The Arkhangel'skiĭ–Tall problem:
a consistent counterexample

by

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Abstract. We construct a consistent example of a normal locally compact metacompact space which is not paracompact, answering a question of A. V. Arkhangel'skiĭ and F. Tall. An interplay between a tower in $\mathcal{P}(\omega)/\mathrm{Fin}$, an almost disjoint family in $[\omega]^{\omega}$, and a version of an $(\omega, 1)$-morass forms the core of the proof. A part of the poset which forces the counterexample can be considered a modification of a poset due to Judah and Shelah for obtaining a Q-set by a countable support iteration.

0. Introduction. In 1971, A. V. Arkhangel'skiĭ [A] proved that every perfectly normal, locally compact, metacompact space is paracompact. This suggests the question, stated in print by Arkhangel'skiĭ (see [AP], Chapter 5, p. 309) and Tall [T] three years later, and oft-repeated since then, whether "perfectly normal" can be reduced to "normal":

PROBLEM. *Is every normal, locally compact, metacompact space paracompact*?

Recall that a space is *metacompact* if every open cover has a point-finite open refinement. Standard topological arguments show that if there is a counterexample to the problem, then there is one which is not collectionwise Hausdorff (CWH). Bing's famous Example G [Bi] is a ZFC example of a normal space which is not CWH, and Michael's metacompact subspace of this example (see [Mi]), which is not locally compact, shows that the assumption of local compactness is essential for this problem.

S. Watson [W$_1$] obtained the first consistency result on the problem: he showed that in Gödel's constructible universe L, normal locally compact spaces are collectionwise Hausdorff, and so $V = L$ implies that the answer is "yes". The answer is also positive in a model obtained by adding supercompact many Cohen or random reals, because there normal locally compact spaces are collectionwise normal [B]. P. Daniels [D] showed the answer is positive in ZFC if "metacompact" is strengthened to "boundedly metacompact", i.e., every open cover has an open refinement such that for some positive integer n each point is in at most n elements of the refinement. In a forthcoming paper [GK], the authors show that $\mathrm{MA}_{\sigma\text{-centered}}(\omega_1)$ also implies a positive answer to the problem, and that $\mathrm{MA}(\omega_1)$ implies that the answer is positive even if "metacompact" is weakened to "metalindelöf" (i.e., every open cover has a point-countable open refinement).

In this paper we complete the solution to the problem by showing that, if ZFC is consistent, then it is consistent with ZFC that there is a normal locally compact metacompact space which is not paracompact. This result, as well as the aforementioned MA results, had its origins in the study of a paper of Watson [W$_2$] in which he constructs consistent examples of normal locally compact metalindelöf spaces which are not paracompact.

The rough idea of Watson's examples is to take a normal locally compact non-collectionwise Hausdorff space of the form $D \cup I$, where D is a closed discrete set and I is a set of isolated points (e.g., the Cantor tree space over a Q-set), replace the isolated points with copies of suitably chosen compact spaces, and define a neighborhood of a point in $d \in D$ to be a tail of a suitably chosen sequence of clopen sets in the compact spaces corresponding to a sequence of isolated points converging to d.

By deciding in advance that the space of the form $D \cup I$ that we are going to use is a space obtained in a standard way from an almost disjoint family of subsets of ω, so $I = \omega$ for our example, and also deciding that we will replace an isolated point n with the space $\omega_1^{\leq n}$ of sequences of length $\leq n$ of countable ordinals endowed with a natural compact Hausdorff topology, we are able to reduce the problem of obtaining a counterexample to a certain combinatorial statement. This is done in Section 1. In Section 2 we prove these combinatorics relatively consistent with ZFC.

Our proof is rather lengthy, so we give here a brief description of the key ideas. The set for our example is $\omega_1 \cup \bigcup_{n<\omega} K_n$, where K_n is a copy of $\omega_1^{\leq n}$. The K_n's are disjoint clopen subsets of the example, and are given the compact Hausdorff topology generated by sets of the form

$$V(\sigma) = \{\tau \in \omega_1^{\leq n} : \sigma \subseteq \tau\}$$

and their complements. Each $\alpha \in \omega_1$ is associated with a certain $X_\alpha \subset \omega$ such that the collection $\mathcal{X} = \{X_\alpha : \alpha < \omega_1\}$ is almost disjoint. To define

a neighborhood of the point α in the closed discrete set ω_1, we first choose $\sigma_{\alpha,n} \in \omega_1^{\leq n}$ for $n \in X_\alpha$. Then the kth basic (cl)open neighborhood of α is $\{\alpha\} \cup \bigcup_{n \in X_\alpha \setminus k} V(\sigma_{\alpha,n})$, where we consider $V(\sigma_{\alpha,n})$ as a subset of K_n. The space is Hausdorff because the X_α's are almost disjoint. Since the $V(\sigma)$'s are compact and point-finite, the resulting space is locally compact and metacompact. The set ω_1 is an uncountable closed discrete subset of the space. Since the space has a σ-compact dense subset, every locally finite collection of open sets must be countable, and it follows that the space cannot be paracompact.

So it remains to make the space normal somehow. We get the closed discrete set ω_1 to be normalized (i.e., every pair of disjoint subsets of it can be put into disjoint open sets) by making $\mathcal{X}$ "separated" in the following sense:

$$(*) \quad \forall A \subset \omega_1 \; \exists u \subset \omega \; [\forall \alpha \in A \; (X_\alpha \subset^* u) \,\&\, \forall \alpha \in \omega_1 \setminus A \; (X_\alpha \cap u =^* \emptyset)].$$

Separated families can easily be obtained from a Q-set (see, e.g., [vD]), and in fact condition $(*)$ implies that $\mathcal{X}$ is a Q-set in the Cantor set 2^ω, where a subset of ω is identified with its characteristic function.

Having the closed discrete set normalized is not enough for normality of the space, however. We must also be able to separate pairs H, K of disjoint closed sets where $H = \omega_1$. This normality turns out to be the more complicated one to analyze and obtain. The way we choose the sequences $\sigma_{\alpha,n}$ in $\omega_1^{\leq n}$ is important: we use a family $\mathcal{F} = \bigcup_{n<\omega} \mathcal{F}_n$ of finite subsets of ω_1 having the following properties (among others... —see Definition 5; $\mathcal{F}$ may be considered a weakening of Velleman's $(\omega, 1)$-morass [V]):

   (i) $\mathcal{F}$ is directed and cofinal in $([\omega_1]^{<\omega}, \subseteq)$.
   (ii) $|F| = n$ for every $F \in \mathcal{F}_n$.
   (iii) If $F, G \in \mathcal{F}_n$ and $\alpha \in F \cap G$, then $F \cap \alpha = G \cap \alpha$.
   (iv) If $F \in \mathcal{F}_n$, $G \in \mathcal{F}_m$, $\alpha \in F \cap G$, and $n < m$, then $\alpha \cap F \subseteq \alpha \cap G$.

We then define $\sigma_{\alpha,n}$ to be the increasing enumeration of $F \cap (\alpha + 1)$, where $\alpha \in F \in \mathcal{F}_n$ (if there exists $F \in \mathcal{F}_n$ containing α). It follows from this definition that:

   (a) If $\beta < \alpha$ and both $\sigma_{\beta,n}$ and $\sigma_{\alpha,n}$ are defined, then $\sigma_{\beta,n} \subset \sigma_{\alpha,n}$.
   (b) The function $m(\cdot, \alpha) : \alpha \to \omega$, where $m(\beta, \alpha)$ is the least m such that β and α are both contained in some $F \in \mathcal{F}_m$, is finite-to-one.
   (c) If $n \geq m(\beta, \alpha)$ and $\alpha \in \bigcup \mathcal{F}_n$, then both $\sigma_{\beta,n}$ and $\sigma_{\alpha,n}$ are defined and $\sigma_{\beta,n} \subset \sigma_{\alpha,n}$.
   (d) If $\beta < \delta < \alpha$ and $k \geq \max\{m(\beta, \delta), m(\delta, \alpha)\}$, then $\alpha \in F \in \mathcal{F}_k$ implies $\beta, \delta \in F$.

The properties of this "coloring" m of $[\omega_1]^2$ are reminiscent of some colorings of Todorčević (see [To]). The advantage of using the above method of choosing the $\sigma_{\alpha,n}$'s is that we can now state just in terms of m and the

almost disjoint family $\mathcal{X}$ the final combinatorial condition needed to make the space normal (see Proposition 1 and the Reduction Lemma 9):

$$(**) \qquad \forall A \subseteq \omega_1 \ \exists B \in [A]^\omega \ \forall \alpha \in A \left[X_\alpha \subseteq^* \bigcup_{\beta \in B} (X_\beta - m(\beta, \alpha)) \right].$$

If one translates condition $(**)$ to see what it means for the topology of the space, it says (a bit more than) "given any collection of neighborhoods of some subset A of the closed discrete set ω_1, some countable subcollection covers a neighborhood of every point (except for the point itself) of A". So it is a kind of "hereditarily Lindelöf" property. The finite-to-one function also enables one to "chop off" the countable subcollection so that only finitely many meet any fixed K_n, hence the union has no limit points outside of the closed discrete set. It is just what is needed for separating pairs of closed sets, one of which is contained in the closed discrete set and the other disjoint from it.

Our task then is to build an almost disjoint collection $\mathcal{X}$ and a collection $\mathcal{F}$ of finite subsets of ω_1 so that $(*)$ and $(**)$ are satisfied. Now for $(*)$ essentially we need a Q-set, but it turns out that the usual methods for producing a Q-set destroy $(**)$. For example, $\mathrm{MA}(\omega_1)$ destroys $(**)$, and so does any uncountable finite support iterated forcing. So we turn to a method due to Judah and Shelah [JS] for forcing a Q-set by a countable support iteration. The first stage of the Judah–Shelah poset is a countably closed poset adding a tower (i.e., an almost increasing family) $\{Y_\alpha : \alpha < \omega_1\}$ of subsets of ω. Then the "rings" $X_\alpha = Y_{\alpha+1} \setminus Y_\alpha$ of the tower form an almost disjoint family. This first stage is followed by an ω_2-stage iteration, each factor of which "separates" this almost disjoint family with respect to some subset A of ω_1 that has appeared by that stage, and so that after ω_2 steps the family has property $(*)$. Assuming CH, the poset is proper and ω_2-c.c., so cardinals are preserved. The continuum is ω_2 in the extension.

An $(\omega, 1)$-morass, and hence a collection $\mathcal{F}$ satisfying the conditions (i)–(iv) above, exists in ZFC. However, using an $\mathcal{F}$ from the ground model seems not to give us enough control to prove $(**)$. So we add to the first coordinate of the Judah–Shelah poset another factor which adds a generic $\mathcal{F}$. Finally, it turns out that the growth of the functions $g_\alpha : X_\alpha \to \omega$, where $g_\alpha(n)$ is the length of $\sigma_{\alpha,n}$, needs to be controlled. By (ii) in the list of properties of $\mathcal{F}$, $g_\alpha(n) \leq n$. If nothing is done about it, these g_α's will be unbounded below the identity, and we can show that this destroys $(**)$ (in this particular forcing extension...; we know of no intrinsic reason why this should be true in general). So we add a third coordinate to the first factor of the forcing which essentially makes the g_α's the left half of a Hausdorff gap. This turns out to give us just enough control over everything to prove that $(**)$ holds in the final model.

1. Reduction to combinatorics

PROPOSITION 1. *Suppose we have an almost disjoint family $\{X_\alpha : \alpha \in \omega_1\}$ of infinite subsets of ω, and for each $\alpha \in \omega_1$ and $n \in X_\alpha$ we have an assigned $\sigma_{\alpha,n} \in \omega_1^{\leq n}$ such that the following conditions hold*:

(i) *For every $A \subseteq \omega_1$, there is $u \subseteq \omega$ such that $u \cap X_\alpha$ is finite if $\alpha \notin A$, and $X_\alpha - u$ is finite if $\alpha \in A$.*

(ii) *For every $n \in \omega$ and $\sigma \in \omega_1^{\leq n}$, the set $\{\alpha : \sigma_{\alpha,n} = \sigma\}$ is finite.*

(iii) *For every $A \subseteq \omega_1$ there is a countable $B \subseteq A$ and a finite-to-one function $f : B \to \omega$ such that for every $\alpha \in A$, for sufficiently large $n \in X_\alpha$, there is $\beta \in B$ with $n \in X_\beta - f(\beta)$ and $\sigma_{\alpha,n} \supseteq \sigma_{\beta,n}$.*

Then there is a normal locally compact metacompact space which is not paracompact.

P r o o f. First we define a compact Hausdorff topology on the set $\omega_1^{\leq n}$. If $\sigma \in \omega_1^{\leq n}$, let $V(\sigma) = \{\tau \in \omega_1^{\leq n} : \tau \supseteq \sigma\}$. The $V(\sigma)$'s and their complements form a subbase for a Hausdorff topology on $\omega_1^{\leq n}$.

We show that $\omega_1^{\leq n}$ is compact. First, $\omega_1^{\leq 0}$ is a single point. If $\alpha \in \omega_1$, it is easy to see that the clopen set $V(\langle\alpha\rangle)$ in $\omega_1^{\leq n+1}$ is a copy of $\omega_1^{\leq n}$. Also, every neighborhood of the empty sequence in $\omega_1^{\leq n+1}$ contains all but finitely many $V(\langle\alpha\rangle)$'s. So $\omega_1^{\leq n+1}$ is the one-point compactification of ω_1-many copies of $\omega_1^{\leq n}$. It follows by induction that each $\omega_1^{\leq n}$ is compact.

Let us note that each $V(\sigma)$ is a clopen, hence compact, subset of $\omega_1^{\leq n}$. Also, $\tau \in V(\sigma)$ if and only if $\sigma \subseteq \tau$, so the collection of all $V(\sigma)$'s is point-finite.

Now we define the space X. Let K_n be a copy of $\omega_1^{\leq n}$ such that $K_0, K_1, K_2, \dots$ are disjoint. The set for X is $\omega_1 \cup \bigcup_{n \in \omega} K_n$. Each K_n is a clopen subspace of X with the topology described above. The kth neighborhood of the point $\alpha \in \omega_1$ is the set

$$U(\alpha, k) = \{\alpha\} \cup \bigcup \{V(\sigma_{\alpha,n}) : n \in X_\alpha - k\},$$

where by $V(\sigma_{\alpha,n})$ we mean the copy of $\{\tau \in \omega_1^{\leq n} : \tau \supseteq \sigma_{\alpha,n}\}$ in K_n.

The space X is clearly locally compact, and ω_1 is a closed discrete subset of X. We prove X is not paracompact. Suppose it were. Then ω_1 would have a discrete separation in X, so there would exist a function $g : \omega_1 \to \omega$ such that $\{U(\alpha, g(\alpha)) : \alpha < \omega_1\}$ is a closed discrete collection. For each α, find $n(\alpha) \in X_\alpha - g(\alpha)$. There is $n \in \omega$ and an infinite subset W of ω_1 with $n(\alpha) = n$ for every $\alpha \in W$. Then $\{V(\sigma_{\alpha,n}) : \alpha \in W\}$ is an infinite closed discrete collection of sets in the compact space K_n, contradiction.

We show X is metacompact. Let $\mathcal{U}$ be any open cover of X. For each $\alpha < \omega_1$, let $g(\alpha) \in \omega$ be such that $U(\alpha, g(\alpha))$ is contained in some member

of $\mathcal{U}$. If $\sigma \in K_n \cap U(\alpha, g(\alpha))$, then $\sigma_{\alpha,n} \subseteq \sigma$. It follows that the collection $\mathcal{V} = \{U(\alpha, g(\alpha)) : \alpha < \omega_1\}$ is point-finite. For each $n \in \omega$, there is a finite collection $\mathcal{W}_n$ of clopen subsets of K_n refining $\mathcal{U}$ and covering K_n. Then $\mathcal{V} \cup \bigcup_{n \in \omega} \mathcal{W}_n$ is a point-finite clopen refinement of $\mathcal{U}$.

It remains to prove X is normal. Let H and J be disjoint closed sets. By a standard subtraction argument it suffices to show that H can be covered by countably many open sets whose closures miss J. Since K_n is a compact clopen subspace of X, there is a clopen subset of K_n containing $H \cap K_n$ and missing $J \cap K_n$. Thus it remains to cover $H \cap \omega_1$. Let $u \subseteq \omega$ be such that $X_\alpha \cap u$ is finite for every $\alpha \in H \cap \omega_1$, and $X_\alpha - u$ is finite for every $\alpha \notin H$. For each $\alpha \in H \cap \omega_1$, choose $g(\alpha) \in \omega$ such that $g(\alpha) \supseteq X_\alpha \cap u$ and $U(\alpha, g(\alpha)) \cap J = \emptyset$.

Let $A_m = \{\alpha \in H \cap \omega_1 : g(\alpha) = m\}$. We will finish the proof by showing that A_m is contained in an open set V whose closure misses J. Let $B \subseteq A_m$ and $f : B \to \omega$ be as in condition (iii) applied with $A = A_m$. For $\beta \in B$, let $h(\beta) = \max\{f(\beta), m\}$, and let $V = A_m \cup \bigcup\{U(\beta, h(\beta)) : \beta \in B\}$.

To show V is open, consider $\alpha \in A_m$. There is $k \in \omega - m$ such that, for every $n \in X_\alpha - k$, there is $\beta \in B$ with $n \in X_\beta - f(\beta)$ and $\sigma_{\alpha,n} \supseteq \sigma_{\beta,n}$. Thus the copy of $V(\sigma_{\alpha,n})$ in K_n is contained in the copy of $V(\sigma_{\beta,n})$, which in turn is contained in V (since $n \geq \max\{f(\beta), m\} = h(\beta)$). It follows that $U(\alpha, k) \subseteq V$.

Finally, we show that the closure of V misses J. Since $h(\beta) \geq m = g(\beta)$ for $\beta \in B$, it is clear that V misses J. Since f is finite-to-one on B, for fixed n only finitely many $U(\beta, h(\beta))$'s for $\beta \in B$ meet K_n. So $K_n \cap V$ is clopen. It remains to prove that if $\alpha \in J \cap \omega_1$, then α is not in the closure of V. There is $k \in \omega$ such that $X_\alpha - k \subseteq u$. By the definition of m and since $h(\beta) \geq m$ for $\beta \in B$, we have

$$\left(\bigcup\{X_\beta - h(\beta) : \beta \in B\}\right) \cap u = \emptyset.$$

It follows that $U(\alpha, k) \cap V = \emptyset$. ∎

2. Proving consistency. By Proposition 1, the following proposition completes the proof that it is consistent with ZFC for there to be an example of a normal, locally compact, metacompact, nonparacompact space.

PROPOSITION 2. *Assuming the consistency of ZFC, the following statement is consistent with ZFC: There is an almost disjoint family $\mathcal{X} = \{X_\alpha : \alpha < \omega_1\}$ of infinite subsets of ω and a family $\Sigma = \{\sigma_{\alpha,n} : \alpha \in \omega_1,\ n \in X_\alpha\}$ of finite sequences of countable ordinals such that*:

(i) *For every $A \subseteq \omega_1$ there is $u \subseteq \omega$ such that $u \cap X_\alpha$ is finite if $\alpha \notin A$, and $X_\alpha - u$ is finite if $\alpha \in A$.*

(ii) *Each $\sigma_{\alpha,n} \in \omega_1^{\leq n}$, and $\sigma_{\alpha,n} \neq \sigma_{\alpha',n'}$ for $\alpha \neq \alpha'$.*

(iii) *For every $A \subseteq \omega_1$ there is a countable $B \subseteq A$ and a finite-to-one function $f : B \to \omega$ such that for every $\alpha \in A$, for sufficiently large n in X_α there is $\beta \in B$ with $n \in X_\beta - f(\beta)$ and $\sigma_{\alpha,n} \supseteq \sigma_{\beta,n}$.*

We will call the property of $\mathcal{X}$ expressed in (i) the *Q-set property*, and the property of Σ and $\mathcal{X}$ expressed in (iii) the *hereditary Lindelöf property*.

The rest of this section is devoted to proving Proposition 2. This is done using the method of iterated forcing. In what follows we first define forcing notions we will be dealing with, then we begin establishing their properties and finally we prove that the families as in Proposition 2 exist in the generic extension obtained by using previously examined forcing notions.

We assume that the ground model satisfies CH. First we force with a countably closed forcing P. We call this forcing the initial forcing. This forcing has two groups of coordinates, the first one denoted by P_1. Then we force with an iteration with countable supports denoted by $Q_{\omega_2} = (Q_\alpha, Q^\alpha)_{\alpha < \omega_2}$. The entire forcing $P * Q_{\omega_2}$ will be denoted by R.

The first group of coordinates of P, which is itself a forcing notion denoted by P_1, forces the family $\{X_\alpha : \alpha < \omega_1\}$. The conditions are of the form $p = (\mathcal{X}_p, \mathcal{Y}_p, \alpha_p)$, where $\alpha_p < \omega_1$ is a limit ordinal, $\mathcal{X}_p = \{X_\alpha^p : \alpha < \alpha_p\}$, $\mathcal{Y}_p = \{Y_\alpha^p : \alpha < \alpha_p\}$ and the following hold:

(1) All X_α^p's and Y_α^p's are infinite and co-infinite subsets of ω.
(2) The X_α^p's form an almost disjoint family of subsets of ω.
(3) The Y_α^p's form a strictly almost increasing family of subsets of ω.
(4) For every $\beta < \alpha \leq \alpha_p$ we have $X_\beta \subseteq^* Y_\alpha$ and X_α is almost disjoint from Y_β for $\beta < \alpha < \alpha_p$ and $X_\alpha \cap Y_\alpha = \emptyset$.

The order is given by $p \leq q$ if and only if $\mathcal{X}_p \supseteq \mathcal{X}_q$, $\mathcal{Y}_p \supseteq \mathcal{Y}_q$, and $\alpha_p \geq \alpha_q$.

FACT 3. *(CH) P_1 is σ-closed and satisfies the ω_2-c.c.* ∎

DEFINITION 4. Suppose f, g are partial functions from ω into ω. Then $f <^+ g$ will mean that

$$\forall n < \omega \ \exists i \in \omega \ \forall j \in (\mathrm{dom}(g) \cap \mathrm{dom}(f) - i) \ [g(j) - f(j) > n].$$

We also let Id denote the identity function on ω.

Before defining the second group of coordinates, it will be convenient to define a weakening of the notion of an $(\omega, 1)$-morass (see [V]):

DEFINITION 5. Suppose that $s \in P_1$. A family $\mathcal{F} = \bigcup_{n < \omega} \mathcal{F}_n$ is called an *s-frame* if and only if the following conditions hold:

(1) If $a \in \mathcal{F}_n$, then $a \subset \alpha_s$ and $|a| = n$.
(2) $\mathcal{F}$ is directed.
(3) If $\beta \in a, b \in \mathcal{F}_n$ for some $n < \omega$, then $a \cap \beta = b \cap \beta$.

(4) If $a \in \mathcal{F}_n$, $b \in \mathcal{F}_m$, $\beta \in a \cap b$ and $n < m$ then $a \cap \beta \subseteq b \cap \beta$.

(5) For every $\beta < \alpha_s$ and every $n \in \omega - Y_\beta^s$, $\mathcal{F}_n$ covers β (by which we simply mean $\beta \in \bigcup \mathcal{F}_n$).

(6) For every $k \in \omega$ and $\beta < \alpha_s$ there is $m < \omega$ such that for all $n > m$ we have $n - |a \cap (\beta + 1)| > k$ for $a \in \mathcal{F}_n$ and $\beta \in a$.

If $\mathcal{F}$ is an s-frame and $\beta \in a \in \mathcal{F}_n$, we denote $a \cap (\beta + 1)$ by $a_\beta(\mathcal{F})(n)$. In this case we also denote $|a_\beta(\mathcal{F})(n)|$ by $\eta_\beta(\mathcal{F})(n)$.

By $m(\beta_1, \beta_2)$ we mean the minimal integer m such that there is $a \in \mathcal{F}_m$ such that $\beta_1, \beta_2 \in a$ for $\beta_1, \beta_2 < \alpha_s$.

If $s_1, s_2 \in P_1$ and $s_1 \leq s_2$ and $\mathcal{F}^i$ are s_i-frames respectively, then we say that $\mathcal{F}^1$ is an *end-extension* of $\mathcal{F}^2$ if and only if $\mathcal{F}^1 \supseteq \mathcal{F}^2$ and

$$\forall n < \omega \ \forall a \in \mathcal{F}_n^1 \ \exists b \in \mathcal{F}_n^2 \ \exists \beta \in \alpha_{s_2} \ (a \cap \alpha_{s_2} = b \cap \beta).$$

FACT 6. *Let $s \in P_1$ and let $\mathcal{F}$ be an s-frame.*

(1) *If $\mathcal{F}_n$ covers β_1 and $\beta_1, \beta_2 < \alpha_s$ then the objects $m(\beta_1, \beta_2)$, $a_{\beta_1}(\mathcal{F})(n)$ and $\eta_{\beta_1}(\mathcal{F})(n)$ are well defined.*

(2) *For each $\beta < \alpha_s$ the sequence $\{a_\beta(\mathcal{F})(n) : \mathcal{F}_n \text{ covers } \beta\}$ is non-decreasing and cofinal in $[\beta]^{<\omega}$ and the partial function $\eta_\beta(\mathcal{F}) : \{n : \mathcal{F}_n \text{ covers } \beta\} \to \omega$ is nondecreasing (and has unbounded range if β is infinite); moreover, $\eta_\beta(\mathcal{F}) <^+ \mathrm{Id}$.*

(3) *For each $\beta_1 < \beta_2 < \alpha_s$ and $n \geq m(\beta_1, \beta_2)$, if $\mathcal{F}_n$ covers both β_1 and β_2, then $a_{\beta_2}(\mathcal{F})(n) \cap (\beta_1 + 1) = a_{\beta_1}(\mathcal{F})(n)$ and $\eta_{\beta_1}(\mathcal{F})(n) < \eta_{\beta_2}(\mathcal{F})(n)$.*

(4) *For every $\beta < \alpha_s$ and $k < \omega$, if $\mathcal{F}_n$ covers both β and $\beta + k$, then $\eta_\beta(\mathcal{F})(n) + k \geq \eta_{\beta+k}(\mathcal{F})(n)$.*

(5) *Suppose that $(s_i : i < \omega)$ is a decreasing sequence of conditions of P_1 and $(\mathcal{F}^i : i < \omega)$ is a sequence of respective s_i-frames. If for every $i < \omega$ the frame $\mathcal{F}^{i+1}$ end-extends the frame $\mathcal{F}^i$, then $\bigcup \mathcal{F}_i$ is a $(\bigcup_{i<\omega} \mathcal{X}_i, \bigcup_{i<\omega} \mathcal{Y}_i, \sup(\alpha_{s_i} : i < \omega))$-frame which end-extends all frames $\mathcal{F}^i$.*

P r o o f. (1) For β_1 covered by $\mathcal{F}_n$ and $\beta_1, \beta_2 < \alpha_s$ the existence of $a_{\beta_1}(\mathcal{F})(n)$ and so of $\eta_{\beta_1}(\mathcal{F})(n)$ follows from the definition, and their uniqueness from Definition 5(3). Now $m(\beta_1, \beta_2)$ is well defined because there are $b_1, b_2 \in \mathcal{F}$ such that $\beta_1 \in b_1$ and $\beta_2 \in b_2$ by 5(5) and then there is $a \in \mathcal{F}$ such that $b_1 \cup b_2 \subseteq a$ by 5(2).

(2) The nondecreasingness follows from Definition 5(4). The cofinality is proved as follows: take $x \in [\beta]^{<\omega}$; by the directedness of $\mathcal{F}$ and 5(5), we will find $a \in \mathcal{F}$ such that $x, \{\beta\} \subseteq a$; now $x \subseteq a_\beta(\mathcal{F})(n)$ for n such that $a \in \mathcal{F}_n$. It is an easy consequence of the above that $\eta_\beta(\mathcal{F})$ is nondecreasing and unbounded if β is infinite. $\eta_\beta(\mathcal{F}) <^+ \mathrm{Id}$ follows from 5(6).

(3) Fix β_1, β_2 and n as in (3). By the definition of $m(\beta_1, \beta_2)$ there is $a \in \mathcal{F}_m$ such that $\beta_1, \beta_2 \in a$ and $m \leq n$, so by 5(4), we see that $\beta_1 \in a_{\beta_2}(\mathcal{F})(n)$

and now the first part of (3) follows from 5(3). The second part follows from the first.

(4) As $n \geq m(\beta, \beta + k)$, it follows from 5(3) that $a_\beta(\mathcal{F})(n)$ may differ from $a_{\beta+k}(\mathcal{F})(n)$ only by elements of the form $\beta + 1, \ldots, \beta + k$.

(5) Clear. ∎

R e m a r k. We introduced the notion of an s-frame so that the assumptions of the following lemma are relatively simple.

EXTENSION LEMMA 7. *Suppose that* $s = (\mathcal{X}_s, \mathcal{Y}_s, \alpha_s) \in P_1$ *and that* $\mathcal{F}$ *is an s-frame and* $(b_i, \beta_i, n_i)_{i<\omega}$ *is a sequence satisfying the following conditions*:

(i) $b_i \subseteq b_{i+1} \in [\alpha_s]^{<\omega}$ *for all* $i < \omega$.

(ii) *For each* $i < \omega$ *we have*

$$a_{\beta_i}(\mathcal{F})(n_i) = b_i.$$

(iii) *The sequence* $(\beta_i : i < \omega)$ *is nondecreasing and unbounded in* α_s.

(iv) *The sequence* $(n_i : i < \omega)$ *is increasing and unbounded in* ω *and almost disjoint from all* $(Y^s_\beta : \beta < \alpha_s)$.

(v) $(|b_i| : i < \omega) <^+ \mathrm{Id}$ *and* $(n_i - |b_i| : i < \omega)$ *is nondecreasing.*

Then there is an $s' \leq s$ *such that* $\alpha_{s'} = \alpha_s + \omega$ *and* $Y^{s'}_{\alpha_s} = \omega - \{n_i : i < \omega\}$ *and there is an s'-frame* $\mathcal{G}$ *such that* $\mathcal{G}$ *is an end-extension of* $\mathcal{F}$ *and for every* $i > i_0$,

$$a_\alpha(\mathcal{G})(n_i) = b_i \cup \{\alpha\},$$

where i_0 *is such that for* $i > i_0$ *we have* $n_i - |\beta_i| > 0$.

P r o o f. Let $s' \in P_1$ be an extension of s such that $Y^{s'}_{\alpha_s} = \omega - \{n_i : i < \omega\}$, and each of the $Y^{s'}_{\alpha_s+k}$'s strictly includes $\omega - \{n_i : i < \omega\}$; this can be accomplished using assumption (iv). Define $f(i) = n_i - |b_i|$ for $i \in \omega$. Put $d_i = b_i \cup [\alpha, \alpha + f(i))$. Now we are ready to define $\mathcal{G}$. We put $\mathcal{G}_n = \mathcal{F}_n$ for $n \notin \{n_i : i < \omega\}$ and

$$\mathcal{G}_{n_i} = \mathcal{F}_{n_i} \cup \{(d_i - \{\max(d_i)\}) \cup \{\gamma\} : \gamma \geq \max(d_i),\ \gamma \geq \alpha\}.$$

So now we have to check that all the clauses of Definition 5 hold.

(1) $n_i = |b_i| + n_i - |b_i| = |b_i \cup [\alpha_s, \alpha_s + f(i))| = |d_i| = |d_i - \{\max(d_i)\}| + 1$.

(2) Note that the sequence $(b_i \cup [\alpha_s, \alpha_s + f(i))$ is cofinal in $[\alpha_s + \omega]^{<\omega}$. Indeed, fix $x \in [\alpha_s + \omega]^{<\omega}$. By (iii) and the fact that f is unbounded in ω find i such that $\beta_i > \max(x \cap \alpha)$ and $x - \alpha_s \subseteq [\alpha, \alpha + f(i))$; now use Fact 6(2) to find $j_0 \in \omega$ such that for $j > j_0$ if there is $a_{\beta_i}(\mathcal{F})(j)$, then $x \subseteq a_{\beta_i}(\mathcal{F})(j)$. Now take $n_k > j_0, n_i$. By assumption (i) we have $x \subseteq a_{\beta_i}(\mathcal{F})(n_k) \cup [\alpha_s, \alpha_s + f(i)) \subseteq a_{\beta_k}(\mathcal{F})(n_k) \cup [\alpha_s, \alpha_s + f(k)) \subseteq d_k$. Now note that $d_j \in \mathcal{G}_{n_j}$ for j such that $f(j) > 0$.

(3) Let $a, b \in \mathcal{G}_n - \mathcal{F}_n$. As all such a, b consist of a common part and distinct maximums, 5(3) is clear in this case. So consider $a \in \mathcal{F}_n$ and $b \in \mathcal{G}_n - \mathcal{F}_n$ and let $n = n_i$. First note that $b \cap \alpha_s = a_\beta(\mathcal{F})(n_i)$ (where β is either β_{n_i} if $f(i) > 0$ or the previous element of b_i if $f(i) = 0$), for some $\beta < \alpha_s$. Then, by the definition of $a_{\beta_i}(\mathcal{F})(n_i)$ and by (ii), we conclude that there is $e \in \mathcal{F}_{n_i}$ such that $\beta \in e$ and $e \cap (\beta + 1) = b \cap \alpha_s$. Now as $a \in \mathcal{F}_{n_i}$, we have $e \cap (\beta + 1) = a \cap (\beta + 1)$ by 5(3). Hence $a \cap \beta = b \cap \beta$.

(4) By 5(3), for $\gamma \geq \alpha_s$ and $n = n_i$, $i < \omega$, there is a unique $a_\gamma(\mathcal{G})(n_i)$ and for no other $n \in \omega$ does $\mathcal{G}_n$ cover γ. Calculating $a_\gamma(\mathcal{G})(n_i)$ we get $a_\gamma(\mathcal{G})(n_i) = b_i \cup [\alpha_s, \alpha_s + f(i) - 1) \cup \{\gamma\}$ for i such that $\alpha_s + f(i) \leq \gamma$. So $a_\gamma(\mathcal{G})(n_i) = b_i \cup [\alpha_s, \gamma]$ for $\gamma < \alpha_s + f(n)$. By assumptions (i) and (v), both $(b_i : i < \omega)$ and $([\alpha_s, \alpha_s + f(i) - 1) : i < \omega)$ are nondecreasing, so (4) follows. Note that we also proved that $a_{\alpha_s}(\mathcal{G})(n_i) = b_i \cup \{\alpha_s\}$.

(5) Follows from the construction of s' and $\mathcal{G}$.

(6) For $\gamma \geq \alpha_s + f(i)$ we have $n_i - \eta_\gamma(\mathcal{G})(n_i) = 0$ and for $\gamma = \alpha_s + k < \alpha_s + f(i)$ we have $n_i - \eta_\gamma(\mathcal{G})(n_i) = n_i - (|b_i| + k) = f(i) - k$. Thus the function in (6) is nondecreasing by assumption (v). The other properties follow directly from the construction. ∎

DEFINITION 8.   Suppose that H is a directed subset of P_1 and that $\sup\{\alpha_s : s \in H\} = \omega_1$ and $\mathcal{F}_s$ for $s \in H$ are s-frames such that if $s_1 \leq s_2, s_3$ then $\mathcal{F}_{s_1} \supseteq \mathcal{F}_{s_2}, \mathcal{F}_{s_3}$. Moreover, suppose $\mathcal{F} = \bigcup\{\mathcal{F}_s : s \in H\}$.

We define $\Sigma(\mathcal{F}) = \{\sigma_{\alpha,n} : \alpha < \omega_1, \ n \in X_\alpha^s, \ s \in H\}$, where

$$\sigma_{\alpha,n} = \langle \alpha_1, \ldots, \alpha_r \rangle$$

with $\alpha_1, \ldots, \alpha_r$ being the increasing enumeration of $a_\alpha(\mathcal{F})(n)$.

REDUCTION LEMMA 9. *Suppose $\mathcal{F}$ and H are as in Definition 8, $\mathcal{X} = \bigcup\{\mathcal{X}_s : s \in H\}$ and that for every uncountable $A \subseteq \omega_1$ we have $\delta < \omega_1$ such that*

$$X_\alpha \subseteq^* \bigcup_{\beta \in A \cap \delta} X_\beta - m(\beta, \delta)$$

for every $\alpha \in A$. Then $\mathcal{X}$ and $\Sigma(\mathcal{F})$ have the hereditary Lindelöf property (property (iii) of Proposition 2).

Proof. Fix H and $\mathcal{F}$ as in Definition 8. Let $A \subseteq \omega_1$. Find δ as in the reduction lemma. We claim that the hereditary Lindelöf property is witnessed by $B = A \cap \delta$ and $f = m(\cdot, \delta)$.

The function $m(\cdot, \delta)$ is finite-to-one by Definition 5(1), (3), and is well defined on the entire δ by Fact 6(1).

Now take any $\alpha \in A$ (we may assume that $\alpha \geq \delta$) and $n \in X_\alpha$ such that:

   (i) $n \geq m(\delta, \alpha)$.
   (ii) $n \notin Y_\delta$.
   (iii) $n \in \bigcup_{\beta \in A \cap \delta} X_\beta - m(\beta, \delta)$.

A sufficiently large $n \in X_\alpha$ has these properties as $Y_\delta \subseteq^* Y_\alpha$ and $X_\alpha \cap Y_\alpha = \emptyset$ and by the assumption of the lemma. Using (iii) find $\beta \in A \cap \delta$ such that $n \in X_\beta - m(\beta, \delta)$.

Now we are left with the proof of the fact that $\sigma_{\beta,n} \subseteq \sigma_{\alpha,n}$. By the facts that $X_\alpha \cap Y_\alpha = X_\beta \cap Y_\beta = \emptyset$ and Definition 5(5), we know that $\mathcal{F}_n$ covers both α and β. Also by (ii) and 5(5), it covers δ. Thus by Fact 6(3) applied twice, using both (i) and (iii), we conclude that

$$a_\beta(\mathcal{F})(n) = a_\alpha(\mathcal{F})(n) \cap (\beta + 1).$$

Thus the definition of $\Sigma(\mathcal{F})$ implies that $\sigma_{\beta,n} \subseteq \sigma_{\alpha,n}$. $\blacksquare$

Now we are ready to define the second group of coordinates of the forcing P. This group depends on the first one so in fact we are defining the entire P. The conditions of P are of the form (s, t), where $s \in P_1$ and t is of the form $t = (\mathcal{F}_t, \alpha_t, \Psi_t)$, where:

(1) $\alpha_t = \alpha_s$.
(2) $\mathcal{F}_t$ is an s-frame.
(3) Ψ_t is a countable family of partial functions from ω into ω.
(4) For every $f \in \Psi_t$, $\mathrm{dom}(f) \supseteq Y_\beta$ for some $\beta < \alpha_t$, and $0 <^+ f$.
(5) For every $\beta < \alpha_t$ and every $f \in \Psi_t$ we have $\eta_\beta(\mathcal{F}_t) <^+ f$.

The order is defined by $(s_1, t_1) \leq (s_2, t_2)$ if and only if:

(6) $s_1 \leq_{P_1} s_2$.
(7) $\mathcal{F}_{t_1}$ end-extends $\mathcal{F}_{t_2}$.
(8) $\Psi_{t_2} \supseteq \Psi_{t_1}$.

R e m a r k. Adding a collection of functions as a side condition makes the family of $\eta_\beta(\mathcal{F})$'s strictly dominated below every function which dominates it. In particular, it is going to form a Hausdorff gap together with Ψ. We have found that the hereditary Lindelöf property fails without such a side condition.

FACT 10. (CH) P is σ-closed and satisfies the ω_2-c.c.

P r o o f. Fact 6(5). $\blacksquare$

DENSITY LEMMA 11. *Suppose that $(s, t) \in P$ and $\alpha \in \omega_1$. There is $(s', t') \leq (s, t)$ with $(s', t') \in P$ such that $\alpha_{s'} = \alpha_{t'} \geq \alpha$.*

P r o o f. The proof is by induction on $\alpha < \omega_1$. If α is a limit ordinal, then we can apply the inductive hypothesis and Fact 6(5). If $\alpha = \beta + k$ for some $\beta < \omega_1$, we will use the Extension Lemma 7. By the inductive hypothesis and the fact that α_s is always a limit ordinal, we can assume that $\alpha_s = \beta$. Now in order to use the Extension Lemma we have to find $(b_i, \beta_i, n_i)_{i<\omega}$ satisfying the assumptions of that lemma.

We construct this sequence by induction on $i < \omega$. Given $b_{i'}, \beta_{i'}, n_{i'}$ for $i' < i$ we find β_i so that (iii) will hold for the entire sequence $(\beta_i : i < \omega)$. Now we have to find n_i, and b_i will be determined by (ii). We should do it so that (i), (iv) and (v) will hold.

By Fact 6(2), for n_i large enough so that $(\mathcal{F}_t)_{n_i}$ covers β_i, the condition (i) will hold, so choose such an n_i which moreover satisfies the following conditions:

(∗) $n_i - |a_{\beta_i}(\mathcal{F}_t)(n_i)| > i, n_{i'} - |b_{i'}|$ for $i' < i$.

(∗∗) $n_i \notin Y^s_{\beta_{i'}}$ for $i' < i$.

(∗∗∗) $\psi_{i'}(n_i) - |a_{\beta_i}(\mathcal{F}_t)(n_i)| > i$ for $i' < i$, where $\{\psi_{i'} : i' < \omega\} = \Psi_t$.

This can be accomplished: (∗) by Fact 6(2), (∗∗) since $X^s_{\beta_i} \subseteq^* Y^s_{\beta_i} - \bigcup_{i < i'} Y^s_{\beta_{i'}}$ and by 5(5), and (∗∗∗) by the definition of a condition in P.

So we obtain $\mathcal{G} \supseteq \mathcal{F}_t$ and $s' \leq s$ as in the Extension Lemma. Put $(s', t') = (s', (\mathcal{G}, \beta + \omega, \Psi_t))$. To make sure that it is a condition of P, we need to check that for every $\gamma < \beta + \omega$ and for every $f \in \Psi_t$ we have $\eta_\gamma(\mathcal{G}) <^+ f$. For $\gamma < \beta$ this follows from the fact that $(s, t) \in P$, and for $\gamma = \beta$ it follows from (∗∗∗). For $\gamma = \beta + k$ it follows from the fact for $\gamma = \beta$ and from Fact 6(4). ∎

The definition of the iteration takes place in V^P. We define $P(A)$ for $A \subseteq \omega_1$; then we will run A through all subsets of ω_1 which appear in some intermediate model and we will iterate these $P(A)$'s with countable supports. Using the standard argument and Fact 17 proved later one can take care of all subsets of ω_1 in the extension. A condition p of $P(A)$ satisfies the following requirements:

(1) $p : \mathrm{dom}(p) \to 2$.

(2) $\mathrm{dom}(p) \subseteq^* Y_{i(p)}$ for some $i(p) < \omega_1$.

(3) For all $j < i(p)$ we have $X_j \subseteq^* \mathrm{dom}(p)$, and if $j \in A$, then $X_j \subseteq^* p^{-1}(\{1\})$, while if $j \notin A$, then $X_j \subseteq^* p^{-1}(\{0\})$.

We say $p \leq q$ if and only if $p \supseteq q$. This was a single step. When we think about the iteration the conditions of Q_{ω_2} have two coordinates, the first will run through ω_2 and the second through ω; thus if we say $p(\xi)(n)$ we mean the condition p at the ξth stage of the iteration (i.e., a name for a partial function on ω) evaluated at $n \in \omega$.

DEFINITION 12. Let $r = (s, t, p) \in R$ ($= P * Q_{\omega_2}$). Let G be a finite subset of ω_2 and X a set of integers. We say that r *avoids* (G, X) if and only if there is a countable set $S \subseteq \omega_2$ such that (s, t) forces that the support of p is included in S and that

$$\forall \xi \in G \; \forall n \in X \; [p{\upharpoonright}\xi \Vdash n \notin \mathrm{dom}(p(\xi))].$$

If $X = \{n\}$, for some $n \in \omega$, then we write (G, n) instead of $(G, \{n\})$.

Note that, for convenience, we added the requirement of deciding a superset of the support of the condition of Q_{ω_2}. Thus, if we are talking of a condition $(s,t,p) \in R$ avoiding some (G,n) we will identify this countable set with the support of p, denoted by $\mathrm{supp}(p)$.

FACT 13. *Suppose $r = (s,t,p) \in R$ and suppose that for some $0 < k < \omega$ we are given $G_1, \ldots, G_k \in [\omega_2]^{<\omega}$ and $n_1 < \ldots < n_k$ such that r avoids (G_i, n_i) for $i \le k$. Then for every sequence $(\sigma_i : i < k)$ where $\sigma_i \in 2^{G_i}$ there is a condition $r[\sigma_1, n_1; \ldots; \sigma_k, n_k] = (s,t,p[\sigma_1, n_1; \ldots; \sigma_k, n_k])$ such that:*

(0) $r[\sigma_1, n_1; \ldots; \sigma_k, n_k] \le r$.
(1) For each $i < k$ and $\xi \in G_i$ we have

$$(s,t,p[\sigma_1, n_1; \ldots; \sigma_k, n_k]{\restriction}\xi) \Vdash p[\sigma_1, n_1; \ldots; \sigma_k, n_k](\xi)(n_i) = \sigma_i(\xi).$$

(2) For every $q \le r[\sigma_1, n_1; \ldots; \sigma_k, n_k]$ there is $r' \le r$ such that r' avoids $(G_1, n_1), \ldots, (G_k, n_k)$ and

$$r'[\sigma_1, n_1; \ldots; \sigma_k, n_k] \le q.$$

(3) The set

$$\{r[\mu] : \mu \in 2^{G_1} \times \{n_1\} \times \ldots \times 2^{G_k} \times \{n_k\}\}$$

is a maximal antichain below r.

P r o o f. For $(\sigma_1, n_1; \ldots; \sigma_k, n_k) \in 2^{G_1} \times \{n_1\} \times \ldots \times 2^{G_k} \times \{n_k\}$ and $r = (s,t,p) \in R$ we put $p[\sigma_1, n_1; \ldots; \sigma_k, n_k]$ to be a P-name for a condition of Q_{ω_2} such that (s,t) forces that $p{\restriction}\xi$ forces

$$p[\sigma_1, n_1; \ldots; \sigma_k, n_k](\xi)(n) = \sigma_i(\xi)$$

if $\xi \in G_i$ and $n = n_i$ and otherwise

$$p[\sigma_1, n_1; \ldots; \sigma_k, n_k](\xi)(n) = p(\xi)(n).$$

Such names can be found by the maximum principle (see, e.g., [K]). Since (s,t,p) avoids $(G_1, n_1), \ldots, (G_k, n_k)$, the condition $r[\sigma_1, n_1; \ldots; \sigma_k, n_k] = (s,t,p[\sigma_1, n_1; \ldots; \sigma_k, n_k])$ is an element of R.

Now (0) and (1) follow from the definition, and (2) and (3) are standard. ∎

LEMMA 14. *Let $\alpha \in \omega_1$. Let $G_1, \ldots, G_k, G_{k+1} \in [\omega_2]^{<\omega}$ and $n_1, \ldots, n_k < \omega$. Suppose that $r \in R$ avoids (G_j, n_j) for $j \le k$. Then there are $r' \le r$, and there is $m < \omega$ and $\alpha' > \alpha$ such that r' avoids (G_j, n_j) for $j \le k$ and it avoids $(G_{k+1}, \omega - (Y_{\alpha'}^{r'} \cup m))$.*

P r o o f. By extending the first two coordinates, using Lemma 11, we can assume that $\alpha_s > \alpha$. By induction on $\eta \le \omega_2$ we prove that the lemma holds for $P * Q_\eta$. Of course for η limit it is clear, so consider $\eta = \eta_0 + 1$. We may assume that $\eta_0 \in G_{k+1}$.

Let $m_0 = \sum_{i \leq k} 2^{|G_i - \{\xi_0\}|}$ and fix an enumeration $\{\mu(j) : j \leq m_0\} = 2^{G_1 - \{\eta_0\}} \times \{n_1\} \times \ldots \times 2^{G_k - \{\eta_k\}} \times \{n_k\}$. We will construct a decreasing sequence $r(j) = (s(j), t(j), p(j)) \leq (s, t, p{\upharpoonright}(\eta_0 + 1))$ in $P * Q_{\eta_0}$ such that $r(j)$ avoids all $(G_i - \{\eta_0\}, n_i)$ for $i \leq k$ and we construct sequences $(\alpha(j) : j < m_0)$ and $(m(j) : j \leq m_0)$ such that

$$r(j)[\mu(j)] \Vdash \mathrm{dom}(p(\eta_0)) - m(j) \subseteq Y_{\alpha(j)}.$$

To construct the next objects $r(j+1), \alpha(j+1), m(j+1)$ we just extend the condition $r(j)[\mu(j+1)]$ to a condition $(s(j+1), t(j+1), q)$ which decides $\alpha(j+1)$ and $m(j+1)$ as above and $\alpha_{s(j+1)} \geq \alpha(j+1)$ (using Density Lemma 11); then we apply Fact 12(2) to find $p(j+1)$.

Now we apply the inductive hypothesis for $(s(m_0), t(m_0), p(m_0)){\upharpoonright}(\eta_0 + 1)$ and α, obtaining $r_1 = (s_1, t_1, p_1)$, α_1 and m_1. Now possibly extending the first two coordinates of r_1 to s' and t' find $\alpha' > \alpha, \alpha_1, \alpha(j)$ for $j \leq m_0$ and $m > m_1, m(j)$ for $j \leq m_0$ such that

$$(Y_{\alpha_1}^{s'} \cup Y_{\alpha(1)}^{s'} \cup \ldots \cup Y_{\alpha(m_0)}^{s'}) - m \subseteq Y_{\alpha'}^{s'}.$$

By the appropriate choice of s' the above sets are determined in the ground model, so m as above can be found. Now α, m and $r' = (s', t', p_1 \widehat{\ } p(\eta_0))$ work. ∎

DENSITY LEMMA 15. *Suppose that* $r \in R$ *and* $G_1, \ldots, G_n \in [\omega_2]^{<\omega}$ *and* $n_1, \ldots, n_k \in \omega$ *and* r *avoids* $(G_1, n_1), \ldots, (G_k, n_k)$, *and suppose* $\xi \in \omega_2$ *and* $\alpha \in \omega_1$. *Then there is* $r' = (s', t', p') \leq r$ *such that* r' *avoids* $(G_1, n_1), \ldots, (G_k, n_k)$ *and*

$$(s', t') \Vdash p'{\upharpoonright}\xi \Vdash i(p'(\xi)) \geq \alpha.$$

Proof. The proof makes use of Fact 13, similar to the way it is done in Lemma 14, to decide $i(p'(\xi))$ up to a finite set. It also uses the fact that if $p \in P(A)$ and $\alpha \in \omega_1$, then there is $p' \leq p$ such that $p' \in P(A)$ and $i(p') \geq \alpha$. The details are left to the reader. ∎

FUSION LEMMA 16. *Suppose that* $(r_i, G_i, n_i, b_i, \beta_i)_{i < \omega}$ *satisfies the following conditions*:

(1) *For all* $i < \omega$, $r_i = (s_i, t_i, p_i) \in R$ *and* $r_{i+1} \leq r_i$.
(2) $\bigcup_{i < \omega} G_i \supseteq \bigcup_{i < \omega} \mathrm{supp}(p_i)$.
(3) *For all* $i < \omega$ *we have* $G_i \subseteq G_{i+1}$.
(4) *For all* $i < \omega$, r_i *avoids* (G_j, n_j) *for* $j \leq i$.
(5) *For all* $i < \omega$ *the condition* (s_{i+1}, t_{i+1}) *forces that*

$$\forall \xi \in G_i \; [p_{i+1}{\upharpoonright}\xi \Vdash \alpha_{s_i} \leq i(p_{i+1}(\xi)) \leq \alpha_{s_{i+1}}].$$

(6) $(b_i, \beta_i, n_i)_{i < \omega}$ *satisfies the assumption of the Extension Lemma 7 for* $s = \bigcup_{i < \omega} s_i$ *and* $\mathcal{F} = \bigcup_{i < \omega} \mathcal{F}_{t_i}$.
(7) *For all* $f \in \Psi_{t_i}$ *we have* $(|b_i| : i < \omega) <^+ (f(n_i) : i < \omega)$.

Then there is $(s^*, t^*, p^*) \in R$ such that:

(8) *For each $i < \omega$ we have $(s^*, t^*, p^*) \leq (s_i, t_i, p_i)$.*
(9) *For each $i < \omega$, (s^*, t^*, p^*) avoids (G_i, n_i).*
(10) $\alpha_{s^*} = \delta + \omega$ *and* $Y_\delta = \omega - \{n_i : i < \omega\}$, *where* $\delta = \sup\{\alpha_{s_i} : i < \omega\}$.
(11) *For each $i < \omega$ we have $a_\delta(\mathcal{F}_{t^*})(n_i) = b_i \cup \{\delta\}$.*

If moreover we are given

(12) $(\psi(n_i) : i < \omega)$ *such that* $(|b_i| : i < \omega) <^+ (\psi(n_i) : i < \omega)$,

then we can also assume

(13) $\psi \in \Psi_{t^*}$.

P r o o f. $s = (\bigcup_{i<\omega} \mathcal{X}_{s_i}, \bigcup_{i<\omega} \mathcal{Y}_{s_i}, \delta)$ is a condition of P_1. By Fact 6(5) the family $\mathcal{F} = \bigcup_{i<\omega} \mathcal{F}_{t_i}$ is an s-frame with $\alpha_s = \delta$. Put $t = (\mathcal{F}, \bigcup_{i<\omega} \Psi_i, \delta)$. So $(s, t) \in P$ and also $(s, t) \leq (s_i, t_i)$ for $i < \omega$.

Now by (6) and the Extension Lemma 7, there is $s^* \leq s$ and an s^*-frame $\mathcal{G}$ which end-extends $\bigcup_{i<\omega} \mathcal{F}_{t_i}$ such that (10) and (11) are satisfied. So put $t^* = (\mathcal{G}, \Psi, \delta + \omega)$, where $\Psi = \bigcup_{i<\omega} \Psi_i$. To make sure that (s^*, t^*) is a condition of P we need to prove that for every $\beta < \delta + \omega$ and for every $f \in \Psi$ we will have $\eta_\beta(\mathcal{G}) <^+ f$. By Fact 6(4) and 6(3) it is enough to prove that $\eta_\delta(\mathcal{G}) <^+ f$ for each $f \in \Psi$. This follows from our assumption (7) and already noted condition (11). Clearly assuming also (12) we can add ψ to Ψ_{t^*} to obtain (13).

To obtain p^*, by induction on $\xi \leq \omega_2$ we will define $p^*(\xi)$ such that (s^*, t^*) forces that

$$p^* {\upharpoonright} \xi \Vdash p^*(\xi) \in Q_\xi, \ \mathrm{dom}(p(\xi)) \cap \{n_i : \xi \in G_i\} = \emptyset, \ p^*(\xi) \leq p_i(\xi) \text{ for } i < \omega.$$

This in turn implies that $(s^*, t^*) \Vdash p_i \geq p^* \in Q_{\omega_2}$ and that (s^*, t^*, p^*) avoids (G_i, n_i) for $i < \omega$, and so (8) and (9) will hold.

We begin the construction. Since, by the inductive assumption $p^* {\upharpoonright} \xi \leq p_i {\upharpoonright} \xi$ for $i < \omega$, we have

$$p^* {\upharpoonright} \xi \Vdash p_i(\xi) \subseteq p_{i+1}(\xi),$$

it follows that $p^* {\upharpoonright} \xi$ forces that $p(\xi) = \bigcup_{i<\omega} p_i(\xi)$ is a partial function from ω into 2 such that for every $\beta < \sup(i(p_i(\xi)) : i < \omega) = \delta$ (by (5)) we have $X_\beta \subseteq^* p(\xi)^{-1}(\{0\})$ if $\beta \notin A_\xi$, and $X_\beta \subseteq^* p(\xi)^{-1}(\{1\})$ if $\beta \in A_\xi$. Also since

(∗) $\qquad\qquad\qquad p_i {\upharpoonright} \xi \Vdash n_i \notin \mathrm{dom}(p_i(\xi))$

for all i such that $\xi \in G_i$ and ξ belongs to almost all G_i by (2) and (3), we conclude that (s^*, t^*) forces that

$$p {\upharpoonright} \xi \Vdash \mathrm{dom}(p(\xi)) \subseteq^* Y_\delta.$$

We put $p^*(\xi) = p(\xi)$ and we conclude that $(s^*, t^*, p^*) \in R$ and that (8) holds, and (∗) implies that (9) holds. ∎

FACT 17. (CH) R *is a proper notion of forcing which preserves cardinals and* $R \Vdash 2^{\omega_1} = \omega_2$.

P r o o f. First let us show that R is proper. This could be done by appropriately modifying the argument given in [JS], since our forcing is the same except for two side conditions added to the first factor. For the benefit of the reader, we outline a somewhat different argument here which uses the machinery we have developed so far. Let $M \prec H(\nu)$ be a countable elementary submodel for ν large enough and let $r \in M \cap R$. Let $(D_i : i < \omega)$ be an enumeration of all predense subsets of R which are elements of M. We construct a fusion sequence $(r_i, G_i, n_i)_{i<\omega}$ satisfying (1)–(5) of the Fusion Lemma 16 such that $r_0 = r$ and D_i is predense below r_i. Each r_i, G_i is in M. Given $r_{i'}, G_{i'}, n_{i'}$ for $i' < i$, in order to construct r_i, G_i, n_i, first choose G_{i+1} so that (2), (3) of Lemma 16 will be satisfied in the end. Now extend r_{i-1} to find (b_i, β_i, n_i) so that (4), (6), and (7) of the Fusion Lemma 16 will be satisfied in the end. Use Lemma 14 to get n_i so that (4) holds; simultaneously choose β_i so that β_i, n_i, and $b_i = a_{\beta_i}(\mathcal{F}_{t_i})(n_i)$ will satisfy (6) and (7) in the end... this may be done as in the proof of Density Lemma 11. Next use Lemma 15 repeatedly for $\xi \in G_i$ and $\alpha = \alpha_{s_{i-1}}$ and Lemma 11, so that the assumption (5) of Lemma 16 is satisfied.

Now we have $r'_i \le r'$ and $G_1, \ldots, G_i$ and $n_1, \ldots, n_i$ such that r'_i avoids all the sets $(G_1, n_1), \ldots, (G_i, n_i)$ and we have taken care of the conditions (1)–(7) of Lemma 16. So now our task is to find $r_i \le r'_i$ avoiding $(G_1, n_1), \ldots \ldots, (G_i, n_i)$ such that D_i is predense below r_i. For this we use repeatedly Fact 13(2) so that for the obtained condition r_i, for every $\mu \in 2^{G_1} \times \{n_1\} \times \ldots \times 2^{G_i} \times \{n_i\}$ we have $r_i[\mu] \le d$ for some $d \in D_i$. Then r_i is as required. Now the fusion r^* obtained using Lemma 16 is an (R, M)-generic stronger condition than r, i.e., R is proper. In particular, ω_1 is preserved.

In order to prove the rest of the fact, we need to prove that for every $\eta < \omega_2$, the forcing $P * Q_\eta$ has a dense set of size ω_1. If we know this, then we can conclude that R has the ω_2-c.c. (this is well known; see, e.g., [J; Cor. 7.10]), and so cardinals are preserved. Also $R \Vdash 2^{\omega_1} = \omega_2$ follows in the standard way. Following [BL], we say that a condition $r = (s, t, p) \in R$ is *determined* if and only if there are sequences $(G_i, n_i)_{i<\omega}$ and a function f_r mapping

$$\{(\mu, \xi) \in 2^{G_1 \cap \xi} \times \{n_1\} \times \ldots \times 2^{G_k \cap \xi} \times \{n_k\} \times \{\xi\} :$$
$$k < \omega, \ \xi \in \mathrm{supp}(p), \ \xi \in G_k\}$$

into $2^{<\omega}$, and such that:

    (i) $\mathrm{supp}(r) \subseteq \bigcup_{i<\omega} G_i$.
    (ii) r avoids (G_i, n_i) for all $i < \omega$.

(iii) For every $\xi \in \mathrm{supp}(p)$ such that $\xi \in G_k$ and $(\mu, \xi) \in \mathrm{dom}(f_r)$ we have

$$(s, t) \Vdash p[\mu] \restriction \xi \Vdash p[\mu](\xi) \restriction n_k = f_r(\mu, \xi).$$

First we note that the collection of all determined conditions is dense in $P * Q_\eta$. To construct a determined condition below a condition $r \in P * Q_\eta$ one uses Fusion Lemma 16, and at each step of the construction, one uses Lemma 14 and Fact 13(2) repeatedly, to decide the values of $p(\xi) \restriction n_k$. The details are left to the reader, since they are standard.

Finally, note that if $r_1, r_2 \in P * Q_\eta$ are determined and f_{r_1} and f_{r_2} are the same, then $p \leq q$ and $q \leq p$. The proof of this fact is by induction on $\eta \leq \omega_2$, using Fact 13(3) and the fact that if $\pi_1, \pi_2 \in 2^\omega$ are distinct then there is $k < \omega$ such that $\pi_1 \restriction n_k \neq \pi_2 \restriction n_k$. So by CH, practically there are ω_1 determined conditions in $P * Q_\eta$, as required. ∎

PROPOSITION 18. *Suppose that $r \in R$ and $\dot{A}$ are such that $r \Vdash \dot{A} \subseteq \omega_1$. Then there are $r^* \leq r$ and $\delta \in \omega_1$ such that r^* forces that*

$$\forall \alpha \in \dot{A} \; \exists k \in \omega \; \forall n \in X_\alpha - k \; \exists \beta \in \dot{A} \cap \delta \; (n \in X_\beta \; \& \; m(\beta, \delta) \leq n).$$

P r o o f. Fix a countable model $M \prec H(\nu)$ for ν large enough. Assume that M contains all relevant objects like $r, \dot{A}$, and so forth. Let $M \cap \omega_1 = \delta = \{\delta_i : i < \omega\}$ and $M \cap \omega_2 = \{\varrho_i : i < \omega\}$.

We will construct a few sequences which will yield the extension r^* witnessing the hereditary Lindelöf property at $\dot{A}$. The sequences are:

(1) $r_i = (s_i, t_i, p_i)$ for $i < \omega$ are conditions in R.
(2) $(G_i : i < \omega) \subseteq [M \cap \omega_2]^{<\omega}$.
(3) $(n_i : i < \omega) \subseteq \omega$.
(4) $(b_i : i < \omega) \subseteq [\delta]^{<\omega}$.
(5) $(\psi(n_i) : i < \omega) \subseteq \omega$.

We will construct these sequences so that the assumptions of the Fusion Lemma will be satisfied, which will then be applied to obtain the desired extension. The sequences will be constructed in V. Each term will be constructed in M, we will leave M for V only to acquire knowledge about next terms of $\{\delta_i : i < \omega\}$ and $\{\varrho_i : i < \omega\}$, so that the obtained sequences diagonalize formulas over M. The construction will be carried out by induction on $i < \omega$, hence the main steps will be executed in M. The input of each step is an element of M and by elementarity of M we can assume that the output is an element of M.

Let $t_i = (\mathcal{F}_i, \alpha_i, \Psi_i)$ and fix some enumeration $\Psi_i = (\psi_i^l : l < \omega)$. Fix some enumeration $\{\xi_k^j : k < \omega\} = \mathrm{supp}(p_j)$. At each $i < \omega$ the ith terms of the above sequences are required to satisfy the following conditions:

To satisfy conditions (1)–(5) of the Fusion Lemma:

(6) $r_i = (s_i, t_i, p_i) \in R$ and $r_{i+1} \leq r_i$.
(7) $\varrho_i \in G_i$.
(8) $G_i \subseteq G_{i+1}$.
(9) r_i avoids $(G_{i'}, n_{i'})$ for $i' \leq i$.
(10) The condition (s_i, t_i) forces that
$$\forall \xi \in G_i \ [p_i \restriction \xi \Vdash \alpha_{s_i} \leq i(p_{i+1}(\xi)) \leq \alpha_{s_{i+1}}].$$

To make sure condition (6) of the Fusion Lemma, i.e., the assumptions of the Extension Lemma 7 are satisfied:

(11) $b_i \subseteq b_{i+1}$.
(12) There is $\beta_i < \delta$ such that $a_{\beta_i}(\mathcal{F}_{s_i})(n_i) = b_i$.
(13) $\beta_i > \delta_i$, and $\beta_i > \beta_{i'}$ for $i' < i$.
(14) $s_i \Vdash n_i \notin Y_{\alpha_{s_{i'}}}$ for $i' < i$.
(15) $n_i - |b_i| > i$, and $n'_i - |b_{i'}|$ for $i' < i$.

To make sure that condition (12) of the Fusion Lemma holds:

(16) $\psi(n_i) - |b_i| > i$.

To make sure that condition (7) of the Fusion Lemma holds:

(17) $\psi_{i'}^{i''}(n_i) - |b_i| > i$ for $i', i'' < i$.

To make the proof of the hereditary Lindelöf property work:

(18) r_i forces that: If
$$\alpha \in \dot{A} \ \& \ n_i \in X_\alpha \ \& \ a_\alpha(\mathcal{F})(n_i) \supseteq b_i$$
and
$$\forall \beta \in \dot{A} \ (n_i \in X_\beta \Rightarrow \beta \notin b_i)$$
then
$$\eta_\alpha(\mathcal{F})(n_i) > \psi(n_i).$$

So suppose we are at stage $i \in \omega$, that is, we are given $r_i, G_i, n_i, b_i, \psi(n_i)$ and we are aiming at constructing $r_{i+1}, G_{i+1}, n_{i+1}, b_{i+1}, \psi(n_{i+1})$. In order to do so we will be applying the following lemma, in which $\mathcal{F}$, Y_α's, and X_α's denote the generic frame, the elements of the generic tower, and the elements of the generic almost disjoint family.

LEMMA 19. *Suppose we are given* $r = (s, t, p) \in R$ *and* $G_1, \ldots, G_i, G_{i+1} \in [\omega_2]^{<\omega}$, $b \in [\alpha_t]^{<\omega}$ *and* $n_1, \ldots, n_i \in \omega$, *an integer-valued function* f, *an integer* l *and an infinite set* X *such that*:

(a) $X \subseteq \omega - Y_{\max(b)}$.
(b) (s, t, p) *avoids* $(G_1, n_1), \ldots, (G_i, n_i), (G_{i+1}, X)$.
(c) (s, t) *forces that for every* $\beta \in \omega_1$ *we have* $\eta_\beta(\mathcal{F}) <^+ f$.

Then there are $r' = (s', t', p') \leq r$ *and* $j \in X$ *and* c *such that*:

(d) $b \subseteq c = a_{\max(c)}(\mathcal{F}_{t'})(j)$.

(e) r' *avoids* $(G_1, n_1), \ldots, (G_i, n_i), (G_{i+1}, j)$.

(f) r' *forces that*: If

$$\alpha \in \dot{A} \ \& \ j \in X_\alpha \ \& \ a_\alpha(\mathcal{F})(j) \supseteq c$$

and

$$\forall \beta \in \dot{A} \ (j \in X_\beta \Rightarrow \beta \notin c)$$

then

$$\eta_\alpha(\mathcal{F})(j) > l + |c|.$$

(g) $|c| < f(j)$.

P r o o f. We may assume that $b \subseteq a_{\max(b)}(\mathcal{F}_t)(j)$ for all $j \in X \subseteq \omega - Y_{\max(b)}$ by taking $j \in X$ large enough (by 6(2)).

Put $m_0 = 2^{|G_1|} + \ldots + 2^{|G_{i+1}|}$. We will construct sequences $\langle r(j) \rangle_{j \in X}$ and $\langle c(j) \rangle_{j \in X}$ satisfying:

(h) $r(j) \leq (s, t, p)$ and $r(j)$ avoids $(G_1, n_1), \ldots, (G_i, n_i), (G_{i+1}, j)$.

(i) $c(j) \in (\mathcal{F}_{t(j)})_j$ and $c(j) \supseteq b$.

For an appropriate j we will in the end put $r' = r(j)$ and c an appropriate subset of $c(j)$.

Given $j \in X$ the construction of $r(j)$ and $c(j)$ takes $m_0 + 1$ steps. In each step $1 \leq k \leq m_0 + 1$ we produce $r(k, j)$ satisfying (h) and $\gamma(k, j)$, $c(k, j)$ and $\mu(k, j)$ such that:

(j) $\gamma(0, j) = \max(b) \leq \gamma(1, j) \leq \ldots \leq \gamma(m_0, j) \leq \max(c(j)) = \gamma(m_0 + 1, j)$ such that the above γ's are elements of $c(j)$.

(k) $c(k, j) = a_{\gamma(k, j)}(\mathcal{F}_{t(k, j)})(j)$.

(l) $\mu(1, j), \ldots, \mu(m_0, j)$ are distinct elements of $2^{G_1} \times \{n_1\} \times \ldots \times 2^{G_i} \times \{n_i\} \times 2^{G_{i+1}} \times \{j\}$.

In the beginning, before the first step, we put $r(0, j) = (s, t, p)$, $\gamma(0, j) = \max(b)$, $c(0, j) = b$. To find the next $r(k + 1, j), \gamma(k + 1, j)$, $c(k + 1, j)$, $\mu(k + 1, j)$ for $k + 1 \leq m_0$, we consider several cases.

C a s e 1: *There is* $r'(k + 1, j)$ *in* R *satisfying* (h) *and such that there exists some* $\mu(k + 1, j) \notin \{\mu(1, j), \ldots, \mu(k, j)\}$ *such that*

$$r'(k + 1, j)[\mu(k + 1, j)] \nVdash \forall \beta \in \dot{A} \ (j \in X_\beta \Rightarrow \beta \notin c(k, j)).$$

In this case apply Fact 13 to find $r(k + 1, j)$ satisfying (h) such that $r(k + 1, j)[\mu(k + 1, j)]$ forces that there is $\beta \in \dot{A} \cap c(k, j)$ such that $j \in X_\beta$, and put $\gamma(k + 1, j) = \gamma(k, j)$.

C a s e 2: *Case 1 does not hold and for each* $\mu \notin \{\mu(1, j), \ldots, \mu(k, j)\}$, $r(k, j)[\mu]$ *forces that*

$$\forall \beta \in \dot{A} \ (j \in X_\beta \Rightarrow c(k, j) \nsubseteq a_\beta(\mathcal{F})(j)).$$

In this case we choose some $c(j + 1, k) \in (\mathcal{F}_{t(j,k)})_j$ including $c(j, k)$ and we put $r(k + 1, j) = r(k, j)$.

C a s e 3: *Cases 1 and 2 do not hold.*

Choose $\gamma(k + 1, j)$ for which there exists $\mu \notin \{\mu(1, j), \ldots, \mu(k, j)\}$ such that there is $r(k + 1, j)$ satisfying (h) such that $r(k + 1, j)[\mu]$ forces that

$$\gamma(k + 1, j) \in \dot{A} \, \& \, j \in X_{\gamma(k+1,j)} \, \& \, c(k, j) \subseteq a_{\gamma(k+1,j)}(\mathcal{F}_{t(k+1,j)})(j).$$

The choice is done so that the value of $\eta_{\gamma(k+1,j)}(\mathcal{F}_{t(k+1,j)})(j)$ is minimal. We put $\mu(k + 1, j) = \mu$ as above and $c(k + 1, j)$ such that (k) is satisfied.

Since at each step we consider less μ's than in the previous step, the construction will terminate in $m_0 + 1$ steps, since there are only m_0 many sequences in $2^{G_1} \times \ldots \times 2^{G_{i+1}}$. In the last, $(m_0 + 1)$th step we find some $c(m_0 + 1, j) = c(j) \in (\mathcal{F}_{t(m_0,j)})_j$ and $\gamma(m_0 + 1, j)$ satisfying (j) and (k) and $r(j) = r(m_0, j)$.

First let us argue that $r(j)$ forces that for each $0 \leq k \leq m_0$, if

(m) $$\alpha \in \dot{A} \, \& \, j \in X_\alpha \, \& \, c(k, j) \subseteq a_\alpha(\mathcal{F})(j)$$

and

(n) $$\forall \beta \in \dot{A} \, (j \in X_\beta \Rightarrow \beta \notin c(k, j)),$$

then

(o) $$\eta_\alpha(\mathcal{F})(j) \geq \eta_{\gamma(k+1,j)}(\mathcal{F})(j).$$

By Fact 13(3) it is enough to note that this is forced by the conditions of the form $r(k', j)[\mu(k', j]$ for $1 \leq k' \leq m_0$ (since by (l), the μ's are distinct and there are enough of them). This is proved in different ways depending on the relationship between k and k'.

C a s e $k = k'$. Since $k' \geq 1$, suppose that we are at the stage when we are given $c(k - 1, j), r(k - 1, j), \gamma(k - 1, j)$, and $\mu(k, j), c(k, j), r(k, j), \gamma(k, j)$ are chosen according to the above algorithm. If at this stage we are in Case 1, then $r(k, j)[\mu(k, j)]$ forces that there is $\beta \in c(k - 1, j)$ such that $j \in X_\beta$, thus (n) is false (because the c's are increasing) so "(m) & (n) $\Rightarrow$ (o)" is true. If we are in Case 2, $r(k, j)[\mu(k, j)]$ forces that (m) is false, so again "(m) & (n) $\Rightarrow$ (o)" is true. Finally, if we are in Case 3 and $r(k, j)[\mu(k, j)]$ does not force "(m) & (n) $\Rightarrow$ (o)", then there is $r'(k, j) \leq r(k, j)$ satisfying (h) and forcing its negation. The negation of (n) cannot be forced because we would be in Case 1, not Case 3. By the description of Case 3, $r(k, j)[\mu(k, j)]$ forces that (m) holds, so $r'(k, j)[\mu(k, j)]$ forces that there is α as in (m) but (o) fails for this α. This contradicts the minimality of $\eta_{\gamma(k,j)}(\mathcal{F})(j)$.

C a s e $k' < k$. If at the stage at which we are constructing $c(k', j)$ we are in Case 1 or Case 3, this means that

$$r(k', j)[\mu(k', j)] \Vdash \exists \beta \in \dot{A} \ [j \in X_\beta \ \& \ \beta \in c(k', j)],$$

and since $c(k', j) \subseteq c(k, j)$ we conclude that $r(k', j)[\mu(k', j)]$ forces that (n) is false, so "(m) & (n) $\Rightarrow$ (o)" is true. If at this stage we are in Case 2, then note that we are in Case 2 in all the following stages, in particular in the stage when objects with the subscript k are constructed, and note that in this situation we put $c(k, j) = c(k + 1, j) = \ldots = c(k', j)$ (because $c(k', j) \in (\mathcal{F}_{t(k', j)})_j)$, thus the formulas (m), (n), (o) are the same for k and k' and so the proof is the same as for $k = k'$.

C a s e $k < k'$. We can assume that at the stage when we construct $r(k + 1, j)$, $c(k + 1, j)$, $\gamma(k + 1, j)$ we are not in Case 1 because if it happens then $c(k, j) = c(k + 1, j)$ and we may assume that we are considering the next stage, and if $k + 1 = k'$ we are done by the first case. If any extension of $r(k', j)[\mu(k', j)]$ forces that (m) is true, then we are not in Case 2. So we can assume that we are in Case 3 at this stage. Now if any extension of the above condition forces that (o) is false, it contradicts the choice of $\mu(k, j)$ which is chosen from outside $\{\mu(1, j), \ldots, \mu(k-1, j)\}$ (so, $\mu(k', j)$ is eligible) so that $\gamma(k, j)(\mathcal{F})(j)$ is minimal.

This completes the proof of the fact that $r(j, m_0) \leq r(j)$ forces the implication "(m) & (n) $\Rightarrow$ (o)". So the (f) part of the statement of the lemma holds for all $j \in X$ and $r' = r(j)$ if $l + |c(k, j)| < \eta_{\gamma(k+1, j)}(\mathcal{F})(j)$, if we put $c = c(k, j)$.

Now to satisfy (g) of the statement of the lemma, and make sure that $l + |c(k, j)| < \eta_{\gamma(k+1, j)}(\mathcal{F})(j)$, we look for a nice $j_0 \in X$, namely such that there is $k_0 \leq m_0$ such that $r(j_0)$ forces that

$$|c(k_0, j_0)| < f(j_0) \ \& \ |\eta_{\gamma(k_0+1, j_0)}(\mathcal{F})(j_0) - \eta_{\gamma(k_0, j_0)}(\mathcal{F})(j_0)| > l.$$

A j_0 and k_0 like this can be found. To see this, first note that

$$c(j) \cap (\max(b) + 1) = a_{\max(b)}(\mathcal{F}_{t(j)})(j) = a_{\max(b)}(\mathcal{F}_t)(j).$$

Since $|c(j)| = j$ and $\eta_{\max(b)}(\mathcal{F}_t) <^+ \text{Id}$ (recall Fact 6(2)), $|c(j) - c(j) \cap (\max(b) + 1)|$ increases with j. Also, $f(j) - \eta_{\max(b)}(\mathcal{F}_t)(j)$ increases with j. So choose j_0 large enough so that both of these numbers are greater than $(m_0 + 1) \cdot l$. Then since $\gamma(0, j_0) = \max(b)$ and $\gamma(m_0 + 1, j_0) = \max(c(j_0))$, for some k we must have $|\eta_{\gamma(k+1, j_0)}(\mathcal{F})(j_0) - \eta_{\gamma(k, j_0)}(\mathcal{F})(j_0)| > l$, and if we let k_0 be the least such k we will have $|c(k_0, j_0)| < f(j_0)$ as well.

Once we have these j_0 and k_0 put

$$c = c(k_0, j_0), \quad j = j_0, \quad r' = r(j_0).$$

Now (g) holds and r' forces that $\eta_{\gamma(k_0+1, j_0)}(j_0) > \eta_{\gamma(k_0, j_0)}(j_0) + l = |c| + l$. ∎

Continuation of the proof of Proposition 18. Now let us see how Lemma 19 is used to construct the next elements of our sequence $(r_i, n_i, G_i, b_i, \beta_i)_{i<\omega}$. First find G_{i+1} such that (7) and (8) are satisfied. Now apply Lemma 14 for α which is bigger than $\alpha_{s_1}, \ldots, \alpha_{s_i}, \delta_{i+1}, \beta_1, \ldots, \beta_i = \max(b_i)$ and for $r = r_i$. This lemma gives $\alpha_0 > \alpha$, $m < \omega$ and $r' = (s', t', p') \leq r = r_i$ as in the statement of Lemma 14. By the choice of α we can assume (increasing m if necessary) that

$$(Y_{\alpha_{s_1}} \cup \ldots \cup Y_{\alpha_{s_i}} \cup Y_{\max(b_i)}) - m \subseteq Y_{\alpha'}^{s'}$$

and that the following function is nonnegative for $j \in X = \omega - (Y_{\alpha'}^{s'} \cup m)$:

$$f(j) = \min\{\phi(j) - (i+1) - (\max\{i' - |b_{i'}| : i' < i+1\}) :$$
$$\phi \in \Phi \,\&\, j \in \mathrm{dom}(\phi)\},$$

where $\Phi = \{\psi_1^1, \ldots, \psi_i^1, \ldots, \psi_1^i, \ldots, \psi_i^i, \mathrm{Id}\}$. Now note that $r = r_0$, $G_1, \ldots$ $\ldots, G_i, G_{i+1}$, $b = b_i \cup \{\delta_{i+1}\}$, $n_1, \ldots, n_i$, the function f and $l = i+1$ and $X = \omega - (Y_{\alpha'}^{s'} \cup m)$ satisfy the assumptions of Lemma 19. Indeed, (a) follows from the choice of m, (b) follows from the properties of r' and X obtained by Lemma 14, and (c) follows from the fact that $\psi_{i'}^{i''} \in \Psi_{t_i}$ for $i', i'' < i+1$. So apply Lemma 19 and find $r_{i+1} \leq r'$ such that condition (10) is satisfied (using the Density Lemma 15), and put $n_{i+1} = j, b_{i+1} = c, \beta_{i+1} = \max(b_{i+1}), \psi(n_{i+1}) = |b_{i+1}| + (i+1)$.

We have to check that conditions (6)–(18) are satisfied. Conditions (6)–(8) already follow from the choice of G_{i+1}, (9) follows from (e) of Lemma 19, and the choice of r_{i+1}. Condition (10) follows from the choice of r_{i+1}. Conditions (11) and (12) follow from (d) of Lemma 19, condition (13) follows from the choice of b and condition (d) of Lemma 19. Condition (14) follows from the choice of m and X. Condition (15) follows from the fact that $f(n_{i+1}) \leq n_j - (n_i' - |b_{i'}|), n_i$ for $i' < i+1$, so by (g) of Lemma 19, we have $|b_{i+1}| < f(n_{i+1})$ so

$$n_{i+1} - |b_{i+1}| > n_i' - |b_{i'}|, i+1$$

for $i' < i+1$. Condition (16) follows from the definition of $\psi(n_{i+1})$. Condition (17) follows from the fact that $f(n_{i+1}) \leq \psi_{i'}^{i''}(n_{i+1}) - (i+1)$ for $i'', i' < i+1$, so by (g) of Lemma 19, we have $|b_{i+1}| < f(n_{i+1})$. Condition (18) follows from (f) of Lemma 19.

Now suppose we have constructed the sequence $(r_i, G_i, n_i, b_i, \beta_i)_{i<\omega}$ satisfying (1)–(18). By (6)–(10) the $(r_i, G_i, n_i)_{i<\omega}$ satisfies the assumptions (1)–(5) of the Fusion Lemma, and by (11)–(17) the assumptions (6), (7), and (12) of the Fusion Lemma are also satisfied. Thus we deduce the existence of $r^* = (s^*, t^*, p^*)$ satisfying (8)–(11) and (13). Now suppose $r'' \leq r^*$ forces that $\alpha \in \dot{A} - \delta$. We will extend r'' so that the obtained extension satisfies the statement of Proposition 18.

The only thing we require from the extension $r^{**} \leq r''$ is that it decides $k \in \omega$ such that:

(p) $X_\alpha^{r^{**}} - k \subseteq \omega - Y_\delta^{r^{**}}$.
(r) $\eta_\alpha(\mathcal{F})(n) < \psi(n)$ for $n \in X_\alpha^{r^{**}} - k$.

Note that by (p), $\psi(n)$ in the condition (r) is defined. We will show that for each $n \in X_\alpha^{r^{**}} - k$ we have

$$r^{**} \Vdash \exists \beta \in \dot{A} \cap \delta \;\; [n \in X_\beta \;\&\; m(\beta, \delta) \leq n].$$

Since $r^{**} \leq r^*$, we have $\omega - Y_\delta^{r^{**}} = \{n_i : i < \omega\}$, so for $n \in X_\alpha^{r^{**}} - k$ find $i < \omega$ such that $n = n_i$ (using (p)), and apply property (18) for $r^{**} \leq r^* \leq r_i$, to conclude that r^{**} forces that either

$$\forall \beta \in \dot{A} \; (n \in X_\beta \Rightarrow \beta \notin b_i)$$

is false, or

$$\eta_\alpha(\mathcal{F}_{r^{**}})(n) > \psi(n)$$

holds. This is because the first displayed formula of condition (18) is true by Fact 6(3) because $n > m(\delta, \alpha)$ and $\mathcal{F}_n$ covers both β and α (by the choice of n), so $a_\alpha(\mathcal{F})(n) \cap \delta = a_\delta(\mathcal{F})(n) \cap \delta = b_i$.

By (r), the second of the above displayed formulas is false, so r^{**} forces that there is $\beta \in \dot{A}$ such that $n \in X_\beta$ and $\beta \in b_i$. In particular, $\beta < \delta$, and by the definition of the function m and by (12) we have $m(\beta, \delta) \leq n$, as required in the statement of Proposition 18. ∎

Proof of Proposition 2. Let $\dot{H}$ be a P-name for a generic filter in R. Consider the R-names

$$\mathcal{X} = \bigcup \{\mathcal{X}_s : (s, \emptyset, \emptyset) \in H\}, \quad \mathcal{F} = \bigcup \{\mathcal{F}_t : \exists s \in P_1 \; [(s, t, \emptyset) \in H]\}.$$

We claim that Proposition 2 is witnessed by $\mathcal{X}$ and $\Sigma(\mathcal{F})$ (see Definition 8). To verify condition (i) of the proposition, note that for each R-name $\dot{A}$ for a subset of ω_1 there is a stage $\xi < \omega_2$ of the iteration $\dot{Q}_{\omega_2}$ such that $\dot{A} = \dot{A}_\xi$. Using a density argument we conclude that

$$\dot{u} = \bigcup \{n \in \omega : p(\xi)(n) = 1, \; \exists (s, t) \in P \; [(s, t, p) \in \dot{H}]\}$$

satisfies condition (i) of Proposition 2.

To get condition (ii), note that it is implied by the definition of $\Sigma(\mathcal{F})$, for if $\alpha \neq \alpha'$, then the last terms of $\sigma_{\alpha,n}$ and $\sigma_{\alpha',n'}$ differ, as they are α and α' themselves respectively, thus $\sigma_{\alpha,n} \neq \sigma_{\alpha',n'}$.

To conclude the last hereditary Lindelöf property, apply the Reduction Lemma 9, and Proposition 18, to $\mathcal{X}$. ∎

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