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# Each nowhere dense nonvoid closed set in $\mathbb{R}^n$ is a $\sigma$-limit set

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**Abstract.** We discuss main properties of the dynamics on minimal attraction centers ($\sigma$-limit sets) of single trajectories for continuous maps of a compact metric space into itself. We prove that each nowhere dense nonvoid closed set in $\mathbb{R}^n$, $n \ge 1$, is a $\sigma$-limit set for some continuous map.

**1. Introduction.** We study dynamical systems which are defined by iterations of continuous maps of a compact metric space $X$ into itself: for a given map $f : X \to X$, each point $x \in X$ has a *trajectory* $\{f^n(x)\}_{n=0}^{\infty}$ under iterations of $f$. The *$\omega$-limit set* of a trajectory is the set of its limit points; it is usually denoted by $\omega_f(x)$, and for the maps under consideration, it is nonvoid, closed, and strictly invariant (i.e., $f(\omega_f(x)) = \omega_f(x)$; see e.g. [6]). It was established by Sharkovskiĭ [7] that the dynamics on $\omega$-limit sets is incompressible in the following sense:

($\omega$) *If $U$ is a proper relatively open subset of an $\omega$-limit set, then the closure of $f(U)$ is not contained in $U$.*

This property indicates a dependence between the topological structure of $\omega$-limit sets and the dynamics on them. As an example we cite the following statement from [7]:

($\omega'$) *No isolated point of an infinite $\omega$-limit set is periodic; if an $\omega$-limit set is finite, then it is a cycle.*

The present paper has to do with sets that can be *$\sigma$-limit sets* (i.e., *minimal attraction centers* of single trajectories) for a map $f : X \to X$. We use the notation $\sigma_f(x)$ to denote such a set. For $x \in X$, define $\sigma_f(x)$ to be the smallest closed set $F$ in $X$ such that for every open set $U$ containing $F$, $\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1} 1_U(f^i(x)) = 1$, where $1_U$ is the indicator function of $U$.

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(For the original definition we refer to Hilmy [4], Krylov and Bogolyubov [5] and recall that the $\sigma$-limit set of any trajectory is a nonvoid closed invariant subset of its $\omega$-limit set [6].) A point which belongs to its $\sigma$-limit set will be referred to as a *$\sigma$-recurrent point* and the set of all $\sigma$-recurrent points of a map $f$ will be denoted by $\mathrm{Rec}_\sigma(f)$. We use also the notation $\mathrm{Fix}(f)$ for the set of fixed points of $f$.

From the standpoint of dynamical systems and related ergodic theory, finite invariant measures play an important role. In topological dynamics the existence of finite invariant measures for continuous flows and cascades on a metric compact $X$ was established by Bogolyubov and Krylov [5]. The supports of finite invariant measures are connected in a certain way with the behavior of trajectories in $X$: all these measures are supported by the minimal attraction center [6]. This is also true in particular for Bowen–Sinai–Ruelle measures, which arise in a natural way when we are interested in statistical predictions. The support of such a measure is an *attractor* for a significant (with respect to the Lebesgue measure on $X$) set of trajectories and it could be useful to have a priori information about the admissible topological structure of this attractor.

In the present paper we formulate general properties of the dynamics of continuous maps on $\sigma$-limit sets and discuss restrictions on the topological structure of such sets. In particular, unlike the $\omega$-limit sets, isolated points of which cannot return, each isolated point of any $\sigma$-limit set must return and be periodic. It has been proved before (see [8]) that a nonvoid closed subset of $\mathbb{R}^1$ is a $\sigma$-limit set for a continuous map of $\mathbb{R}^1$ into itself if and only if it is either a union of finitely many disjoint nondegenerate closed intervals or a nowhere dense set. This result is closely related to the results of [1, 2]. In fact, the $\omega$-limit sets and the minimal attraction centers have the same characterization in $\mathbb{R}^1$. In higher dimensions, however, this is not the case: for instance, property $(\omega')$ above implies easily that the union of a line segment and an isolated point fails to be an $\omega$-limit set for continuous maps in $\mathbb{R}^2$. On the other hand, Theorem 2 below asserts that any nonvoid nowhere dense compact set in $\mathbb{R}^n$, $n > 1$, is a $\sigma$-limit set for a continuous map of $\mathbb{R}^n$.

To obtain a complete characterization of $\sigma$-limit sets in $\mathbb{R}^n$, it remains to describe the admissible compact sets with nonvoid interior. We will address this case in a subsequent paper because such $\sigma$-limit sets must coincide with the corresponding $\omega$-limit sets (Proposition 1) and hence the problem is closely related to a similar problem for $\omega$-limit sets in $\mathbb{R}^n$.

**2. Dynamics on $\sigma$-limit sets.** In this section we prove the following property of the dynamics on minimal attraction centers and then formulate some consequences of this property.

($\sigma$) *The $\sigma$-recurrent points are dense in the minimal attraction center of any trajectory.*

Proof. This can be proved by the method used in [5] for the minimal attraction center of all trajectories of the system. On the other hand, ($\sigma$) can be derived from general known results: for example from the facts that every ergodic measure is supported by its generic points (i.e., points such that the average of point masses along the orbit converges in the weak$^*$ topology to that measure, see e.g. Furstenberg's book [3]), and next, that every invariant measure is an integral average of ergodic measures (since these are the extreme points among invariant measures). ∎

From property ($\sigma$) we see that minimal attraction centers of trajectories of continuous maps may split into a family of smaller invariant subsets, which are $\sigma$-limit sets of other $\sigma$-recurrent points and which can be considered as independent blocks linked by an outer trajectory. We use this observation in order to prove in the next section that any nowhere dense compact set in $\mathbb{R}^n$ is a $\sigma$-limit set for a continuous map.

To end this section let us formulate some simple consequences of properties ($\sigma$) and ($\omega$) characterizing the interconnection between the dynamics and topological structure of $\sigma$-limit sets of continuous maps.

Proposition 1. *For continuous maps of a compact metric space into itself, the following statements hold*:

(A) *if a $\sigma$-limit set has an isolated point, then this point is periodic*;

(B) *if the $\omega$-limit set of a trajectory is infinite, then its $\sigma$-limit set is contained in the set of limit points of the $\omega$-limit set*;

(C) *if the $\sigma$-limit set of a trajectory does not coincide with its $\omega$-limit set, then this $\sigma$-limit set is nowhere dense*;

(D) *if the $\sigma$-limit set of a trajectory contains an open set, then the trajectory eventually hits this open set and hence the $\sigma$-limit set coincides with the $\omega$-limit set of this trajectory*;

(E) *if the $\sigma$-limit set of a trajectory coincides with its $\omega$-limit set and the latter is infinite, then both are perfect, i.e., they have no isolated points*; *in particular, the $\sigma$-limit set of any $\sigma$-recurrent point is perfect.*

Proof. By ($\sigma$) an isolated point in a $\sigma$-limit set must be $\sigma$-recurrent. It can return only to itself and hence it must be periodic. This proves statement (A).

Now observe that every isolated point of a set is also isolated in any subset containing this point. By ($\omega'$) no isolated point of an infinite $\omega$-limit set is periodic. Hence according to (A), such a point cannot belong to a

$\sigma$-limit set which is contained in this $\omega$-limit set. This proves (B), which, in turn, immediately implies (E).

Statement (D) is an easy consequence of the definition and basic properties of $\omega$-limit sets and $\sigma$-limit sets. Since any $\sigma$-limit set is closed, statement (C) follows from (D). ∎

**3. Admissible structure of $\sigma$-limit sets and the realization problem.** By a *continuum* we mean a compact connected set containing at least two points. A space is *locally connected* if every neighborhood of every point in it contains a connected neighborhood of the point.

The following proposition provides some trivial restrictions on the admissible topological structure of $\sigma$-limit sets for continuous maps in locally connected spaces, for example, in the Euclidean spaces $\mathbb{R}^n$, $n \geq 1$.

PROPOSITION 2. *For continuous maps of a locally connected compact metric space into itself, any $\sigma$-limit set is either a nowhere dense set or a union of finitely many disjoint continua, at least one of which has a nonempty interior.*

Proof. Suppose that $\sigma_f(x)$ is not nowhere dense. Since $\sigma_f(x)$ is closed, it contains an open set $U$. In this case by Proposition 1(D), $\sigma_f(x) = \omega_f(x)$ and without loss of generality we can assume that $x \in U \subset \sigma_f(x)$. Since the space under consideration is supposed to be locally connected, metric and compact, we can also assume that $U$ is connected. Let $C_0$ be the connected component of $\sigma_f(x)$ containing $x$. Then $x \in U \subset C_0 \subset \sigma_f(x)$.

Since $x$ must return to $U$ and connected subsets of $\sigma_f(x)$ are mapped into connected subsets, there exists $n \geq 1$ such that $f^n(C_0) \subset C_0$ and $f^i(C_0) \cap C_0 = \emptyset$ for $0 < i < n$. Let $C_i$ be the connected component of $\sigma_f(x)$ which contains $f^i(C_0)$. The components $C_i$, $0 \leq i < n$, are disjoint and $\sigma_f(x) = \bigcup_{0 \leq i < n} C_i$ because the trajectory of $x$ is dense in $\sigma_f(x)$. Finally, note that each $C_i$, $0 \leq i < n$, must contain at least two points because otherwise the trajectory of $x$ would be eventually periodic and hence its $\sigma$-limit set would be finite. ∎

Now suppose that a closed subset $S$ of a compact metric space $X$ satisfies the restrictions stated in Proposition 2. Is it a $\sigma$-limit set for a continuous map of $X$ into itself? For the real line $\mathbb{R}^1$ we can answer this question positively as follows. (Notice that in what follows we consider noncompact spaces $\mathbb{R}^n$, $n \geq 1$, just for the sake of convenience. In fact, since so far the whole theory only applies to compact spaces, we have to define the required continuous map on a compact subset $X \supset S$ in $\mathbb{R}^n$ first, and then extend it continuously onto the whole $\mathbb{R}^n$ in a suitable way. For the continuous map

$f$ in the proof of Theorem 2 below, the extension can evidently be defined to be the identity outside the below described compact set $X = P^{(0)}$.)

THEOREM 1 ([8]). *Let $S$ be a nonvoid closed subset of the real line $\mathbb{R}^1$. Then $S$ is a $\sigma$-limit set for a continuous map of $\mathbb{R}^1$ into itself if and only if it is either a nowhere dense set or a union of finitely many nondegenerate closed intervals.*

In higher dimensions the answer is negative in general. For instance, consider the union of two disjoint continua with nonvoid interiors, each with a different finite number of arcwise connected components. By property ($\omega$) these continua must be mapped onto each other, and lack of arcwise connected subsets in a continuum will lead to a contradiction. However, for nowhere dense sets, the following statement holds.

THEOREM 2. *Any nowhere dense nonvoid closed set in $\mathbb{R}^n$, $n \geq 1$, is a $\sigma$-limit set for some continuous map of $\mathbb{R}^n$ into itself.*

Proof. The idea of the proof is quite similar to that in [8] for $\mathbb{R}^1$ but, as usual, it is much harder to describe any method in higher dimensions. In order to simplify the exposition we consider mainly the case $n = 2$. So let $S$ be a nowhere dense nonvoid compact set in $\mathbb{R}^2$. We divide the proof into several steps.

Step 1. We are going to imbed $S$ in a perfect nowhere dense set $S^*$. Let $P^{(0)}$ be a closed rectangular set in $\mathbb{R}^2$ (a hyperparallelepiped in $\mathbb{R}^n$) which contains $S$ and whose sides are parallel to the coordinate axes (hyperplanes). Since $S$ is nowhere dense in $P^{(0)}$, we can choose a sufficiently small open square (hypercube) $C^{(0)}$ near the center of $P^{(0)}$ such that $S \cap C^{(0)} = \emptyset$ and such that the sides of $C^{(0)}$ are also parallel to coordinate axes. Extending the sides of the square $C^{(0)}$ up to sides of the rectangle $P^{(0)}$, we decompose $P^{(0)} - C^{(0)}$ into 8 rectangular sets (or $3^n - 1$ hyperparallelepipeds in $\mathbb{R}^n$) which we denote by $P_k^{(1)}$, where $1 \leq k \leq K_1$. Now repeating the reasoning for each $P_k^{(1)}$, we can define the next generation of open squares $C_k^{(1)}$ and of rectangular sets $P_{k'}^{(2)}$, where $1 \leq k' \leq K_2$. Continuing this process, we see that the size of $P_k^{(m)}$ tends to zero as $m \to \infty$ and hence the open squares $C_k^{(m)}$ are dense in $P^{(0)}$. The required perfect nowhere dense set $S^*$ is defined to be the complement of the open set $\bigcup C_k^{(m)}$ in $P^{(0)}$. We notice the following property of $S^*$:

CLAIM 1. *If $U$ is an open set and $U \cap S^* \neq \emptyset$, then there exist $m$ and a set $P_k^{(m)}$ such that $P_k^{(m)} \subset U$.*

Step 2. Let us construct a continuous expanding map $f$ for which $S^* \subset \mathrm{Fix}(f)$. The map $f$ will be defined in such a way that $f(x) = x$ for

$x \in S^*$, $f(C^{(0)}) = P^{(0)}$ and, for $m > 0$, $f(C_k^{(m)}) = P_{k'}^{(m-1)}$, where $P_{k'}^{(m-1)}$ is the rectangular set of the $(m-1)$th generation which contains the open square $C_k^{(m)}$. On $C_k^{(m)}$, $f$ is defined as follows. Let $I_k^{(m)}$ be a smaller closed square contained in $C_k^{(m)}$ with sides parallel to those of $C_k^{(m)}$. View the interior of $C_k^{(m)}$ as an elastic film which is attached to the borders of $C_k^{(m)}$ and $I_k^{(m)}$. The border of $C_k^{(m)}$ is a fixed frame but the border of $I_k^{(m)}$ is an elastic frame. Now let us stretch the border of $I_k^{(m)}$ onto the border of $P_{k'}^{(m-1)}$ without rotation in such a way that vertices are mapped to vertices. Then the square $I_k^{(m)}$ will be mapped onto $P_{k'}^{(m-1)}$ and the annular area $C_k^{(m)} - I_k^{(m)}$ will be mapped onto the annular area $P_{k'}^{(m-1)} - C_k^{(m)}$. The definition of $f$ on $C_k^{(m)}$ can be formalized; the map so defined is continuous and has the following two properties.

CLAIM 2. *For $m > 0$, one has $f(P_k^{(m)}) = f(C_k^{(m)}) = f(I_k^{(m)}) = P_{k'}^{(m-1)}$, where $P_{k'}^{(m-1)}$ is the rectangular set of the $(m-1)$-th generation which contains $P_k^{(m)}$. For $m = 0$, one has $f(P^{(0)}) = f(C^{(0)}) = f(I^{(0)}) = P^{(0)}$.*

CLAIM 3. *Each closed square $I_k^{(m)}$ is mapped linearly onto its image in a one-to-one way.*

Step 3. Now we can indicate a point $x$ for which $\sigma_f(x) = S$. Let $\delta_i = 1/(i+1)$ for $i = 1, 2, \ldots$ Since $S$ is compact, we can find a sequence of finite $\delta_i$-nets $S_i = \{s_1^{(i)}, \ldots, s_{N_i}^{(i)}\} \subset S$ such that $S_i \subset S_{i+1}$ for all $i \geq 1$. We arrange all points of $\bigcup_{i=1}^{\infty} S_i$ in a sequence $\{s_j\}_{j=1}^{\infty}$ (with repetitions) by juxtaposition of the elements of the consecutive sets $S_i$. Formally, every positive integer $j$ can be uniquely represented in the form $j = n(j) + \sum_{k<i(j)} N_k$, where $1 \leq n(j) \leq N_{i(j)}$. Then $s_j = s_{n(j)}^{(i(j))}$. Set $\varepsilon_j = \delta_{i(j)}$. The $s_j$ form a dense subset of $S$, and for any $j \geq 1$ there are infinitely many $k > j$ with $s_k = s_j$ and $\varepsilon_k < \varepsilon_j$.

For $i > 0$, let $U_i$ be the $\varepsilon_i$-neighborhood of $s_i$. For each $i$, by Claim 1 we can find a rectangular set $P^{(m_i')} \subset U_i$ (in order to simplify the notation we have dropped the subscript $k$ in $P_k^{(m_i')}$). By Claim 2 we have $f^{m_i'}(P^{(m_i')}) = P^{(0)}$. Observe that for any $P^{(m_i)} \subset P^{(m_i')}$ with $m_i > m_i'$, we also have $f^{m_i - m_i'}(P^{(m_i)}) = P^{(m_i')}$ and $f^j(P^{(m_i)}) \subset P^{(m_i')}$ for all $0 \leq j \leq m_i - m_i'$. By Claim 3 there exists a closed rectangular subset $J^{(m_i)}$ of the square $I^{(m_i)}$ in $P^{(m_i)}$ such that $J^{(m_i)}$ is linearly mapped onto $P^{(0)}$ by $f^{m_i}$. We have $f^j(J^{(m_i)}) \subset U_i$ for all $0 \leq j \leq m_i - m_i'$. Observe that $m_i'$ only depends on the neighborhood $U_i$, and $m_i$ can be arbitrarily large.

Now let $F_1 = J^{(m_1)}$ and, for $i > 1$, let $F_i$ be a closed rectangular subset of $F_{i-1}$ such that $f^{m_1 + \ldots + m_{i-1}}(F_i) = J^{(m_i)}$. The set $F_i$ is well defined by

Claim 3. We obtain a sequence $\{F_i\}$ of nested closed sets whose intersection must contain a point $x$. Let us prove that by choosing appropriate $m_i$, we can obtain the required equality $\sigma_f(x) = S$.

Let $\varepsilon > 0$ and let $U(\varepsilon)$ be the $\varepsilon$-neighborhood of $S$. Evidently, there is $i_0 \geq 1$ such that $U_i \subset U(\varepsilon)$ for all $i > i_0$. Then

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} 1_{U(\varepsilon)}(f^i(x)) \geq \liminf_{n\to\infty} \frac{1}{n'}\sum_{i=i_0}^{n}(m_i - m_i'),$$

where $n' = \sum_{i=1}^n m_i$. If we choose $\{m_i\}_{i=1}^\infty$ such that $m_i'/m_i \to 0$ and $m_i/\sum_{j=1}^i m_j \to 1$ as $i \to \infty$, then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} 1_{U(\varepsilon)}(f^i(x)) = 1$$

for any $\varepsilon > 0$. This means $\sigma_f(x) \subseteq S$.

On the other hand, recall that for any $i \geq 1$, there are infinitely many $k > i$ for which $s_k = s_i$, and $\varepsilon_k \to 0$ as $k \to \infty$. Hence, with the above choice of $m_i$, for any fixed $i$ we obtain

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} 1_{U_i}(f^j(x)) > 0$$

because it is bounded from below by the asymptotic upper bound of $(m_k - m_k')/\sum_{j=1}^k m_j$ taken over all $k$ with $s_k = s_i$. This proves that all $s_i$ belong to $\sigma_f(x)$ and hence $S \subseteq \sigma_f(x)$. Thus $\sigma_f(x) = S$ and the proof is complete. ∎

Remark. In fact, we have proved that for any nowhere dense nonvoid closed set $S$ in $\mathbb{R}^n$, there is a continuous map $f$ which realizes all nonvoid closed subsets of $S$ as $\sigma$-limit sets. Note also that by our construction all these $\sigma$-limit sets are realized by everywhere dense (in $P^{(0)}$) trajectories. On the other hand, each of these $\sigma$-limit sets is "totally disconnected" from the dynamical viewpoint because it disintegrates into fixed points.

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