FUNDAMENTA
MATHEMATICAE
149 (1996)

On Haar null sets

by

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Abstract. We prove that in Polish, abelian, non-locally-compact groups the family of Haar null sets of Christensen does not fulfil the countable chain condition, that is, there exists an uncountable family of pairwise disjoint universally measurable sets which are not Haar null. (Dougherty, answering an old question of Christensen, showed earlier that this was the case for some Polish, abelian, non-locally-compact groups.) Thus we obtain the following characterization of locally compact, abelian groups: Let G be a Polish, abelian group. Then the σ-ideal of Haar null sets satisfies the countable chain condition iff G is locally compact. We also show that in Polish, abelian, non-locally-compact groups analytic sets cannot be approximated up to Haar null sets by Borel, or even co-analytic, sets; however, each analytic Haar null set is contained in a Borel Haar null set. Actually, we prove all the above results for a class of groups which is much wider than the class of all Polish, abelian groups, namely for Polish groups admitting a metric which is both left- and right-invariant.

Let G be a Polish abelian group. Christensen [C] calls a universally measurable set $A \subseteq G$ *Haar null* if there exists a probability Borel measure μ on G such that $\mu(g + A) = 0$ for all $g \in G$. It was proved in [C] that in case G is locally compact a universally measurable set is Haar null iff it is of Haar measure zero. Also, the union of a countable family of Haar null sets is Haar null, that is, Haar null sets constitute a σ-ideal. One of the first questions asked by Christensen in [C] was whether any family of mutually disjoint, universally measurable sets which are not Haar null is countable, as is the case when the group is Polish locally compact. This was answered in the negative by Dougherty [D] who constructed such uncountable families, for example, in all infinite dimensional Banach spaces. (Haar null sets are called "shy" in [D] following the terminology of [HSY].) This gives rise to the question whether the existence of such uncountable families characterizes non-locally-compact, Polish, abelian groups. We prove that this is indeed the case, that is, a Polish, abelian group is not locally compact iff

1991 *Mathematics Subject Classification*: 28C10, 43A05, 28A05.

there exists an uncountable family of universally measurable or, equivalently, closed, pairwise disjoint sets which are not Haar null. We also consider the problem of approximating sets modulo Haar null sets. We show that in each non-locally-compact, Polish, abelian group there exists an analytic set A such that $A\triangle B$ is not Haar null for any co-analytic set B; but each analytic Haar null set is contained in a Borel Haar null set. (This last statement answers a question of Dougherty [D, p. 86].) Additionally, we prove that for any $\alpha < \omega_1$ there exists $A \in \boldsymbol{\Sigma}^0_\alpha$ such that $A\triangle B$ is not Haar null for any $B \in \boldsymbol{\Pi}^0_\alpha$.

The definition of Haar null sets was extended by Topsøe and Hoffmann-Jørgensen [TH-J] and Mycielski to all Polish groups. A universally measurable set $A \subseteq G$ is said to be Haar null if there exists a Borel probability measure μ such that $\mu(gAh) = 0$ for all $g, h \in G$. Haar null sets are still closed under countable unions and coincide with Haar measure zero sets in locally compact groups. We prove all our results for Polish groups which admit an invariant metric. (A metric d on G is *invariant* if $d(g_1hg_2, g_1kg_2) = d(h, k)$ for any $g_1, g_2, h, k \in G$.) This class of groups contains properly all Polish, abelian groups, since each metric group G admits a left-invariant metric which, obviously, is invariant when G is abelian. Any invariant metric on a Polish group is automatically complete.

By $\mathrm{cl}(A)$ we denote the closure of A. $\mathbb{N}$ stands for the set of all natural numbers (and $0 \in \mathbb{N}$) and $2^{\mathbb{N}}$ for the countable infinite product of $\{0, 1\}$ with the product topology. By 2^n, for $n \in \mathbb{N}$, we denote the set of all sequences of 0's and 1's of length n indexed by $\{0, \ldots, n-1\}$. For $x \in 2^{\mathbb{N}}$, by $x|n$, for some $n \in \mathbb{N}$, we denote the sequence $(x(0), \ldots, x(n-1))$; in particular, $x|0 = \emptyset$.

First, we prove the following purely topological theorem.

THEOREM. *Assume G is a Polish, non-locally-compact group admitting an invariant metric. Then there exists a closed set $F \subseteq G$ and a continuous function $\phi: F \to 2^{\mathbb{N}}$ such that for any $x \in 2^{\mathbb{N}}$ and any compact set $K \subseteq G$ there is $g \in G$ with $gK \subseteq \phi^{-1}(x)$.*

Proof [1]. Let d be an invariant metric on G. Recall that d is complete. By $B(r)$, $r > 0$, we denote the ball with radius r centered at e, the identity element of G. For $A, B \subseteq G$, we write $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

Let (Q_k), $k \in \mathbb{N}$, be a sequence of finite subsets of G such that $\bigcup_k Q_k$ is dense in G and $Q_k \subseteq Q_{k+1}$.

[1] I would like to thank the referee for suggesting simplifications in this proof.

CLAIM. *Given $\varepsilon > 0$ there are $g_k \in G$, $k \in \mathbb{N}$, and $\delta > 0$ such that*

(i) $g_k \in B(\varepsilon)$;
(ii) $d(g_k Q_k, \bigcup_{i<k} g_i Q_i) \geq \delta$.

Proof of the claim. Since G is not locally compact, we can find $\delta > 0$ and an infinite set $D \subseteq B(\varepsilon)$ whose points are at distance at least 2δ from each other. Let $g_0 = e$. Assume g_i have already been chosen for $i < k$. If for every $g \in D$ we could find $a \in Q_k$ and $b \in \bigcup_{i<k} g_i Q_i$ with $d(ga, b) < \delta$, then there would exist distinct $g, g' \in D$ with the same pair $a \in Q_k$, $b \in \bigcup_{i<k} g_i Q_i$. But then $d(g, g') = d(ga, g'a) \leq d(ga, b) + d(b, g'a) < 2\delta$, contradicting $d(g, g') \geq 2\delta$. Thus there is $g \in D$ with $d(gQ_k, \bigcup_{i<k} g_i Q_i) \geq \delta$. Let $g_k = g$.

Using repeatedly the claim, we can recursively choose ε_n, δ_n, and $g_k^n \in G$, $k, n \in \mathbb{N}$, so that

(iii) $g_k^n \in B(\varepsilon_n)$;
(iv) $d(g_k^n Q_k, \bigcup_{i<k} g_i^n Q_i) \geq 3\delta_n$;
(v) $\sum_{m>n} \varepsilon_m < \delta_n/2$.

Now let

$$F = \bigcap_n \bigcup_k g_k^n Q_k \operatorname{cl}(B(\delta_n))$$

and, for $x \in 2^{\mathbb{N}}$,

$$F_x = \bigcap_n \bigcup_{k \equiv x(n) \pmod 2} g_k^n Q_k \operatorname{cl}(B(\delta_n)).$$

By (iv), for fixed n, the sets $g_k^n Q_k B(\delta_n)$ are disjoint and at distance at least δ_n from each other, so

$$\operatorname{cl}\Big(\bigcup_k g_k^n Q_k B(\delta_n)\Big) = \bigcup_k g_k^n Q_k \operatorname{cl}(B(\delta_n)).$$

Thus, we see that F is closed. Also, F is the disjoint union of the sets F_x. Now define $\phi : F \to 2^{\mathbb{N}}$ by letting $\phi(g)$ be equal to the unique x with $g \in F_x$. To prove that ϕ is continuous, it is enough to see that the preimages of basic clopen subsets of $2^{\mathbb{N}}$ are closed. But for $\tau \in 2^m$, $m \in \mathbb{N}$, we have

$$\phi^{-1}(\{x \in 2^{\mathbb{N}} : x|m = \tau\}) = F \cap \bigcap_{n<m} \bigcup_{k \equiv \tau(n) \pmod 2} g_k^n Q_k \operatorname{cl}(B(\delta_n)).$$

This set is closed since $\bigcup_{k \equiv \tau(n) (\bmod 2)} g_k^n Q_k \operatorname{cl}(B(\delta_n))$ is by an argument as above for F.

Let $K \subseteq G$ be compact. Given $x \in 2^{\mathbb{N}}$, we construct $h_n \in B(\varepsilon_n)$ such that $hK \subseteq F_x$ for $h = \ldots h_2 h_1 h_0$. Suppose we have already found h_m for

$m < n$. Note that $\bigcup_k Q_k B(\delta_n/2) = G$ because $\bigcup_k Q_k$ is dense. Since the union is increasing, for k large enough, we have

$$h_{n-1}h_{n-2}\ldots h_0 K \subseteq Q_k B(\delta_n/2).$$

Choose such a k with $k \equiv x(n) \pmod 2$, and let $h_n = g_k^n$. By (iii), $h_n \in B(\varepsilon_n)$.

For each $n \in \mathbb{N}$, we have $h_{n-1}h_{n-2}\ldots h_0 K \subseteq Q_k B(\delta_n/2)$, with k as in the definition of h_n, so

$$(*) \qquad h_n h_{n-1}h_{n-2}\ldots h_0 K \subseteq g_k^n Q_k B(\delta_n/2).$$

Since $h_m \in B(\varepsilon_m)$ for $m > n$, it is easy to check, using the invariance of d, that the sequence $(h_m h_{m-1}\ldots h_{n+1})_{m>n}$ is Cauchy. So, since d is complete, it converges, and by (v),

$$(**) \qquad d(\ldots h_{n+2}h_{n+1}, e) < \sum_{m>n} \varepsilon_m < \delta_n/2.$$

Now, a quick calculation using $(*)$ and $(**)$ and the invariance of d gives

$$hK \subseteq B(\delta_n/2) g_k^n Q_k B(\delta_n/2) = g_k^n Q_k B(\delta_n/2) B(\delta_n/2) \subseteq g_k^n Q_k \,\mathrm{cl}(B(\delta_n)).$$

Since this works for all n, $hK \subseteq F_x$. This finishes the proof of the theorem.

Corollary. *Let G be a Polish group admitting an invariant metric. Then each family of universally measurable or, equivalently, closed, pairwise disjoint sets which are not Haar null is countable iff G is locally compact.*

Proof. ($\Leftarrow$) If G is locally compact, Haar null sets coincide with sets of Haar measure zero (see [C] and [TH-J]). Since G is Polish, Haar measure is σ-finite.

($\Rightarrow$) Assume G is not locally compact. Since for any Borel probability measure on G there is a compact set of positive measure, it follows that the sets $\phi^{-1}(x)$, $x \in 2^{\mathbb{N}}$, from the Theorem are not Haar null.

Proposition. *Let G be a Polish group.*

(i) *If $A \subseteq G$ is analytic and Haar null, then there exists a Borel set $B \subseteq G$ which is Haar null and $A \subseteq B$.*

(ii) *Assume that G is not locally compact and admits an invariant metric. Then there exists an analytic set A such that $A \triangle B$ is Haar null for no co-analytic set B. For any $\alpha < \omega_1$ there exists $A \in \boldsymbol{\Sigma}^0_\alpha$ such that $A \triangle B$ is Haar null for no $B \in \boldsymbol{\Pi}^0_\alpha$.*

Proof. If $Z \subseteq X \times Y$, then, as usual, $Z_x = \{y \in Y : (x,y) \in Z\}$ for $x \in X$.

(i) Let A be analytic and Haar null. Let μ be a probability Borel measure witnessing it. Then the family of sets

$$\Phi = \{X \subseteq G : X \in \boldsymbol{\Sigma}^1_1 \text{ and } \mu(g_1 X g_2) = 0\ \forall g_1, g_2 \in G\}$$

is $\boldsymbol{\Pi}^1_1$ on $\boldsymbol{\Sigma}^1_1$, that is, for any $\boldsymbol{\Sigma}^1_1$ set $P \subseteq Y \times G$, with Y a Polish space, the set $\{y \in Y : P_y \in \Phi\}$ is $\boldsymbol{\Pi}^1_1$. To check this, let $P \subseteq Y \times G$ be $\boldsymbol{\Sigma}^1_1$, with Y Polish. Define $\widetilde{P} \subseteq G \times G \times Y \times G$ by

$$(g_1, g_2, y, g) \in \widetilde{P} \ \text{ iff } \ g \in g_1 P_y g_2.$$

Then we have $\widetilde{P} \in \boldsymbol{\Sigma}^1_1$. It follows from [K, Theorem 29.26] that $\{(g_1, g_2, y) : \mu(\widetilde{P}_{(g_1,g_2,y)}) = 0\}$ is $\boldsymbol{\Pi}^1_1$, whence so is

$$\{y \in Y : \mu(\widetilde{P}_{(g_1,g_2,y)}) = 0\ \forall g_1, g_2 \in G\} = \{y \in Y : P_y \in \Phi\}.$$

Now, since $A \in \Phi$, by (the dual form of) the First Reflection Theorem (see [K, Theorem 35.10 and the remarks following it]), there exists a Borel set B with $B \supset A$ and $B \in \Phi$, so B is as required.

(ii) Let F and $\phi : F \to 2^{\mathbb{N}}$ be as in the Theorem. The argument below is essentially the same as Balcerzak's argument in the proof of Lemma 2.1 from [B]. Let $\Lambda =$ co-analytic sets or $\Lambda = \boldsymbol{\Pi}^0_\alpha$ for some $\alpha < \omega_1$. Let $U \subseteq 2^{\mathbb{N}} \times G$ be universal for $\Lambda | G$, that is, $U \in \Lambda$ and $\{B \subseteq G : B \in \Lambda\} = \{U_x : x \in 2^{\mathbb{N}}\}$. Put

$$A = (G \setminus F) \cup \bigcup_{x \in 2^{\mathbb{N}}} (\phi^{-1}(x) \setminus U_x).$$

Note that $A = (G \setminus F) \cup \{g \in F : (\phi(g), g) \notin U\}$ whence, since ϕ is continuous and F is closed, $G \setminus A \in \Lambda$. Also, for any $x \in 2^{\mathbb{N}}$, we have $A \triangle U_x \supset \phi^{-1}(x)$. Thus, $A \triangle B$ is not Haar null for any $B \in \Lambda$.

Remark. Proposition (i) can also be deduced from a theorem of Dellacherie. If μ witnesses that an analytic set A is Haar null, put $\widetilde{\mu}(X) = \sup\{\mu^*(gXh) : g, h \in G\}$, where $X \subseteq G$ and μ^* is the outer measure induced by μ. Then it is easy to check that $\widetilde{\mu}$ is what is called in [De] a caliber. Thus, since $\widetilde{\mu}(A) = 0$, by [De, Theorem 2.4], there exists a Borel set $B \supset A$ with $\widetilde{\mu}(B) = 0$, that is, $\mu(gBh) = 0$ for any $g, h \in G$.

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Received 29 August 1994;
in revised form 20 July 1995