

Iterations of rational functions: which hyperbolic components contain polynomials?

by

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*Dedicated to the memory of Wiesław Szlenk,
my teacher and friend*

Abstract. Let H^d be the set of all rational maps of degree $d \geq 2$ on the Riemann sphere, expanding on their Julia set. We prove that if $f \in H^d$ and all, or all but one, critical points (or values) are in the basin of immediate attraction to an attracting fixed point then there exists a polynomial in the component $H(f)$ of H^d containing f . If all critical points are in the basin of immediate attraction to an attracting fixed point or a parabolic fixed point then f restricted to the Julia set is conjugate to the shift on the one-sided shift space of d symbols. We give *exotic* examples of maps of an arbitrary degree d with a non-simply connected completely invariant basin of attraction and arbitrary number $k \geq 2$ of critical points in the basin. For such a map $f \in H^d$ with $k < d$ there is no polynomial in $H(f)$. Finally, we describe a computer experiment joining an exotic example to a Newton's method rational function (for a polynomial) with a 1-parameter family of rational maps.

Introduction. In the space Q^d of rational maps of degree $d \geq 2$ of the Riemann sphere $\overline{\mathbb{C}}$ denote by H^d the set of maps which are expanding on their Julia set. Expanding means that there exists $n > 0$ such that for every z in the Julia set, $|(f^n)'(z)| > 1$, derivative in the standard spherical Riemann metric. We call $z \in \overline{\mathbb{C}}$ a *critical point* if $f'(z) = 0$. We call v a *critical value* if $v = f(z)$ for a critical point z .

In Section 1 we prove the following:

THEOREM A. *Let $f \in H^d$. Suppose that all, or all but one, critical values of f are in a basin $B(f)$ of immediate attraction to one attracting*

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f-fixed point p . Then the component $H(f)$ of H^d containing f also contains a polynomial.

(The critical values in Theorem A are counted without multiplicities. However, throughout the paper, critical points are counted with multiplicities.)

COROLLARY B. *If all critical points of f are in $B(f)$ then f restricted to the Julia set $J(f)$ is conjugate to the full one-sided shift.*

THEOREM C. *Let $f \in Q^d$, p be a parabolic fixed point and $B(f)$ be a basin of immediate attraction to p adjacent to p such that $f(B(f)) = B(f)$ (which is equivalent to $f'(p) = 1$). Suppose that all critical points of f are in $B(f)$. Then f restricted to $J(f)$ is conjugate to the full one-sided shift.*

The proof of Theorem C is similar to that of Corollary B so it will only be sketched.

It is much easier to prove that, under the assumptions of Corollary B or Theorem C, $J(f)$ is Cantor, $f|_{J(f)}$ is conjugate to a topological Markov chain and $f^n|_{J(f)}$ is conjugate to a 1-sided shift for some $n > 0$, than to prove that $f|_{J(f)}$ itself is conjugate to a full 1-sided shift. See Remarks 1.5 and 1.6.

The questions answered in Corollary B and Theorem C were posed to me by John Milnor. In the case of the basin of a sink he suggested to join critical values to the sink along trajectories of the gradient flow, Morse curves, described below. This was a fruitful idea. After proving Theorem A and Corollary B, I learned that these facts were proved by P. Makienko ⁽¹⁾ in his PhD paper but stayed unpublished ⁽¹⁾. Corollary B is also stated in [GK] but proved only for $d = 2$.

Note that Corollary B and Theorem C depend on holomorphic phenomena. Indeed, there exists a 1-sided topological Markov chain T with each point having 10 preimages which has the same ζ -function as the full 1-sided shift S_{10} on 10 symbols and whose sufficiently high power T^n is conjugate to S_{10}^n , but T itself is not conjugate to S_{10} [Boy].

If B , the immediate basin of attraction to an attracting fixed point, is simply connected, then the number of critical points of f in B is equal to $\deg(f|_B) - 1$ (because f pulled back to the unit disc \mathbb{D} by a Riemann mapping is a Blaschke product which has $\deg(f|_B) - 1$ critical points in \mathbb{D} , and the same number outside).

If B is the basin of attraction to ∞ for a polynomial and B is not simply connected, then the number of critical points of f in B is at least $\deg(f|_B)$ (including ∞ as a $(\deg(f) - 1)$ -multiple critical point).

⁽¹⁾ Recently his proof appeared in the preprint [M]. It influenced our revised version of the paper. Also our Theorem C is proved in [M].

Surprisingly this is false in general. The “proof” that if the basin is not simply connected then it contains at least as many critical points as the degree of f on it, given in [P], is erroneous for degree larger than 2: the corresponding Lemma in [P] is false.

In Section 2, with the use of the quasiconformal surgery technique [D], we prove the existence of *exotic basins*:

THEOREM D. *There exists a rational function f of an arbitrary degree $d \geq 3$ with a completely invariant (i.e. invariant under f^{-1}) non-simply connected basin of immediate attraction to an attracting fixed point, and with an arbitrary number k , $2 \leq k \leq 2d - 2$, of critical points in the basin.*

The conclusion is that the assumption that all, or all but one, critical values are in the basin in Theorem A, is essential. Namely, for $k \leq d - 1$ Theorem D yields

COROLLARY E. *For every $d \geq 3$ there exists $f \in H^d$ with a completely invariant basin of immediate attraction to an attracting fixed point such that $H(f)$ contains no polynomial.*

Neither in Theorem A nor in Corollary E does it matter that $f \in H^d$. Only the basin $B(f)$ matters. The reader will find appropriate precise assertions in Sections 1 and 2.

I checked with computer that $f(z) = z^2 + c + b/(z - a)$ for $c = -3.121092$, $a = 1.719727$ and $b = .3142117$ is an exotic example for $d = 3$ whose existence is asserted in Corollary E. I am grateful to Ben Bielefeld and Scott Sutherland for their help in producing several computer pictures of Julia sets for such f 's and related pictures in the parameter space. Ben invented the parametrization in which the pictures were done. Our 1-parameter families join exotic examples of the above type with $d = 3$, having two superattracting fixed points and a critical point of period 2, to functions having three superattracting fixed points. It is easy to see (and well known) that the latter functions must be Newton's method rational functions for degree 3 polynomials.

(If $f \in Q^d$ has d superattracting fixed points then in appropriate holomorphic coordinates on $\overline{\mathbb{C}}$ it is Newton's for a degree d polynomial. Hint to the proof: Change first the coordinates on $\overline{\mathbb{C}}$ by a homography so that the unique repelling fixed point becomes ∞ .)

1. Rearranging critical values. Proofs of Theorem A and Corollary B. Let $B(f)$ be the immediate basin of attraction to an attracting fixed point p for a rational map $f \in H^d$. Suppose $f'(p) \neq 0$.

In this section we shall make the following types of perturbation of such maps in H^d :

1. *Perturbation along a curve γ .* We have in mind the following construction: Suppose there is a curve $\gamma = \gamma(t)$, $t \in [0, 1]$, embedded in $B(f)$ with $p \notin \gamma$. Take a small neighbourhood U of γ , $U \subset B(f)$, disjoint from a neighbourhood of p . Let g_t for $t \in [0, 1]$ be a diffeomorphism of $\overline{\mathbb{C}}$ so that $g_t(\gamma(0)) = \gamma(t)$, g_t is the identity outside U , $g_0 = \text{id}$ and g_t smoothly depends on t . We obtain the homotopy $h_t = g_t \circ f$. Note that though we are talking about a perturbation “along γ ” we change the map in a neighbourhood of $f^{-1}(\gamma)$.

If the following assumption holds:

$$(1) \quad p \notin U, \quad h_t^n(U) \rightarrow p,$$

then of course the basin of attraction to p is the same for h_t as for f .

Then we construct an invariant measurable L^∞ conformal structure for each h_t as follows: We take the standard structure on a small neighbourhood of p and then pull it back by h_t^{-n} . On the complement we take the standard structure. Now we integrate this structure (we refer to the Measurable Riemann Mapping Theorem [B], [AB]), and h_t in the new coordinates gives a homotopy f_t through maps in H^d . See [D] for this technique.

2. *Small C^1 -perturbation.* If a map g is C^1 close to f on U such that $\text{cl}U \subset B(f) \setminus \{p\}$, $f = g$ outside U , and near critical points, g differs from f only by affine parts, then clearly g is homotopic to f also through small perturbations satisfying the same conditions as g . The condition (1) holds automatically. As before we introduce new conformal structures, integrate them and obtain a perturbation homotopic to f through maps in H^d .

3. *Blaschke type perturbation.* Let $U \subset B(f)$ be an open topological disc containing p , with smooth boundary not containing critical points, such that $f(\text{cl}U) \subset U$, $f : U \rightarrow f(U)$ is a proper map and $d' := \deg(f|_U) \geq 2$. Then we construct a 1-parameter family of maps joining in $H(f)$ the map f to a map having p as a d' -multiple fixed point as follows:

Let R_1, R_2 be Riemann maps from U , resp. $f(U)$, to the unit disc \mathbb{D} such that $R_i(p) = 0$, $i = 1, 2$. Let $a_1 = 0$, $a_2, \dots, a_{d'}$ be the R_1 -images of the $f|_U$ -preimages of p . Let

$$B_t = \lambda z \prod_{i=2}^{d'} \frac{z - ta_i}{1 - t\bar{a}_i z}, \quad |\lambda| = 1, \quad t \in [0, 1].$$

We set $h_t = R_2^{-1} \circ B_t \circ R_1$. Here λ is chosen so that $h_1 = f$. It is useful to write $h_t = R_1^{-1} \circ g_t \circ R_1$, where $g_t = R_1 \circ R_2^{-1} \circ B_t$. Change h_t in U close to ∂U by a smooth isotopy so that h_t and f coincide on ∂U for all t . We extend h_t outside U by f to the whole $\overline{\mathbb{C}}$. As in the previous cases we pull back the standard conformal structure from $R_1(f(U))$ to $f(U)$ by R_1^{-1} ,

extend it by h_t^{-n} , complete on $\overline{\mathbb{C}} \setminus \bigcup h_t^{-n}(f(U))$ with the standard structure and integrate.

Let Φ_f conjugate f to $z \mapsto \lambda z$, where $\lambda := f'(p) \neq 0$, in a neighbourhood of p (i.e. $\Phi_f(f(z)) = \lambda \Phi_f(z)$). Extend Φ_f to $B(f)$ by $\Phi_f(z) = \lambda^{-n} \Phi_f(f^n(z))$. Define $G_f(z) = |\Phi_f(z)|^2$.

If $f \in H^d$ and $g \in H(f)$ then we write p_g for the point z with $g(z) = z$ such that (g, z) belongs to the component of the Cartesian product $H(f) \times \overline{\mathbb{C}}$ containing (f, p_f) . There exist $B(g)$ and Φ_g (provided $g'(p_g) \neq 0$) as above.

Now we can formulate

MAIN LEMMA. *For every $f \in H^d$ there exists $g \in H(f)$ such that $g'(p_g) \neq 0$ and there exists $a > 0$ such that all critical values for critical points in $B(g)$ are in a component ∂ of $\{G_g = a\}$ which is a topological circle separating p_g from the Julia set $J(g)$.*

Proof. By small perturbations (types 2 and 3) we assure that the sink p_f is neither a critical point nor a critical value for a critical point in $B = B(f)$, all critical points in B are simple and their forward trajectories are pairwise disjoint.

At the end it may occur useful also to have $\frac{1}{\pi} \text{Arg } f'(p)$ irrational. We assure this by a type 3 perturbation where $\text{Arg } \lambda$ is the parameter.

The critical points of $G = G_f$ are

- 1) the fixed point p and its iterated pre-images (these are minimum points with $G = 0$),
- 2) the critical points of f and their iterated pre-images (these are saddle points for G).

Denote the set of points in 1) by M and the set of points in 2) by S .

For every $q \in M$ let $A(q)$ denote the basin of attraction to q for the flow of the vector field $-\text{grad } G$. (Note that the boundary of $A(q)$ is the union of a set of some points from S and their stable separatrices.)

Denote by $r(q)$ the least non-negative integer such that $f^{r(q)}(q) = p$. For every $z \in A(q)$ or $X \subset A(q)$ write $r(z) := r(q)$ and $r(X) := r(q)$.

Observe that for every $z \in B(f)$ there exists a curve γ joining z to p consisting of critical points of G and of trajectories of $\text{grad } G$ where this field is non-zero (i.e. γ goes from z to a critical point, say a minimum, then to a saddle, then to a minimum etc. until it reaches p). Let $\gamma(z)$ denote a curve as above intersecting the least possible number of $A(q)$'s. Denote the number of these $A(q)$'s by $s(z)$ or $s_f(z)$ and call the curve a *Morse curve*.

OBSERVATIONS. 1. $r(f(z)) = r(z) - 1$ if $r(z) \geq 1$.

2. $s(f(z)) \leq s(z)$.

Observation 2 follows from the fact that f maps trajectories of $\text{grad } G$ to trajectories of $\text{grad } G$.

The plan is now to move all critical values to the same level $G = a$ in $A(p)$.

We shall do this for each critical point separately so that we do not move the critical values moved before to level a in $A(p)$. We move each critical value $f(c)$ step by step so that after each step, $s(f(c))$ decreases and $f(c)$ is in some $A(q)$. When $f(c) \in A(p)$ we move $f(c)$ along the trajectory of $\text{grad } G$ to level a as described in 1.

Take $c \in B(f)$, a critical point for f . By a small perturbation in a neighbourhood of c we obtain $f(c) \in A(q)$. We can assume that $q \neq p$. This is correct because:

(a) The perturbation is above the G -level of $f(c)$ so it does not change the part of the stable manifolds of a saddle to which $f(c)$ might have belonged, in a neighbourhood of $f(c)$ and between it and the saddle.

(b) The change of coordinates integrating the new invariant conformal structure is conformal there. This is so because the structure is non-standard only in some places above $f(c)$. Thus the change of coordinates maps the gradient lines of the old G to the gradient lines of the new one and stable manifolds (separatrices) to separatrices.

Now let γ be the part of a Morse curve $\gamma(f(c))$ joining $f(c)$ to the first saddle $\omega \in \gamma(f(c))$.

Observe that for every $j > 0$ we have $f^j(\gamma) \cap \gamma = \emptyset$.

Indeed, we have $f^j(\gamma \cap A(q)) \cap \gamma = \emptyset$ by Observation 1. By the same observation, $f^j(\omega) \in \text{cl } A(f^j(q))$ cannot be in $\gamma \cap A(q)$. Finally, $f^j(\omega) \rightarrow p$ so $f^j(\omega) \neq \omega$.

As $f^j(\gamma) \rightarrow p$ we can find a neighbourhood U of γ so that $f^j(U) \cap U = \emptyset$ for all $j > 0$. Take in U a curve γ' joining $f(c)$ to a point z close to ω with $z \in A(q')$ and $s(q') = s(q) - 1$. Take care to choose γ' disjoint from $V =$ the set of all critical values under iterates of f except $f(c)$.

Now perturb f along γ' as described in 1. Do it with g_t different from identity only in a neighbourhood of γ' small enough to be disjoint from V . The condition (1) is of course satisfied.

Now let us explain why the new map f_1 has the property

$$(2) \quad s_{f_1}(f_1(c_{f_1})) = s_f(f(c)) - 1.$$

Here c_{f_1} is the old c in the new coordinates; it is a critical point for f_1 . We use the fact that the part of the domain where we changed f to h_t is in the basins $A(f^{-1}(q))$, where by Observation 2, $s \geq s(q)$. So we did not change G in the basins and the part of the Morse curve $\gamma(f(c))$ beyond

ω goes through, where $s < s(q)$. We also changed f in a neighbourhood of $f^{-1}(\omega)$. This does not change G below a neighbourhood of ω because $G(f^{-j}(\omega)) > G(\omega)$, $j = 1, 2, \dots$ (Compare this with the argument (a).) This does not change G below the part of $\gamma(f(c))$ beyond a neighbourhood of ω either, because this change is close to the set $s \geq s(q)$.

The change of h_t to f_t does not affect (2). Indeed, also by the above arguments the new measurable conformal structure is the standard one below ω and the part of $\gamma(f(c))$ beyond ω . So the change of coordinates maps the gradient lines of the old G to the gradient lines of the new one as in (b).

When $f(c) \in A(p)$ we first make a small perturbation so that the gradient line of G passing through $f(c)$ does not intersect the forward trajectories of other critical values which are already in $A(p)$. Next we move $f(c)$ along γ which is a piece of the gradient line of G passing through $f(c)$ joining it to $\{G = a\}$ in $A(p)$. We succeed because $\frac{1}{\pi} \text{Arg}(f^n)'(p)$ is irrational so all the curves $f^n(\gamma)$, $n \geq 0$, are pairwise disjoint and the conformal structure does not change below these curves.

After a sequence of consecutive perturbations as above we obtain a rational mapping g with all the critical values on one level $\{G = a\}$, more precisely, on its component ∂ intersecting $A(p)$.

Otherwise $\text{cl } D_a$ contains an S -type critical point x for G_g . Then there exists $n > 0$ such that $y = g^n(x)$ is a critical value for g . Hence $G_g(y) = |g'(p_g)|^n G_g(x) < G_g(x) \leq a$. This contradicts $G_g(y) = a$ just achieved.

The proof of the Main Lemma is finished. ■

Proof of Theorem A. Let f be already as g in the Main Lemma. By perturbing along curves one obtains additionally all critical values for critical points from $f^{-1}(B(f)) \setminus B(f)$ also in ∂ . (*A posteriori* we will see that under the assumptions of Theorem A we have $f^{-1}(B(f)) \setminus B(f) = \emptyset$, i.e. $B(f)$ is completely invariant.)

Denote again the domain of $\overline{\mathbb{C}} \setminus \partial$ containing p by D_a . Denote the complementary open topological disc $\overline{\mathbb{C}} \setminus \text{cl } D_a$ by D'_a .

OBSERVATIONS. 3. *There is at most one critical value for f in D'_a . So the components of $f^{-1}(D'_a)$ are topological discs D'^j , $j = 1, \dots, \widehat{d}$ where $\widehat{d} \leq d$, with closures in D'_a . (In particular, $f^{-1}(\text{cl } D_a)$ is connected and hence $B(f)$ is completely invariant.)*

4. *The closures of D'^j intersect each other or “self-intersect” only at critical points of f and $\text{cl } f^{-1}(D'_a)$ is connected (see Fig. 1).*

If the latter were false then $f^{-1}(D_a)$ would contain a non-simply connected component V . But f maps V onto the disc D_a so V would contain a critical point, hence D_a would contain a critical value. This would contradict the assumption that all critical values are on level a .

After a small perturbation moving (exposing) critical values of some $d-1$ critical points towards below a , the set $\text{cl } f^{-1}(D'_a)$ consists of \widehat{d} closed discs intersecting one another in at most 1 point, whose union is connected and simply connected (Fig. 2).

So for $\varepsilon > 0$ small enough the set $f^{-1}(\partial D_{a-\varepsilon})$ (where $D_{a-\varepsilon} := D_a \cap \{G < a - \varepsilon\}$) is a topological circle ∂_ε bounding a topological disc $U \ni p$ and under f it winds d times onto $\partial D_{a-\varepsilon}$. Of course $f(\text{cl } U) = \text{cl } D_{a-\varepsilon} \subset U$.

After performing a Blaschke type perturbation and a holomorphic change of coordinates on $\overline{\mathbb{C}}$ we arrive at a polynomial. ■

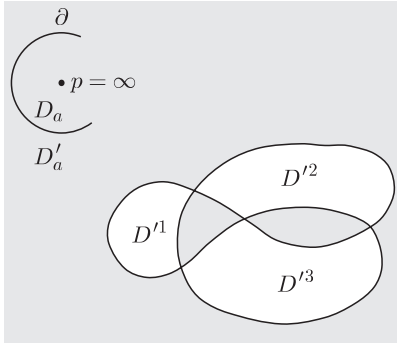
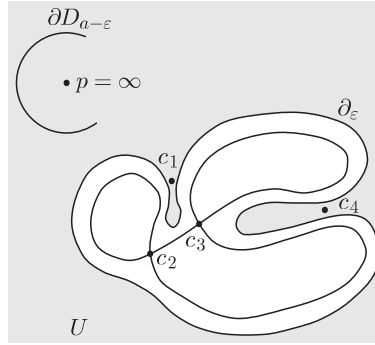


Fig. 1

Fig 2. Critical points c_1 and c_4 are exposed towards p

Proof of Corollary B. This corollary follows from Theorem A because its assertion is an open property in H^d and it is true and easy for polynomials.

Indeed, if f is a polynomial, then with the help of a small perturbation we guarantee that for each critical value v_j , $j = 1, \dots, d-1$ (different from ∞), the trajectory γ_j for $\text{grad } G$ where G is Green's function in the basin of attraction to ∞ , with pole at ∞ , goes from v_j up to ∞ . In other words, γ_j does not go to any critical point for G . Then there are no critical values for f in the topological disc

$$U = \overline{\mathbb{C}} \setminus A, \quad \text{where } A = \bigcup_{j=1, \dots, d-1, n \geq 0} f^n(\gamma_j).$$

Since $f(A) \subset A$, we have a collection of branches $g_j : U \rightarrow U$ of f^{-1} . Denote $g_j(U)$ by U_j . Then each $z \in J(f)$ is coded by the sequence of symbols j_n , $n = 0, 1, \dots$, where $f^n(z) \in U_{j_n}$.

For each sequence (j_n) the family of maps $\Psi_n = g_{j_0} \circ g_{j_1} \circ \dots \circ g_{j_n}$ is a normal family of maps on U . It is easy to find a slightly smaller topological

disc $U' \subset U$ so that $U' \supset J(f)$ and

(3) each g_j maps $\text{cl}U'$ into U' .

Hence the sequences $\Psi_n|_{U'}$ converge uniformly to points in $J(f)$. This proves that the coding is one-to-one. ■

Remark 1.1. In fact, a proof of Corollary B is contained in the proof of Theorem A. Indeed, already Observation 3 gives a partition of the Julia set J into $J \cap D'^j$, and $J \ni z \mapsto (j_n)$ such that $f^n(z) \in D'^{j_n}$ gives a conjugacy to the one-sided shift on the space of d symbols.

Observe that for this proof of Corollary B, there was no need to refer to the Measurable Riemann Mapping Theorem. Namely, there was no need to integrate every time a new conformal structure to obtain a rational mapping. One could just work with smooth maps until the properties described in Observation 3 were reached.

Remark 1.2. One can prove Theorem A by perturbing f along a curve $\hat{\gamma}$ close to the Morse curve, dragging the critical value to a small neighbourhood of p in just one step instead of doing it step by step decreasing $s(f(c))$. The property (1) is satisfied by Observations 1 and 2. (Every point of $\hat{\gamma}$ can come close to $\hat{\gamma}$ under f^j only further, i.e. if $f^j(\hat{\gamma}(\tau))$ is close to $\hat{\gamma}(\tau')$ then $\tau' > \tau$. The more so for f^j replaced by h_t^j .)

Remark 1.3. Makienko [M] proved the following Proposition which corresponds to our Main Lemma:

If all critical values v_1, \dots, v_m for critical points in the basin $B(f)$ of immediate attraction to an attracting fixed point p_f have disjoint forward orbits then there exists a topological disc $U \subset B(f)$ containing p_f such that $f|_U$ is injective, $f(\text{cl}U) \subset U$ and all v_j , $j = 1, \dots, m$, belong to the annulus $U \setminus f(U)$.

One can easily change U so that all v_j belong to $\partial f(U)$. There is a quasiconformal conjugacy Φ of f on U to $z \mapsto \lambda z$ on the unit disc \mathbb{D} , $|\lambda| < 1$. So as usual one can pull back the standard conformal structure on \mathbb{D} by Φ^{-1} to U , spread it by Φ^{-n} , $n = 1, 2, \dots$, to the whole basin of ∂_f and complete on the complement of the basin by the standard structure. After integration of this structure the new map satisfies the properties asserted in the Main Lemma.

Thus: Makienko's Proposition (preceded by a small perturbation) + the Measurable Riemann Mapping Theorem give the Main Lemma.

Conversely, g from the assertion of the Main Lemma is conjugate to f (provided f is already after the first perturbation in the proof of the Main Lemma). So Makienko's $f(U)$ can be defined as the image under the conjugacy of the disc bounded by ∂ . Observe that for that we did not need

to construct new holomorphic structures along the proof. (Compare this with Remark 1.1.)

Thus Makienko's Proposition is the topological heart of the Main Lemma.

Remark 1.4. Neither in the Main Lemma nor in Theorem A does one need to assume $f \in H^d$. The proofs work for an arbitrary rational map f with $\deg f \geq 2$ if one replaces in the statements $H(f)$ by Teich_J . The latter denotes the set of all rational maps g such that there exists

- 1) a connected open domain $\mathcal{P} \ni 0$ in the complex plane,
- 2) a family of quasiconformal homeomorphisms $h_\lambda : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$, $\lambda \in \mathcal{P}$, with $h_0 = \text{id}$ such that for each z the point $h_\lambda(z)$ depends holomorphically on z (such a family is usually called a *holomorphic motion*),
- 3) a family of rational maps f_λ , $\lambda \in \mathcal{P}$, with $f_0 = f$ and $f_{\lambda_0} = g$ for some $\lambda_0 \in \mathcal{P}$ such that for every λ the map h_λ conjugates f with f_λ between ε -neighbourhoods of their Julia sets (ε not depending on λ).

Proof of Theorem C. One first proves an analogue to the Main Lemma:

If $f(z) = z + a(z-p)^{t+1} + o((z-p)^{t+1})$ with $a \neq 0$ then $H_1(z) = \lambda/z^t$ for some $\lambda \neq 0$ conjugates f on a "petal" $P \subset B(f)$ to $F(z) = z + 1 + o(1)$ for z with large real part. (See [DH2] for the precise description.) Next conjugate F smoothly to $z \mapsto z + 1$ by H_2 . Define $\Phi_f = H_2 \circ H_1$ on P and extend it to $B(f)$ by $\lim_{n \rightarrow \infty} (\Phi_f \circ f^n(z) - n)$. Define $G_f(z) := 1/\exp \Re \Phi_f(z)$ on $B(f)$.

Then after a small perturbation such that all critical points in B are simple and their forward orbits are disjoint, one can find a quasiconformally conjugate $g \in Q^d$ such that all critical values v_j for g -critical points in $B(g)$ are in the component of $\{G_g = a\}$ bounding a "petal". (See Fig. 3.)

The proof of this parabolic version of the Main Lemma is the same as that of the Main Lemma except that there are no M -critical points for G_f in $B(f)$. One can think of M -critical points as belonging to $\partial B(f)$, forming precisely the set $\partial B(f) \cap \bigcup_{n \geq 0} f^{-n}(\{p\})$. So the perturbation is not along Morse curves in basins $A(q)$ but along curves in $B(f) \cap A(q)$ joining directly consecutive S -critical points in $\partial A(q)$.

Repeat now the proof of Corollary B:

(The assumptions imply $t = 1$.)

For f already as g above we take curves γ_j , $j = 1, \dots, 2d - 2$, joining v_j to p . We can take as γ_j 's the Φ_f^{-1} -preimages of horizontal lines in $\overline{\mathbb{C}}$. Then $f(\gamma_j) \subset \gamma_j$. So for $A = \bigcup_{j=1}^{d-1} \gamma_j$ the set $\overline{\mathbb{C}} \setminus A$ is a topological disc and for the branches g_j of f^{-1} we have $g_j(U) \subset U$, $j = 1, \dots, d$.

We find $U' \subset U$ such that (3) holds, except that $\partial U' \cap \partial U = \{p\}$ and $g_{j_0}(p) = p$ for the branch g_{j_0} with $g_{j_0}(p) = p$ (see Fig. 3). Since $(g_{j_0}^n)' \rightarrow 0$

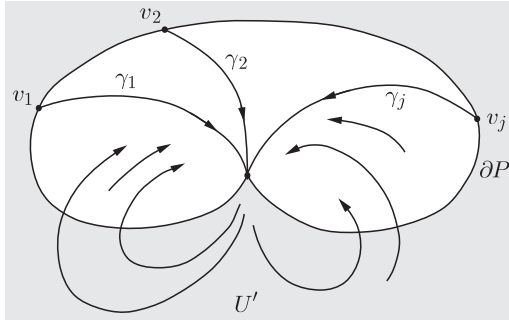


Fig. 3

on U' we conclude that $\bigcap_n g_{j_0}^n(\text{cl } U')$ is only one point p . ■

Remark 1.5. As mentioned in the introduction, the fact that under the assumptions of Corollary B or Theorem C, $J(f)$ is a Cantor set, is much easier than the fact that $f|_{J(f)}$ is conjugate to the one-sided shift.

Indeed, it is easy to prove that $f^N|_{J(f)}$ is conjugate to a one-sided shift, for an integer $N > 0$. Just take a small topological disc $D \subset B(f)$ so that $D \ni p$ (or a small petal in the parabolic case) such that $f(\text{cl } D) \subset D$. For each critical value v_j and every $n \geq 0$ such that $f^n(v_j) \notin D$ join $f^n(v_j)$ to D by an embedded curve $\gamma_{j,n}$ so that these curves are mutually disjoint. Take N such that $f^N(\gamma_{j,n}) \subset D$ for each j, n . Consider now $U = \overline{\mathbb{C}} \setminus A$, where $A = D \cup \bigcup_{j,n} \gamma_{j,n}$. Next proceed as in the proof of Corollary B with f replaced by f^N ⁽²⁾.

Remark 1.6. It is also easy to prove that $f|_{J(f)}$ is conjugate to a one-sided topological Markov chain.

Indeed, let $D_0 = D$ as in Remark 1.5. Let $D_n, n = 1, 2, \dots$, be defined recursively: D_n is the component of $f^{-1}(D_{n-1})$ containing D_{n-1} . Let $N \geq 0$ be large enough that there are no critical values in $\overline{\mathbb{C}} \setminus D_N$. Let U_1, \dots, U_K denote all the components of $\overline{\mathbb{C}} \setminus \text{cl } D_N$. These are topological discs because D_N is connected. Consider the family of topological discs $g_j(U_k)$ for all branches $g_j, j = 1, \dots, d$, of f^{-1} and $k = 1, \dots, K$. Then for every sequence of pairs $(j_n, k_n), n = 0, 1, \dots$, such that $g_{j_n}(U_{k_n}) \subset U_{k_{n+1}}$ there exists precisely one point $z \in J(f)$ such that $f^n(z) \in g_{j_n}(U_{k_n})$ for every $n \geq 0$.

Remark 1.7 ⁽³⁾. Considering the situation in Theorem A such that p is already not critical (say all f -critical points belong to $B(f)$ and $\text{Arg } 2\pi f'(p)$ is rational), John Milnor asked whether it is possible to find a set A which is the union of parts of trajectories of $\text{grad } G$, contains all critical values, is

⁽²⁾ I owe this proof to K. Barański.

⁽³⁾ Before reading this remark the reader is advised to read Section 2.

compact in $B(f)$, f -forward invariant, connected and simply connected (a tree).

This would immediately allow us to prove Corollary B with the set A as here.

Unfortunately, the answer is negative. We can modify an exotic example of Section 2 (the case $d = 3$, Fig. 5) so that $f(c_4) = f^2(c_3) = a$, the pole. Then the set of saddles for G is $S = \bigcup_{n \geq 0} f^{-n}(c_2)$. So every curve Γ built from pieces of trajectories of $\text{grad } G$ joining $f(c_4)$ to ∞ passes through a point of $f^{-n}(\{c_2\})$ for some n (in fact, $n > 0$). Hence $f^n(\Gamma)$ joins ∞ to ∞ passing through c_2 , hence it is a loop, i.e. A is not a tree.

(Even the assumption $f(c_4) \neq a \neq f^2(c_3)$ does not help if $f(c_4), f^2(c_3)$ are close to a . Γ must still leave the basin $A(a)$ for $\text{grad } G$, passing through a point in $\bigcup_{n \geq 0} f^{-n}(\{c_2\})$, and for typical $f(c_4)$, first meets the point a .)

Remark 1.8 ⁽³⁾. Though in the Main Lemma we can arrange to have all critical values for critical points in $B(f)$ on one component of a level of G , it is sometimes not so for critical points (unless all critical points for f , or all but one, are in $B(f)$ as in Theorem A, see Fig. 2). Again we modify an exotic example of Section 2. Here the degree of f is 5. Start with a cubic polynomial P which has degree 1 on ∂A_1 and degree 2 on ∂A_2 (see Fig. 3, Section 2), and P maps the critical point $a \in A_2$ to the critical point at the self-intersection of the figure 8 line $\partial A_1 \cup \partial A_2$, which escapes to ∞ .

Consider now the function $z \mapsto P(z) + b/(z - a)^2$ with b a small real positive number. We obtain the picture of Figure 4 (do the surgery as in Section 2 to find appropriate a and b). For a final example split a into two different poles, which gives an f -critical point c_8 between them (Fig. 4). Make a Blaschke type perturbation close to ∞ to have ∞ not critical and move $f(c_8), f(c_1), f(c_2)$ to one level with $f(c_3)$.

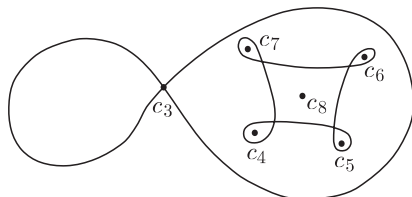


Fig. 4. $f^2(c_4) = c_4, f^2(c_6) = c_6, f(c_5) = c_5, f(c_7) = c_7$

2. Exotic basins

Proof of Theorem D, case $d = 3$. We start with the geometric description of an exotic example of degree 3, illustrated in Fig. 5.

Start with a quadratic polynomial $P(z) = z^2 + c$ with the critical point $c_2 = 0$ escaping to $c_1 = \infty$, which is an attracting fixed point of multipli-

city 2. The level $\partial = \{G = t\}$ of Green's function of the basin of attraction to ∞ with pole at ∞ , containing c_2 , is a figure 8. Now we change the map on B_2 , one of the two discs B_1, B_2 bounded by ∂ , as follows:

Draw two little discs D_1, D_2 in B_2 , intersecting each other. Let $D_1 \cap D_2$ be mapped 1-to-1 onto $\overline{\mathbb{C}} \setminus (B_1 \cup B_2)$. Let $D_1 \setminus D_2$ go onto B_1 and $D_2 \setminus D_1$ go onto B_2 , both via proper maps with degree 2. So there are critical points $c_3 \in D_1 \setminus D_2$ and $c_4 \in D_2 \setminus D_1$. On D_2 this map f is quadratic-like so we can do anything there, for example $f(c_4) = c_4$. On D_1 the map f^2 is quadratic-like so we can assume $f^2(c_3) = c_3$.

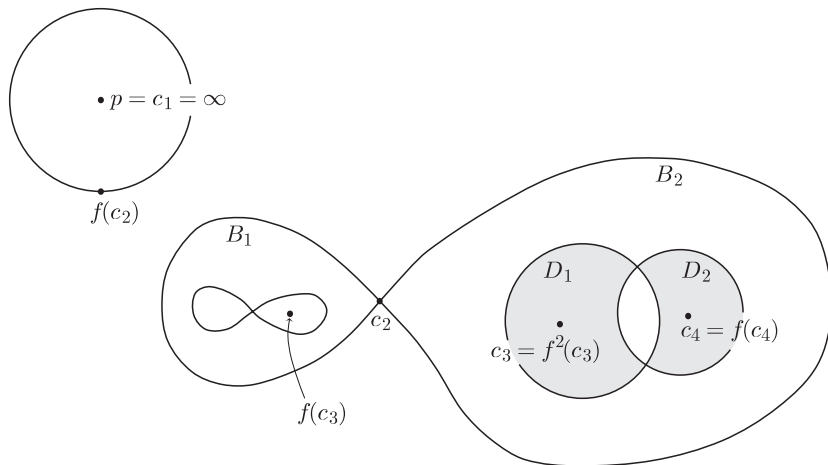


Fig. 5

The required rational function is obtained from this topological picture by the quasiconformal surgery technique [D]. We now explain it in more detail.

We need the following lemma which generalizes Douady–Hubbard’s theorem that a polynomial-like mapping is quasiconformally conjugate to a polynomial [DH1]:

LEMMA 2.1. *Let $U \subset \overline{\mathbb{C}}$ be an open set (not necessarily connected or simply connected) with boundary being a union of smooth Jordan curves. Let $F_1 : U \rightarrow U$ be a holomorphic map such that $F_1(\text{cl}U) \subset U$. (We denote the continuous extension of F_1 to $\text{cl}U$ by the same symbol.) Let $V \subset \overline{\mathbb{C}}$ be homeomorphic to $\overline{\mathbb{C}} \setminus \text{cl}U$ by a homeomorphism h_1 which extends to an orientation preserving homeomorphism of $\overline{\mathbb{C}}$. (Again we do not assume V is connected or simply connected.) Let $F_2 : V \rightarrow \overline{\mathbb{C}}$ be a holomorphic map. Also, suppose that the boundary of V is smooth and denote by F_2 the continuous extension of the original F_2 to $\text{cl}V$. Suppose the family of curves consisting of the components of ∂U and their F_1 -images has the same*

combinatorics in $\overline{\mathbb{C}}$ as the family of curves consisting of the components of ∂V and their F_2 -images, in the sense that

1. There exists a standard orientation preserving homeomorphism $h_2 : \partial U \cup F_1(\partial U) \rightarrow \partial V \cup F_2(\partial V)$ such that the boundary of each component K of $\overline{\mathbb{C}} \setminus (\partial U \cup F_1(\partial U))$ is mapped to the boundary of a component of $\overline{\mathbb{C}} \setminus (\partial V \cup F_2(\partial V))$ and the map extends Q -quasiconformally on K with a constant Q independent on K .

2. For each component ∂ of ∂U the map h_2 maps ∂ to $h_1(\partial)$, $F_1(\partial)$ to $F_2(h_1(\partial))$ and moreover there is a homeomorphism (lift) $\tilde{h}_2 : \partial \rightarrow h_1(\partial)$ such that on ∂ we have $h_2 \circ F_1 = F_2 \circ \tilde{h}_2$.

Then there exists a rational map $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ and an open set $W \subset \overline{\mathbb{C}}$ such that f is quasiconformally conjugate to F_1 on W and quasiconformally conjugate to F_2 on $\overline{\mathbb{C}} \setminus W$.

Note that the curves $\partial \rightarrow F_1(\partial)$ can intersect each other or self-intersect. In particular, ∂ can map to a Jordan curve with degree larger than 1.

Proof. We replace h_2 on ∂U by the lift \tilde{h}_2 and then extend it from $\partial U \cup F_1(\partial U)$ to a quasiconformal homeomorphism $h : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$. Define $F : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ to be F_2 on $\text{cl} V$ and $h \circ F_1 \circ h^{-1}$ on $\overline{\mathbb{C}} \setminus V$.

Let μ_0 denote the standard conformal structure on $\overline{\mathbb{C}}$. Take $\mu_1 = h_*(\mu_0|_U)$ on $\overline{\mathbb{C}} \setminus \text{cl} V$. (Think about μ_1 as a field of ellipses, up to multiplication by a positive function.) For each $z \in V$ define $\mu_1(z)$ as a pull-back $F_*^{-n}(\mu_1)(F^n(z))$, where $n \geq 0$ is such that $F^n(z) \in \overline{\mathbb{C}} \setminus \text{cl} V$. If no such n exists, take $\mu_1(z) = \mu_0(z)$. This is correct due to the crucial property $F(\overline{\mathbb{C}} \setminus \text{cl} V) \subset \overline{\mathbb{C}} \setminus \text{cl} V$. As F_2 is holomorphic, μ_1 is in L^∞ !

Now integrate μ_1 . In the new coordinates F changes to the rational map f we looked for. ■

Now we construct F_1 and F_2 satisfying the assumptions of Lemma 2.1. This is illustrated in Fig. 6.

Take the polynomial $P(z) = z^2 + c$, $c < -2$. Define F_1 by adding to P a term $b/(z - a)$ for $a = \sqrt{-c} \in P^{-1}(0) \cap B_2$ ($a > 0$). Let $b > 0$ be so small that for our F_1 the level $\hat{\partial} = \{\hat{G} = t_0\}$ containing the F_1 -critical point \hat{c}_2 close to $c_2 = 0$ is a figure 8 close to ∂ . Here \hat{G} is defined analogously to Green's function or to G in Section 1: on the basin of attraction to ∞ by F_1 , one defines $\hat{G}(z) = \lim_{n \rightarrow \infty} 2^{-n} \log |F_1^n(z)|$. Denote discs bounded by appropriate parts of $\hat{\partial}$ close to B_1, B_2 by \hat{B}_1, \hat{B}_2 respectively.

It is easy to compute that $\hat{c}_2 = b/(2a^2) + o(b)$ and for the two other F_1 -critical points \hat{c}_3, \hat{c}_4 we have $F_1(\hat{c}_{3,4}) = \mp 2\sqrt{2a}\sqrt{b} + o(\sqrt{b})$. So $F_1(\hat{c}_3) \in \hat{B}_1$ and $F_1(\hat{c}_4) \in \hat{B}_2$.

Let $0 < t_2 < t_1 < t_0$ with $t_2 \approx t_1 \approx t_0$ and denote by K_2, K_1 the topological discs both in \widehat{B}_1 bounded by $\{\widehat{G} = t_2\}$, resp. $\{\widehat{G} = t_1\}$. Denote by K'_4 the topological disc in \widehat{B}_2 bounded by $\{\widehat{G} = t_1\}$. Finally, denote the component of $F_1^{-1}(K_1)$ in \widehat{B}_2 by K_3 , and the component of $F_1^{-1}(K'_4)$ in \widehat{B}_2 by K_4 . We have $\widehat{c}_3 \in K_3$ and $\widehat{c}_4 \in K_4$.

Define $U := \overline{\mathbb{C}} \setminus (K_2 \cup K_3 \cup K_4)$. We have $F_1(\text{cl } U) \subset U$. So F_1 and U satisfy the assumptions of Lemma 2.1. Now we need to define F_2 :

Set $F_2(z) := z^2$ on a geometric disc $L_4 = \{|z| < r_4\}$, $r_4 > 1$. Take a disc $L_3 = \{|z - z_0| < r_3\} \subset F_2(L_4) \setminus \text{cl } L_4$ and define $\widetilde{F}_2(z) = (z - z_0)^2 + z_0$. One finds large r_4, r_3 such that $\widetilde{F}_2(L_3) \supset \text{cl } F_2(L_4)$. Pick in $\widetilde{F}_2(L_3) \setminus F_2(L_4)$ two discs $L_2 \subset L_1$ of the form $L_1 = \{|z - z_1| < r_1\}$ and $L_2 = \{|z - z_1| < r_2\}$ with $r_2 < r_1$. Take an affine holomorphic map $\Psi : L_1 \rightarrow \widetilde{F}_2(L_3)$ (onto). Define $F_2 = \Psi^{-1} \circ \widetilde{F}_2$ on L_3 and $F_2 = \Psi|_{L_2}$ on L_2 . We take r_2 so close to r_1 that $\Psi(L_2) \supset \text{cl } F_2(L_4)$.

Now take $V = L_2 \cup L_3 \cup L_4$ and F_2 defined on V as above.

The assumptions of Lemma 2.1 are satisfied. So we can “glue” F_1 and F_2 in one rational mapping f .

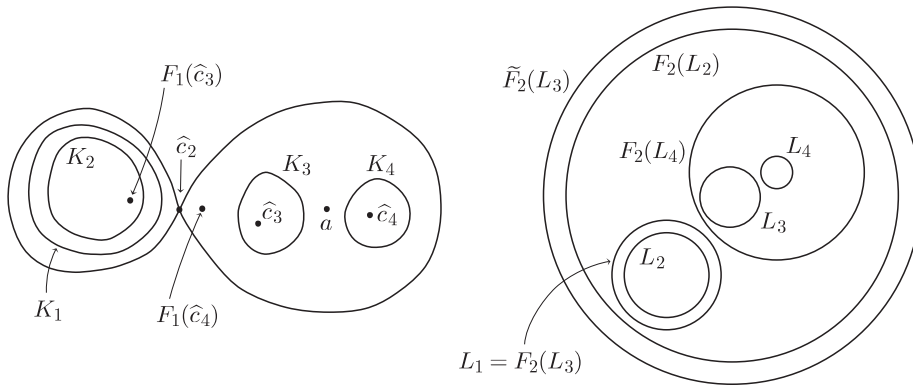


Fig. 6

Observe finally that $J(f)$ is disconnected because $F_1^n(\widehat{c}_2) \rightarrow \infty$ and moreover $F_1^n(l) \rightarrow \infty$, where $l = \{\Im z = \Im \widehat{c}_2\}$ (\Im denotes imaginary part). The line l separates K_2 from say K_4 . Both K_1 and K_4 intersect $J(f)$ (in the coordinates after the integration of μ_1) so the intersections belong to different components of $J(f)$.

The degree of f on the basin of attraction to ∞ is 3 because such is the degree of F_1 on U . Only two critical points: ∞ and the one corresponding to \widehat{c}_2 belong to the basin, because $\widehat{c}_{3,4}$ do not escape under iteration by F_2 . Theorem D is thus proved for $d = 3$.

Remark 2.2. Observe that in appropriate holomorphic coordinates on $\overline{\mathbb{C}}$ we have $f(z) = z^2 + c + b/(z - a)$. Indeed, after subtracting from the f constructed above the principal part of the Laurent series expansion at the pole, we are left with a quadratic polynomial. By an affine holomorphic change of coordinates we make the polynomial be $z^2 + c$.

Remark 2.3. One should be careful in the above construction because not every branched cover of $\overline{\mathbb{C}}$ preserves a conformal structure. Above, an annulus \mathcal{A} in B_2 containing c_3 and c_4 is mapped in a proper way by f to the disc $D' = \{G < t'\}$, $t' > t$, i.e. a disc in $\overline{\mathbb{C}}$ bounded by level t' of Green's function G , outside the figure 8 level $\{G = t\}$.

Instead of mapping c_3 into D_1 so that $F^2(c_3) = c_3$ we can map \mathcal{A} onto D' in a proper way so that $f(c_3) = c_3$ and $f(c_4) = c_4$. This will be a topological branched cover. However, it does not admit a holomorphic invariant structure.

Indeed, for a holomorphic f , either $f^n(c_3)$ or $f^n(c_4)$ must escape from \mathcal{A} . Otherwise we consider the branch g of f^{-1} on $D' \setminus \mathcal{A}$ mapping $\partial D'$ to the external component of $\partial \mathcal{A}$. All moduli of the annuli $P_n := g^n(D' \setminus \mathcal{A})$ are the same, and positive, so $\text{mod } \bigcup_n P_n = \infty$ and hence $D' \setminus \bigcup_n P_n$ consists of only one point and contains $c_3 \neq c_4$, a contradiction ⁽⁴⁾.

Another argument is that such an f would have 3 superattracting fixed points. So it would be a Newton's method rational function of a degree 3 polynomial (see the introduction). But the basin of attraction to ∞ is not simply connected. This contradicts a theorem that the basins of immediate attraction to the attracting fixed points for Newton's method are simply connected [P].

Proof of Theorem D, the general case $d \geq 3$. We shall realize holomorphically the picture of Fig. 7:

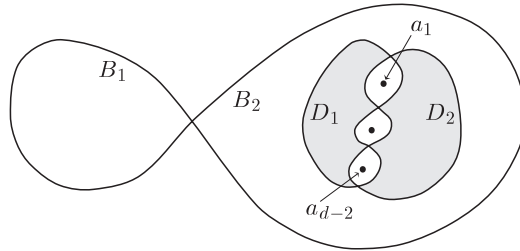


Fig. 7

In Fig. 7, D_j is mapped properly onto B_j for $j = 1, 2$. Each D_j contains $d - 2$ critical points. The points a_1, \dots, a_{d-2} are poles.

⁽⁴⁾ This proof was obtained in a discussion with K. Barański after the preprint version of this paper appeared. It will appear in a wider context in [Ba].

We proceed similarly to the case $d = 3$. Let

$$F_1(z) = z^2 + c + b \sum_{m=1}^{d-2} \frac{1}{z - a_m}.$$

Take $a_m = \sqrt{-c} + imT$ for a real constant T with $0 < T \ll 1$, in particular T small enough that all a_m are well in B_2 .

For $b > 0$ small, there is a small annulus around each pole a_m , containing two critical points

$$\widehat{c}_{m,3}, \widehat{c}_{m,4} = a_m \mp \sqrt{b/(2a_m)} + o(\sqrt{b}).$$

The corresponding critical values

$$v_{m,3} = F_1(\widehat{c}_{m,3}), \quad v_{m,4} = F_1(\widehat{c}_{m,4})$$

are

$$a_m^2 + c \mp 2\sqrt{2a_m}\sqrt{b} + o(\sqrt{b}) = 2imT\sqrt{-c} + (-m^2T^2 \mp 2\sqrt{2a_m}\sqrt{b} + o(\sqrt{b})).$$

(Computing $\widehat{c}_{m,3(4)}$ and $v_{m,3(4)}$ we need only consider $z^2 + c + b/(z - a_m)$. Other terms $b/(z - a_t)$ make only an $O(b)$ contribution.)

The F_1 -critical point close to 0 is $O(b)$.

Take K_1, K'_4 from the case $d = 3$ slightly modified, larger than the original ones: Let l be the line (parabola) $(2iT\sqrt{-c}\tau, -T^2\tau^2)$ for $\tau > 0$. Observe that the critical values $v_{m,3}$ are to the left of l , and $v_{m,4}$ to the right of l . We extend K_1, K'_4 to $\widehat{K}_1, \widehat{K}'_4$ almost reaching l to capture $v_{m,3}, v_{m,4}$ respectively (see Fig. 8).

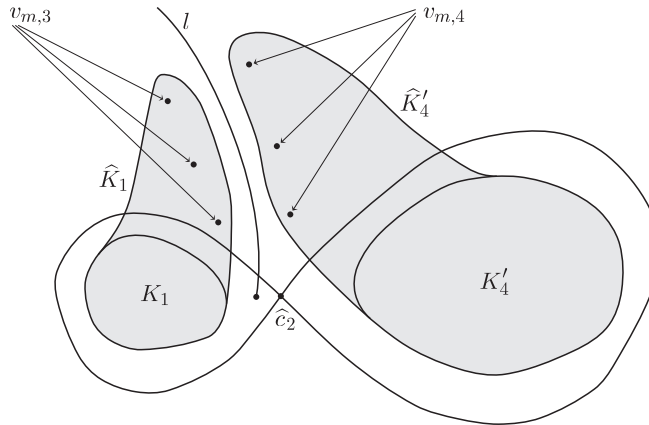


Fig. 8

Consider the topological discs

$$K_{m,3} = \text{Comp } F_1^{-1}(K_1), \quad K_{m,4} = \text{Comp } F_1^{-1}(K'_4),$$

where Comp means the component containing $\widehat{c}_{m,3}$, $\widehat{c}_{m,4}$ respectively. Let K_2 be as in the case $d = 3$, such that $F_1(K_2) \supset \text{cl } K'_4$. Finally, set

$$U := \overline{\mathbb{C}} \setminus \left(K_2 \cup \bigcup_{m=1}^{d-2} (K_{m,3} \cup K_{m,4}) \right).$$

The rest of the construction of f is the same as for $d = 3$. When we define quadratic-like maps \widetilde{F}_2 on $L_{m,3}$ and F_2 on $L_{m,4}$ we have a complete freedom of choosing quadratic polynomials we glue in, in particular, we can have the corresponding critical points escaping or not. (In particular, if no critical point escapes we have the most surprising case $k = 2$ of the assertion of Theorem D.) ■

For completeness of exposition we prove the following simple facts (the first of them was already stated in the introduction):

PROPOSITION 2.4. (a) *Let $f \in H^d$ be a polynomial with $B(f)$, the basin of attraction to ∞ , not simply connected. Then $B(f)$ contains at least one critical point different from ∞ .*

(b) *More generally, if $f \in Q^d$ and for a pair of topological discs A, A_1 with $\text{cl } A \subset A_1$ the map $f|_A : A \rightarrow A$ is proper of degree $d' \leq d$, then A_1 is in the basin of attraction $B(f)$ to an attracting fixed point, and if $B(f)$ is not simply connected then it contains at least d' critical points. This concerns in particular the case $d' = d$ in which $f|_{\overline{\mathbb{C}} \setminus \text{cl } A_1} : \overline{\mathbb{C}} \setminus \text{cl } A_1 \rightarrow \overline{\mathbb{C}} \setminus \text{cl } A$ is polynomial-like.*

Proof. (a) Take a topological disc $D = \{G > a\}$ around ∞ (cf. proof of Corollary B or Remarks 1.5, 1.6). If there are no critical points in $B(f)$ (except ∞) then $f^{-n}(D)$ is an increasing sequence of topological discs, so $B(f) = \bigcup_{n \geq 0} f^{-n}(D)$ is a topological disc, and hence $B(f)$ is simply connected.

(Note that in Remark 2.2 we already used the argument that if there is only one critical value for a proper map $f : W_1 \rightarrow W_2$ where W_2 is a topological disc, then W_1 is also a topological disc.)

(b) The proof is similar. There are $d' - 1$ critical points in A_1 and there must be one in $B(f) \setminus A_1$.

(One can also deduce (b) from (a) using a Blaschke type perturbation, Section 1.) ■

PROPOSITION 2.5. *Every non-simply connected immediate basin of attraction to an attracting or parabolic fixed point p (with $f'(p) = 1$) contains at least 2 different critical values for critical points in the basin.*

This complements Theorem D: the integer k cannot be less than 2.

Proof. Consider the sets D_n defined in Remark 1.6. As $B(f) = \bigcup_{n \geq 0} D_n$ is not simply connected, there exists n such that D_n is simply connected and D_{n+1} is not. Then D_n contains at least two different critical values for critical points in D_{n+1} . ■

(Almost the same argument proves the above for periodic basins, with period larger than 1.)

Proof of Corollary E. Let $f \in H^d$ be as in Theorem D for $d \geq 3, k = 2$. Consider an arbitrary $g \in H(f)$. Then there exist a real continuous 1-parameter family of homeomorphisms $h_t : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ and a real 1-parameter family of maps $f_t \in H(f)$ having the same properties as h_λ and f_λ in Remark 1.4. (There exist complex families precisely as in Remark 1.4, but we do not need them here.) Then $\sharp(\text{Crit}(f_t) \cap B(f_t))$ is constant because critical points cannot be too close to $J(t)$, where $|f'| > 0$ uniformly, and they move continuously with t , so they cannot jump between components of $\overline{\mathbb{C}} \setminus J(f_t)$. So by Proposition 2.4, $B(g)$ cannot be the basin of attraction to ∞ for a polynomial. But the degree of f , and hence of g , on any other invariant basin B_1 is less than d . (Otherwise $\partial B_1 = J(f)$ would be connected and it is not because $B(f)$ is not simply connected.) So g cannot be a polynomial. ■

Note that it follows from Proposition 2.4(b) and the above proof that no $g \in H(f)$ has a polynomial-like restriction of degree d .

3. A 1-parameter family of functions joining an exotic $z \mapsto z^2 + c + b/(z - a)$ to a Newton's method rational function. Let $f(z) = z^2 + c + b/(z - a)$. Then $f'(z) = 2z - b/(z - a)^2$. The equation for the critical points in $\overline{\mathbb{C}}$ is

$$2z(z - a)^2 = b.$$

Suppose that $w = c_4$ is an f -fixed critical point (see Fig. 5, Section 2). (This restricts the number of parameters to 2.) We obtain

$$(w^2 - w + c)(w - a) = -b, \quad 2w(w - a)^2 = b.$$

Let $a = kw$. We parametrize f by k and w . We obtain

$$a = kw, \quad b = 2w^3(1 - k)^2, \quad c = w^2(2k - 3) + w.$$

The critical points are $u = c_2, v = c_3, w$, where

$$u, v = w \left(-\frac{1}{2} + k \mp \frac{1}{2} \sqrt{4k - 3} \right).$$

Given a parameter $k < 1$ sufficiently close to 1, one finds w such that $f^2(v) = v$ and the trajectory of u escapes to ∞ . For $k = .85$ one finds $w \approx 1.88053$. This is an exotic example as in Fig. 5, Section 2. The reason is that the geometry is as in Fig. 6, Section 2, so the basin B of attraction

to ∞ is connected, i.e. the immediate basin is completely invariant. The picture is similar to that in Fig. 10d.

(It is not clear to me whether just the escape of u to ∞ proves the connectedness of B . One should be careful because for f being only a topological branched cover this is not so; see the example in Remark 2.3.)

In the rest of this section we discuss the change of dynamics for varying parameter w .

For $k = .85$ the number $w = 1.88053$ is in the principal part of a Mandelbrot-like set $M(c_3)$, symmetric with respect to real w 's, pronged to the left. For $w \in M(c_3)$, the Julia set for quadratic-like $f^2|_{D_1}$ (see Fig. 5, Section 2) is connected and we still have exotic maps.

Now let us decrease w . It leaves $M(c_3)$ at $w \approx 1.86874$ and below that w the trajectory of v escapes from D_1 . It need not escape to ∞ . There is a sequence of intervals where $f^{2n}(v)$ hits B_w , the basin of immediate attraction to w , as n decreases to 2. Later on, after escape, again $f^4(v) \in B_w$ but $f^2(v) < u$ (before, it was between w and v). This happens at $w \approx 1.63045$ (see Fig. 9).

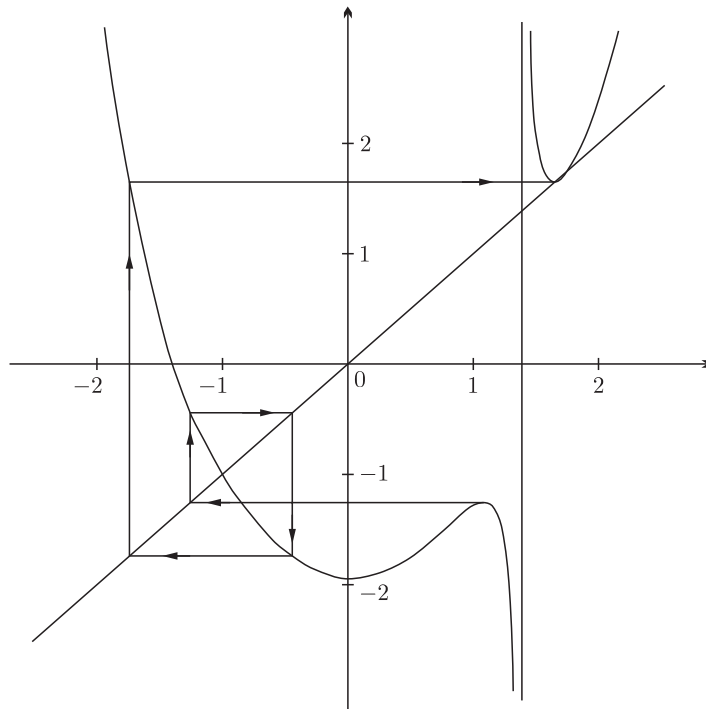


Fig. 9. $k = .85, w = 1.63045$

At some parameter w the trajectory of $u = c_2$ stops to escape to ∞ . It hits B_w . But next with further decrease of w it can again escape to ∞ .

Starting from $w \approx 1.541549$ the trajectory of u neither escapes to ∞ nor to w . The parameter w is in a Mandelbrot-like set $M(c_2)$ pronged towards the right. In fact, at this parameter $f^2(u) \in (v, a)$. Only after some further decrease of w do we arrive at $f^2(u) \in (u, v)$, so that one has a unimodal map $f : (f(u), v) \mapsto (f(u), v)$.

$w \approx .713615$ is in the principal part of $M(c_2)$ and f is Newton's. Then $f^n(c_3) \rightarrow c_2 = f(c_2)$. The number $w \approx .303151$ is still in the principal part of $M(c_2)$ and f is Newton's but now $f^n(c_2) \rightarrow c_3 = c_3$ ⁽⁵⁾.

Let us now present pictures from this experiment for $k = .81$ ⁽⁶⁾.

In Fig. 10, $k = .81$, white is the basin of attraction to w , grey the basin of attraction to ∞ , black is the complement. For Newton's f (Fig. 10a), black contains both c_2 and c_3 , so it has a connected interior and approaches the only repelling fixed point in two channels. Let w grow. For $w \approx 1.37$ the black Newton's basin has bifurcated to period 4, Fig. 10b.

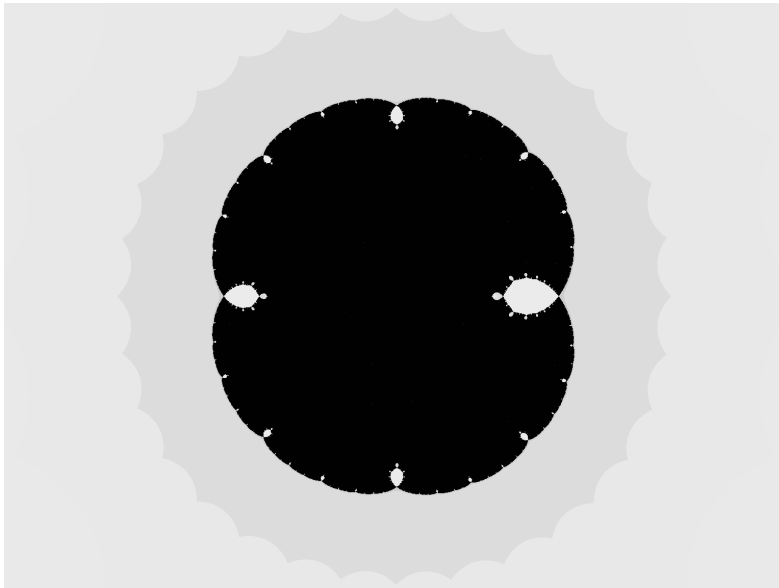


Fig. 10a. $k = .81$, $w = .63$, window $-2-2i, 2+2i$. Iteration of a Newton's method rational map for a polynomial. Black, white and grey are basins of attraction to the zeros of the polynomial.

⁽⁵⁾ This description comes from a computer made picture in 9 colors, showing whether c_2, c_3 escape to ∞ , to w or behave differently.

⁽⁶⁾ Added July 1995: I received recently a letter from Prof. Kiyoko Nishizawa (Josai University, Japan) containing the description of a variety of Newton's maps. I slightly corrected the estimated values of the Newton's parameters w for $k = .85$ according to his computations.

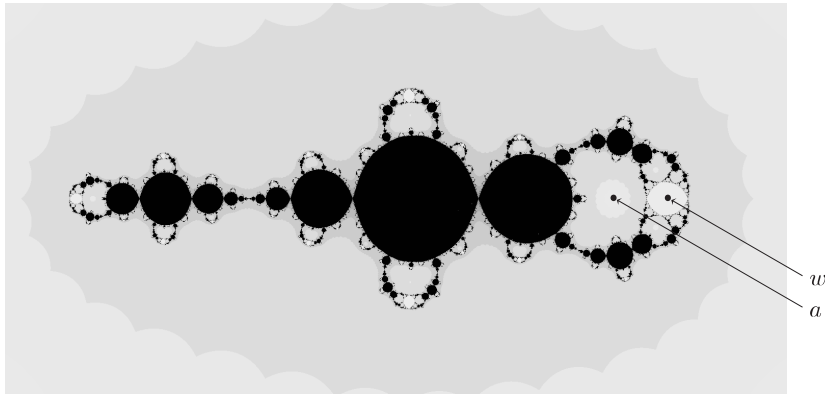


Fig. 10b. $k = .81, w = 1.37$, window $-2 - 2i, 2 + 2i$. The black Newton's basin has bifurcated to a period 4 immediate basin and its preimages.

For $w \approx 1.4961$ (Fig. 10c), w is already in $M(c_3)$ but $u = c_2$ does not escape to ∞ . It is in the basin of w . The basin of ∞ is not connected. This is so because the immediate basin (and the whole basin too) contains only 1 critical point: ∞ . So it is simply connected (see Proposition 2.5). Hence f has only degree 2 on this immediate basin.

For $w \approx 1.51545$, u escapes to ∞ . The basin of ∞ becomes connected. This is one of our exotic examples: see Fig. 10d.

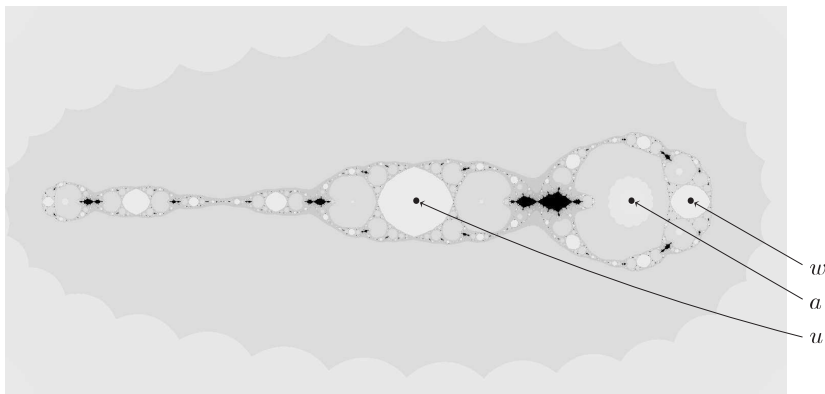


Fig. 10c. $k = .81, w = 1.49$, window $-2 - 2i, 2 + 2i$. The map is still not exotic because the trajectory of the critical point u is attracted to w .

QUESTION 3.1. In the set of Newton's method rational functions NP_λ for the polynomials $P_\lambda(z) = z^3 + (\lambda - 1)z - \lambda$ there exist Mandelbrot-like sets of "exotic" parameters for which the critical point different from the zeros of P_λ converges to a periodic attracting orbit different from these zeros

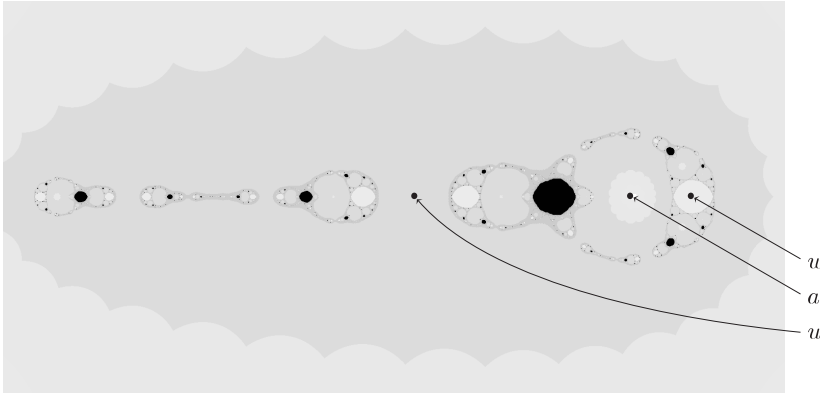


Fig. 10d. $k = .81, w = 1.51545$, window $-i, 2 + i$. This is an exotic map. The pattern is as in Figs. 5, 6. The union of black and white does not separate the plane any more, u escapes to ∞ .

[CGS]. Do these sets move to $M(c_3)$ sections of the set of our exotic maps when we change parameters from Newton's to our exotic ones ⁽⁷⁾?

QUESTION 3.2. Describe precisely how the dynamics bifurcates (i.e. describe the limit behaviour of the trajectories of c_2 and c_3) for real parameters k, w . This question concerns the iteration of a real map having 2 critical points, namely our f restricted to $(-\infty, a)$. (The right branch from a to ∞ does not take part in the recurrence because for $z > a$ and every $n \geq 0$, $f^n(z) \geq a$.)

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⁽⁷⁾ Added July 1995: One can find a family $f_{\mu,\lambda}$ of rational maps of degree 3, $\mu \in (0, 1)$, so that for each $\mu < 1/2$ the maps $f_{\mu,\lambda}$ are Newton's "exotic" (or in their hyperbolic component), and for $\mu > 1/2$ our exotic. See [Ba].

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