On strongly Hausdorff flows

by

Hiromichi Nakayama (Hiroshima)

Abstract. A flow of an open manifold is very complicated even if its orbit space is Hausdorff. In this paper, we define the strongly Hausdorff flows and consider their dynamical properties in terms of the orbit spaces. By making use of this characterization, we finally classify all the strongly Hausdorff $C^1$-flows.

1. Introduction. Let $M$ be a connected manifold (maybe open), and $\varphi_t$ a non-singular flow of $M$. The flow $\varphi_t$ is called strongly Hausdorff if, for any point sequences $\{p_n\}$ and $\{q_n\}$ ($n = 1, 2, \ldots$) converging to $p$ and $q$ respectively and satisfying $\varphi_{t_n}(p_n) = q_n$ for some positive numbers $t_n$, there is a number $t$ ($t \geq 0$) such that $\varphi_t(p) = q$. The product flow of a product manifold $N \times \mathbb{R}$, defined by $\varphi_t(z, s) = (z, s + t)$ ($z \in N, s, t \in \mathbb{R}$), is an example of a strongly Hausdorff flow. Furthermore, a flow tangent to a generalized Seifert fibration (see §3) is also an example of a strongly Hausdorff flow whose orbits are all periodic. The following main theorem shows that any strongly Hausdorff flow turns out to be one of the above examples up to conjugation, where two flows $\varphi_t$ and $\psi_t$ are conjugate if there exists a diffeomorphism $f$ such that $f \varphi_t f^{-1} = \psi_t$ for any $t \in \mathbb{R}$.

Theorem. Every strongly Hausdorff $C^1$-flow is either conjugate to a product flow or tangent to a generalized Seifert fibration.

We prove this theorem in §3. In §2, we study the dynamical properties of strongly Hausdorff flows: the relation between strongly Hausdorff flows and Hausdorff flows, and the coincidence of the non-wandering set and the set of periodic points for strongly Hausdorff flows.

The author wishes to thank A. Marin for his helpful comments.

2. Strongly Hausdorff flows. Let $M$ be a connected $m$-dimensional manifold, and $\varphi_t$ a non-singular flow of $M$. Denote by $\pi$ the quotient map

---

1991 Mathematics Subject Classification: Primary 58F25; Secondary 58F18.
which maps each orbit to a point, and by \( M/\varphi_t \) the quotient space, which is called the orbit space. A flow \( \varphi_t \) is Hausdorff if the orbit space \( M/\varphi_t \) is Hausdorff with respect to the quotient topology. The notion of strongly Hausdorff flows is stronger than that of Hausdorff flows by the following lemma:

**Lemma 1.** Every strongly Hausdorff flow is Hausdorff.

**Proof.** Suppose that \( M/\varphi_t \) is not Hausdorff. Then there exist points \( p \) and \( q \) of \( M \) contained in distinct orbits such that, for any neighborhoods \( U \) and \( V \) of \( p \) and \( q \) respectively, the saturation of \( U = \bigcup_{t \in \mathbb{R}} \varphi_t(U) \) intersects \( V \). Thus there are point sequences \( \{p_n\} \) and \( \{q_n\} \) converging to \( p \) and \( q \) respectively satisfying \( \varphi_{t_n}(p_n) = q_n \) for some numbers \( t_n \). By taking a subsequence of \( \{t_n\} \), we can assume that the \( t_n \) are either all positive or all negative. If they are all negative, then we exchange the roles of \( p \) and \( q \). Thus we can further assume that the \( t_n \) are all positive. If \( \varphi_t \) is strongly Hausdorff, then there is a number \( t \) satisfying \( \varphi_t(p) = q \). However, this contradicts the fact that \( p \) and \( q \) are not contained in the same orbit. Thus every strongly Hausdorff flow is Hausdorff.

For any Hausdorff flow, every orbit is a closed set because it is the inverse image of one point with respect to the quotient map \( \pi \). In case of a compact manifold, every orbit is periodic and hence the flow is strongly Hausdorff because the notion of strongly Hausdorff flows coincides with that of Hausdorff flows if the orbits are all periodic. However, a Hausdorff flow is not always strongly Hausdorff in case of a non-compact manifold:

**Example 1.** Let \( \mathcal{F} \) be a foliation of \( S^3 \) tangent to the Hopf fibration. For any point \( p \) of \( S^3 \), the restriction of \( \mathcal{F} \) to \( S^3 - p \) induces a (complete) flow of \( \mathbb{R}^3 \) which is not strongly Hausdorff but Hausdorff (see Corollary of Lemma 2).

**Lemma 2.** For any strongly Hausdorff flow, the non-wandering set \( \Omega(\varphi_t) \) coincides with the set of periodic points.

**Proof.** Let \( p \) be a point of the non-wandering set of a strongly Hausdorff flow \( \varphi_t \), i.e. for any neighborhood \( U \) of \( p \) and any positive number \( T \), \( \bigcup_{t > T} \varphi_t(U) \) intersects \( U \). Let \( F : D^{m-1} \times I \to M \) (\( I = [-\varepsilon, \varepsilon], \varepsilon > 0 \)) be a flow box of \( \varphi_t \) around \( p \) (\( p = F(0, 0) \)) defined by \( F(x, t) = \varphi_t(F(x, 0)) \), where \( F(D^{m-1} \times \{0\}) \) is transverse to \( \varphi_t \). Since \( p \) is non-wandering, there exist point sequences \( \{p_n\} \) and \( \{q_n\} \) \( (n = 1, 2, \ldots) \) both converging to \( p \) and a sequence \( \{t_n\} \) such that \( \varphi_{t_n}(p_n) = q_n \) and \( t_n > 4\varepsilon \). For sufficiently large \( n \), both \( p_n \) and \( q_n \) are contained in \( F(D^{m-1} \times I) \). Denote by \( (x_n, t_n) \in D^{m-1} \times I \) (resp. \( (x'_n, t'_n) \in D^{m-1} \times I \)) the inverse image of \( p_n \) (resp. \( q_n \)) with respect to \( F \). Since \( \{p_n\} \) (resp. \( \{q_n\} \)) converges to \( p \), \( \{F(x, \varepsilon)\} \) (resp. \( \{F(x', -\varepsilon)\} \)) converges to \( F(0, \varepsilon) \) (resp. \( F(0, -\varepsilon) \)). By the definition of a strongly Hausdorff flow, there exists a positive number \( t \) such that \( \varphi_t(F(0, \varepsilon)) = F(0, -\varepsilon) \).
Since \( F(0, -\varepsilon) \) is connected with \( F(0, \varepsilon) \) by the orbit contained in the flow box, \( p \) is a periodic point.

By definition, the periodic points are non-wandering. Thus \( \Omega(\varphi_t) \) is the set of periodic points.

Since the non-wandering set is closed by definition, the following corollary holds:

**Corollary.** For any strongly Hausdorff flow, the set of periodic points is closed.

3. Periodic flows. First we define the generalized Seifert fibration precisely (see [1]). Let \( K \) be a finite subgroup of the orthogonal group \( O(m-1) \), and \( \psi : \pi_1(S^1) \to K \) a surjective homomorphism. By using the covering map from \( S^1 \) to \( S^1 \) corresponding to the kernel of \( \psi \), we obtain an action of \( K \) on \( S^1 \) by \( k \cdot s = \gamma \cdot s \) \((k \in K, s \in S^1)\) where \( \psi(\gamma) = k \) and \( (s \mapsto \gamma \cdot s) \) is the covering transformation. Let \( N \) be the quotient space of \( D^{m-1} \times S^1 \) obtained by identifying \((z, k \cdot s) \) with \((k \cdot z, s) \) for \( z \in D^{m-1}, k \in K, s \in S^1 \). Denote by \( q \) the quotient map. Then \( N \) is foliated by circles of the form \( q(z, S^1) \) \((z \in D^{m-1})\). A **generalized Seifert fibration** of an \( m \)-dimensional manifold is a foliation by circles such that each circle has a saturated neighborhood diffeomorphic to the above local model on \( N \).

Next we prove the main theorem by using the following Lemma 3, which can be shown in the same way as in the proof of Epstein’s theorem ([1], Theorem 4.1).

**Lemma 3.** The set of periodic points of any Hausdorff flow is an open set.

**Proof.** Let \( K \) be a compact neighborhood of a periodic orbit \( O \). Denote by \( \partial K \) the boundary of \( K \). We claim that \( K - \pi^{-1}(\partial K) \) is an open neighborhood of \( O \) consisting of periodic points.

Since \( \partial K \) is compact and \( M/\varphi_t \) is Hausdorff, \( \pi^{-1}(\partial K) \) is closed. Hence \( K - \pi^{-1}(\partial K) = \text{int} K - \pi^{-1}(\partial K) \) is an open set containing \( O \).

Let \( p \) be an arbitrary point of \( K - \pi^{-1}(\partial K) \). The orbit passing through \( p \) does not intersect \( \partial K \). Hence the orbit is entirely contained in \( K \). As we stated above, the orbit of any Hausdorff flow is a closed set. Since the orbit passing through \( p \) is contained in the compact set \( K \), it is also compact, and hence periodic (see Corollary (2.36) of [2], [3]).

**Proof of Theorem.** By Lemmas 1–3, the orbits of any strongly Hausdorff flow are all periodic if its non-wandering set is not empty. Then the manifold admits a generalized Seifert fibration whose fibers consist of the periodic orbits by Epstein’s Theorem 4.3 of [1].

Suppose that the non-wandering set is empty. Then, for any point \( p \in M \), there are a neighborhood \( U \) and a positive number \( T \) such that \( U \) does not
intersect $\bigcup_{t \geq T} \phi_t(U)$. Since $\phi_t$ has no periodic points, we can construct an embedding $\tilde{F} : D^{m-1} \times [0, T] \to M$ satisfying $F(z, t) = \phi_t(F(z, 0))$ and $F(D^{m-1} \times \{0\}) \subset U$ by joining flow boxes. We define the extension $\tilde{F} : D^{m-1} \times \mathbb{R} \to M$ by $\tilde{F}(z, t) = \phi_t(F(z, 0))$. Then $\tilde{F}$ is an embedding because, if not, there are distinct $z_1, z_2 \in D^{m-1}$ and $s \geq 0$ such that $\phi_s(F(z_1, 0)) = F(z_2, 0)$, which contradicts the fact that either $\bigcup_{t \geq T} \phi_t(U)$ is disjoint from $U$ or $F$ is an embedding. Therefore the orbit space $M/\phi_t$ is a (Hausdorff) manifold and $\pi$ is a fiber bundle. Since any orientable $\mathbb{R}$-bundle is trivial, the fiber bundle $\pi$ is trivial. Thus the flow $\phi_t$ is conjugate to the product flow of $(M/\phi_t) \times \mathbb{R}$.

Remark. As Vogt stated in his paper [4], no Euclidean space admits a Seifert fibration. In particular, any strongly Hausdorff flow of $\mathbb{R}^3$ is conjugate to the product flow. However, a Hausdorff flow of $\mathbb{R}^3$ is not always conjugate to the product flow even if it has no periodic orbits. For example, Marin showed that the flow of $\mathbb{R}^3$ induced from the vector field $X(x, y, z) = (-2xy, x^2 - y^2, 4 - x^2)$ is a Hausdorff flow without periodic orbits whose non-wandering set is not empty. In fact, this flow is tangent to the open cylinders $\{(x, y, z) : (x - r)^2 + y^2 = r^2\}$ containing the $z$-axis, on which the flow is a so-called slope component. Hence this flow is Hausdorff. On the other hand, the orbit passing near the origin and contained in a larger open cylinder comes back more closely to the origin. Thus this flow has a non-wandering point.

References


FACULTY OF INTEGRATED ARTS AND SCIENCES
HIROSHIMA UNIVERSITY
HIGASHI-HIROSHIMA, 739, JAPAN
E-mail: NAKAYAMA@MIS.HIROSHIMA-U.AC.JP

Received 25 February 1995;
in revised form 20 October 1995