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Ramsey, Lebesgue, and Marczewski sets and the Baire property

by

Patrick Reardon (Durant, Okla.)

Abstract. We investigate the completely Ramsey, Lebesgue, and Marczewski σ-algebras and their relations to the Baire property in the Ellentuck and density topologies. Two theorems concerning the Marczewski σ-algebra (s) are presented.

THEOREM. *In the density topology D, (s) coincides with the σ-algebra of Lebesgue measurable sets.*

THEOREM. *In the Ellentuck topology on $[\omega]^{\omega}$, $(s)_0$ is a proper subset of the hereditary ideal associated with (s).*

We construct an example in the Ellentuck topology of a set which is first category and measure 0 but which is not $B_{\rm r}$-measurable. In addition, several theorems concerning perfect sets in the Ellentuck topology are presented. In particular, it is shown that there exist countable perfect sets in the Ellentuck topology.

0. Introduction. We are interested in the σ-algebras B of Borel sets, L of Lebesgue measurable sets, (s) of Marczewski measurable sets, $B_{\rm w}$ of sets with the Baire property in the wide sense, $B_{\rm r}$ of sets with the Baire property in the restricted sense, and CR of sets which are completely Ramsey. B, $B_{\rm w}$, $B_{\rm r}$, and (s) have a well-defined meaning in any topological space, and we are particularly interested in the Euclidean, Ellentuck, and density topologies.

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Now for some definitions. Let (X,T) be a topological space. Then

$$(s)=\{M\subseteq X:(\forall\text{perfect }P)(\exists\text{perfect }Q\subseteq P)(Q\subseteq M\text{ or }Q\subseteq M^{\mathrm{c}})\},$$
$$B_{\mathrm{w}}=\{M\subseteq X:M=U\triangle F,\ \text{where }U\text{ is open and }F\text{ is first category}\},$$
$$B_{\mathrm{r}}=\{M\subseteq X:(\forall\text{perfect }P)(M\text{ is }B_{\mathrm{w}}\text{-measurable (rel }P))\}.$$

To define the σ-algebra CR, we first define what we mean by $[F,u]$. For each $F\in[\omega]^{<\omega}$ and each $u\in[\omega]^{\omega}$, $[F,u]=\{S\in[\omega]^{\omega}:F\subseteq S\subseteq F\cup u\}$. Many authors also stipulate that $\max(F)<\min(u)$. However, this affects neither the σ-algebra CR (defined below) nor the Ellentuck topology which is generated by the sets $[F,u]$. Therefore we choose to leave out this requirement as it simplifies many of the proofs in this paper. Define

$$CR=\{M\subseteq[\omega]^{\omega}:(\forall\ [F,u])(\exists v\in[u]^{\omega})([F,v]\subseteq M\text{ or }[F,v]\subseteq M^{\mathrm{c}})\}.$$

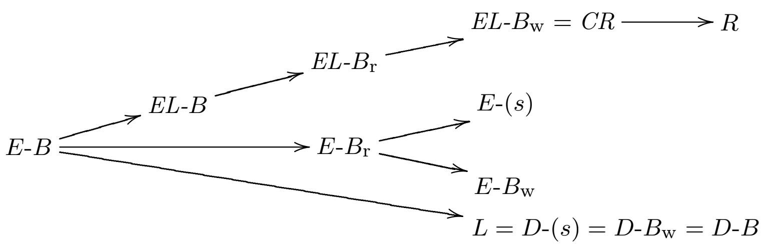
The Ramsey sets are defined by

$$R=\{M\subseteq[\omega]^{\omega}:(\exists v\in[\omega]^{\omega})([\emptyset,v]^{\omega}\subseteq M\text{ or }[\emptyset,v]^{\omega}\subseteq M^{\mathrm{c}})\}.$$

$CR\subseteq R$ but the Ramsey sets do not form a σ-algebra (see [C]).

In arbitrary topological spaces it is known that $B\to B_{\mathrm{r}}\to B_{\mathrm{w}}$ [K, p. 93]. Scheinberg [Sc] has shown that in the density topology, $B=L$. Oxtoby extended this result by showing that $B_{\mathrm{w}}=L$ [O, p. 89]. In this paper we show that in the density topology, $(s)=L$ as well. We also show that in the Ellentuck topology, $B_{\mathrm{r}}\neq B_{\mathrm{w}}$, a result similar to that obtained in the Euclidean topology. We note here that Marczewski has shown in [M] that for complete separable metric spaces, $B_{\mathrm{r}}\to(s)$. This result also holds in the density topology but it is not known if $B_{\mathrm{r}}\to(s)$ in the Ellentuck topology.

Suppose T is a topology on X and P is a property which has meaning in any topological space. We use the notation T-P to refer to the class of subsets of (X,T) which satisfy property P, and we denote the Euclidean, Ellentuck, and density topologies by E, EL, and D, respectively. This yields the following diagram:



With the exception of relationships involving EL-B and EL-B_{r}, counterexamples exist which show that these are the only implications which hold (see [Br], [BrCo], [C], or [W]). The only unknown directions are whether EL-B

or EL-B_r imply E-B_r, E-(s), E-B_w, or L, and whether EL-B_r implies EL-B. In Section 2 we construct an example of a set $M \in (CR_0 \cap L_0) \setminus EL\text{-}B_r$. In addition, an easy cardinality argument shows that $EL\text{-}B \setminus E\text{-}B \neq \emptyset$.

Many important σ-algebras have definitions similar to that for (s). It is well known that M is CR-measurable iff for every set $[F, u]$ there is a set $[G, v] \subseteq [F, u]$ such that $[G, v] \subseteq M$ or $[G, v] \subseteq M^c$. Burstin showed in [Bu] that M is L-measurable iff for every perfect set P of positive measure there is a perfect set $Q \subseteq P$ of positive measure such that $Q \subseteq M$ or $Q \subseteq M^c$. In Section 3 we generalize Burstin's result to characterize measurability under complete non-atomic Borel measures on complete separable metric spaces. Finally, J. B. Brown in a private communication has shown that M is B_w-measurable iff for every locally residual G_δ set P there is a locally residual G_δ set $Q \subseteq P$ such that $Q \subseteq M$ or $Q \subseteq M^c$.

The Ellentuck topology (EL). There has been considerable interest over the past thirty years or so in infinite versions of Ramsey's theorem. This has led to the definition of the Ramsey sets and the investigation of their relationship to the σ-algebras mentioned above. Galvin and Prikry have shown that E-Borel sets are Ramsey [GP]. They did this by defining the σ-algebra CR of completely Ramsey sets which is a subclass of R and then showing that E-open implies CR-measurable. Silver extended their result from E-Borel sets to E-analytic sets [Si], and the proof was greatly simplified by Ellentuck [E] and independently by Louveau [L], both of whom showed that $CR = EL\text{-}B_w$.

The Euclidean topology on $[\omega]^\omega$ is just the relative product topology from ω^ω, where $[\omega]^\omega$ is embedded in ω^ω as the set of all increasing sequences. Another way of looking at this, which is quite useful, is to identify points of $[\omega]^\omega$ with their characteristic functions. This embeds $[\omega]^\omega$ in 2^ω as a dense G_δ. In fact it is just 2^ω minus the left endpoints and E is just the order topology on this set. We will say that a set $M \subseteq [\omega]^\omega$ is Lebesgue-measurable (or L-measurable) iff $\{\chi_A \in 2^\omega : A \in M\}$ is measurable in the usual product measure on 2^ω.

The Ellentuck topology on $[\omega]^\omega$ is that generated by sets of the form $[F, u]$. This topology refines E and Plewik has shown it is not normal [P]. Moreover, it does not satisfy the countable (or even 2^ω) chain condition, and thus is not compact nor even Lindelöf.

In investigating the above-mentioned σ-algebras in the EL-topology, one of the primary difficulties encountered was in constructing examples showing that certain of these classes are not contained in the others. In the E-topology, Bernstein-type sets are quite useful for this purpose. However, the usual construction of a Bernstein set fails in the EL-topology because there are too many EL-closed sets. In Section 1, we show that in fact there are no EL-Bernstein sets. This follows from the surprising fact that every EL-dense set contains a countable EL-perfect set. We also show that there is a set in the hereditary ideal associated with EL-(s) that is not EL-$(s)_0$-measurable. This is somewhat unusual when compared with similar statements for either of the topologies E or D.

In Section 2, we give an example of a set which has measure zero and is EL-first category (i.e. CR_0) but is not EL-$B_{\rm r}$- measurable. The construction is an adaptation of the construction of Vitali to the space $[\omega]^\omega$ with the Ellentuck topology. After we got this example, we discovered that certain E-Bernstein sets have this property as well. We have included the Vitali-type example as well as a theorem on E-Bernstein sets as the techniques employed are quite different.

The density topology (D). Goffman and Waterman defined the density topology in 1961 [GW]. This topology is of major interest to real analysts because the approximately continuous functions are precisely the D-E continuous functions, i.e. functions which are continuous when the domain is given the density topology and the range is given the Euclidean topology [GW]. This has applications in the theory of real functions, for example, since every approximately continuous function is of Baire class 1 and every bounded approximately continuous function is a derivative.

The fact that D is a topology was not shown until a second paper by Goffman, Neugebauer and Nishiura [GNN]. In that paper, they show that D is completely regular but not normal. In Tall's excellent survey paper [T], he states that D is neither separable nor first countable, but is hereditarily Baire and satisfies the countable chain condition.

The density topology D on the real line is defined as follows. Let λ denote Lebesgue measure. A measurable set M has density 1 at p iff

$$\lim_{h\to 0} \lambda(M\cap[p-h,p+h])/(2h)=1.$$

A set $M\subseteq\mathbb{R}$ is said to be D-open if it is L-measurable and has density 1 at each of its points. The Lebesgue Density Theorem implies that every measurable set is the union of a D-open set and a set of measure zero.

In Section 3 we show that the σ-algebra D-(s) coincides precisely with L. This parallels J. C. Oxtoby's result [O, Sec. 22] that the σ-algebra D-$B_{\rm w}$ is equal to L.

1. EL-perfect sets. This section contains several theorems concerning perfect sets in the Ellentuck topology. In particular, we show that for every infinite $\kappa\le 2^\omega$ there exists an EL-perfect set of cardinality κ which is EL-first category relative to itself, that every EL-dense set contains a countable EL-perfect set, that there are no EL-Bernstein sets, and that EL-dense subsets of sets which are EL-perfect and EL-first category relative to themselves contain countable EL-perfect sets.

1.1. THEOREM. *Every EL-open set contains a countable EL-perfect set.*

Proof. Every EL-open set contains a basic EL-open set, say $[F,u]$. Let $P=\{F\cup v : v\subseteq u$ and v contains a final segment of $u\}$. P is a countable subset of $[F,u]$ and contains no EL-isolated points. If x is an EL-limit point of P, then for every finite initial segment G of x, $[G,x]$ intersects P. But this implies that $x\in P$. Therefore P is EL-closed. ∎

1.2. COROLLARY. *EL-$(s)_0$ is not a σ-ideal.*

It is known that every set of cardinality $< 2^\omega$ is both E-$(s)_0$- and CR_0-measurable. Although Theorem 1.1 implies that not every set of cardinality $< 2^\omega$ is EL-$(s)_0$-measurable, we leave open the question of whether every set of cardinality $< 2^\omega$ is EL-(s)-measurable.

1.3. THEOREM. *For every infinite cardinal $\kappa\le 2^\omega$, there exists an EL-perfect set of cardinality κ which is EL-first category relative to itself.*

Proof. Construct an almost disjoint collection of subsets of ω as follows. First construct a binary tree T by setting $n <_T 2n$ and $n <_T 2n+1$ for every $n\in\mathbb{N}$. Every chain in T corresponds to a subset of ω and by a *branch* in T we mean an infinite subset of ω corresponding to a maximal chain in T. Let $A=\{u_\alpha : \alpha<\kappa\}$, where for each α, u_α is a branch in T and $\alpha\ne\beta$ implies $u_\alpha\ne u_\beta$. Then A is an almost disjoint collection of subsets of ω and $\{[u_\alpha] : \alpha<\kappa\}$ is a pairwise disjoint collection of EL-open subsets of $[\omega]^\omega$. For each α, let $P_\alpha=\{w : w\subseteq u_\alpha$ and w contains a final segment of $u_\alpha\}$. Each P_α is a countable EL-perfect set and if $\alpha\ne\beta$, then P_α and P_β are disjoint. Let $P=\bigcup_{\alpha<\kappa}P_\alpha$. Then P has cardinality κ and contains no EL-isolated points. Suppose x is an EL-limit point of P and for every u_β, $x\not\subseteq u_\beta$. Choose P_α such that $[x]\cap P_\alpha$ is non-empty. Let j denote a positive integer in $x\setminus u_\alpha$ and let k denote a positive integer in $x\cap u_\alpha$ such that $j<k$. Let $F=\{z\in x : z\le k\}$. Since j and k must necessarily occur on different branches, no subset of any branch can contain F as an initial segment. Therefore, $[F,x]\cap P$ is empty, a contradiction. It follows that for some β, $x\subseteq u_\beta$. Thus $[x]$ intersects P_β and we have $x\in P_\beta$. Hence P is EL-closed.

Now for each $\alpha<\kappa$, enumerate $P_\alpha=\{p_{\alpha,n} : n<\omega\}$. For each $k<\omega$, let $N_k=\{p_{\alpha,k} : \alpha<\kappa\}$. Each N_k is EL-nowhere dense (rel P) and $\bigcup N_k = P$. ∎

Particularly in the study of the Marczewski measure algebra, we are often faced with the question of whether or not a given subset of an arbitrary perfect set contains a perfect subset. Partial results in this direction are given by Theorems 1.5 and 1.9 below.

We adopt the following notations. Suppose I is a finite binary sequence. Then I^- is the initial segment of I of length $|I|-1$. If I and J are finite binary sequences, then by $I^\wedge J$ we mean the unique binary sequence of length $|I|+|J|$ that has I as an initial segment and J as a final segment. (We often abuse this notation and write $I^\wedge 0$ for $I^\wedge\langle 0\rangle$ and $I^\wedge 1$ for $I^\wedge\langle 1\rangle$.)

1.4. Lemma. *Suppose $\{[F_I, x_I] : I \in \Sigma \subseteq 2^{<\omega}\}$ is a collection of basic EL-open sets such that for every $I \in \Sigma$, F_I is a finite initial segment of x_I and there exist distinct $H, K \in \Sigma$ such that H and K extend I, $x_K = x_I$, and $[F_H, x_H]$ and $[F_K, x_K]$ are disjoint subsets of $[F_I, x_I]$. Then $\{x_I : I \in \Sigma\}$ is EL-dense-in-itself.*

Proof. Let $[F, u]$ be an arbitrary basic *EL*-open set and suppose $x_I \in [F, u]$, where $I \in \Sigma$. Without loss of generality, we may assume that $F \subseteq F_I$. Choose extensions H and K of I as above. Then x_H is a point of $[F_I, x_I] \subseteq [F, u]$ distinct from x_I. ∎

1.5. Theorem. *If M is EL-dense, then M contains a countable EL-perfect set.*

Proof. We suppose M is *EL*-dense and recursively define a countable *EL*-perfect set $D = \{x_I : I \in 2^{<\omega}\}$. For $n = 0$, we consider sequences $I \in 2^0$. Thus $I = \emptyset$ and we choose $x_\emptyset \in M$ and set $F_\emptyset = \emptyset$. For $n = 1$, define x_I and F_I for $I \in 2^1 = \{\langle 0\rangle, \langle 1\rangle\}$ as follows. Write $x_\emptyset \setminus F_\emptyset = \{u_{\emptyset,t} : t < \omega\}$. Set $x_{\langle 0\rangle} = x_\emptyset$ and $F_{\langle 0\rangle} = F_\emptyset \cup \{u_{\emptyset,0}\}$. Choose $u_{\emptyset,m(1)} > u_{\emptyset,n(1)} > u_{\emptyset,0}$. Set $F_{\langle 1\rangle} = F_\emptyset \cup \{u_{\emptyset,n(1)}, u_{\emptyset,m(1)}\}$ and by *EL*-density of M, choose $x_{\langle 1\rangle} \in [F_{\langle 1\rangle}, x_\emptyset \setminus (\max(F_{\langle 1\rangle})+1)] \cap M$.

Now suppose $k < \omega$ is arbitrary and for every $I \in 2^k$, F_I and x_I have been defined such that $x_I \in M$, F_I is a finite initial segment of x_I, and if I and J both end in a 1 and $I \neq J$, then $x_I \not\subseteq x_J$ and $x_J \not\subseteq x_I$. We define $x_{I^\wedge 0}$, $F_{I^\wedge 0}$, $x_{I^\wedge 1}$, and $F_{I^\wedge 1}$ as follows. Enumerate the set of all k-term binary sequences $\{S(i) : i < 2^k\}$. For each $S(i)$, write $x_{S(i)} \setminus F_{S(i)} = \{u_{S(i),t} : t < \omega\}$. Set $x_{S(i)^\wedge 0} = x_{S(i)}$ and $F_{S(i)^\wedge 0} = F_{S(i)} \cup \{u_{S(i),0}\}$. For $0 \le i \le 2^k - 1 = H$, choose $u_{S(i),n(i)}, u_{S(i),m(i)} \in x_{S(i)} \setminus F_{S(i)}$ so that the following inequality is satisfied:

$$\max\{u_{S(i),0} : i < 2^k\} < u_{S(0),n(0)} < \ldots < u_{S(i),n(i)} < \ldots < u_{S(H),n(H)} < \ldots < u_{S(0),m(0)} < \ldots < u_{S(i),m(i)} < \ldots < u_{S(H),m(H)}.$$

Set $F_{S(i)^\wedge 1} = F_{S(i)} \cup \{u_{S(i),n(i)}, u_{S(i),m(i)}\}$. Now let $x_{S(i)^\wedge 1} \in [F_{S(i)^\wedge 1}, x_{S(i)} \setminus (\max(F_{S(i)^\wedge 1})+1)] \cap M$.

Clearly, $x_{S(i)^\wedge 1} \in M$ and has $F_{S(i)^\wedge 1}$ as an initial segment. Thus the first two conditions of the induction hypothesis are satisfied for all $I \in 2^{k+1}$. If

$I, J \in 2^{k+1}$ both end in a 1 and $I \neq J$, then for some i and j, $I = S(i)^\wedge 1$ and $J = S(j)^\wedge 1$. We may assume without loss of generality that $i < j$. Thus $F_I = F_{S(i)} \cup \{u_{S(i),n(i)}, u_{S(i),m(i)}\}$ and $F_J = F_{S(j)} \cup \{u_{S(j),n(j)}, u_{S(j),m(j)}\}$, where $\max(F_{S(i)} \cup F_{S(j)}) < u_{S(i),n(i)} < u_{S(j),n(j)} < u_{S(i),m(i)} < u_{S(j),m(j)}$. It follows that $x_I \not\subseteq x_J$ and $x_J \not\subseteq x_I$ and so the third condition of the induction hypothesis is satisfied. To complete the construction we set $D = \{x_I : I \in 2^{<\omega}\}$.

It remains to show that D is *EL*-perfect. First we note that D is *EL*-dense-in-itself since $\{[F_I, x_I] : I \in 2^{<\omega}\}$ satisfies the hypotheses of Lemma 1.4. To show that D is *EL*-closed, set $B = \bigcap_{n<\omega} \bigcup \{[F_I, x_I] : I \in 2^n\}$. B is *EL*-closed and contains D. Let y be an element of $B \setminus D$. Then there exists a collection $\{C(k) : k < \omega\}$ of finite binary sequences such that $y \in \bigcap_{k<\omega} [F_{C(k)}, x_{C(k)}]$ and for every k, $C(k)$ ends in a 1. We will show that $[y] \cap D = \emptyset$.

It is easy to see that $y \subseteq \bigcap_{k<\omega} x_{C(k)}$. Suppose x_J is an arbitrary element of D. We consider two cases. Suppose for some $h < \omega$, $C(h)$ is an extension of J. By construction of D we have $x_{C(h)} \subseteq x_J$. Since $\bigcap_{k<\omega} x_{C(k)}$ is a decreasing intersection, y must be a proper subset of $x_{C(h)}$ and it follows that $x_J \notin [y]$. On the other hand, suppose for every $k < \omega$, $C(k)$ is not an extension of J. Let h be some integer such that $|C(h)| > |J|$ and let J' be an extension of J of length $|C(h)|$ that ends in a 1. Then $x_{J'}$ is a proper subset of x_J and by construction of D, we have $x_{J'} \not\subseteq x_{C(h)}$. Since $y \subseteq x_{C(h)}$, it follows that $x_J \notin [y]$. Thus $[y] \cap D = \emptyset$, which implies that D is *EL*-closed. Hence D is the desired countable *EL*-perfect set. ■

1.6. Corollary. *Every E-Bernstein subset of* $[\omega]^\omega$ *contains a countable EL-perfect set.*

Proof. Each basic *EL*-open set contains an *E*-perfect set (in fact is *E*-homeomorphic to the irrationals), so *E*-Bernstein sets are *EL*-dense. ■

1.7. Corollary. *There are no EL-Bernstein sets.*

Proof. A set that meets every *EL*-perfect set must be *EL*-dense. ■

1.8. Corollary. *If U is EL-open and M is EL-dense in U, then M contains a countable EL-perfect set.*

Proof. Since U is *EL*-open there is some $[G, v] \subseteq U$ and an *EL*-homeomorphism $h : [G, v] \to [\omega]^\omega$. Now just use the fact that homeomorphisms preserve density, cardinality, and perfect sets. ■

Thus we see that if a set is *EL*-dense in an *EL*-open set, i.e. a "big" set with respect to category, then it contains a countable *EL*-perfect set. It is an open question whether *EL*-dense subsets of arbitrary *EL*-perfect sets contain *EL*-perfect subsets. In Theorem 1.9 below, however, we are able to show that if the *EL*-perfect set in question is also *EL*-first category relative to itself, i.e. a "small" set with respect to category, then every *EL*-dense subset does contain a countable *EL*-perfect set.

We use this result to show that the hereditary ideal corresponding to the σ-algebra of *EL*-(s)-measurable sets properly contains the collection of *EL*-$(s)_0$ sets. This result is remarkable because in E or D, the hereditary ideal is always equal to the collection of singular sets for all of the more widely studied σ-algebras with which the author is familiar, including (s).

We also use this result to show that any *EL*-perfect set which is *EL*-first category relative to itself can have no *EL*-Bernstein subdivision.

1.9. THEOREM. *If P is EL-perfect and EL-first category* (*rel to itself*) *and M is EL-dense in P, then M contains a countable EL-perfect set.*

Proof. Write $P = \bigcup_{k\ge 0} N_k$, where each N_k is *EL*-nowhere dense (rel P) and suppose M is *EL*-dense in P. We may assume without loss of generality that for all k, N_k is *EL*-closed and $N_k \subseteq N_{k+1}$. We construct a countable *EL*-perfect set $D = \{x_I : I \in 2^{<\omega}\}$ in $M \cap P$ by recursion on the length of $I \in 2^{<\omega}$. For $|I| = 0$, choose $x_\emptyset \in M \cap (P \setminus N_0)$ and let $F_\emptyset = \emptyset$. For $|I| = 1$, set $x_0 = x_\emptyset$ and choose $x_1 \in M \cap (([F_\emptyset, x_\emptyset] \cap P) \setminus N_1)$ distinct from x_0. Next choose finite initial segments F_0 and F_1 of x_0 and x_1, respectively, which extend $F_\emptyset$, separate x_0 and x_1, and such that $[F_1, x_1] \cap N_1 = \emptyset$.

Now suppose $k < \omega$ is arbitrary and for every $I \in 2^k$, F_I and x_I have been defined such that $x_I \in M \cap P$, F_I is a finite initial segment of x_I, F_I and F_J separate x_I and x_J whenever $I \ne J$, and $[F_I, x_I] \cap N_k = \emptyset$ if I ends in a 1. For each I, set $x_{I^\wedge 0} = x_I$ and choose $x_{I^\wedge 1} \in M \cap (([F_I, x_I] \cap P) \setminus N_{k+1})$ distinct from $x_{I^\wedge 0}$. Next choose finite initial segments $F_{I^\wedge 0}$ and $F_{I^\wedge 1}$ of $x_{I^\wedge 0}$ and $x_{I^\wedge 1}$, respectively, which extend F_I, separate $x_{I^\wedge 0}$ and $x_{I^\wedge 1}$, and such that $[F_{I^\wedge 1}, x_{I^\wedge 1}] \cap N_{k+1} = \emptyset$. Now let $D = \{x_I : I \in 2^{<\omega}\}$.

It is easy to see that D is *EL*-dense-in-itself. To show that D is *EL*-closed, we first establish that for all $j \ge 0$,

$$A_j = \bigcup_{I \in 2^j} \{[F_{I^\wedge Z(n)}, x_{I^\wedge Z(n)}] : n \ge 0\} \cup \{x_I : I \in 2^j\}$$

is *EL*-closed, where $Z(n)$ denotes a 1 preceded by n many zeros. (For example, $Z(0) = 1$, $Z(1) = 01$, $Z(2) = 001$, etc.) Suppose $y \in \bigcup\{[F_I, x_I] : I \in 2^j\} \setminus A_j$ and choose the unique I such that $y \in [F_I, x_I]$. Let $F_{I^\wedge J}$ be a finite initial segment of x_I such that $y \notin [F_{I^\wedge J}, x_I]$. Thus all but finitely many "intervals" of $A_j \cap [F_I, x_I]$ lie within $[F_{I^\wedge J}, x_I]$. It follows that y is not a limit point of A_j and hence that A_j is *EL*-closed. Thus every limit point of D must belong to A_j.

Recalling that the N_k's are nested, it is easy to see that our construction guarantees $A_j \cap (N_j \setminus D) = \emptyset$ for each $j < \omega$. Therefore $N_j \setminus D$ contains no

EL-limit points of D. But $\bigcup_{j\geq 0}(N_j \setminus D) = (\bigcup_{j\geq 0} N_j) \setminus D = P \setminus D$ and it follows that D is *EL*-closed. ∎

1.10. Corollary. *EL-$(s)_0$ is a proper subset of $H(EL$-$(s))$.*

Proof. It is obvious that every EL-$(s)_0$-measurable set belongs to $H(EL$-$(s))$. To show that the converse does not hold, suppose P is countable and *EL*-perfect. It suffices to prove that $P \in H(EL$-$(s))$. Let S be a subset of P and suppose Q is an arbitrary *EL*-perfect set. If $Q \cap S$ contains a subset M which is *EL*-dense-in-itself, then M is an *EL*-dense subset of $\mathrm{Cl}_{EL}(M)$, which in turn is countable and *EL*-perfect. Thus M contains a countable *EL*-perfect set. On the other hand, if $Q \cap S$ is *EL*-scattered, then it is *EL*-nowhere dense (rel Q). Let $[F, u] \cap Q$ be a relative *EL*-open subset of Q which misses $Q \cap S$. But $[F, u] \cap Q$ is *EL*-perfect. Thus s is EL-(s)-measurable and it follows that $P \in H(EL$-$(s))$. ∎

1.11. Corollary. *If P is EL-perfect and EL-first category relative to itself, then there is no EL-Bernstein subdivision of P.*

Proof. A set that meets every *EL*-perfect (rel P) set must be *EL*-dense in P. ∎

2. Non-measurable sets. Since there are $2^{\mathfrak{c}}$ many *EL*-open sets it is easy to see that *EL-B* properly includes *E-B*. We now prove two theorems which guarantee the existence of sets which are *EL*-first category and have measure zero, but which are not EL-B_{r}-measurable. The construction of a non-EL-B_{r}-measurable set in Example 2.3 is modeled after the classical construction of Vitali [K, p. 91]. In Theorem 2.4, we show that certain *E*-Bernstein sets fail to be EL-B_{r}-measurable. We include both results as the techniques used are quite different.

Set $\mathrm{Ev} = \{2n : n < \omega\}$ and $\mathrm{Od} = \{2n+1 : n < \omega\}$. Let $M = \{x \in [\omega]^\omega : \mathrm{Ev} \subseteq x\}$. M is *E*-perfect but is not quite *EL*-perfect, since every element $p \in M$ that contains only finitely many odd integers, say the set $G \in [\mathrm{Od}]^{<\omega}$, can be *EL*-isolated by the *EL*-open set $[G \cup \{2, 4, \ldots, \max(G)+1\}, p]$. The set $P = \{x \in M : \mathrm{Od} \cap x \text{ is infinite}\}$ is just M minus these *EL*-isolated points. P is *EL*-perfect, *EL*-first category, has measure zero, and is both *EL*- and *E*-homeomorphic to $[\omega]^\omega$.

In Vitali's construction on the real line, two numbers are said to be equivalent if their difference is rational. This produces 2^ω many equivalence classes, each of which is countable. Another important fact used in that construction is that A and $A + x = \{a + x : a \in A\}$ are of the same category for any set A and any real number x. The connection between addition, the closure of $\mathbb{R}$ under $+$, the translation invariance of category, and the cardinality of the rationals is then cleverly exploited to produce a non-$B_{\rm w}$-measurable set. We adapt this argument to the space P with the relative EL-topology by defining an equivalence relation on P, a notion of addition (or translation) under which P is closed, and a countable set of addends for the space $[\omega]^\omega$ that are similarly related.

For all $s,t\in\mathcal{P}(\omega)$, define $s\equiv t$ if and only if χ_s and χ_t are eventually equal. It is easy to verify that $\equiv$ is an equivalence relation. For all $s,F\in\mathcal{P}(\omega)$, define $s\oplus F=\chi^{-1}[\chi_s+\chi_F]$, where $+$ is the usual pointwise mod 2 addition in 2^ω. We also define $A\oplus F=\{s\oplus F: s\in A\}$ for all $A\subseteq\mathcal{P}(\omega)$. Finally, we take $[\mathrm{Od}]^{<\omega}$ as the countable set of addends.

2.1. Lemma. *If F is a finite set of odds and $A\subseteq P$, then A and $A\oplus F$ have the same EL-category* (*rel P*).

Proof. It suffices to show that for each $F\in[\mathrm{Od}]^{<\omega}$, $(\cdot)\oplus F$ preserves EL-open (rel P) sets. Let $F\subseteq\mathrm{Od}$ be finite and suppose $[G,v]\cap P\neq\emptyset$. Then $[G,v]\oplus F=[G\oplus F,v\oplus F]$, where $G\oplus F$ and $v\oplus F$ differ from G and v, respectively, by some finite set of odds. Moreover, it is clear that Ev is a subset of $[G\oplus F,v\oplus F]$. Thus $([G,v]\oplus F)\cap P$ is non-empty and EL-open relative to P. ■

2.2. Lemma. *If F is a finite set of odds, then $P\oplus F=P$.*

Proof. Suppose $F\subseteq\mathrm{Od}$ is finite. For every $q\in P$, $\mathrm{Ev}\subseteq q\oplus F$ and $|\,\mathrm{Od}\cap(q\oplus F)|=\omega$. Thus for all $q\in P$, $q\oplus F\in P$ and it follows that $P\oplus F\subseteq P$. It is also easy to see that for every $q\in P$, $(q\oplus F)\oplus F=q$. Hence $(P\oplus F)\oplus F=P$ and it follows that $P\subseteq P\oplus F$. ■

2.3. Example. *There exists a subset of $[\omega]^\omega$ which is EL-first category* (*i.e. CR_0*) *and L_0 but which is not EL-$B_{\rm r}$-measurable.*

Construction. It is easy to see that the restriction of $\equiv$ to $P\times P$ is an equivalence relation on P. We denote the restriction by $\equiv_P$. By the axiom of choice, there is a set $V_0\subseteq P$ which contains exactly one representative of each $\equiv_P$-equivalence class. Since P is EL-first category and has measure zero, the same is true of V_0. List $[\mathrm{Od}]^{<\omega}=\langle F_1,F_2,F_3,\ldots\rangle$ and define $V_n=V_0\oplus F_n$. Observe that $s\equiv t$ iff $(\exists F\in[\mathrm{Od}]^{<\omega})(t=s\oplus F)$. Thus $P\subseteq\bigcup_{n<\omega}V_n$ and Lemma 2.2 implies $\bigcup_{n<\omega}V_n\subseteq P$. Therefore $P=\bigcup_{n<\omega}V_n$.

Since P is not EL-first category relative to itself, there exists a positive integer K such that V_K is EL-second category (rel P). By Lemma 2.1 it follows that V_0 is also EL-second category (rel P). If V_0 is EL-$B_{\rm r}$-measurable, then there exists some $[G,u]$ such that V_0 is EL-residual (rel P) in $[G,u]\cap P$. Let $g\in u$ be an odd integer which is greater than $\max(G)$ and choose N such that $F_N=\{g\}$.

By Lemmas 2.1 and 2.2, V_N is EL-residual (rel P) in $([G,u]\cap P)\oplus F_N=([G,u]\oplus F_N)\cap(P\oplus F_N)=([G,u]\oplus F_N)\cap P$, which, in turn, is an EL-second category subset of $[G,u]\cap P$. But this is a contradiction since V_0 and V_N are disjoint. Therefore we conclude that V_0 is not EL-$B_{\rm r}$-measurable. ■

2.4. THEOREM. *Every E-Bernstein subset of P is EL-first category and has measure zero but is not* EL-$B_{\rm r}$*-measurable.*

Proof. Let $h:[\omega]^\omega\to P$ be an EL-homeomorphism and let $B\subseteq[\omega]^\omega$ be a Bernstein set. We show that $h(B)$ is not EL-$B_{\rm r}$-measurable. Ellentuck [E] has shown that a set $Q\subseteq[\omega]^\omega$ is EL-first category if and only if it is EL-nowhere dense. Both B and $B^{\rm c}$ meet every E-perfect set in $[\omega]^\omega$ and hence are EL-categorically dense in $[\omega]^\omega$. Thus $h(B)$ and $h(B^{\rm c})$ are likewise EL-categorically dense (rel P). It follows that $h(B)$ is not EL-$B_{\rm r}$-measurable. ■

3. Marczewski measurable sets in the density topology. We now turn our attention to the Marczewski measurable sets in the density topology and their relation to the Lebesgue measurable subsets of the real line. Oxtoby in [O, Sec. 22] has shown the σ-algebras L and D-$B_{\rm w}$ coincide. In this section, we show that the σ-algebra D-(s) is equal to L as well. We will need two lemmas, one of which is a characterization of non-Lebesgue measurable sets which was first proved by Burstin [Bu] in 1914.

3.1. LEMMA [Bu]. *If M is a set of reals, then M is non-Lebesgue measurable iff there exists an E-perfect set P of positive measure such that for every E-perfect $A\subseteq P$ of positive measure, A intersects both M and $M^{\rm c}$.*

3.2. LEMMA. *If M has positive Lebesgue measure, then for every $\varepsilon>0$, there exists a D-perfect set $Q\subseteq M$ such that $\lambda(Q)>\lambda(M)-\varepsilon$.*

Proof. Suppose that M has positive Lebesgue measure and let $\varepsilon>0$ be arbitrary. Let P denote an E-perfect subset of M with positive Lebesgue measure such that $\lambda(P)>\lambda(M)-\varepsilon$. Let Q denote the set of points of P where P has Lebesgue density 1. Q is D-open and has the same measure as P. Now set $R=D_{\rm cl}(Q)$ and note that R is D-perfect, has the same measure as P, and since P is E-closed, $R\subseteq P\subseteq M$. ■

3.3. THEOREM. *The σ-algebra D-(s) is equal to L.*

Proof. We first suppose that M is in D-$(s)\setminus L$. By Lemma 3.1 there is an E-perfect set P of positive measure such that for every perfect $A\subseteq P$ of positive measure, A intersects both M and $M^{\rm c}$. By Lemma 3.2, P must contain a D-perfect set, say Q. But by D-(s)-measurability of M, we may obtain a D-perfect subset of Q, say R, where $R\subseteq M$ or $R\subseteq M^{\rm c}$. But this is impossible since R has positive measure (measure zero sets are D-discrete). Therefore D-$(s)\subseteq L$.

For the other direction, suppose M is Lebesgue measurable and P is D-perfect. In case $P\cap M$ has positive measure, we may apply Lemma 3.2

to get a D-perfect subset of $P \cap M$. If, on the other hand, $\lambda(P \cap M) = 0$, then $P \cap M^c$ has positive measure and we may obtain a D-perfect subset of $P \cap M^c$ in similar fashion. Hence M is D-(s)-measurable. ∎

3.4. Corollary. $H(D\text{-}(s)) = D\text{-}(s)_0$.

Proof. It is obvious that every D-$(s)_0$-measurable set belongs to $H(D\text{-}(s))$. Conversely, suppose $M \in H(D\text{-}(s)) = H(L)$. Thus $\lambda(M) = 0$ and so for every D-perfect set P, $P \setminus M$ contains a D-perfect set. It follows that M is D-$(s)_0$-measurable. ∎

3.5. Corollary. *If $A \in \{B, B_r, L, (s), B_w\}$ and A_0 is the corresponding collection of A-singular sets*, *then $H(D\text{-}A) = D\text{-}A_0$.*

Proof. Oxtoby [O] has shown that $D\text{-}B_w = D\text{-}L$ and Scheinberg [Sc] has shown that $D\text{-}L = D\text{-}B$. Thus all the σ-algebras listed above collapse to a single class which can be represented by D-(s). The result now follows trivially from Corollary 3.4. ∎

We note here that Lemmas 3.1 and 3.2 generalize to any complete non-atomic Borel measure μ on a complete separable metric space X and the corresponding μ-density topology T. Thus M is μ-measurable iff M is T-(s) measurable. The proof of a generalized version of Lemma 3.1 is given below as Lemma 3.6. The proof of a generalized Lemma 3.2 follows the same lines as the proof of Lemma 3.2 but with Lebesgue measure, Lebesgue density, and D (the density topology) replaced by μ-measure, μ-density, and T (the μ-density topology). See [O, p. 88] for definitions and background theorems.

3.6. Lemma. *Suppose μ is a complete non-atomic Borel measure on a complete separable metric space X and let T denote the* (*metric*) *topology on this space. If $M \subseteq X$*, *then M is non-μ-measurable iff there exists a T-perfect set P of positive μ-measure such that for every T-perfect $A \subseteq P$ of positive μ-measure*, *A intersects both M and M^c.*

Proof. Suppose that M is not μ-measurable and let G_1 and G_2 be T-G_δ sets containing M and M^c, respectively. Further suppose that $\mu^*(G_1) = \mu^*(M)$ and $\mu^*(G_2) = \mu^*(M^c)$, where μ^* denotes the outer measure induced by μ. Clearly $G_1 \cap G_2$ has positive μ-measure, for otherwise M would be μ-measurable. Let P be a T-perfect subset of $G_1 \cap G_2$ of positive measure. Now suppose $Q \subseteq P$ is T-perfect and has positive μ-measure. If $Q \subseteq M$, then $M^c \subseteq G_2 \setminus Q$ and we have $\mu^*(M^c) \le \mu(G_2 \setminus Q) < \mu(G_2) = \mu^*(M^c)$, a contradiction. A similar contradiction arises if we assume $Q \subseteq M^c$. Therefore Q intersects both M and M^c.

For the other direction, suppose M is μ-measurable, let P be a T-perfect set of positive μ-measure, and assume $\mu(M \cap P) > 0$. Let F be a T-F_σ subset of $M \cap P$ which has the same μ-measure as $M \cap P$. Let $C \subseteq F$ be T-closed with $\mu(C) > 0$. Thus C is uncountable and can be written as the union of a T-perfect set Q and a countable set N. Of course $Q \subseteq M$. The argument goes through just as well if $\mu(M \cap P) = 0$, for then $\mu((M \cap P)^{c}) > 0$. ∎

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Department of Mathematics
S.E. Oklahoma State University
Durant, Oklahoma 74701
U.S.A.
E-mail: reardon@marcie.sosu.edu

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