique compact leaf T^2 of $\mathcal{F}$ provides us with a complete transversal. The holonomy pseudogroup $\mathcal{H}_T$ is generated by two maps h_+ and h_- given by

$$h_+(t) = \begin{cases} t, & t < 0, \\ \lambda(t), & t \geq 0, \end{cases} \qquad h_-(t) = \begin{cases} \lambda(t), & t < 0, \\ t, & t \geq 0, \end{cases}$$

where λ is a map contracting T to the point $t_0 = 0$. If $\varepsilon > 0$ is arbitrarily small, then the maps $h_+^{-m}|(-\varepsilon/2, \varepsilon/2)$, $h_-^{-m}|(-\varepsilon/2, \varepsilon/2)$, $h_+^{-(m+1)}|(-\varepsilon/2, \varepsilon/2)$ and $h_-^{(m+1)}|(-\varepsilon/2, \varepsilon/2)$, where $m \in \mathbb{N}$ is large enough, generate $\mathcal{H}_T$, so $H_\varepsilon^s(\mathcal{H}_T) \leq 4\varepsilon^s$ and $H^s(\mathcal{H}_T) = 0$ for all $s > 0$.

2. The Hirsch foliation $\mathcal{F}$ [Hi] is obtained from the foliation of the solid torus $N = D^2 \times S^1 = \{(z, w) \in \mathbb{C}^2 : |z| \leq 1 \text{ and } |w| = 1\}$ by the slices $w = \text{const}$ in the following way: Map N into itself by $f : (z, w) \mapsto \left(\frac{1}{2}w + \frac{1}{4}z, w^2\right)$ and glue together (by the map $f|\partial N$) the boundary components of $N \setminus \operatorname{Int} f(N)$. The leaves of $\mathcal{F}$ are obtained by gluing together suitable slices $w = \text{const}$. The holonomy pseudogroup $\mathcal{H}$ of $\mathcal{F}$ is isomorphic to $\mathcal{G}(h)$, where $h : S^1 \to S^1$, $h(z) = z^2$. Therefore, $\dim_{\mathrm{H}}^{\pitchfork}\mathcal{F} = 0$ and $\mathrm{dl}_{\mathrm{H}}^{\pitchfork}\mathcal{F} = 1$.

Remark. In a separate paper [IW], T. Inaba and the author generalize the observations made above to prove (among other results) the following:

For any codimension-one non-minimal C^2*-foliation* $\mathcal{F}$ *one has*

$$\dim_{\mathrm{H}}^{\pitchfork}\mathcal{F} = \dim_{\mathrm{H}}(C \cap T), \tag{2.3.5}$$

where T *is a complete transversal and* C *the union of all the compact leaves of* $\mathcal{F}$. *If* $\mathcal{F}$ *is minimal with non-trivial holonomy, then*

$$\dim_{\mathrm{H}}^{\pitchfork}\mathcal{F} = 0. \tag{2.3.6}$$

There are examples of minimal codimension-one foliations without holonomy for which the transverse Hausdorff dimension is either 0 or 1. Also, there are examples showing that the above is not true for C^1-foliations.

2.4. *Invariant subspaces.* If $Y \subset X$ is closed and $\mathcal{G}$-invariant, i.e. $g(D_g \cap Y) \subset Y$ for any $g \in \mathcal{G}$, then the maps $g|D_g \cap Y$, $g \in \mathcal{G}$, generate the pseudogroup $\mathcal{G}|Y$ acting on Y. If $A \subset \mathcal{G}$ generates $\mathcal{G}$, then $A|Y = \{g|D_g \cap Y : g \in A\}$ generates $\mathcal{G}|Y$. Obviously, $\operatorname{diam}(D_g \cap Y) \leq \operatorname{diam} D_g$ for any g. The following is immediate.

2.4.1. PROPOSITION. *For any closed* $\mathcal{G}$*-invariant set* $Y \subset X$,

$$\dim_{\mathrm{H}} \mathcal{G}|Y \leq \dim_{\mathrm{H}} \mathcal{G}. \tag{2.4.1}$$ ∎

Since also $\dim_{\mathrm{H}} Y \leq \dim_{\mathrm{H}} X$, one cannot expect any general relation between $\mathrm{dl}_{\mathrm{H}}\,\mathcal{G}$ and $\mathrm{dl}_{\mathrm{H}}\,\mathcal{G}|Y$.

2.4.2. Corollary. *If $\mathcal{F}'$ is a subfoliation of a foliation $\mathcal{F}$, then*

$$\dim_{\mathrm{H}}^{\pitchfork} \mathcal{F}' \geq \dim_{\mathrm{H}}^{\pitchfork} \mathcal{F}. \tag{2.4.2}$$

Proof. If T and T', $T \subset T'$, are complete transversals for $\mathcal{F}$ and $\mathcal{F}'$, respectively, and $\mathcal{H}$ and $\mathcal{H}'$ are the corresponding holonomy pseudogroups, then $\mathcal{H}'|T$ is a subpseudogroup of $\mathcal{H}$. Therefore, the result follows directly from Propositions 2.3.2 and 2.4.1 together with the definition (2.3.3) of the transverse Hausdorff dimension. ∎

We say that Y has the *property of unique extension* (UEP) with respect to $\mathcal{G}$ whenever the equality $g|Y \cap V = \mathrm{id}$, V being an open subset of X, implies that $g = \mathrm{id}$ on an open (in X) neighbourhood V' of $Y \cap V$.

2.4.3. Proposition. *If $X = Y_1 \cup \ldots \cup Y_N$, where all Y_i's are closed $\mathcal{G}$-invariant and have UEP with respect to $\mathcal{G}$, then*

$$\dim_{\mathrm{H}} \mathcal{G} = \sup_i \dim_{\mathrm{H}} \mathcal{G}|Y_i. \tag{2.4.3}$$

Again, in spite of the equality

$$\dim_{\mathrm{H}} X = \sup_i \dim_{\mathrm{H}} Y_i, \tag{2.4.4}$$

one cannot expect $\mathrm{dl}_{\mathrm{H}}\,\mathcal{G}$ and $\sup_i \mathrm{dl}_{\mathrm{H}}\,\mathcal{G}|Y_i$ to be related in general.

Proof. The inequality "$\geq$" in (2.4.3) follows immediately from (2.4.1).

Let $s > \sup_i \dim_{\mathrm{H}} \mathcal{G}|Y_i$, $\varepsilon, \eta > 0$. For any i, $H^s_\varepsilon(\mathcal{G}|Y_i) = 0$, so there are generating sets $A_i \in \mathcal{A}_{\mathcal{G}|Y_i}(\varepsilon)$ for which $H_s(A_i) < \eta$. For any i and $h \in A_i$ choose $\widetilde{h} \in \mathcal{G}$ such that $\widetilde{h}|Y_i = h$ and let $\overline{h} = \widetilde{h}|D_h(\delta)$, where $\delta = \operatorname{diam} D_h$ and $Z(\delta) = \{x \in X : \varrho(z, Z) < \delta\}$ for any subset Z of X. Clearly, $\operatorname{diam} D_{\bar{h}} \leq 3 \operatorname{diam} D_h \leq 3\varepsilon$.

Put

$$A = \{\overline{h} : h \in A_i \text{ and } i = 1, \ldots, N\}.$$

Then A generates $\mathcal{G}$. (More precisely, $A \in \mathcal{A}_{\mathcal{G}}(3\varepsilon)$.) In fact, if $g \in \mathcal{G}$ and $x \in D_g \cap Y_i$, then $g|Y_i \cap V = h_1^{e_1} \circ \ldots \circ h_m^{e_m}|Y_i \cap V$, where $h_1, \ldots, h_m \in A_i$, $e_1 \ldots, e_m \in \{\pm 1\}$ and V is an open neighbourhood of x. From the UEP for Y_i it follows that $g|V' = \overline{h}_1^{e_1} \circ \ldots \circ \overline{h}_m^{e_m}|V'$ for another open neighbourhood V' of x.

Since

$$H_s(A) \leq 3^s \sum_{i=1}^N H_s(A_i) < 3^s N\eta,$$

we have $H^s_{3\varepsilon}(\mathcal{G}) = 0$ and $H^s(\mathcal{G}) = 0$. This proves the inequality "$\leq$" in (2.4.3) and ends the proof of the proposition. ∎

Now, we exhibit a pseudogroup $\mathcal{G}$ for which both $\dim_{\mathrm{H}} \mathcal{G}$ and $\mathrm{dl}_{\mathrm{H}} \mathcal{G}$ are positive and irrational. A pseudogroup like this could be called *fractal*.

EXAMPLE. Let $X = [0,1]$, $Y \subset X$ be the standard $(1/3)$-Cantor set and $Y \setminus X = \bigcup_m I_m$, I_m being the gaps of Y. Define $f : X \to X$ by

$$f(x) = \begin{cases} x & \text{for } x \in Y, \\ h_m^{-1}(h_m(x)^2) & \text{for } x \in I_m, \end{cases}$$

where h_m is the unique increasing linear map of I_m onto $(0,1)$. The map f is a homeomorphism, so it generates a pseudogroup $\mathcal{G} = \mathcal{G}(f)$. Since Y is $\mathcal{G}$-invariant and $\mathcal{G}|Y = \mathcal{G}(\mathrm{id}_Y)$,

$$\dim_{\mathrm{H}} \mathcal{G} \ge \dim_{\mathrm{H}}(\mathcal{G}|Y) = \dim_{\mathrm{H}} Y = \log 2/\log 3.$$

On the other hand, given $\varepsilon = 3^{-k}$ and $s > 0$, the number of gaps I_m of length $\ge \varepsilon$ equals 2^{k-1} and for each of them one can find four maps defined on domains of diameter less than $(\varepsilon/2^{k-1})^{1/s}$ and generating $\mathcal{G}|\overline{I}_m$ (compare Example 2 of Section 2.2). Removing all such gaps we remain with 2^k closed intervals K_i of length ε. The set $A = \{f|K_i : i = 1, \ldots, 2^k\}$ generates $\mathcal{G}|\bigcup_i K_i$. It follows that

$$H^s_\varepsilon(\mathcal{G}) \le 2^k \varepsilon^s + 4\varepsilon \to 0 \quad \text{as } k \to \infty$$

for all $s > \log 2/\log 3$. Therefore, $\dim_{\mathrm{H}} \mathcal{G} \le \log 2/\log 3$ and finally,

$$\dim_{\mathrm{H}} \mathcal{G} = \log 2/\log 3 \quad \text{and} \quad \mathrm{dl}_{\mathrm{H}} \mathcal{G} = 1 - \log 2/\log 3.$$

Clearly, given $r \in (0,1)$, one can modify the example to create a pseudogroup $\mathcal{G}$ on $[0,1]$ (or on S^1) with $\mathrm{dl}_{\mathrm{H}} \mathcal{G} = r$. Also, it is not difficult to make this example C^∞-differentiable.

3. Invariant measures

3.1. *Some existence results.* Given a pseudogroup $\mathcal{G}$ acting on a compact space X, $\mathcal{M}(X,\mathcal{G})$ denotes the space of all $\mathcal{G}$-invariant Borel probability measures on X. So, if $\mu \in \mathcal{M}(X,\mathcal{G})$, $g \in \mathcal{G}$ and $A \subset D_g$ is a Borel set, then $\mu(g(A)) = \mu(A)$. If $\mathcal{H} = \mathcal{H}_T$ is the holonomy pseudogroup of a foliation $\mathcal{F}$, then members of $\mathcal{M}(T,\mathcal{H}_T)$ are called *transverse invariant measures* (in the sense of Plante [Pl]). If $f : X \to X$, then $\mathcal{M}(X,f)$ denotes the set of all Borel probability measures on X which are f-invariant in the sense that $\mu(f^{-1}A) = \mu(A)$ for any $A \subset X$.

If $f : X \to X$ is a homeomorphism, then, by the classical Krylov–Bogolyubov Theorem ([Wa], p. 152), the space $\mathcal{M}(X,\mathcal{G}(f)) = \mathcal{M}(X,f)$ is non-empty, compact and convex in $\mathcal{M}(X)$, the space of all Borel probability measures on X. In fact, $\mathcal{M}(X,f) \ne \emptyset$ for any continuous transformation f of X, but $\mathcal{M}(X,\mathcal{G}(f)) \nsubseteq \mathcal{M}(X,f)$ for some local homeomorphisms $f : X \to X$.

In general, the space $\mathcal{M}(X,\mathcal{G})$ may be empty. In [Pl], we can find the following condition sufficient for $\mathcal{G}$ to admit non-trivial invariant measures.

Given a finite symmetric generating set $A \subset \mathcal{G}$ (and $x \in X$) let $N(n,A) = \#\{g_1 \circ \ldots \circ g_n : g_i \in A\}$ $(N(n,x,A) = \#\{g(x) : g = g_1 \circ \ldots \circ g_n,\ g_i \in A\})$. Then $\mathcal{G}$ *has non-exponential growth* (*at* x) whenever

$$\liminf_{n\to\infty} \frac{1}{n} \log N(n,A) = 0 \qquad \Big(\liminf_{n\to\infty} \frac{1}{n} \log N(n,x,A) = 0\Big).$$

If $\mathcal{G}$ *has non-exponential growth* (*at a point* x), *then there exists* $\mu \in \mathcal{M}(X,\mathcal{G})$ (*supported in the closure of* $\mathcal{G}(x)$, *the* $\mathcal{G}$*-orbit of* x).

Another result of this sort can be found in [GLW]:

If the geometric entropy $h(\mathcal{F})$ *of a foliation* $\mathcal{F}$ *vanishes, then* $\mathcal{M}(X,\mathcal{H}) \neq \emptyset$, $\mathcal{H}$ *being the holonomy pseudogroup of* $\mathcal{F}$.

Recall that $h(\mathcal{F})$ is defined (up to a positive factor) as the entropy $h(\mathcal{H})$ of its holonomy pseudogroup $\mathcal{H}$ generated by the collection $\mathcal{H}_1$ of the holonomy maps corresponding to the overlapping charts of a fixed nice covering $\mathcal{U}$, and

$$h(\mathcal{H}) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(n,\varepsilon),$$

where $N(n,\varepsilon)$ is the maximal cardinality of (n,ε)-separated subsets Y of T, the complete transversal of $\mathcal{F}$ determined by $\mathcal{U}$; a set Y is said to be (n,ε)-*separated* whenever for any distinct points x and y of Y, there exist $g_1,\ldots,g_n \in \mathcal{H}_1$ such that $\varrho(g(x),g(y)) \geq \varepsilon$ for $g = g_1 \circ \ldots \circ g_n$. Although $h(\mathcal{F})$ depends on the choice of a Riemannian structure on M, its vanishing does not. $h(\mathcal{F}) = 0$ iff $\mathcal{F}$ has non-exponential expansion growth in the sense of [Eg1].

3.2. *s-continuous measures.* Fix $s \geq 0$. A Borel probability measure μ on a compact metric space X is said to be s-*continuous* if there exist positive constants c and ε_0 such that

$$\mu(A) \leq c(\operatorname{diam} A)^s \tag{3.2.1}$$

for any Borel set $A \subset X$ with $\operatorname{diam} A \leq \varepsilon_0$. Clearly, all measures are 0-continuous, and an s-continuous measure is s'-continuous for any $s' < s$. The smallest upper bound of the set of all the exponents s satisfying (3.2.1) could be considered as the Hausdorff dimension $\dim_{\mathrm{H}} \mu$ of the measure μ. Obviously, measures with positive Hausdorff dimension have no atoms.

Examples. 1. The Lebesgue measure λ on $\mathbb{R}^n$ is n-continuous and has $\dim_{\mathrm{H}} \lambda = n$.

2. A smooth measure $\mu = f dV$, where dV is the volume form and $f \in L^\infty(M)$, on a compact oriented Riemannian manifold M is n-continuous, $n = \dim M$. In fact, since M is compact, it has bounded geometry and

there exists $a > 0$ such that $\operatorname{Vol} B(x,r) \le ar^n$ for all $x \in M$ and $r > 0$. Therefore, if $A \subset M$, $r = \operatorname{diam} A$ and $x_0 \in A$, then $A \subset B(x_0, r)$ and

$$\mu(A) = \int_A f\,dV \le \int_{B(x_0,r)} f\,dV \le \|f\|_\infty ar^n,$$

so the inequality (3.2.1) holds with $s=n$ and $c=a\|f\|_\infty$. Again, $\dim_{\mathrm{H}} \mu = n$.

3. If $X = X_1 \times X_2$ and $\mu = \mu_1 \times \mu_2$, where μ_i is a Borel probability measure on X_i, $i = 1, 2$, and μ_1 is s_1-continuous then μ is s_1-continuous as well. In fact, if $Y \subset X$ is a Borel set and $Y_x = \{w \in X_1 : (w, x) \in Y\}$, $x \in X_2$, then

$$\mu(Y) = \int_{X_2} \mu_1(Y_x)\,d\mu_2(x) \le c \int_{X_2} (\operatorname{diam} Y_x)^{s_1}\,d\mu_2(x) \le c(\operatorname{diam} Y)^{s_1}$$

for a suitable constant c.

In the same way, if both μ_i's are (respectively) s_i-continuous, then μ is $(s_1 + s_2)$-continuous.

4. Let $X = \{0, 1, \ldots, k-1\}^{\mathbb{Z}}$, $k > 1$, $\varrho(x,y) = \sum_{n=-\infty}^{\infty} 2^{-|n|}|x_n - y_n|$, when $x = (x_n)$ and $y = (y_n)$, and $\tau : X \to X$ be the two-sided Bernoulli shift, $\tau((x_n)) = (y_n)$ with $y_n = x_{n+1}$ for all $n \in \mathbb{Z}$. Let μ be the unique τ-invariant measure with maximal entropy: $\mu([x_j, \ldots, x_{j+m-1}]) = k^{-m}$, where $j \in \mathbb{Z}$, $m \in \mathbb{N}$ and $[x_j, \ldots, x_{j+m-1}]$ is the "rectangle" consisting of all the sequences $y = (y_n)$ for which $y_j = x_j, \ldots, y_{j+m-1} = x_{j+m-1}$. It is easy to see that μ is s-continuous with $s = \dim_{\mathrm{H}}(X, \varrho) = 2\log k/\log 2$.

Note that the situation is quite different for the one-sided shift $\tau_0 : Y \to Y$, $Y = \{0, 1, \ldots, k-1\}^{\mathbb{N}}$, the distance function ϱ_0 and the τ_0-invariant measure μ_0 defined analogously to ϱ and μ above. Here, μ_0 is τ_0-invariant in the sense that $\mu_0(\tau_0^{-1}A) = \mu_0(A)$ for all A but it is not $\mathcal{G}(\tau_0)$-invariant: $\tau_0([x_1]) = Y$ for any $x_1 \in \{0, 1, \ldots, k-1\}$, so $\mu_0(\tau_0[x_1]) \ne \mu_0([x_1])$. Moreover, $\mathcal{G}(\tau_0)$ is equivalent to the pseudogroup generated by a system of k similarities, so $\operatorname{dl_H} \mathcal{G}(\tau_0) = \dim_{\mathrm{H}} Y = \log k/\log 2$ (compare Example 3 of Section 2.2 and Example 2 of Section 3.3).

5. Let (X, ϱ) and τ be as in Example 4. Let $A = (a_{ij})$, $a_{ij} \in \{0,1\}$, $i, j = 0, 1, \ldots, k-1$, be an irreducible matrix [Ga], $X_A = \{(x_n) \in X : a_{x_n x_{n+1}} = 1 \text{ for all } n \in \mathbb{Z}\}$, $\tau_A = \tau|X_A$ and μ_A be the τ_A-invariant Parry measure [Pa] on X_A. Then μ_A is s-continuous with $s = 2\log\lambda/\log 2$, λ being the largest positive eigenvalue of A.

3.3. *A loss estimate from above.* Again, fix $s > 0$.

3.3.1. Proposition. *If a pseudogroup $\mathcal{G}$ acting on a compact space X admits an s-continuous invariant Borel probability measure μ, then*

$$\dim_{\mathrm{H}} \mathcal{G} \ge s \quad \text{and} \quad \operatorname{dl_H} \mathcal{G} \le \dim_{\mathrm{H}} X - s. \tag{3.3.1}$$

In other words,

(3.3.2) $$\dim_{\mathrm{H}} \mathcal{G} \geq \sup\{\dim_{\mathrm{H}} \mu : \mu \in \mathcal{M}(X, \mathcal{G})\}$$

and

(3.3.3) $$\mathrm{dl}_{\mathrm{H}} \mathcal{G} \leq \inf\{\dim_{\mathrm{H}} X - \dim_{\mathrm{H}} \mu : \mu \in \mathcal{M}(X, \mathcal{G})\}.$$

Proof. If ε is small enough and $A \in \mathcal{A}_{\mathcal{G}}(\varepsilon)$, then, by (1.1.1) and $\mathcal{G}$-invariance of μ,

(3.3.4) $$1 = \mu(X) \leq \sum_{g \in A} (\mu(D_g) + \mu(R_g)) = 2 \sum_{g \in A} \mu(D_g),$$

and s-continuity of μ implies that

(3.3.5) $$1 \leq 2cH_s(A),$$

where $c > 0$ satisfies (3.2.1). Consequently, $H^s_{\varepsilon}(\mathcal{G}) \geq (2c)^{-1} > 0$ for any sufficiently small $\varepsilon > 0$ and $H^s(\mathcal{G}) > 0$. This implies (3.3.1). ■

3.3.2. Corollary. *If a pseudogroup $\mathcal{G}$ acts on a compact manifold M and admits a smooth invariant measure*, *then*

(3.3.6) $$\dim_{\mathrm{H}} \mathcal{G} = \dim M \quad \textit{and} \quad \mathrm{dl}_{\mathrm{H}} \mathcal{G} = 0. \ ■$$

Remark. Inequalities (3.3.4) and (3.3.5) could be replaced by

$$l(A) = \sum_{g \in A} \mu(D_g) \leq cH_s(A),$$

where $l(A)$ is Levitt's cost of generating. Since $l(A) \geq 1 - e(\mathcal{G})$, where $e(\mathcal{G}) = \int_X N(x)^{-1} d\mu(x)$, $N(x)$ is the cardinality of the $\mathcal{G}$-orbit of $x \in X$ and $1/\infty = 0$ [Le], we get the inequality

$$1 - e(\mathcal{G}) \leq cH^s(\mathcal{G}),$$

which implies (3.3.1) provided $e(\mathcal{G}) < 1$, i.e. if the set of points fixed under the action of $\mathcal{G}$ has measure strictly less than 1.

Examples. 1. Since any measure is $\mathcal{G}(\mathrm{id}_X)$-invariant, our proposition implies that $s \leq \dim_{\mathrm{H}} X$ if X admits s-continuous probability measures.

2. The pseudogroup $\mathcal{G}(\tau)$ generated by the two-sided shift τ discussed in Example 4 of Section 3.2 satisfies

$$\mathrm{dl}_{\mathrm{H}} \mathcal{G}(\tau) = 0.$$

Also, by Example 5 of Section 3.2, for any irreducible matrix A with largest positive eigenvalue λ one has the inequality

$$\dim_{\mathrm{H}} \mathcal{G}(\tau_A) \geq 2 \log \lambda / \log 2.$$

3. For any finitely generated subgroup G of $\mathrm{SL}(n, \mathbb{Z})$ one has $\mathrm{dl}_{\mathrm{H}} \mathcal{G} = 0$, where $\mathcal{G}$ is the pseudogroup of local diffeomorphisms of the n-torus T^n generated by G. In fact, all the elements of $\mathcal{G}$ preserve the canonical volume form on T^n.

4. Any pseudogroup $\mathcal{G}$ of local isometries of a Riemannian manifold M satisfies the equality

$$\dim_{\mathrm{H}} \mathcal{G} = \dim M$$

since $\mathcal{G}$ preserves the volume element on M. Consequently,

$$\dim_{\mathrm{H}}^{\pitchfork} \mathcal{F} = \operatorname{codim} \mathcal{F}$$

for any Riemannian foliation $\mathcal{F}$. (Note (Section 4.1) that the situation is more complicated in the case of pseudogroups of local isometries of arbitrary metric spaces.)

5. The geodesic flow (X_t) of a Riemannian manifold M acts on the unit tangent bundle SM and preserves the Liouville measure (i.e. the volume form induced by the Sasaki metric, [Kl], Chapter 3) on SM. Since the corresponding vector field $X = (dX_t/dt)|_{t=0}$ has norm one, X_t preserves the volume element in the bundle $T^{\perp}\mathcal{F}_0$, the orthogonal complement in TSM of the bundle $T\mathcal{F}_0$ tangent to the 1-dimensional foliation $\mathcal{F}_0$ of SM by the orbits of (X_t). Therefore, $\mathrm{dl}_{\mathrm{H}}^{\pitchfork} \mathcal{F}_0 = 0$ in this case. A similar result holds for 1-dimensional foliations by the orbits of geodesic flows of transversely minimal foliations $\mathcal{F}$ (of arbitrary dimension) of compact Riemannian manifolds. Such geodesic flows preserve a suitable volume element ([W1], see [W2] for a slightly more general result). Also, holonomy maps of transversely minimal foliations $\mathcal{F}$ preserve the volume element in the bundle $T^{\perp}\mathcal{F}$, so $\mathrm{dl}_{\mathrm{H}}^{\pitchfork} \mathcal{F} = 0$ for such $\mathcal{F}$.

4. Further examples

4.1. *Local isometries.* A pseudogroup $\mathcal{G}$ generated by isometries $g_i : D_i \to R_i$, $i = 1, \ldots, N$, consists of local isometries: if $g \in \mathcal{G}$ and $x \in D_g$, then there exists a neighbourhood U of x such that $g|U$ maps isometrically U onto $g(U)$. Since $\mathcal{G}$ is closed under unions of maps, it can contain maps which do not map isometrically the whole domains onto the ranges (see Figure 1, where g rotates the vertical segment on the left and fixes all the points of the horizontal segment and of the vertical segment on the right).

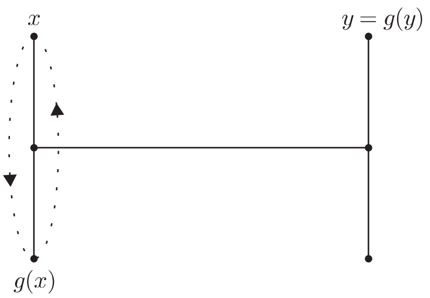

Fig. 1

So, in general, the maps of $\mathcal{G}$ do not preserve the diameters of subsets of their domains. However, we have the following.

4.1.1. LEMMA. *Pseudogroups of local isometries of a space X preserve all the Hausdorff measures H^s, $s > 0$, on X.*

Proof. Since the Hausdorff measures H^s are regular (in the sense of [Ru]), it is sufficient to show that

$$H^s(g(K)) = H^s(K) \tag{4.1.1}$$

whenever $g \in \mathcal{G}$ and $K \subset D_g$ is compact.

To this end, cover K by finitely many subsets $U_1, \ldots, U_m$ of D_g open and such that $g|U_i$ is an isometry for any i. Let λ be the Lebesgue number of the covering (U_i) and ε be a positive number less than λ. For any covering $\mathcal{V} = \{V_1, \ldots, V_n\}$ of K by sets of diameter less than ε and any $j \leq n$ there exists i such that $V_j \subset U_i$. Therefore, $g|V_j$ is an isometry, $\operatorname{diam} g(V_j) = \operatorname{diam} V_j$ for any j and $H_s(g(\mathcal{V})) = H_s(\mathcal{V})$. Consequently,

$$H^s_\varepsilon(g(K)) = H^s_\varepsilon(K) \tag{4.1.2}$$

for any $\varepsilon < \lambda$. Obviously, (4.1.2) implies (4.1.1). ∎

The above lemma together with Propositions 2.1.1 and 3.3.1 implies directly the following.

4.1.2. PROPOSITION. *If $s_0 = \dim_{\mathrm{H}} X$ and the Hausdorff measure H^{s_0} is non-trivial, finite and s_0-continuous, then*

$$\dim_{\mathrm{H}} \mathcal{G} = \dim_{\mathrm{H}} X \tag{4.1.2}$$

for any pseudogroup $\mathcal{G}$ of local isometries of X. ∎

Remarks. (i) The measure H^n on a compact n-dimensional Riemannian manifold M satisfies the assumptions of Proposition 4.1.2, so the observations of Example 4 in Section 3.3 follow directly from this proposition.

(ii) Obviously, there exist metric spaces for which the assumptions of Proposition 4.1.2 are not satisfied. For example, if

$$X = \bigcup_{n=1}^{\infty} X_n \cup [0,1] \subset \mathbb{R}^2,$$

where X_n is the union of 2^n segments of length $2^{-(n+1)}$ attached to $[0,1]$

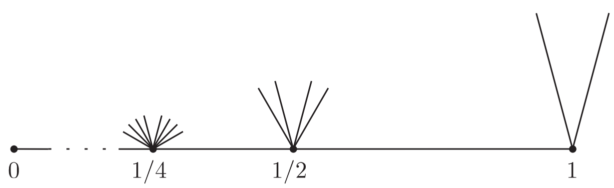

Fig. 2

at the point $(2^{-(n-1)},0)$ (Figure 2), then $\dim_{\mathrm{H}} X = 1$ and $H^1(X_n) \geq 2^{n-1} \operatorname{diam} X_n$, so H^1 is not 1-continuous. (Moreover, we have $H^1(X_n) \geq 2^{s(n-1)}(\operatorname{diam} X_n)^s$ for any $s > 0$, so $\dim_{\mathrm{H}} H^1 = 0$ in this case. Also, it is easy to modify the example to get a space X with $H^1(X) < \infty$.) This simple example suggests that calculation of the Hausdorff dimension for pseudogroups of local isometries of "wild" metric spaces could be rather difficult.

4.2. *Hyperbolic groups.* First, let us collect definitions and facts about hyperbolic metric spaces and hyperbolic groups needed to formulate and prove the result. In general, we follow the terminology and notation of [GH1].

A metric space (X,ϱ) is *δ-hyperbolic* ($\delta \geq 0$) if

$$(x|y)_w \geq \min\{(x|z)_w, (z|y)_w\} - \delta$$

for all $w,x,y,z \in X$, where $(x|y)_w$ is the *Gromov product* defined by

$$(x|y)_w = \tfrac{1}{2}\left(\varrho(x,w) + \varrho(y,w) - \varrho(x,y)\right).$$

(X,ϱ) is *hyperbolic* whenever it is δ-hyperbolic for some δ.

A metric space (X,ϱ) is *geodesic* if for any x and y in X there exists a geodesic segment joining x to y, i.e. a curve $c : [0,d] \to X$ such that $d = \varrho(x,y)$, $x = c(0)$, $y = c(d)$ and $\varrho(c(s),c(t)) = |s-t|$ for all s and t. (X,ϱ) is *proper* if all the closed balls in X are compact. By the Hopf–Rinow Theorem [GLP], a geodesic space is proper iff it is locally compact and complete.

A finitely generated group Γ is said to be *hyperbolic* if its Cayley graph $G(\Gamma,S)$ with the word metric d_S determined by a finite symmetric generating set $S \subset \Gamma$ is hyperbolic for some (equivalently, any) S. A hyperbolic group Γ is *non-elementary* if it is infinite and contains no cyclic subgroups of finite index. A Cayley graph of any group Γ is geodesic and proper.

Any isometry γ of a hyperbolic geodesic proper metric space (X,ϱ) is either *elliptic* (when all the orbits of γ are bounded), or *hyperbolic* (when all the orbits are quasi-isometric to $\mathbb{Z}$), or *parabolic* (otherwise).

If (X,ϱ) is hyperbolic, geodesic and proper, then ∂X, the *boundary* of X, is defined as the space of equivalent *quasi-rays*, i.e. quasi-isometric maps of $\mathbb{R}_+$ (or $\mathbb{Z}_+$) into X. When equipped with a suitable topology and a metric d, ∂X becomes a compact metric space of finite Hausdorff dimension. A possible definition for d is

$$d(a,b) = \inf\Big\{\sum_{i=0}^{n-1} \varrho_\eta(a_i,a_{i+1}) : a_0,a_1,\ldots,a_n \in \partial X,\\ a_0 = a,\ a_n = b \text{ and } n \in \mathbb{N}\Big\}, \tag{4.2.1}$$

where

$$\varrho_\eta(a,b) = \exp(-\eta(a|b)) \tag{4.2.2}$$

and

$$(a|b) = \sup \liminf_{i,j\to\infty} (x_i|y_j)_w \tag{4.2.3}$$

for $a, b \in \partial X$, (x_i) and (y_j) being sequences of points of X converging, respectively, to a and b, and w being an arbitrarily fixed base point for all the Gromov products involved. (In (4.2.1) and (4.2.2), $\eta > 0$ is an arbitrarily fixed constant.)

Any isometry γ of X extends uniquely to a quasi-isometry (denoted again by γ) of ∂X. An isometry γ is hyperbolic iff $\gamma : \partial X \to \partial X$ has exactly two fixed points a_1 and a_2 such that $\partial X \setminus \{a_i\}$ can be equipped with a complete metric d_γ, compatible with the topology of ∂X and such that

$$d_\gamma(\gamma(b_1), \gamma(b_2)) = \Phi_{a_i}(\gamma) d_\gamma(b_1, b_2), \tag{4.2.4}$$

where

$$\Phi_{a_i}(\gamma) = \lim_{n\to\pm\infty} \left(\frac{d(\gamma^n b_1, \gamma^n b_2)}{d(b_1, b_2)}\right)^{1/n} \qquad (b_1, b_2 \in \partial X \setminus \{a_i\}, b_1 \neq b_2) \tag{4.2.5}$$

is the *force* of γ at a_i. Note that $\Phi_{a_i}(\gamma) \neq 1$ and $\Phi_{a_1}(\gamma) < 1$ whenever $\Phi_{a_2}(\gamma) > 1$. Moreover, if U_i, $i = 1, 2$, are arbitrary open neighbourhoods of a_i in ∂X and, for instance, $\Phi_{a_1}(\gamma) < 1$, then there exists $n_0 \in \mathbb{N}$ such that

$$\gamma^n(\partial X \setminus U_1) \subset U_2 \quad \text{and} \quad \gamma^{-n}(\partial X \setminus U_2) \subset U_1 \tag{4.2.6}$$

for all $n \geq n_0$.

Any group Γ acts isometrically on its Cayley graph by left (or right) translations. If Γ is hyperbolic and non-elementary, its element γ is elliptic iff it is of finite order. There are no parabolic elements of Γ while hyperbolic elements have to exist.

Now, let Γ be a finitely generated group of isometries of a hyperbolic geodesic proper metric space (X, ϱ). Assume that Γ admits a finite generating set $S = \{\gamma_0, \gamma_1, \ldots, \gamma_N\}$ containing a hyperbolic isometry γ_0.

Let $a, b \in \partial X$ be fixed points of γ_0 such that $\Phi_a(\gamma_0) < 1$ and $\Phi_b(\gamma_0) > 1$. Fix $\varepsilon > 0$ and let $U = B(a, \varepsilon/2)$ and $V = B(b, \varepsilon/2)$ be open balls in $(\partial X, d)$ centred at a and b, respectively. Assume that ε is small enough to have $\overline{U} \cap \overline{V} = \emptyset$. By (4.2.6), we can choose $n_0 \in \mathbb{N}$ such that $\gamma_0^n(\partial X \setminus U) \subset V$ and $\gamma_0^{-n}(\partial X \setminus V) \subset U$ for any $n \geq n_0$. Fix $n \geq n_0$ and let

$$\begin{aligned} A = {} & \{\gamma_0^n|U, \gamma_0^{n+1}|U, \gamma_0^{-n}|V, \gamma_0^{-(n+1)}|V\} \\ & \cup \{\gamma_i \circ \gamma_0^n|U, \gamma_i \circ \gamma_0^{-n}|V : i = 1, \ldots, N\}. \end{aligned}$$

Let $\mathcal{G}(\Gamma)$ be the pseudogroup generated by Γ on ∂X. The set A generates $\mathcal{G}(\Gamma)$,

$$H^s_\varepsilon(\mathcal{G}(\Gamma)) \le (4+2N)\varepsilon^s \quad \text{and} \quad H^s(\mathcal{G}(\Gamma)) = 0$$

for all $s > 0$ and $\varepsilon > 0$ small enough. In this way we proved the following.

4.2.1. Proposition. *If a group Γ of isometries of a hyperbolic geodesic proper metric space (X, ϱ) contains a hyperbolic element, then*

$$\dim_{\mathrm{H}} \mathcal{G}(\Gamma) = 0 \quad \textit{and} \quad \mathrm{dl}_{\mathrm{H}}\, \mathcal{G}(\Gamma) = \dim_{\mathrm{H}} \partial X < \infty. \ \blacksquare \tag{4.2.7}$$

4.2.2. Corollary. *Equalities* (4.2.7) *hold for any non-elementary hyperbolic group Γ.* ■

Remark. Non-elementary hyperbolic groups provide more contraction (or expansion) than needed to get (4.2.7). In fact, we have the following.

4.2.3. Proposition. *Any non-elementary hyperbolic group Γ admits a finite generating set consisting of hyperbolic elements only.*

Proof. Let $S = \{g_1, \ldots, g_N\}$ be any symmetric set generating Γ. Denote by d_S the word metric in the Cayley graph $X = G(\Gamma, S)$ and let $|g| = d_S(e, g)$ be the corresponding norm of $g \in \Gamma$.

Take a hyperbolic element h of Γ and let

$$L(h) = \lim_{n\to\infty} \frac{1}{n}|h^n|. \tag{4.2.8}$$

(Note that since $|h^{m+n}| \le |h^m| + |h^n|$, the limit in (4.2.8) exists by an elementary argument which can be found, for example, in [Wa], Thm. 4.9.) Obviously,

$$L(h) > 0, \quad L(ghg^{-1}) = L(h) \quad \text{and} \quad L(h^k) = |k| L(h) \tag{4.2.9}$$

for any $k \in \mathbb{Z}$ and $g \in \Gamma$.

Moreover, up to a constant factor which depends only on the choice of a metric on ∂X, $L(h)$ equals $\log \Phi_a(h)$, where $a \in \partial X$ is a point fixed by h. In fact, there exist points b_n and b'_n ($n \in \mathbb{N}$) such that $|b_n| = |b'_n| = n$ and $(b_n|b'_n)_e = 0$ for all n. Let $b = \lim_{n\to\infty} b_n$ and $b' = \lim_{n\to\infty} b'_n$. Then $b, b' \in \partial X$ and $b \ne b'$ so, without loosing generality, we may assume that $b \ne a \ne b'$. For all m and n we have $(h^m b_n | h^m b'_n)_e = 2|h^m|$, so if the metric d on ∂X is given by (4.2.1)–(4.2.3), then $d(b, b') = 1$ and

$$\begin{aligned} \log \Phi_a(h) &= \lim_{m\to\infty} \frac{1}{m} \log d(h^m b, h^m b') \\ &= \lim_{m\to\infty} \frac{-2\eta|h^m|}{m} = -2\eta L(h). \end{aligned} \tag{4.2.10}$$

Take $k \in \mathbb{N}$ and let $h_0 = h^k$. Let $A = \{h_0, g_1 h_0, \ldots, g_N h_0\}$. Obviously, A generates Γ. If k and $R > 0$ are large enough, and $1 \le i \le N$, then—by (4.2.4), (4.2.9) and (4.2.10)—one of the maps $g_i h_0$, $(g_i h_0)^{-1}$ contracts

$B_j = \overline{B}(a_j, R)$, $j = 1, 2$, into itself, $a_1, a_2 \in \partial X$ being the points fixed by h. By the completeness argument, each of the maps $g_i h_0$, $i = 1, \ldots, N$, admits exactly two fixed points $a_1^i, a_2^i \in \partial X$. By the classification of elements of Γ, $g_i h_0$ is hyperbolic for any i. ■

4.3. *Rational maps.* Let J be the Julia set of a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Assume that f has no critical points on J. Since J is f-invariant, $f|J$ induces a pseudogroup on J.

4.3.1. Proposition. $\dim_{\mathrm{H}} \mathcal{G}(f|J) = 0$.

Proof. Recall (see, for example, [Be], Section 6.9) that J can be defined as the closure of the set of all the repelling periodic points of f. For any repelling periodic point $x \in J$ choose a disc $D(x)$ centred at x and such that $f^{n(x)}$ is conjugate to $\lambda(x) \cdot \mathrm{id}$, where $n(x)$ is the period of x and $\lambda(x) = (f^{n(x)})'(x)$. Cover J by finitely many discs $D(x_1), \ldots, D(x_N)$ and put $m = n(x_1) \cdot \ldots \cdot n(x_N)$. Then $F = f^m$ fixes all the points x_j.

Take positive numbers s and ε, and exponents k_j, $j = 1, \ldots, N$, such that $D(x_j) \subset F^{k_j}(D(x_j, \varepsilon/2))$. The maps

$$F^{k_j}|D(x_j, \varepsilon/2) \cap J, F^{k_j+1}|D(x_j, \varepsilon/2) \cap J, \qquad j = 1, \ldots, N,$$

generate the pseudogroup $\mathcal{G}(F|J)$. It follows that $H^s_\varepsilon(\mathcal{G}(F|J)) \le 2N\varepsilon^s$, $H^s(\mathcal{G}(F|J)) = 0$ and $\dim_{\mathrm{H}} \mathcal{G}(F|J) = 0$.

Since $\mathcal{G}(F|J)$ is a subpseudogroup of $\mathcal{G}(f|J)$, the statement follows from Proposition 2.3.1. ■

Note that $\dim_{\mathrm{H}} J \ge \log d / \log K_0 > 0$, where d is the degree of f and $K_0 = \max_J |f'|$ ([Be], Section 10.3). Therefore, $\mathrm{dl}_{\mathrm{H}}(f|J) > 0$.

4.4. *Attractors.* Let $\mathcal{G}$ be, as usual, a pseudogroup acting on X. An *attractor* for $\mathcal{G}$ is a compact set K such that

$$K \subset D_g, \quad R_g \subset D_g \quad \text{and} \quad \bigcap_{n \ge 0} g^n(D_g) = K \tag{4.4.1}$$

for some $g \in \mathcal{G}$. Let $\mathcal{G}_K$ be the set of all $g \in \mathcal{G}$ which satisfy (4.4.1). Obviously, $g^n \in \mathcal{G}_K$ for any $n \in \mathbb{N}$ and $g \in \mathcal{G}_K$.

The family $\mathcal{D}_K = \{D_g : g \in \mathcal{G}_K\}$ is partially ordered by inclusion. Unions of ordered subfamilies of $\mathcal{D}_K$ are called *basins of attraction* of K. Note that for any basin of attraction B and any $x \in B$ there exist $g \in \mathcal{G}_K$, $y \in K$ and $m_n \in \mathbb{N}$ such that $m_n \to \infty$ and $g^{m_n}(x) \to y$ as $n \to \infty$. In other words, any $x \in B$ admits $g \in \mathcal{G}_K$ such that its ω-limit set $\omega_g(x)$ is contained in K.

4.4.1. Proposition. *If $K_1, \ldots, K_m$ are attractors for $\mathcal{G}$ and $B_1, \ldots, B_m$ are their basins of attraction, then*

$$\dim_{\mathrm{H}} \mathcal{G} \le \dim_{\mathrm{H}} \Big(\bigcup_{i=1}^m K_i \cup \Big(X \setminus \bigcup_{i=1}^m B_i \Big) \Big). \tag{4.4.2}$$

P r o o f. Let $Y = \bigcup K_i \cup (X \setminus \bigcup B_i)$. Take $s > \dim_{\rm H} Y$, $\varepsilon > 0$, $\eta > 0$ and a finite open covering $\mathcal{U}$ of Y by sets of diameter less than ε and such that $H_s(\mathcal{U}) < \eta$. Also, fix a finite set A_0 generating $\mathcal{G}$.

For any $j \le m$ there exists $g_j \in \mathcal{G}_{K_j}$ such that $B_j \setminus \bigcup \mathcal{U} \subset D_{g_j}$ and $R_{g_j} \subset \bigcup \mathcal{U}$. Put

$$A = \{h|U,\ h \circ g_j^{-1}|U,\ g_j^{-1}|U : h \in A_0,\ U \in \mathcal{U},\ j \le m\}.$$

Clearly, the maps of A generate $\mathcal{G}$ and

$$H_s(A) \le (2m+1) \cdot \#A_0 \cdot H_s(\mathcal{U}) < (2m+1)\eta \cdot \#A_0.$$

Therefore, $H^s_\varepsilon(\mathcal{G}) = 0$, $H^s(\mathcal{G}) = 0$ and $\dim_{\rm H} \mathcal{G} \le \dim_{\rm H} Y$. ∎

4.5. *Smale horseshoe.* Let $f : S^2 \to S^2$ be the "horseshoe" described in [Sm], Section I.5. The non-wandering set Ω of f consists of two isolated fixed points, one of them, say p_0, contracting, the other one, say q_0, expanding, and of a compact hyperbolic invariant set Λ with periodic points dense in it.

4.5.1. Proposition. $\dim_{\rm H} \mathcal{G}(f) = \dim_{\rm H} \Lambda$.

P r o o f. Take s, η and $\varepsilon > 0$, two neighbourhoods $V_0 \ni p_0$ and $W_0 \ni q_0$ of diameter less than ε and a finite open covering $\mathcal{U} = \{U_1, \ldots, U_m\}$ of Λ such that $\operatorname{diam} U_i < \varepsilon$ and $H_s(\mathcal{U}) < H^s_\varepsilon(\Lambda) + \eta$. Since f is uniformly hyperbolic on Λ, there exists $N \in \mathbb{N}$ such that

$$f^{-N}(V_0) \cup f^N(W_0) \cup \bigcup_{i=1}^{m} (F^N(U_i) \cup f^{-N}(U_i)) = S^2.$$

For this N, the maps $f^{-N}|V_0, f^{-(N-1)}|V_0, f^N|W_0, f^{N+1}|W_0, f^N|U_i, f^{-N}|U_i, f^{N+1}|U_i, f^{-(N-1)}|U_i$, $i = 1, \ldots, m$, generate $\mathcal{G}(f)$. Therefore,

$$H^s_\varepsilon(\mathcal{G}(f)) \le 4\varepsilon^s + 4H_s(\mathcal{U}) < 4(\varepsilon^s + H^s_\varepsilon(\Lambda) + \eta),$$
$$H^s_\varepsilon(\mathcal{G}(f)) \le 4(\varepsilon^s + H^s_\varepsilon(\Lambda)), \qquad H^s(\mathcal{G}(f)) \le 4H^s(\Lambda)$$

and

$$\dim_{\rm H} \mathcal{G}(f) \le \dim_{\rm H} \Lambda. \tag{4.5.1}$$

On the other hand, by Proposition 2.4.1,

$$\dim_{\rm H} \mathcal{G}(f) \ge \dim_{\rm H} \mathcal{G}(f|\Lambda). \tag{4.5.2}$$

Now $f|\Lambda$ is topologically conjugate to the two-sided shift on $X = \{0, 1, \ldots \ldots, k-1\}^{\mathbb{Z}}$ (for a suitable k which depends on how many times the "horseshoe" intersects the original square Q involved in the construction of f). Moreover, X can be equipped with a metric ϱ for which $\dim_{\rm H} X = \dim_{\rm H} \Lambda$ and the conjugation becomes Lipschitz. (Here, Λ is equipped with the metric induced from the standard Riemannian structure on S^2.) Also, as in Example 4 of Section 3.2, X admits an f-invariant measure μ with

$\dim_H \mu = \dim_H \Lambda$. By Propositions 2.1.1, 2.3.2 and 3.3.1,

$$\dim_H \mathcal{G}(f|\Lambda) = \dim_H \Lambda. \tag{4.5.3}$$

Comparing (4.5.1)–(4.5.3) ends the proof. ∎

Remark. Takashi Inaba brought to our attention the following: The argument similar to that of the first part of the proof of Proposition 4.5.1 shows that

$$\dim_H \mathcal{G}(f) = 0 \tag{4.5.4}$$

when f is a Morse–Smale diffeomorphism of a compact manifold M.

5. Final remarks. 1. A similar notion of a dimension loss could be obtained by replacing $\dim_H \mathcal{G}$ by other Hausdorff-like dimensions constructed for a pseudogroup $\mathcal{G}$ by following the definitions of the packing dimension [TT], entropy dimensions [Ed], entropy indices (called also box-counting dimensions [Fa] or fractal dimensions [Ba]), and others. It is an open question which of the notions obtained this way serves best to describe the dynamics of a pseudogroup action.

2. It would be interesting to establish some relations between the dimension loss and other invariants (entropy or expansion growth, for instance) describing the dynamics of pseudogroup actions. Since $h(\mathcal{F}) = 0$ and $\mathrm{dl}_H^{\pitchfork} \mathcal{F} = 1$ for the Reeb foliation while $h(X_t) > 0$ and $\mathrm{dl}_H^{\pitchfork} \mathcal{F}_0 = 0$ for the geodesic flow (X_t) of a compact negatively curved Riemannian manifold M and the one-dimensional foliation $\mathcal{F}_0$ by the orbits of this flow, one should look for some conditions which would imply relations between the geometric entropy and dimension loss.

3. In various situations, maps and foliations with some singularities are of great interest. (For example, rational maps of $\overline{\mathbb{C}}$ of positive degree always have isolated critical points.) Therefore, one could try to generalize the notion of $\dim_H$ to systems with some (say, isolated) singularities. If the singular set $\Sigma \subset X$ is invariant, the simplest idea is to replace X by $X \setminus \Sigma$ and to modify the definitions of Section 2.1 by admitting countably infinite generating sets.

4. The most classical Anosov systems, geodesic flows of negatively curved Riemannian manifolds and linear maps of tori corresponding to hyperbolic matrices of $\mathrm{SL}(n, \mathbb{Z})$, preserve smooth measures, so the dimension loss of the pseudogroups generated by them vanishes. However, there are plenty of Anosov systems which do not admit smooth invariant measures and which are not Lipschitz equivalent to the systems mentioned above. It would be interesting to study (either estimate or calculate) $\dim_H$ and dl_H of the pseudogroups generated by such systems.

The more general case of the pseudogroup generated by a single Axiom A (with the transversality condition, if necessary) diffeomorphism should be even more interesting. For example, one could ask if (or when)

$$\dim_{\mathrm{H}} \mathcal{G}(f) = \dim_{\mathrm{H}} \mathcal{G}(f|\Omega)$$

or

$$\dim_{\mathrm{H}} \mathcal{G}(f) = \dim_{\mathrm{H}} \Omega,$$

where Ω is the non-wandering set of such a diffeomorphism f.

Acknowledgments. This research was done during the author's stay at the University of Tokyo, where he enjoyed the hospitality of the faculty and staff of the Department of the Mathematical Sciences. Especially, he is indebted to Takashi Tsuboi for arranging the visit and valuable conversations. Also, he enjoyed participating in the Saturday Seminar "Dosemi" at the Tokyo Institute of Technology and discussions with the participants (Takashi Inaba and Shigenori Matsumoto among others). Finally, some ideas were brought to the author's attention during his short visit to France (Toulouse, Dijon and Lyon) and conversations with Gilbert Levitt, Rémi Langevin and Etienne Ghys.

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Received 28 February 1995;
in revised form 16 October 1995