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Selections that characterize topological completeness

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Abstract. We show that the assertions of some fundamental selection theorems for lower-semicontinuous maps with completely metrizable range and metrizable domain actually characterize topological completeness of the target space. We also show that certain natural restrictions on the class of the domains change this situation. The results provide in particular answers to questions asked by Engelking, Heath and Michael [3] and Gutev, Nedev, Pelant and Valov [5].

1. Introduction. Let Y be a completely metrizable space. Michael [12] proved that for any 0-dimensional (in the sense of the covering dimension dim) metrizable space X, any lower-semicontinuous map $X \to \mathcal{F}(Y)$ has a continuous selection (here $\mathcal{F}(Y)$ denotes the collection of nonempty closed subsets of Y).

We shall show that a metrizable space Y satisfying the assertion of Michael's Theorem must be completely metrizable. This is a consequence (see Corollary 3.2) of Theorem 1.1 below, which is the main result of this paper.

Let M be a metric space. We denote by $\mathcal{C}(M)$ the collection of all closed discrete $D \subseteq M$ for which there exists a Cauchy sequence $(d_i)_{i=1}^{\infty}$ in M with $D=\{d_1, d_2, \ldots\}$. Roughly speaking, $\mathcal{C}(M)$ consists of all finite nonempty subsets of M together with all the Cauchy sequences that have no limit. We shall consider $\mathcal{C}(M)$ with the Hausdorff distance with respect to the metric on M.

1.1. Theorem. *$f : M \to N$ be continuous, where M is metric and N is Hausdorff. Assume that all fibers of f are topologically complete. If there*

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exists a continuous map $s : \mathcal{C}(M) \to N$ *such that* $s(D) \in f[D]$ *for every* $D \in \mathcal{C}(M)$ *then* M *is completely metrizable.*

Actually, the proof of Theorem 1.1 shows that if $\mathcal{C}(M)$ is endowed with any "reasonable" topology (such as the Vietoris topology) then the existence of a "selection" for f implies that M is topologically complete. This allows us to answer a question of Engelking, Heath and Michael [3] in the affirmative. They asked whether a metrizable space which admits a continuous selection on the space of all of its nonempty closed subsets, endowed with the Vietoris topology, must always be completely metrizable. Let us recall that they constructed a continuous selection for the Vietoris hyperspace of a 0-dimensional completely metrizable space M by defining a topological well-ordering on M (i.e. a compatible ordering on M with the additional property that each closed subset of M has a least element). So by our results, the existence of a topological well-ordering on a 0-dimensional metrizable space M is equivalent to the existence of a topological (i.e. continuous) selection.

The existence of a continuous selection for closed subsets is a rather special property: among compacta, those which admit such selections are the orderable ones only (van Mill and Wattel [14]). One can, however, relax the continuity condition, obtaining a characterization of completeness in terms of selections. This is demonstrated in §6, by a modification of the proof of Theorem 1.1. Yet another modification of this proof yields a characterization of complete metrizability in terms of compact-valued upper-semicontinuous selections, presented in §7.

We shall also discuss (leaving some natural problems open) the following question. Let Y be metrizable and let $\mathcal{Z}$ be a natural class of 0-dimensional metrizable spaces. When does the assumption that Y has the "$\mathcal{Z}$-selection property", i.e. each lower-semicontinuous map $X \to \mathcal{F}(Y)$ with $X \in \mathcal{Z}$ has a continuous selection, imply that Y is completely metrizable? Our main result provides a positive answer in case $\mathcal{Z}$ is the class of 0-dimensional spaces of density equal to the density of Y.

Using an example due to Stone from [15] we shall show (Corollary 4.2) that there exist non-absolutely Borel spaces Y of weight $\aleph_1$ with the "separable selection property" (i.e., $\mathcal{Z}$ is the class of all separable 0-dimensional spaces).

We also show, using some results of Kanoveĭ and Ostrovskiĭ [8], that the statement "all analytic spaces with the Cantor selection property (i.e., $\mathcal{Z} = \{\text{Cantor set}\}$) are completely metrizable" is independent of the usual axioms for set theory. On the other hand, the case of coanalytic spaces is much simpler. By Hurewicz's classical result, a coanalytic space which is not completely metrizable contains a closed copy of the rationals $\mathbb{Q}$. However, $\mathbb{Q}$ does not have the Cantor selection property (not being Baire [5]), hence the

Cantor selection property for coanalytic spaces is equivalent to topological completeness.

Finally, assuming Martin's Axiom, we construct a subset of the real line not belonging to the σ-algebra generated by the analytic sets but with the Cantor set selection property.

These results answer some questions of Gutev, Nedev, Pelant and Valov [5].

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2. Preliminaries. We denote by $\mathcal{F}(X)$ the collection of all nonempty closed subsets of the topological space X. If X and Y are topological spaces then $F : X \to \mathcal{F}(Y)$ is called *lower-semicontinuous* (for short: l.s.c.) provided that for every open subset $U \subseteq Y$ the set $\{x \in X : F(x) \cap U \neq \emptyset\}$ is open in X. A (*continuous*) *selection* for $F : X \to \mathcal{F}(Y)$ is a (continuous) function $f : X \to Y$ such that $f(x) \in F(x)$ for every $x \in X$.

The *Vietoris topology* on $\mathcal{F}(X)$ is generated by the sets $\{A \in \mathcal{F}(X) : A \cap U \neq \emptyset\}$ and $\{A \in \mathcal{F}(X) : A \subseteq U\}$ with U an arbitrary open subset of X.

We shall state in Lemma 2.1 a criterion for complete metrizability (close to some "complete covers" characterizations originated by Z. Frolík) which we shall use in the proof of Theorem 1.1. The criterion is based on the classical theorem of Montgomery [10, §30, X.3] that locally G_δ-sets in metrizable spaces are G_δ.

Let us call a disjoint family $\mathcal{S}$ of subsets of a metrizable space X an $\mathcal{M}$-family (cf. [10, §30, X(1)]) if there exists a transfinite sequence $G_0, G_1, \ldots, G_\xi, \ldots$, $\xi < \kappa$, of open sets in X such that $\mathcal{S}$ is equal to the collection

$$\Big\{\bigcup \mathcal{S} \cap G_\xi \setminus \bigcup_{\eta<\xi} G_\eta : \xi < \kappa\Big\}.$$

(In Hansell [6] such an $\mathcal{S}$ is called a scattered family.) Evidently, $\mathcal{S}$ is an $\mathcal{M}$-family in X if and only if $\mathcal{S}$ is an $\mathcal{M}$-family in its union $\bigcup \mathcal{S}$. Montgomery's Theorem implies (by transfinite induction) that the union of an $\mathcal{M}$-family of G_δ-sets is a G_δ-set.

2.1. Lemma. *Let $\mathcal{S}_0, \mathcal{S}_1, \ldots$ be disjoint families of G_δ-sets in a metric space M satisfying the following conditions*:

(A) $\mathcal{S}_0 = \{M\}$, $\mathcal{S}_{n+1}$ *refines* $\mathcal{S}_n$ *and* $\operatorname{diam} S \leq 1/n$ *for* $S \in \mathcal{S}_n$, $n \geq 1$;

(B) *if* $S \in \mathcal{S}_n$ *then* $\mathcal{S}(S) = \{T \in \mathcal{S}_{n+1} : T \subseteq S\}$ *is an* $\mathcal{M}$*-family and* $G(S) = S \setminus \bigcup \mathcal{S}(S)$ *is an absolute* G_δ*-set*;

(C) *if* $S_1 \supseteq S_2 \supseteq \ldots$, $S_n \in \mathcal{S}_n$, *then* $\bigcap_n \overline{S}_n \subseteq M$, *where the closure is taken in the completion* M^* *of* M.

Then M *is completely metrizable.*

Proof. One readily sees that given an $\mathcal{M}$-family $\mathcal{S}$ of G_δ-sets in M, each $S \in \mathcal{S}$ can be extended to a G_δ-set S^* in the completion M^* contained in $\overline{S}$ so that $\mathcal{S}^* = \{S^* : S \in \mathcal{S}\}$ is an $\mathcal{M}$-family in M^*. Using this observation, one can get subsequently disjoint families $\mathcal{S}_n^* = \{S^* : S \in \mathcal{S}_n\}$ of G_δ-sets in M^* such that $\mathcal{S}_{n+1}^*$ refines $\mathcal{S}_n^*$ and for each $S^* \in \mathcal{S}_n^*$, $\{T^* \in \mathcal{S}_{n+1}^* : T^* \subseteq S^*\}$ is an $\mathcal{M}$-family.

Let

$$G_n = \bigcup\{G(S) : S \in \mathcal{S}_n\}, \quad H_n = \bigcup \mathcal{S}_n^*.$$

By the remark preceding Lemma 2.1, one checks inductively for $i=0,1,\ldots,n$ that for each $S^* \in \mathcal{S}_{n-i}^*$, the sets $G_n \cap S^*$ and $H_n \cap S^*$ are G_δ in M^*. For $i = n$ this means that G_n and H_n are G_δ-sets in M^*. It is therefore enough to check that

$$M = \bigcap_n (G_0 \cup \ldots \cup G_{n-1} \cup H_n).$$

Since $G_n \subseteq M$ we have to consider an arbitrary point $x \in \bigcap_n H_n$. There are $S_i^* \in \mathcal{S}_i^*$ with $x \in \bigcap_{i\ge1} S_i^*$. Then $S_1 \supseteq S_2 \supseteq \ldots$ and, by (C), $x \in M$. ∎

3. Proof of Theorem 1.1. In this section we prove our main result and derive from it some consequences. The proof given in 3.1 below will be slightly modified in the forthcoming §6. Having this in mind, we shall deal with collections of locally closed sets (Engelking [2]), i.e. sets that are the intersection of an open and a closed set, rather than with G_δ-sets.

3.1. Proof of Theorem 1.1. Let M, N, f and s be as in Theorem 1.1. Let $\mathcal{C}_f$ be the subspace of $\mathcal{C}(M)$ consisting of all $C \in \mathcal{C}(M)$ on which f is injective. We shall define inductively disjoint families $\mathcal{S}_0, \mathcal{S}_1, \ldots$ of nonempty locally closed sets in M satisfying the conditions (A) and (B) in Lemma 2.1, associating in addition with each $S \in \mathcal{S}_n$, $n \ge 1$, a finite set $F(S) \in \mathcal{C}_f$ such that

(1) $f[F(S)] \cap \overline{f[S]} = \emptyset$ and if $C \in \mathcal{C}_f$, $C \subseteq S$ then $s(F(S) \cup C) \in f[C]$,

(2) if $S \in \mathcal{S}_n$, $T \in \mathcal{S}_{n+1}$, $T \subseteq S$, then $F(T) \setminus S = F(S)$ and $F(T) \cap S \neq \emptyset$.

To show that M is completely metrizable, it is enough to verify that (1) and (2) imply that property (C) in Lemma 2.1 holds. To this end, let $S_1 \supseteq S_2 \supseteq \ldots$ with $S_n \in \mathcal{S}_n$, let $F_n = F(S_n)$ and $A = \bigcup_n F_n$. By (1) and (2), $F_1 \subseteq F_2 \subseteq \ldots$, $A \cap S_n \neq \emptyset$ for all n, f is injective on A, and since $A \setminus F_n \subseteq S_n$ and $\operatorname{diam} S_n \to 0$, A is the range of a Cauchy sequence. To check that $\bigcap_n \overline{S}_n \subseteq M$ we make sure that this sequence converges in M.

For suppose otherwise, i.e. $A \in \mathcal{C}_f$. By (1), $f[A] \cap \bigcap_n \overline{f[S_n]} = \emptyset$. But then $s(A) \notin \overline{f[S_n]}$ for some n, and letting $C = A \setminus F_n \subseteq S_n$ we get $C \in \mathcal{C}_f$ and $s(F_n \cup C) \notin f[C]$, which contradicts (1).

It remains to construct the families $\mathcal{S}_n$ and the assignments $F(S)$ for $S \in \mathcal{S}_n$. Suppose that $\mathcal{S}_n$ is defined, fix an arbitrary element $T \in \mathcal{S}_n$ and let $F = F(T)$. For $n = 0$, we put $T = M$ and $F = \emptyset$.

We apply a standard exhaustion procedure. We define a sequence $\emptyset = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_\xi \subseteq \ldots$ of relatively open sets in T with $G_\xi = \bigcup\{G_\lambda : \lambda < \xi\}$ for limit ξ, while moreover

(3) $S_\xi = G_{\xi+1} \setminus G_\xi$ is nonempty and has diameter $\le 1/(n+1)$;
(4) $F_\xi \in \mathcal{C}_f$ is a finite subset of $T \setminus S_\xi$ which is associated with S_ξ (1);
(5) the pair $S = S_\xi$, $F(S) = F \cup F_\xi$ satisfies (1).

This process terminates at some λ, providing the elements of $\mathcal{S}_{n+1}$ contained in T and the associated finite sets. We only have to check that $G(T) = T \setminus G_\lambda$ is an absolute G_δ-set.

Striving for a contradiction, assume that it is not. Then $G(T)$ is not contained in a fiber of f (since the fibers of f are topologically complete), and so there are $a, b \in G(T)$ with $f(a) \ne f(b)$. By (1), $s(F \cup \{a,b\}) \in \{f(a), f(b)\}$ and we may assume without loss of generality that $s(F \cup \{a,b\}) = f(b)$. Let V be a neighborhood of $f(b)$ the closure of which misses $f[F \cup \{a\}]$. By continuity of s, there exists a neighborhood W of b in T with $\operatorname{diam} W \le 1/(n+1)$ and $f[W] \subseteq V$ such that for all $C \in \mathcal{C}_f$ contained in W, $s(F \cup \{a\} \cup C) \in V$, i.e., $s(F \cup \{a\} \cup C) \in f[C]$. But then, setting $G_{\lambda+1} = W \cup G_\lambda$, $S_\lambda = G_{\lambda+1} \setminus G_\lambda$, $F_\lambda = \{a\}$, we would extend the procedure beyond the ordinal λ at which the process terminated.

3.2. Corollary. *Let Y be a metrizable space. The following statements are equivalent.*

(1) *Y is completely metrizable.*

(2) *For every* 0*-dimensional metrizable space X, every lower-semicontinuous function $F : X \to \mathcal{F}(Y)$ admits a continuous selection $s : X \to Y$.*

(3) *For every* 0*-dimensional metrizable space X with density $X \le$ density Y, every lower-semicontinuous function $F : X \to \mathcal{F}(Y)$ admits a continuous selection $s : X \to Y$.*

Proof. As observed in the introduction, the implication (1)⇒(2) is a result due to Michael [12]. Since (2)⇒(3) is trivial, it suffices to consider (3)⇒(1). Fix a 0-dimensional space X which admits a perfect map

(1) In fact, F_ξ is a singleton in this proof. But in the modification in §6, F_ξ can contain more elements.

$f : X \to Y$. (That such a map exists is well known. For a simple construction of f, see Remark 6.2.) The density of X does not exceed the density of Y. Let d be a metric on X with the property that the completion (X^*, d^*) of (X, d) is again 0-dimensional. We will show that the Hausdorff metric with respect to this metric induces a 0-dimensional topology on $\mathcal{C}(X)$ with density not exceeding the density of X. Identifying $C \in \mathcal{C}(X)$ with its closure in X^*, we embed $\mathcal{C}(X)$ topologically in the space $\mathcal{K}(X^*)$ of compact subsets of X^* with the Vietoris topology. Since $\mathcal{K}(X^*)$ is 0-dimensional, the densities of X and X^* agree, and the densities of X^* and $\mathcal{K}(X^*)$ agree, this will establish our claim. That $\mathcal{K}(X^*)$ is 0-dimensional is probably well known, but for the reader's convenience we provide a simple proof of this fact. Let $\mathcal{U}_n$ ($n \in \mathbb{N}$) be a sequence of clopen partitions of X^* such that (1) for every n, $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$, and (2) every $U \in \mathcal{U}_n$ has diameter at most $1/n$. For every finite subcollection $\mathcal{V} \subseteq \mathcal{U}_n$, the set

$$\langle \mathcal{V} \rangle = \{K \in \mathcal{K}(X^*) : (K \subseteq \bigcup \mathcal{V}) \wedge (\forall V \in \mathcal{V} : V \cap K \neq \emptyset)\}$$

is an open and closed subset of $\mathcal{K}(X^*)$. In addition, the collection

$$\mathcal{U}_n^* = \{\langle \mathcal{V} \rangle : \mathcal{V} \subseteq \mathcal{U}_n \text{ finite}\}$$

is a partition of $\mathcal{K}(X^*)$. Moreover, $\bigcup_{n=1}^{\infty} \mathcal{U}_n^*$ is easily seen to be a base for $\mathcal{K}(X^*)$. We conclude that $\mathcal{K}(X^*)$ is 0-dimensional.

Now the function $F : \mathcal{C}(X) \to Y$ defined by $F(D) = f[D]$ is l.s.c. So, F admits a selection. By Theorem 1.1, X is completely metrizable, and so is Y by [2, 4.5.13(e)]. ∎

3.3. Remark. Let X and Y be metrizable spaces and let $f : X \to Y$ be an open surjection. Hausdorff's Theorem from [7] says that if X is completely metrizable then so is Y. We will now sketch a new proof of this fact within the framework of selections. Let M be a 0-dimensional space and let $F : M \to \mathcal{F}(Y)$ be l.s.c. The function $\Phi : M \to \mathcal{F}(X)$ defined by $\Phi(m) = f^{-1}[F(m)]$ is also l.s.c. To see this, simply observe that for an open subset $U \subseteq X$ we have

$$\Phi(m) \cap U \neq \emptyset \Leftrightarrow F(m) \cap f[U] \neq \emptyset.$$

Since X is complete, by Michael's Theorem, Φ has a continuous selection s. But then $f \circ s$ is a continuous selection for F. By Corollary 3.2 we conclude that Y must be completely metrizable.

3.4. Remark. Let us remark that in the proof of Theorem 1.1 we only used the following fact about the topology on $\mathcal{C}(M)$:

- If $F \subseteq M$ is finite, $x \in F$ and $s(F) = f(x)$ then for every neighborhood U of $s(F)$ there exists a neighborhood V of x with $(F \setminus \{x\}) \cap V = \emptyset$ such that for every $C \in \mathcal{C}(M)$ with $C \subseteq V$ we have $s(C \cup (F \setminus \{x\})) \in U$.

This fact also holds true for the Vietoris topology on $\mathcal{C}(M)$. So by taking $M = N$ and f the identity on M, as a corollary to the proof of Theorem 1.1, we deduce that the existence of a continuous selection for the space of all nonempty closed subsets of M, endowed with the Vietoris topology, implies that M is completely metrizable. This answers a question of Engelking, Heath and Michael [3] in the affirmative. For a stronger result in this direction, see Theorem 6.1.

3.5. COROLLARY. *Let X be a metrizable space and let $\mathcal{F}(X)$ denote the collection of all nonempty closed subsets of X endowed with the Vietoris topology. If there is a continuous function $s : \mathcal{F}(X) \to X$ such that $s(A) \in A$ for every $A \in \mathcal{F}(X)$, then X is completely metrizable.* ∎

4. A non-Borel space with the "separable selection property". We shall show in this section that by removing a "free sequence of type ω_1" of closed sets from a complete space we get a space with the "separable selection property". Then, using Stone's space S from [15], we get an example with the properties stated in the title.

4.1. THEOREM. *Let X be a complete metric space and let $\langle F_\xi : \xi < \omega_1\rangle$ be a family of closed subsets of X such that for every limit ordinal $\gamma < \omega_1$ we have*

$$\overline{\bigcup_{\alpha\le\gamma} F_\alpha} \cap \bigcup_{\alpha>\gamma} F_\alpha = \emptyset.$$

Put $E = X \setminus \bigcup_{\alpha<\omega_1} F_\alpha$. Then for every lower-semicontinuous map $\varphi : T \to \mathcal{F}(E)$ defined on a 0-dimensional separable metrizable space T, φ has a continuous selection.

P r o o f. Let d be a complete metric on X. By the 0-dimensionality of T it is enough to check that each $t_0 \in T$ has a neighborhood W such that the l.s.c. map $\varphi\restriction W$ has a continuous selection.

Let Γ denote the set of limit ordinals in ω_1. For $\gamma \in \Gamma$ we let

$$Z_\gamma = \overline{\bigcup_{\alpha\le\gamma} F_\alpha}.$$

Then $Z = \bigcup_{\gamma\in\Gamma} Z_\gamma$ is closed in X, being the union of an increasing sequence of type ω_1 of closed sets in the metric space X.

For each $\xi < \omega_1$ we list the countable collection $\{F_0, \ldots, F_\xi\}$ as $\{F_\xi^n : n \in \mathbb{N}\}$.

Now fix an arbitrary element $x_0 \in \varphi(t_0)$. If $x_0 \notin Z$ then in a neighborhood of t_0, $\varphi(t)$ intersects the open subset $X \setminus Z$ of X, and then we can use Michael's Theorem to get a selection for φ in this neighborhood. We shall assume therefore that $x_0 \in Z$.

We shall define inductively continuous functions $g_i : T \to X$, ordinals $\alpha_1 < \alpha_2 < \ldots < \omega_1$, disjoint open covers $\mathcal{V}_i$ of T with $\mathcal{V}_{i+1}$ refining $\mathcal{V}_i$, and for each $V \in \mathcal{V}_i$, positive numbers $\varepsilon(V)$, such that

(1) if $t \in V$ and $V \in \mathcal{V}_i$ then $d(g_i(t), g_{i+1}(t)) \le \frac{1}{3}\varepsilon(V)$;

(2) if $V \in \mathcal{V}_i$, $U \in \mathcal{V}_{i+1}$ and $U \subseteq V$ then $\varepsilon(U) < \frac{1}{3}\varepsilon(V)$;

(3) $g_i(t) \in \overline{\varphi(t)}$, $g_i(t_0) = x_0$, $g_i(T) \cap Z \subseteq Z_{\alpha_i}$;

(4) if $U \in \mathcal{V}_{i+1}$ then

$$\varepsilon(U) < \tfrac{1}{3}\operatorname{dist}(g_{i+1}[U], L_i),$$

where $L_i = \bigcup\{F^j_{\alpha_k} : j, k \le i\}$;

(5) if $t \in V$, $V \in \mathcal{V}_i$ and $g_{i+1}(t) \ne g_i(t)$, then

$$\operatorname{dist}(g_{i+1}(t), g_i[T] \cap Z) < \varepsilon(V).$$

We start with any continuous selection g_1 for the l.s.c. map $t \mapsto \overline{\varphi(t)}$ satisfying $g_1(t_0) = x_0$, we set $\mathcal{V}_1 = \{T\}$, $\varepsilon(T) = 1$ and we pick α_1 so that $g_1(T) \cap Z \subseteq Z_{\alpha_1}$.

Assume that g_i, $\mathcal{V}_i$, α_i and $\varepsilon(V)$ for $V \in \mathcal{V}_i$ have been defined.

Fix $V \in \mathcal{V}_i$. Let H be a clopen neighborhood of $g_i^{-1}[Z] \cap V$ contained in the open set

$$\big\{t \in V : \operatorname{dist}(g_i(t), g_i[T] \cap Z) < \tfrac{1}{2}\varepsilon(V)\big\}.$$

For each $t \in V \setminus H$ we let $g_{i+1}(t) = g_i(t)$. Consider $G = X \setminus \bigcup\{F^j_{\alpha_k} : j, k \le i\}$ and define the l.s.c. map $\Psi : H \to \mathcal{F}(G)$ by the formula

$$\Psi(t) = \overline{\big\{x \in \varphi(t) : d(x, g_i(t)) < \tfrac{1}{3}\varepsilon(V)\big\}} \cap G.$$

Since G is open in X and hence is completely metrizable, we can find a continuous selection $g_{i+1} : H \to G$ for Ψ with $g_{i+1}(t_0) = x_0$. Split H and $H \setminus V$ into pairwise disjoint open sets U such that $\operatorname{dist}(g_{i+1}[U], L_i) > 0$, where L_i is defined by (4). These sets form the part of $\mathcal{V}_{i+1}$ that refines V. For each of these sets U choose $\varepsilon(U) > 0$ satisfying (4) and (2).

Finally, $\alpha_{i+1} > \alpha_i$ is chosen so that $g_{i+1}[T] \cap Z \subseteq Z_{\alpha_{i+1}}$.

By (1) and (2) the sequence $(g_i)_{i=1}^{\infty}$ converges uniformly to a continuous function $g : T \to X$. By (3),

(6) $g(t) \in \overline{\varphi(t)}$, $g(t_0) = x_0$.

Let us check that

(7) $g[T] \cap Z \subseteq Z_\gamma$, where $\gamma = \sup\{\alpha_i : i = 1, 2, \ldots\}$.

Suppose $g(t) = \lim_{i\to\infty} g_i(t) \in Z$. If $g_i(t) \in Z$, then by (3), $g_i(t) \in Z_{\alpha_i}$, so we are done if this happens for infinitely many i. Otherwise, for infinitely many i, $g_{i+1}(t) \ne g_i(t)$ and, using (5) and (3), we can pick for each such i a point $z_i \in Z_{\alpha_i}$ with $d(g_{i+1}(t), z_i) \le \varepsilon(V_i)$, where $t \in V_i$, $V_i \in \mathcal{V}_i$. Since

$\varepsilon(V_i) \to 0$, by (2), and all accumulation points of the sequence $\{z_i\}$ are in Z_γ, we obtain (7).

Conditions (1) and (4) guarantee that $g[T]$ is disjoint from $\bigcup_{i=1}^{\infty} L_i = \bigcup_{\alpha<\gamma} F_\alpha$, i.e. by (7),

$$g[T] \cap \bigcup_{\alpha<\omega_1} F_\alpha \subseteq F_\gamma.$$

By (6), $g(t_0) = x_0 \notin F_\gamma$. Therefore, $W = T \setminus g^{-1}[F_\gamma]$ is a neighborhood of t_0 such that $g[W] \subseteq E$. From (6) we get $g(t) \in \varphi(t)$ for $t \in W$. ∎

Let us recall a construction due to A. H. Stone. Let $B(\aleph_1)$ be the countable product of the set of countable ordinals with the discrete topology. For each limit ordinal $\gamma \in \Gamma$ choose a sequence $s_\gamma = \{\gamma_i\}$ in ω_1 with $\gamma_i \nearrow \gamma$. Stone [15] proved that the set $S = \{s_\gamma : \gamma \in \Gamma\}$ is non-Borel in $B(\aleph_1)$. Setting $F_\gamma = \{s_\gamma\}$ we get a sequence in $B(\aleph_1)$ such as in Theorem 4.1. Therefore we obtain

4.2. COROLLARY. *Let $S \subseteq B(\aleph_1)$ be Stone's set. Then the non-Borel space $E = B(\aleph_1) \setminus S$ has the property that each lower-semicontinuous map $\varphi : T \to \mathcal{F}(E)$ defined on a 0-dimensional separable metrizable space T has a continuous selection.* ∎

4.3. Remark. If in Theorem 4.1, "ω_1" is replaced by a regular cardinal κ, and we demand that the union $\bigcup\{F_\alpha : \alpha \le \gamma\}$ is F_σ for $\gamma < \kappa$, then the proof shows that the assertion of the theorem is true for all T of density less than κ. Using this fact, and axiom $E(\kappa)$ (cf. Fleissner [4, Definition 3.10]), one can supplement Corollary 4.2 with the following statement: it is consistent with the usual axioms for set theory that for each regular cardinal κ there exists a metrizable non-Borel space of density κ which has the selection property with respect to all 0-dimensional spaces of density less than κ.

4.4. Remark. Let E be a metric space with completion E^* such that $E^* \setminus E$ does not contain any Cantor set. Then for every l.s.c. map $\varphi : X \to \mathcal{F}(E)$ there exists an l.s.c map $\psi : X \to \mathcal{F}(E)$ for which $\psi(x)$ is separable and contained in $\varphi(x)$ for every $x \in X$ (this cannot always be done, even for the spaces E considered in Theorem 4.1). This result can also be used to prove Corollary 4.2, or its extension described in Remark 4.3. In general, the "shrinking theorem" and Theorem 4.1 have different areas of applications. This alternative approach will be discussed in detail elsewhere.

5. Analytic spaces with the "compact selection property". A separable metrizable space X is called *analytic* if it is a continuous image of the space $\mathbb{P}$ of irrational numbers. In this section we are interested in the question whether all analytic spaces with the Cantor selection property are completely metrizable.

The cardinality of the continuum is denoted by $\mathfrak{c}$ throughout. We denote Martin's Axiom by MA (see Kunen [9] for more information). Let us recall that MA guarantees that in any separable completely metrizable space the intersection of fewer than $\mathfrak{c}$ dense open sets is dense.

5.1. THEOREM. (*MA*) *Let X be an uncountable separable completely metrizable space and let $A \subseteq X$ be of cardinality less than $\mathfrak{c}$. Then $Y = X \setminus A$ has the Cantor selection property.*

Proof. Let K denote the Cantor set and let $F : K \to \mathcal{F}(Y)$ be l.s.c. The function $\varphi : K \to \mathcal{F}(X)$ defined by $\varphi(x) = \overline{F(x)}$ is l.s.c. as well. Let $C(K, X)$ denote the collection of all continuous functions from K to X endowed with the compact-open topology. As is well known, $C(K, X)$ is separable and completely metrizable. Define

$$\mathcal{A} = \{f \in C(K, X) : f \text{ is a selection for } \varphi\}.$$

It is easily seen that $\mathcal{A}$ is a closed subset of $C(K, X)$, whence $\mathcal{A}$ is separable and completely metrizable as well. For every $a \in A$ put

$$\mathcal{A}_a = \{f \in \mathcal{A} : a \notin f[K]\}.$$

It is clear that $\mathcal{A}_a$ is an open subset of $\mathcal{A}$ and we claim that it is also dense. To see this, pick an arbitrary element $f \in \mathcal{A}$ and let $\varepsilon > 0$. Fix an open neighborhood U of a of diameter less than ε. The set $V = \{x \in K : F(x) \cap U \neq \emptyset\}$ is open in K since F is l.s.c. Put $Z = X \setminus \{a\}$. Let us define $\psi : K \to \mathcal{F}(Z)$ by

$$\psi(x) = \begin{cases} \overline{F(x) \cap U} \setminus \{a\} & \text{if } x \in V, \\ \{f(x)\} & \text{if } x \in K \setminus V. \end{cases}$$

Then ψ is l.s.c. The space Z is topologically complete and hence we can use Michael's Theorem to get a selection t for ψ. Evidently, t is ε-close to f.

Now by MA, the set $\mathcal{B} = \bigcap_{a \in A} \mathcal{A}_a$ is dense in $\mathcal{A}$. Clearly, every $f \in \mathcal{B}$ is a selection for F. ∎

5.2. Remark. Let us now consider the question of whether all analytic spaces with the Cantor selection property are completely metrizable. The standard Cantor set will be denoted by K throughout. By Martin and Solovay [11, 3.1 and 3.2], it is consistent with $\mathrm{MA} + \neg\mathrm{CH}$ that $K \setminus A$ is analytic, for every subset of A of cardinality $\aleph_1$. Now let A be any subset of K of cardinality $\aleph_1$. Then clearly $K \setminus A$ is not complete since otherwise A would either be countable or of cardinality $\mathfrak{c}$. So in this model $K \setminus A$ is an example of an analytic space which is not topologically complete but yet has the Cantor selection property. On the other hand, Kanoveĭ and Ostrovskiĭ [8] showed that there is also a model of set theory in which every analytic space which is not complete contains a closed copy of $\mathbb{Q}$, the space consisting of all rational numbers. In this model, every analytic space with the Cantor selection property is complete by [5, Theorem 1].

5.3. Remark. Let X be an uncountable compact metrizable space and let $\mathcal{A}$ be a family of fewer than $\mathfrak{c}$ closed nowhere dense sets. With the same proof as in Theorem 5.1 it follows that under MA the space $X \setminus \bigcup \mathcal{A}$ has the Cantor selection property. This allows us to use a Bernstein type argument to construct in the unit interval $\mathbb{I}$ under MA a pair A, B of disjoint dense sets with the Cantor selection property. By [5, Theorem 1] both A, B are of second Baire category at each point of $\mathbb{I}$, and therefore they are not open modulo a first category set (and so, they do not belong to the σ-algebra generated by the analytic sets). Let us sketch the argument.

We endow $\mathcal{F}(\mathbb{I})$ with the Vietoris topology. Since every l.s.c. function $\varphi : K \to \mathcal{F}(\mathbb{I})$ is a Borel map, there are only $\mathfrak{c}$ such maps. List them as $\{\varphi_\alpha : 1 \le \alpha < \mathfrak{c}\}$. Let A_0 and B_0 be disjoint countable dense subsets of $\mathbb{I}$. By transfinite induction on $1 \le \alpha < \mathfrak{c}$, applying the above modified Theorem 5.1 at every step of the construction, it is now possible to construct compact subsets A_α and B_α such that

(1) $\bigcup_{\xi\le\alpha} A_\xi \cap \bigcup_{\xi\le\alpha} B_\xi = \emptyset$;

(2) if for every $t \in K$, $\varphi_\alpha(t) \setminus \bigcup_{\xi\le\alpha} B_\xi$ is dense in $\varphi_\alpha(t)$, then there exists a continuous selection $s_\alpha : K \to A_\alpha$ for φ_α;

(3) if for every $t \in K$, $\varphi_\alpha(t) \setminus \bigcup_{\xi\le\alpha} A_\xi$ is dense in $\varphi_\alpha(t)$, then there exists a continuous selection $t_\alpha : K \to B_\alpha$ for φ_α.

Then $A = \bigcup_{\alpha<\mathfrak{c}} A_\alpha$ and $B = \bigcup_{\alpha<\mathfrak{c}} B_\alpha$ are disjoint dense subsets of $\mathbb{I}$ both having the Cantor selection property. Indeed, if e.g. $\psi : K \to \mathcal{F}(A)$ is l.s.c., then $\varphi : K \to \mathcal{F}(\mathbb{I})$ defined by $\varphi(t) = \overline{\psi(t)}$ is listed as some φ_α which satisfies (2).

6. Selections for $\mathcal{F}(M)$ that characterize topological completeness of M.

In this section we shall consider the space $\mathcal{F}(X)$ of closed nonempty subsets of X with the Vietoris topology. We denote by $\mathcal{D}(X)$ the subspace of $\mathcal{F}(X)$ consisting of discrete subsets.

Given $\mathcal{A} \subseteq \mathcal{F}(X)$ we say that $s : \mathcal{A} \to X$ is a *selection* for $\mathcal{A}$ if $s(A) \in A$ for $A \in \mathcal{A}$ (cf. [3, p. 150]).

Engelking, Heath and Michael [3] proved in Corollary 1.2 that for each completely metrizable 0-dimensional space M, there is a continuous selection for $\mathcal{F}(M)$. Combined with Corollary 3.5 this provides a characterization of complete metrizability in the realm of 0-dimensional spaces.

But, as was mentioned in the introduction, only for linearly ordered compacta there is a continuous selection for the collection of two-point sets, so there is no continuous selection for the family of all two-point sets in the unit circle. There is also no continuous selection for the family $\mathcal{D}(\mathbb{R})$, where $\mathbb{R}$ is the real line (see [3], the remark following the proof of Proposition 5.1).

To characterize completeness in the general situation we are therefore forced to impose a weaker continuity property on the selection.

6.1. Theorem. *For a metrizable space M the following conditions are equivalent*:

(1) *M is completely metrizable.*

(2) *There exists a selection for $\mathcal{D}(M)$ such that its restriction to each set of the form $\mathcal{F}(Z)\cap\mathcal{D}(M)$, with Z closed in M, has a dense set of points of continuity.*

Proof. (1)⇒(2). Let $p : X \to M$ be a continuous one-to-one map from a completely metrizable 0-dimensional space X onto M such that the inverse map $p^{-1} : M \to X$ has the property that its restriction to each closed subset of M has a point of continuity. (Since we did not find a convenient reference for p in the nonseparable case, we briefly comment on this in Remark 6.2.) By the result of Engelking, Heath and Michael there exists a continuous selection $f : \mathcal{F}(X) \to X$ for $\mathcal{F}(X)$. Let $s : \mathcal{F}(M) \to M$ be defined by

$$s(A) = p(f(p^{-1}(A))).$$

Let Z be an arbitrary closed subset of M, let C be the set of all continuity points of p^{-1} restricted to Z and let $\mathcal{A}$ be the collection of all finite subsets of C. One readily checks that s restricted to $\mathcal{F}(Z)$ is continuous at each point $A \in \mathcal{A}$, and since $\mathcal{A}$ is dense in $\mathcal{F}(Z)\cap\mathcal{D}(M)$, we get (2).

(2)⇒(1). We shall slightly modify the proof of Theorem 1.1 in §2. Let us adopt the notation from this proof, setting $f =$ the identity on M, and replacing the collection $\mathcal{C}_f$ by $\mathcal{D}(M)$. Then we follow the reasoning until the last step, where we have to check that $G(T) = T \setminus G_\lambda$ is topologically complete. We shall show that in our case, it is at most a singleton. Assume to the contrary that $G(T)$ contains two distinct points a, b. The set $G(T)$ being locally closed in M, we can find disjoint closed sets K, L in M, with $a \in K$, $b \in L$, $K \cup L \subseteq G(T)$, such that both K, L are regularly closed in the space $G(T)$. Let

$$Z = F \cup K \cup L$$

and let $\mathcal{U}$ be the collection of all $D \in \mathcal{D}(M)$ such that $F \subsetneq D \subseteq Z$ and D intersects both K and L. Then $\mathcal{U}$ is a nonempty open set in $\mathcal{F}(Z)\cap\mathcal{D}(M)$ and by (2), there is $E \in \mathcal{U}$ such that s restricted to $\mathcal{F}(Z)\cap\mathcal{D}(M)$ is continuous at E. Let $s(E) = c$. Then $c \in K \cup L$ (see (1) in §3.1). We may assume without loss of generality that $c \in L$. Let V be a neighborhood of c in Z the closure of which misses the remaining points of E. By the continuity of s at E we can choose a finite set $H \subseteq (K\cup L)\setminus V$, $H\cap K \neq \emptyset$, and a neighborhood W of c in L contained in V such that for all $D \in \mathcal{D}(M)$ with $D \subseteq W$, $s(F\cup H\cup D) \in V$, i.e., $s(F \cup H \cup D) \in D$. Since L is regularly closed in $G(T)$, W has nonempty interior relative to $G(T)$, and there consequently exists an open set U in T such that $\emptyset \neq U \cap G(T) \subseteq W$. But then we set $G_{\lambda+1} = U \cup G_\lambda$, $S_\lambda = G_{\lambda+1} \setminus G_\lambda$, $F_\lambda = H$, obtaining, as in the proof of Theorem 1.1, a contradiction. ∎

6.2. Remark. Let us construct a map $p : X \to M$ used in the proof of the implication (1)⇒(2). Let $\mathbb{P}$ be the set of irrationals, $J(\mathbb{R})$ the hedgehog of density equal to the density of M, and let $J(\mathbb{P})$ be obtained from $J(\mathbb{R})$ by removing from each spine the rational points different from 0. One can easily define a continuous one-to-one map $u : \mathbb{P} \to \mathbb{R}$ onto $\mathbb{R}$ such that its inverse u^{-1} is continuous at each irrational point. The map u applied to every spine induces a map $J(u) : J(\mathbb{P}) \to J(\mathbb{R})$ and the product map $q = J(u)^\infty : J(\mathbb{P})^\infty \to J(\mathbb{R})^\infty$ is continuous, one-to-one, onto, and the inverse q^{-1} has a point of continuity on each closed set. Since M embeds in $J(\mathbb{R})^\infty$ as a closed set, we can take $X = q^{-1}[M]$ and $p =$ the restriction of q to X.

To get a perfect map f such as the one used in the proof of Corollary 3.2, we can start from a continuous map $u : K \to \mathbb{I}$ of the Cantor set onto the unit interval, and then take as f the resulting map p.

7. Compact-valued selections that characterize topological completeness. A map $\psi : X \to \mathcal{F}(Y)$ is *upper-semicontinuous* (for short: u.s.c.) if for each open set U in Y, the set $\{x \in X : \psi(x) \subseteq U\}$ is open.

Michael [13] proved that if Y is completely metrizable, then for every l.s.c. $\varphi : X \to \mathcal{F}(Y)$ with metrizable domain, there exists a compact-valued u.s.c. map $\psi : X \to \mathcal{F}(Y)$ with $\psi(x) \subseteq \varphi(x)$ for every $x \in X$ (Čoban and Michael [1] derived this theorem from the theorem of Michael discussed in the introduction).

We shall show that a metrizable Y satisfying the assertion of this theorem must be topologically complete. More specifically, adopting the notation from the introduction, we have the following

7.1. Theorem. *If there exists an upper-semicontinuous map $s : \mathcal{C}(M) \to \mathcal{F}(M)$ such that for any $A \in \mathcal{C}(M)$, $s(A) \subseteq A$ is finite, then M is completely metrizable.*

Proof. We shall follow closely the proof of Theorem 1.1, adopting the same notation, and setting $f =$ the identity on M; in particular, $\mathcal{C}_f = \mathcal{C}(M)$.

We have to replace the second part of condition (1) in 3.1 by

$$(*) \qquad \text{if } C \in \mathcal{C}(M) \text{ and } C \subseteq S \text{ then } s(F(S) \cup C) \cap C \neq \emptyset.$$

Then the proof in 3.1 runs smoothly also in our case until the last step, with the following obvious modifications following condition (2): $s(A) \subseteq A \setminus \overline{S}_n$, for some n, and for $C = A \setminus F_n$, $s(F_n \cup C) \cap C = \emptyset$, contradicting $(*)$.

In the last step we shall show that $G(T)$ is a singleton. Striving for a contradiction, assume that it is not, and consider the family $\mathcal{E}$ of all non-singleton elements of $\mathcal{C}(M)$ contained in $G(T)$, setting for $E \in \mathcal{E}$ (see $(*)$)

$$t(E) = s(F(T) \cup E) \cap E.$$

If $\mathcal{E}$ contains an element E for which $t(E)$ is a singleton then we can end the proof following the final part of the proof of implication (2)⇒(1) in 6.1 with some evident omissions.

Assume that $t(E)$ is never a singleton, and let a, b be distinct points of $G(T)$. Then $t(\{a, b\}) = \{a, b\}$. Let V be a neighborhood of b disjoint from $F \cup \{a\}$, where $F = F(T)$. Using the upper-semicontinuity of s, choose a neighborhood W of b in T contained in V so that for all $C \in \mathcal{C}(M)$ contained in W, $s(F \cup \{a\} \cup C) \subseteq F \cup \{a\} \cup V$. If, in addition, $C \subseteq G(T)$, then $E = \{a\} \cup C$ is in $\mathcal{E}$, and $t(E)$, being a non-singleton subset of E, must intersect C. Therefore, we can reach a contradiction as in the proof of Theorem 1.1. ■

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