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On the tameness of trivial extension algebras

by

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Abstract. For a finite dimensional algebra A over an algebraically closed field, let $T(A)$ denote the trivial extension of A by its minimal injective cogenerator bimodule. We prove that, if T_A is a tilting module and $B = \operatorname{End} T_A$, then $T(A)$ is tame if and only if $T(B)$ is tame.

Introduction. Let k be an algebraically closed field. In this paper, an algebra A is always assumed to be associative, with an identity and finite dimensional over k. We denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, and by $\underline{\operatorname{mod}}\, A$ the stable module category whose objects are the A-modules, and the set of morphisms from M_A to N_A is $\underline{\operatorname{Hom}}_A(M,N) = \operatorname{Hom}_A(M,N)/\mathcal{P}(M,N)$, where $\mathcal{P}(M,N)$ is the subspace of all morphisms factoring through projective modules. Two algebras R and S are called *stably equivalent* if the categories $\underline{\operatorname{mod}}\, R$ and $\underline{\operatorname{mod}}\, S$ are equivalent. There are several important problems of the representation theory of algebras which are formulated in terms of the stable equivalence of two self-injective algebras (see, for instance, [9, 19]). But few things are known. For instance, it is not yet known whether for two stably equivalent self-injective algebras R and S, the tameness of R implies that of S. We consider this problem in the following context.

Let A be an algebra, and $D = \operatorname{Hom}_k(-,k)$ denote the usual duality on $\operatorname{mod} A$. The *trivial extension* $T(A)$ of A (by the minimal injective cogenerator bimodule DA) is defined to be the k-algebra whose vector space structure is that of $A \oplus DA$, and whose multiplication is defined by

$$(a,q)(a',q') = (aa', aq' + qa')$$

for $a, a' \in A$ and $q, q' \in {}_A(DA)_A$. Trivial extensions are a special class of self-injective (actually, of symmetric) algebras. They have been extensively

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studied in the representation theory of algebras (see, for instance, [2, 3, 8, 10, 12, 17]), in particular, in the context of tilting theory. Following [7], an A-module T_A is called a *tilting module* if $\operatorname{Ext}^2_A(T,-)=0$, $\operatorname{Ext}^1_A(T,T)=0$ and there is an exact sequence

$$0\to A_A\to T'_A\to T''_A\to 0,$$

with T',T'' in the additive category $\operatorname{add} T$ consisting of the direct sums of direct summands of T. A *tilting triple* $(B,{}_BT_A,A)$ consists of an algebra A, a tilting module T_A and $B=\operatorname{End} T$. For a tilting triple (B,T,A), Tachikawa and Wakamatsu have constructed in [17] a stable equivalence functor $\mathcal{S}:\underline{\operatorname{mod}}\,T(A)\to\underline{\operatorname{mod}}\,T(B)$. Our main result is the following:

THEOREM. *Let $(B,{}_BT_A,A)$ be a tilting triple. Then $T(A)$ is tame if and only if $T(B)$ is tame.*

Recall from [5] that an algebra C is called *tame* if, for each $d\in\mathbb{N}$, there is a finite number of $k[t]-C$-bimodules $M_1,\ldots,M_{s(d)}$ which are free and finitely generated left $k[t]$-modules and such that, for all but at most finitely many non-isomorphic indecomposable C-modules X with $\dim_k X=d$, there is an isomorphism $X\xrightarrow{\sim}(k[t]/\langle t-\lambda\rangle)\otimes_{k[t]}M_i$, for some $1\le i\le s(d)$ and $\lambda\in k$. In this case, we let $\mu_C(d)$ denote the least possible $s(d)$. We say that C is *of polynomial growth* (or *domestic*) if $\mu_C(d)\le d^m$ for some $m\in\mathbb{N}$ (or $\mu_C(d)\le K$ for some $K\in\mathbb{N}$, respectively).

The representation theory of the trivial extension of polynomial growth is well known. Namely, the representation-finite trivial extensions are trivial extensions of tilted algebras of Dynkin type (by [8]), the representation-infinite domestic trivial extensions are either trivial extensions of tilted algebras of Euclidean type or quotients of a trivial extension $T(A)$ of some representation-infinite algebra A of Euclidean type $\widetilde{A}_n$ by the group $\mathbb{Z}_2$ (by [3, 10, 11]), and the non-domestic trivial extensions of polynomial growth are trivial extensions of tubular algebras (by [10, 12]). But the representation theory of the tame trivial extensions which are not of polynomial growth is still completely unknown. Our theorem ensures that the tameness of these algebras is preserved under tilting of the original algebra.

The main technique used for the proof of the theorem is the geometric setting developed in [14]. The proof essentially reduces to showing that the construction of $\mathcal{S}$ yields a constructible function on objects $\mathcal{S}:\operatorname{mod} T(A)\to\operatorname{mod} T(B)$. Then one applies [14], (4.3).

1. The Tachikawa–Wakamatsu stable equivalence functor

1.1. Let A be a finite dimensional k-algebra assumed moreover to be basic and connected. We shall use freely, and without further reference, facts about the module category $\operatorname{mod} A$ and the Auslander–Reiten translations

$\tau = D\,\mathrm{Tr}$ and $\tau^{-1} = \mathrm{Tr}\,D$, as in [4, 15]. For tilting theory, we refer the reader to [1]. Recall in particular that if $(B, {}_BT_A, A)$ is a tilting triple, then T_A induces a torsion theory $(\mathcal{T}, \mathcal{F})$ in $\mathrm{mod}\,A$, where $\mathcal{T}$ (or $\mathcal{F}$) denotes the full subcategory of $\mathrm{mod}\,A$ generated by T_A (or cogenerated by τT_A, respectively).

Given an A-module M, the evaluation morphism $\varepsilon_M : \mathrm{Hom}_A(T, M) \otimes_B T \to M$ defined by $f \otimes t \mapsto f(t)$ (where $f \in \mathrm{Hom}_A(T, M)$, $t \in T$) is functorial, and is an isomorphism if and only if $M \in \mathcal{T}$. Similarly, to a B-module X corresponds a functorial morphism $\eta_X : X_B \to \mathrm{Hom}_A(T, X \otimes_B T)$ defined by $x \mapsto (t \mapsto x \otimes t)$ (where $x \in X$, $t \in T$).

Finally, we have canonical isomorphisms $DT \otimes_B T \xrightarrow{\sim} DA$ and $T \otimes_B DT \xrightarrow{\sim} DB$, which we shall consider as identifications throughout this paper.

1.2. *Torsion resolutions*. Let (B, T, A) be a tilting triple, and $(\mathcal{T}, \mathcal{F})$ be the corresponding torsion theory in $\mathrm{mod}\,A$. For an A-module M_A, an exact sequence of the form

$$0 \longrightarrow M \xrightarrow{\alpha_0} V_0 \xrightarrow{\beta_0} T_0 \longrightarrow 0$$

with $V_0 \in \mathcal{T}$ and $T_0 \in \mathrm{add}\,T$ is called a *torsion resolution* for M. By [17], each module M_A admits a torsion resolution

$$0 \longrightarrow M \xrightarrow{\alpha_M} V(M) \xrightarrow{\beta_M} T(M) \longrightarrow 0$$

such that $T(M) = P \otimes_B T$, where P_B is a projective cover of $\mathrm{Ext}^1_A(T, M)$, and which is minimal in the following sense: for any other torsion resolution for M,

$$0 \longrightarrow M \xrightarrow{\alpha'} V' \xrightarrow{\beta'} T' \longrightarrow 0,$$

there exists $T'' \in \mathrm{add}\,T$ such that we have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha'"] \arrow[d, equal] & V' \arrow[r, "\beta'"] \arrow[d, equal] & T' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "{\left[\begin{smallmatrix}\alpha_M \\ 0\end{smallmatrix}\right]}"] & V(M) \oplus T'' \arrow[r, "{\left[\begin{smallmatrix}\beta_M & 0 \\ 0 & 1\end{smallmatrix}\right]}"] & T(M) \oplus T'' \arrow[r] & 0
\end{tikzcd}$$

In fact, the module $V(M)$ is constructed as follows:

(a) If $M \in \mathcal{T}$, then $V(M) = M$.

(b) If $M \in \mathcal{F}$, then $V(M) \xrightarrow{\sim} K \otimes_B T$, where K_B is the kernel of the projective cover morphism $P_B \to \mathrm{Ext}^1_A(T, M)$.

(c) In general, if $0 \to tM \to M \to M/tM \to 0$ is the canonical sequence of M in the torsion theory $(\mathcal{T}, \mathcal{F})$, the torsion resolution of M is obtained as the middle column in the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & tM \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & M/tM \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & tM \arrow[r] & V(M) \arrow[r] \arrow[d] & V(M/tM) \arrow[r] \arrow[d] & 0 \\
 & & T(M) \arrow[r, equal] \arrow[d] & T(M/tM) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Lemma. *Let* $t = \dim_k T$, $s_A = \dim_k A$, $s_B = \dim_k B$ *and* M_A *be an* A*-module with* $d = \dim_k M$. *Then* $\dim_k V(M) \le d + dt^2 s_A^3 s_B$.

Proof. Using the middle row in the above diagram, we see that $\dim_k V(M) = \dim_k tM + \dim_k V(M/tM)$. Since $tM \subseteq M$, we have $\dim_k tM \le d$. Put $N = M/tM$. We have a short exact sequence of B-modules

$$0 \longrightarrow K \longrightarrow P_B \xrightarrow{p} \operatorname{Ext}_A^1(T, N) \longrightarrow 0,$$

where p is a projective cover. Hence $V(N) \xrightarrow{\sim} K \otimes_B T$ yields $\dim_k V(N) \le t \dim_k K$. On the other hand,

$$\dim_k K \le \dim_k P_B \le s_B \dim_k \operatorname{Ext}_A^1(T, N).$$

In order to bound $\dim_k \operatorname{Ext}_A^1(T, N)$, we shall use the Auslander–Reiten formula $\operatorname{Ext}_A^1(T, N) \xrightarrow{\sim} D\,\underline{\operatorname{Hom}}_A(\tau^{-1}N, T)$ and a minimal projective presentation in $\operatorname{mod} A^{\mathrm{op}}$:

$${}_AP_1 \xrightarrow{f_1} {}_AP_0 \xrightarrow{f_0} {}_ADN \longrightarrow 0.$$

Indeed, the latter yields $\dim_k P_1 \le s_A \dim_k \operatorname{Ker} f_0$ and $\dim_k \operatorname{Ker} f_0 \le s_A \dim_k DN = s_A \dim_k N$. Applying the functor $\operatorname{Hom}_A(-, A)$ yields an exact sequence

$$\operatorname{Hom}_A(P_0, A) \longrightarrow \operatorname{Hom}_A(P_1, A) \longrightarrow \tau^{-1}N \longrightarrow 0.$$

Hence, $\dim_k \tau^{-1}N \le \dim_k \operatorname{Hom}_A(P_1, A) \le s_A \dim_k P_1 \le s_A^3 \dim_k N$. On the other hand, since $N = M/tM$, we have $\dim_k N \le \dim_k M = d$. The result then follows from the inequalities

$$\begin{aligned}
\dim_k V(M/tM) &\le t \dim_k K \le ts_B \dim_k \operatorname{Ext}_A^1(T, N) \\
&\le ts_B \dim_k \operatorname{Hom}_A(\tau^{-1}N, T) \le ts_B \dim_k T \dim_k \tau^{-1}N \\
&\le t^2 s_B s_A^3 d. \ \blacksquare
\end{aligned}$$

1.3. *The stable equivalence*. For an algebra A, let $T(A)$ denote the trivial extension of A by DA. We shall use the following equivalent description of $\operatorname{mod} T(A)$ (see [6]). Let $\mathcal{C}$ be the category whose objects are the pairs (M,φ) where M is an A-module and $\varphi : M \otimes_A DA \to M$ is an A-linear map such that $\varphi(\varphi \otimes DA) = 0$, and where a morphism $f : (M,\varphi) \to (M',\varphi')$ is an A-linear map $f : M \to M'$ such that $\varphi'(f \otimes DA) = f\varphi$. Then $\mathcal{C} \xrightarrow{\sim} \operatorname{mod} T(A)$. Throughout this paper, we shall identify these two categories. In particular, any A-module M induces a $T(A)$-module $\left(M \oplus (M \otimes DA), \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]\right)$ that will be denoted by

$$\frac{M}{\widetilde{M \otimes DA}}\,.$$

It is well known that the image of the canonical embedding $\operatorname{mod} A \to \operatorname{mod} T(A)$ actually lies inside the stable category (see, for instance, [17]).

Let (M,φ) be an arbitrary $T(A)$-module. As an A-module, M admits a minimal torsion resolution

$$0 \longrightarrow M \xrightarrow{\alpha_M} V(M) \xrightarrow{\beta_M} T(M) \longrightarrow 0.$$

Now, define a B-linear map

$$\phi_M : M \otimes_A DT_B \longrightarrow \operatorname{Hom}_A(T, V(M)) \oplus [\operatorname{Hom}_A(T, V(M)) \otimes_B DB]$$

by the formula

$$\phi_M = \begin{bmatrix} \operatorname{Hom}_A(T,\alpha_M)\operatorname{Hom}_A(T,-\varphi)\eta_{M\otimes DT} \\ (\varepsilon^{-1}_{V(M)} \otimes DT)(\alpha_M \otimes DT) \end{bmatrix}.$$

The construction of ϕ_M may be visualised as follows:

$$\begin{tikzcd}
\operatorname{Hom}_A(T, M\otimes_A DT \otimes_B T) \arrow[r, "\sim"] & \operatorname{Hom}_A(T, M\otimes_A DA) \arrow[r, "\operatorname{Hom}_A(T,-\varphi)"] & \operatorname{Hom}_A(T,M) \arrow[d, "\operatorname{Hom}_A(T,\alpha_M)"'] \\
M \otimes_A DT \arrow[u, "\eta_{M\otimes DT}"'] \arrow[d, "\alpha_M\otimes DT"'] & & \begin{array}{c}\operatorname{Hom}_A(T,V(M))\oplus \\ {[\operatorname{Hom}_A(T,V(M)) \oplus_B DB]}\end{array} \\
V(M)\otimes_A DT \arrow[rr, "\sim", "\varepsilon^{-1}_{V(M)}\otimes DT"'] & & \operatorname{Hom}_A(T,V(M))\otimes_B T \otimes_A DT \arrow[u, "\sim"]
\end{tikzcd}$$

The source and the target of ϕ_M are each endowed with a natural $T(B)$-module structure. Namely, $M \otimes_A DT$ has the $T(B)$-module structure induced by the morphism

$$\begin{aligned} M \otimes_A DT \otimes_B DB &\xrightarrow{\sim} M \otimes_A DT \otimes_B T \otimes_A DT \\ &\xrightarrow{\sim} M \otimes_A DA \otimes_A DT \xrightarrow{-\varphi\otimes DT} M \otimes_A DT, \end{aligned}$$

while $\operatorname{Hom}_A(T,V(M))\oplus[\operatorname{Hom}_A(T,V(M))\otimes_B DB]$ induces the $T(B)$-module

$$\frac{\widetilde{\operatorname{Hom}_A(T, V(M))}}{\operatorname{Hom}_A(T, V(M)) \otimes_B DB}.$$

One shows that ϕ_M is $T(B)$-linear, so that $\mathcal{S}(M, \varphi) = \operatorname{Coker} \phi_M$ is a $T(B)$-module. We have thus defined a map on objects $\mathcal{S} : \operatorname{mod} T(A) \to \operatorname{mod} T(B)$. One can show that for an A-module $M \in \mathcal{T}$, we have $\mathcal{S}(M, 0) = \operatorname{Hom}_A(T, M)$, that the image of a projective $T(A)$-module is a projective $T(B)$-module, and, finally, that $\mathcal{S}$ is a functor from $\underline{\operatorname{mod}}\, T(A)$ to $\underline{\operatorname{mod}}\, T(B)$. We have the following theorem:

THEOREM [17]. *Let (B, T, A) be a tilting triple. Then the functor $\mathcal{S} : \underline{\operatorname{mod}}\, T(A) \to \underline{\operatorname{mod}}\, T(B)$ is an equivalence.* ∎

2. Geometrisation of the problem

2.1. The proof of our theorem relies on the methods of algebraic geometry and the criteria for tameness developed in [13, 14]. For more details on the constructions used, we refer the reader to [14].

First, we identify our A- and $T(A)$-modules with points of constructible sets in appropriate affine spaces.

Let A be an algebra, and $1 = a_1, a_2, \ldots, a_s$ be a k-basis of A such that $a_i a_j = \sum_{m=1}^{s} \lambda_{ij}^{(m)} a_m$ for all $1 \le i, j \le m$ (the scalars $\lambda_{ij}^{(m)}$ are the so-called *structure constants* of A). Recall that an A-module M of k-dimension d may be identified with a representation $M : A \to \operatorname{End}_k(k^d)$, thus, each of the basis elements a_i corresponds to a $d \times d$ matrix $M(a_i)$. We define, for each $d \in \mathbb{N}$, $\operatorname{mod}_A(d)$ to be the (closed) subset of the affine space $\prod_{i=1}^{s} k^{d \times d}$ consisting of the s-tuples of $d \times d$ matrices $M = (M(a_1), \ldots, M(a_s))$ such that $M(a_1)$ is the identity matrix and $M(a_i)M(a_j) = \sum_{m=1}^{s} \lambda_{ij}^{(m)} M(a_m)$ for all $1 \le i, j \le s$.

Recall that a *vector space category* is a pair $(K, |-|)$ consisting of a k-linear category and a faithful k-linear functor $|-| : K \to \operatorname{mod} k$.

Following [14], we say that the pair $(K, |-|)$ is *geometrisable* if there exists an increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ inducing, for each $d \in \mathbb{N}$, a function $\{X \in K : \dim_k |X| = d\} \to k^{\sigma(d)}$ (denoted by $X \mapsto \widetilde{X}$) satisfying

(G1) the image $K(d)$ is constructible as a subset of the affine space $k^{\sigma(d)}$;

(G2) there is a function $\mu : K(d) \to K$ such that $\dim_k |\mu(\widetilde{X})| = d$ and $\mu(\widetilde{X})^{\sim} = \widetilde{X}$; moreover, $\mu(\widetilde{X}) \xrightarrow{\sim} X$.

EXAMPLES. (a) The module category $\operatorname{mod} A$ is a vector space category taking $|-| : \operatorname{mod} A \to \operatorname{mod} k$ to be the forgetful functor. Let $\sigma(d) = sd^2$, for $d \in \mathbb{N}$. As above, let $1 = a_1, a_2, \ldots, a_s$ be a k-basis of A. The map $M \mapsto \widetilde{M} = (M(a_1), \ldots, M(a_s)) \in k^{\sigma(d)}$ defines a geometrisation of $\operatorname{mod} A$.

(b) Let K_A be the category whose objects are triples (M, f, N), where M, N are A-modules, and $f : M \to N$ is A-linear, and whose morphisms $(h, g) : (M, f, N) \to (M', f', N')$ are pairs of A-linear maps such that $f'h = gf$. Then K_A is a vector space category with $|(M, f, N)| = |M| \oplus |N|$ (where, on the right, $|-| : \operatorname{mod} A \to \operatorname{mod} k$ denotes the forgetful functor, as in (a)). Let $\sigma(d) = (s+1)d^2$. For a triple (M, f, N) with $\dim_k |M| = m$, $\dim_k |N| = n$ and $d = m+n$, we define $(M, f, N)^\sim$ to be the $(s+1)$-tuple of $d \times d$ matrices

$$\left(\begin{bmatrix} M(a_1) & 0 \\ 0 & N(a_1) \end{bmatrix}, \ldots, \begin{bmatrix} M(a_s) & 0 \\ 0 & N(a_s) \end{bmatrix}, \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix}\right).$$

The images of this map form a closed subset $K_A(m, n)$ of $k^{\sigma(d)}$. Thus $K_A(d) = \bigcup_{d=m+n} K_A(m, n)$ is closed in $k^{\sigma(d)}$. Therefore K_A is geometrisable.

2.2. Let us now consider $\operatorname{mod} T(A)$. For $(M, \varphi) \in \operatorname{mod} T(A)$, we set $|(M, \varphi)|$ to equal the underlying k-space $|M|$ of M. Thus $\operatorname{mod} T(A)$ is a vector space category. The following lemma is shown in [14], we only indicate the main steps of the proof.

LEMMA. *The vector space category* $\operatorname{mod} T(A)$ *is geometrisable.*

Sketch of proof. Let $\sigma(d) = 2sd^2$, where $a_1 = 1, \ldots, a_s$ is a basis of A. Let $m \le sd$. Consider the functor $F = - \otimes_A DA : \operatorname{mod} A \to \operatorname{mod} A$. By [14], (2.2), the set $F^{-1}\binom{d}{m} = \{M \in \operatorname{mod}_A(d) : \dim_k |FM| = m\}$ is constructible. The set of pairs (M, φ) in $\operatorname{mod}_A(d) \times k^{d \times m}$ such that $\varphi : \widetilde{FM} \to \widetilde{M}$ is A-linear and $\varphi(\varphi \otimes DA) = 0$ is a constructible subset $C(m)$ of $k^{sd^2} \times k^{d \times m} \subseteq k^{\sigma(d)}$. Since, for any $N \in \operatorname{mod}_A$, $\dim_k |FN| \le s \dim_k |N|$, it follows that $\operatorname{mod}_{T(A)}(d) = \bigcup_{m \le sd} C(m)$ is a constructible subset of $k^{\sigma(d)}$. For $(M, \varphi) \in \operatorname{mod} T(A)$ with $\dim_k |M| = d$, we thus set $(M, \varphi)^\sim = (\widetilde{M}, \varphi) \in \operatorname{mod}_{T(A)}(d)$. ∎

2.3. In 1.3, we defined the function on objects $\mathcal{S} : \operatorname{mod} T(A) \to \operatorname{mod} T(B)$ which induces a stable equivalence $\mathcal{S} : \underline{\operatorname{mod}}\, T(A) \to \underline{\operatorname{mod}}\, T(B)$.

Let $\mathcal{U}, \mathcal{V}$ be two vector space categories. A function on objects $f : \mathcal{U} \to \mathcal{V}$ is called an *object-function* if $X \xrightarrow{\sim} Y$ in $\mathcal{U}$ implies $f(X) \xrightarrow{\sim} f(Y)$ in $\mathcal{V}$. An object-function $f : \mathcal{U} \to \mathcal{V}$, where $\mathcal{U}, \mathcal{V}$ are geometrisable categories, is called *constructible* [14] if, for each $d, m \in \mathbb{N}$, the following are satisfied:

(C1) the set $f^{-1}\binom{d}{m} = \{M \in \mathcal{U}(d) : \dim_k |f(M)| = m\}$ is constructible, and empty for m large enough;

(C2) there exist $c \in \mathbb{N}$ and a constructible subset $C \subseteq f^{-1}\binom{d}{m} \times k^c \times \mathcal{V}(m)$ such that the following diagram (where π_1, π_3 denote the respective projection morphisms) commutes:

$$\begin{tikzcd} C \subseteq f^{-1}\binom{d}{m} \times k^c \times \mathcal{V}(m) \arrow[r, "\pi_3"] \arrow[d, "\pi_1"'] & \mathcal{V}(m) \\ f^{-1}\binom{d}{m} \arrow[ur, "f"'] & \end{tikzcd}$$

in the sense that $f\pi_1(w)^\sim \xrightarrow{\sim} \pi_3(w)^\sim$, for $w \in C$.

(C3) $\pi_1(C) = f^{-1}\binom{d}{m}$.

The following examples are treated in detail in [14].

Examples. (a) The composition of constructible object-functions is constructible.

(b) Let ${}_BX_A$ be a bimodule. Then $- \otimes_B X_A : \operatorname{mod} B \to \operatorname{mod} A$ and $- \otimes_B X_A : K_B \to K_A$ yield constructible object-functions.

(c) Let $f : {}_BX_A \to {}_BY_A$ be a morphism of B-A-bimodules. Then $- \otimes_B f : \operatorname{mod} B \to K_A$ yields a constructible object-function.

(d) The functors $\operatorname{Ker}, \operatorname{Coker} : K_A \to \operatorname{mod} A$ yield constructible object-functions.

(e) Let ${}_BT_A$ be a bimodule and $\eta : 1_{\operatorname{mod} B} \to \operatorname{Hom}_B(T, - \otimes_B T)$ be the functorial morphism defined in 1.1. Then η induces a constructible object-function $\eta : \operatorname{mod} B \to K_B$.

(f) Let ${}_BT_A$ be a bimodule and $\varepsilon : \operatorname{Hom}_A(T, -) \otimes_B T \to 1_{\operatorname{mod} A}$ be the (evaluation) functorial morphism defined in 1.1. Then ε induces a constructible object-function $\varepsilon : \operatorname{mod} A \to K_A$. If T_A is a tilting module and $B = \operatorname{End} T_A$, then ε_M is invertible for each $M \in \mathcal{T}$ and $\varepsilon^{-1} : \mathcal{T} \to K_A$ is also constructible (here, $\mathcal{T}$ denotes, as usual, the torsion class induced by T_A in $\operatorname{mod} A$).

2.4. Proposition. *The object function* $\mathcal{S} : \operatorname{mod} T(A) \to \operatorname{mod} T(B)$ *is constructible.*

Proof. By the examples in 2.3 above, and the construction of $\mathcal{S}$ given in 1.3, it suffices to show that, if $M \in \operatorname{mod} A$ and

$$0 \longrightarrow M \xrightarrow{\alpha_M} V(M) \xrightarrow{\beta_M} T(M) \longrightarrow 0$$

is a minimal torsion resolution of M, then the object-function $s : \operatorname{mod} A \to K_A$ given by $M \mapsto (V(M), \beta_M, T(M))$ is constructible.

Let $(\mathcal{T}, \mathcal{F})$ be the torsion theory in $\operatorname{mod} A$ induced by the tilting triple (B, T, A). We claim that, for any $d \in \mathbb{N}$, the sets $\mathcal{F}(d) = \{M \in \operatorname{mod}_A(d) : M \in \mathcal{F}\}$ and $\mathcal{T}(d) = \{M \in \operatorname{mod}_A(d) : M \in \mathcal{T}\}$ are constructible. Indeed, by [14], (2.5), the functors $F = \operatorname{Hom}_A(T, -)$ and $F' = \operatorname{Ext}^1_A(T, -)$ from $\operatorname{mod} A$

to $\operatorname{mod} B$ induce constructible object-functions. Hence $F^{-1}\binom{d}{0} = \{M \in \operatorname{mod}_A(d) : F(M) = 0\} = \mathcal{F}(d)$ and $F'^{-1}\binom{d}{0} = \{M \in \operatorname{mod}_A(d) : F'(M) = 0\} = \mathcal{T}(d)$ are constructible.

Consider the object-function $c : \operatorname{mod} A \to K_A \times K_A$ defined by $M \mapsto ((tM, i, M), (M, p, M/tM))$, where

$$0 \longrightarrow tM \xrightarrow{i} M \xrightarrow{p} M/tM \longrightarrow 0$$

is the canonical sequence of M in the torsion theory $(\mathcal{T}, \mathcal{F})$. We now claim that c is constructible. Indeed, for $e \le d$, let

$$\begin{aligned} &C(d,e) \\ &\quad = \{(M', i, M, p, M'') \in \mathcal{T}(e) \times k^{e\times d} \times \operatorname{mod}_A(d) \times k^{d\times(d-e)} \times \mathcal{F}(d-e) : \\ &\qquad\qquad \text{the sequence } 0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0 \text{ is exact}\}. \end{aligned}$$

Clearly, $C(d,e)$ is constructible. Therefore c is constructible.

The object-function $f : F \to K_A$ defined by $f(M) = (K \otimes_B T, j \otimes_B T, P_0 \otimes_B T)$, where

$$0 \longrightarrow K \xrightarrow{j} P_0 \xrightarrow{p_0} \operatorname{Ext}^1_A(T, M) \longrightarrow 0$$

is an exact sequence, and P_0 is a projective cover of $\operatorname{Ext}^1_A(T, M)$, is also a constructible object-function. Indeed, the coordinates of f are obtained by composition of the following object-functions:

(a) $\operatorname{Ext}^1_A(T, -) : \operatorname{mod} A \to \operatorname{mod} B$,
(b) $P_0 : \operatorname{mod} B \to \operatorname{mod} B$, such that $P_0(X)$ is a projective cover of X,
(c) $\operatorname{Ker} : K_B \to \operatorname{mod} B$,
(d) $- \otimes_B T : \operatorname{mod} B \to \operatorname{mod} A$,

each of which is constructible (see 2.3 above, or [14]). Hence f is constructible.

We proceed to show that s is constructible. Let $d \in \mathbb{N}$ and $m \le d + dt^2 s_A^3 s_B$ (as in 1.2). Choose also $e \le d$. Consider the set $C(d, e, m)$ of 16-tuples

$$(M, M', i, p, M'', j, V'', \sigma, P, q, L, \alpha, V, \beta, \alpha', \beta')$$

such that

(a) $M \in \operatorname{mod}_A(d), c(M) = ((M', i, M), (M, p, M'')) \in C(d, e)$,

(b) $f(M'') = (V'', \sigma, P)$ and $0 \longrightarrow M'' \xrightarrow{j} V'' \xrightarrow{q} L \longrightarrow 0$ is exact in $\operatorname{mod} A$,

(c) $V \in \operatorname{mod}_A(m)$ and the following diagram is commutative, with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "i"] \arrow[d, equal] & M \arrow[r, "p"] \arrow[d, "\alpha"] & M'' \arrow[r] \arrow[d, "j"] & 0 \\
0 \arrow[r] & M' \arrow[r, "\alpha'"] & V \arrow[r, "\beta'"] \arrow[d, "\beta"] & V'' \arrow[r] \arrow[d, "q"] & 0 \\
 & & L \arrow[r, equal] \arrow[d] & L \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The set $C(d,e,m)$ is constructible and

$$s^{-1}\tbinom{d}{(m,m-d)} = \bigcup_{e\le d} \pi_1 C(d,e,m)$$

is constructible. Moreover, the diagram

$$\begin{tikzcd}
C(d,e,m) \arrow[r, "\pi"] \arrow[d, "\pi_1"'] & K_A((m,m-d)) \\
s^{-1}\tbinom{d}{(m,m-d)} \subseteq \operatorname{mod}_A(d) \arrow[ur, "s"'] &
\end{tikzcd}$$

(where π, π_1 denote the respective projections) commutes. Hence s is constructible. ∎

2.5. Proof of the theorem. As pointed out in the introduction, our theorem immediately follows from the above proposition and the criterion for tameness in [14], (4.3). Indeed, let X be a non-projective $T(B)$-module. Then there exists a $T(A)$-module M and a projective $T(B)$-module P such that $\mathcal{S}(M) \xrightarrow{\sim} X \oplus P$. In the terminology of [14], (4.1), we say that $\mathcal{S}$ constructively almost covers $\operatorname{mod} T(B)$. A direct application of [14], (4.3) completes the proof. ∎

Remark. Let (B,T,A) be a tilting triple. Then $T(A)$ is domestic representation-infinite (or non-domestic of polynomial growth) if and only if $T(B)$ is domestic representation-infinite (or non-domestic of polynomial growth, respectively). This follows from [3, 10, 11, 12].

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