FUNDAMENTA
MATHEMATICAE
149 (1996)

On strongly Hausdorff flows

by

Hiromichi Nakayama (Hiroshima)

Abstract. A flow of an open manifold is very complicated even if its orbit space is Hausdorff. In this paper, we define the strongly Hausdorff flows and consider their dynamical properties in terms of the orbit spaces. By making use of this characterization, we finally classify all the strongly Hausdorff C^1-flows.

1. Introduction. Let M be a connected manifold (maybe open), and φ_t a non-singular flow of M. The flow φ_t is called *strongly Hausdorff* if, for any point sequences $\{p_n\}$ and $\{q_n\}$ $(n = 1, 2, \ldots)$ converging to p and q respectively and satisfying $\varphi_{t_n}(p_n) = q_n$ for some positive numbers t_n, there is a number t $(t \geq 0)$ such that $\varphi_t(p) = q$. The product flow of a product manifold $N \times \mathbb{R}$, defined by $\varphi_t(z, s) = (z, s + t)$ $(z \in N,\ s, t \in \mathbb{R})$, is an example of a strongly Hausdorff flow. Furthermore, a flow tangent to a generalized Seifert fibration (see §3) is also an example of a strongly Hausdorff flow whose orbits are all periodic. The following main theorem shows that any strongly Hausdorff flow turns out to be one of the above examples up to conjugation, where two flows φ_t and ψ_t are *conjugate* if there exists a diffeomorphism f such that $f\varphi_t f^{-1} = \psi_t$ for any $t \in \mathbb{R}$.

THEOREM. *Every strongly Hausdorff C^1-flow is either conjugate to a product flow or tangent to a generalized Seifert fibration.*

We prove this theorem in §3. In §2, we study the dynamical properties of strongly Hausdorff flows: the relation between strongly Hausdorff flows and Hausdorff flows, and the coincidence of the non-wandering set and the set of periodic points for strongly Hausdorff flows.

The author wishes to thank A. Marin for his helpful comments.

2. Strongly Hausdorff flows. Let M be a connected m-dimensional manifold, and φ_t a non-singular flow of M. Denote by π the quotient map

1991 *Mathematics Subject Classification*: Primary 58F25; Secondary 58F18.

which maps each orbit to a point, and by M/φ_t the quotient space, which is called the *orbit space*. A flow φ_t is *Hausdorff* if the orbit space M/φ_t is Hausdorff with respect to the quotient topology. The notion of strongly Hausdorff flows is stronger than that of Hausdorff flows by the following lemma:

LEMMA 1. *Every strongly Hausdorff flow is Hausdorff.*

Proof. Suppose that M/φ_t is not Hausdorff. Then there exist points p and q of M contained in distinct orbits such that, for any neighborhoods U and V of p and q respectively, the saturation of U $(= \bigcup_{t\in\mathbb{R}} \varphi_t(U))$ intersects V. Thus there are point sequences $\{p_n\}$ and $\{q_n\}$ converging to p and q respectively satisfying $\varphi_{t_n}(p_n) = q_n$ for some numbers t_n. By taking a subsequence of $\{t_n\}$, we can assume that the t_n are either all positive or all negative. If they are all negative, then we exchange the roles of p and q. Thus we can further assume that the t_n are all positive. If φ_t is strongly Hausdorff, then there is a number t satisfying $\varphi_t(p) = q$. However, this contradicts the fact that p and q are not contained in the same orbit. Thus every strongly Hausdorff flow is Hausdorff. ∎

For any Hausdorff flow, every orbit is a closed set because it is the inverse image of one point with respect to the quotient map π. In case of a compact manifold, every orbit is periodic and hence the flow is strongly Hausdorff because the notion of strongly Hausdorff flows coincides with that of Hausdorff flows if the orbits are all periodic. However, a Hausdorff flow is not always strongly Hausdorff in case of a non-compact manifold:

EXAMPLE 1. Let $\mathcal{F}$ be a foliation of S^3 tangent to the Hopf fibration. For any point p of S^3, the restriction of $\mathcal{F}$ to $S^3 - p$ induces a (complete) flow of $\mathbb{R}^3$ which is not strongly Hausdorff but Hausdorff (see Corollary of Lemma 2).

LEMMA 2. *For any strongly Hausdorff flow, the non-wandering set $\Omega(\varphi_t)$ coincides with the set of periodic points.*

Proof. Let p be a point of the non-wandering set of a strongly Hausdorff flow φ_t, i.e. for any neighborhood U of p and any positive number T, $\bigcup_{t\ge T} \varphi_t(U)$ intersects U. Let $F : D^{m-1} \times I \to M$ $(I = [-\varepsilon, \varepsilon],\ \varepsilon > 0)$ be a flow box of φ_t around p $(p = F(0,0))$ defined by $F(x,t) = \varphi_t(F(x,0))$, where $F(D^{m-1} \times \{0\})$ is transverse to φ_t. Since p is non-wandering, there exist point sequences $\{p_n\}$ and $\{q_n\}$ $(n = 1,2,\ldots)$ both converging to p and a sequence $\{t_n\}$ such that $\varphi_{t_n}(p_n) = q_n$ and $t_n > 4\varepsilon$. For sufficiently large n, both p_n and q_n are contained in $F(D^{m-1}\times I)$. Denote by $(x_n, t_n) \in D^{m-1}\times I$ (resp. $(x'_n, t'_n) \in D^{m-1} \times I$) the inverse image of p_n (resp. q_n) with respect to F. Since $\{p_n\}$ (resp. $\{q_n\}$) converges to p, $\{F(x_n, \varepsilon)\}$ (resp. $\{F(x'_n, -\varepsilon)\}$) converges to $F(0, \varepsilon)$ (resp. $F(0, -\varepsilon)$). By the definition of a strongly Hausdorff flow, there exists a positive number t such that $\varphi_t(F(0, \varepsilon)) = F(0, -\varepsilon)$.

Since $F(0, -\varepsilon)$ is connected with $F(0, \varepsilon)$ by the orbit contained in the flow box, p is a periodic point.

By definition, the periodic points are non-wandering. Thus $\Omega(\varphi_t)$ is the set of periodic points. ∎

Since the non-wandering set is closed by definition, the following corollary holds:

COROLLARY. *For any strongly Hausdorff flow, the set of periodic points is closed.*

3. Periodic flows. First we define the generalized Seifert fibration precisely (see [1]). Let K be a finite subgroup of the orthogonal group $O(m-1)$, and $\psi : \pi_1(S^1) \to K$ a surjective homomorphism. By using the covering map from S^1 to S^1 corresponding to the kernel of ψ, we obtain an action of K on S^1 by $k \cdot s = \gamma \cdot s$ $(k \in K,\ s \in S^1)$ where $\psi(\gamma) = k$ and $(s \mapsto \gamma \cdot s)$ is the covering transformation. Let N be the quotient space of $D^{m-1} \times S^1$ obtained by identifying $(z, k \cdot s)$ with $(k \cdot z, s)$ for $z \in D^{m-1}$, $k \in K$, $s \in S^1$. Denote by q the quotient map. Then N is foliated by circles of the form $q(z, S^1)$ $(z \in D^{m-1})$. A *generalized Seifert fibration* of an m-dimensional manifold is a foliation by circles such that each circle has a saturated neighborhood diffeomorphic to the above local model on N.

Next we prove the main theorem by using the following Lemma 3, which can be shown in the same way as in the proof of Epstein's theorem ([1], Theorem 4.1).

LEMMA 3. *The set of periodic points of any Hausdorff flow is an open set.*

Proof. Let K be a compact neighborhood of a periodic orbit O. Denote by ∂K the boundary of K. We claim that $K - \pi^{-1}\pi(\partial K)$ is an open neighborhood of O consisting of periodic points.

Since ∂K is compact and M/φ_t is Hausdorff, $\pi^{-1}\pi(\partial K)$ is closed. Hence $K - \pi^{-1}\pi(\partial K) = \operatorname{int} K - \pi^{-1}\pi(\partial K)$ is an open set containing O.

Let p be an arbitrary point of $K - \pi^{-1}\pi(\partial K)$. The orbit passing through p does not intersect ∂K. Hence the orbit is entirely contained in K. As we stated above, the orbit of any Hausdorff flow is a closed set. Since the orbit passing through p is contained in the compact set K, it is also compact, and hence periodic (see Corollary (2.36) of [2], [3]). ∎

Proof of Theorem. By Lemmas 1–3, the orbits of any strongly Hausdorff flow are all periodic if its non-wandering set is not empty. Then the manifold admits a generalized Seifert fibration whose fibers consist of the periodic orbits by Epstein's Theorem 4.3 of [1].

Suppose that the non-wandering set is empty. Then, for any point $p \in M$, there are a neighborhood U and a positive number T such that U does not intersect $\bigcup_{t\geq T}\varphi_t(U)$. Since φ_t has no periodic points, we can construct an embedding $F : D^{m-1}\times[0,T]\to M$ satisfying $F(z,t)=\varphi_t(F(z,0))$ and $F(D^{m-1}\times\{0\})\subset U$ by joining flow boxes. We define the extension $\widetilde{F} : D^{m-1}\times\mathbb{R}\to M$ by $\widetilde{F}(z,t)=\varphi_t(F(z,0))$. Then $\widetilde{F}$ is an embedding because, if not, there are distinct $z_1,z_2\in D^{m-1}$ and $s\geq 0$ such that $\varphi_s(F(z_1,0))=F(z_2,0)$, which contradicts the fact that either $\bigcup_{t\geq T}\varphi_t(U)$ is disjoint from U or F is an embedding. Therefore the orbit space M/φ_t is a (Hausdorff) manifold and π is a fiber bundle. Since any orientable $\mathbb{R}$-bundle is trivial, the fiber bundle π is trivial. Thus the flow φ_t is conjugate to the product flow of $(M/\varphi_t)\times\mathbb{R}$. ∎

Remark. As Vogt stated in his paper [4], no Euclidean space admits a Seifert fibration. In particular, any strongly Hausdorff flow of $\mathbb{R}^3$ is conjugate to the product flow. However, a Hausdorff flow of $\mathbb{R}^3$ is not always conjugate to the product flow even if it has no periodic orbits. For example, Marin showed that the flow of $\mathbb{R}^3$ induced from the vector field $X(x,y,z)=(-2xy,x^2-y^2,4-x^2)$ is a Hausdorff flow without periodic orbits whose non-wandering set is not empty. In fact, this flow is tangent to the open cylinders $\{(x,y,z):(x-r)^2+y^2=r^2\}$ containing the z-axis, on which the flow is a so-called slope component. Hence this flow is Hausdorff. On the other hand, the orbit passing near the origin and contained in a larger open cylinder comes back more closely to the origin. Thus this flow has a non-wandering point.

References

[1] D. B. A. Epstein, *Foliations with all leaves compact*, Ann. Inst. Fourier (Grenoble) 26 (1) (1976), 265–282.

[2] M. C. Irwin, *Smooth Dynamical Systems*, Academic Press, London, 1980.

[3] H. Nakayama, *Some remarks on non-Hausdorff sets for flows*, in: Geometric Study of Foliations, World Scientific, Singapore, 1994, 425–429.

[4] E. Vogt, *A foliation of* $\mathbb{R}^3$ *and other punctured* 3*-manifolds by circles*, I.H.E.S. Publ. Math. 69 (1989), 215–232.

FACULTY OF INTEGRATED ARTS AND SCIENCES
HIROSHIMA UNIVERSITY
HIGASHI-HIROSHIMA, 739, JAPAN
E-mail: NAKAYAMA@MIS.HIROSHIMA-U.AC.JP

Received 25 February 1995;
in revised form 20 October 1995