FUNDAMENTA
MATHEMATICAE
149 (1996)

The box-counting dimension for geometrically finite Kleinian groups

by

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Abstract. We calculate the box-counting dimension of the limit set of a general geometrically finite Kleinian group. Using the "global measure formula" for the Patterson measure and using an estimate on the horoball counting function we show that the Hausdorff dimension of the limit set is equal to both: the box-counting dimension and packing dimension of the limit set. Thus, by a result of Sullivan, we conclude that for a geometrically finite group these three different types of dimension coincide with the exponent of convergence of the group.

1. Introduction. In this paper we determine various fractal dimensions for limit sets of Kleinian groups. We show that the three concepts of Hausdorff, packing and box-counting dimension, when applied to the limit set of a general geometrically finite Kleinian group, all lead to the same number, namely the exponent of convergence of the group. That such a result should hold was already conjectured for some time. The conjecture was explicitly stated by Davies and Mandouvalos in their paper [6] on upper and lower bounds of the essential spectrum of the Laplacian for regions with fractal boundary.

As already mentioned, we study the limit set $L(G)$ of a non-elementary, geometrically finite Kleinian group G. These groups are discrete subgroups of $\mathrm{Con}(N)$, the group of all orientation preserving Möbius transformations acting on the $(N+1)$-dimensional unit ball D^{N+1} and leaving the hyperbolic metric d in D^{N+1} invariant.

The group G is *geometrically finite* if there exists a fundamental polyhedron for its action on D^{N+1} which has a finite number of faces. This definition includes the cases where G is cofinite, that is, G has parabolic

1991 *Mathematics Subject Classification*: 20H10, 30F40.

Research of the first author supported by the SFB 170 at the University of Göttingen.
Research of the second author supported by the SFB 170 at the University of Göttingen and by NSF Grant DMS 9303888.

elements and the limit set is the boundary S^N of D^{N+1}, and the case where G is convex cocompact, that is, G acts cocompactly on the convex hull of $L(G)$, and also the 1-dimensional case where G is a finitely generated Fuchsian group.

The *limit set* $L(G)$ of a Kleinian group G is the complement of the region of discontinuity of the extended action of G on S^N. It is well known that for a geometrically finite G the perfect set $L(G)$ splits disjointly into the set of parabolic fixed points and the set of conical limit points. In fact, this last property turns out to be equivalent to the property of G being geometrically finite ([3], [4]). Unless G is cofinite, the set $L(G)$ is an extremely complicated structured set which has proved to be an inspiring object for the theory of fractal sets.

In his pioneering work [11], Patterson laid the foundation for a comprehensive study of the limit set in terms of measure theory and in particular in terms of fractal dimensions. His work was certainly motivated by results of the classical theory of metric Diophantine approximation. By constructing a measure on the limit set, which is now called the *Patterson measure*, he was able to show that for "essentially" every finitely generated Fuchsian group G the *exponent of convergence* $\delta = \delta(G)$ is equal to $\dim_{\mathrm{H}}(L(G))$, the Hausdorff dimension of the limit set (in fact, his proof introduces an unnecessary complication in the "parabolic case" and if this is avoided the results are actually seen to be valid for all finitely generated Fuchsian groups). In [15] Sullivan generalized the construction of Patterson to the Kleinian group case. He proved that $\delta(G)$ and $\dim_{\mathrm{H}}(L(G))$ also coincide for all geometrically finite Kleinian groups G.

The determination of the Hausdorff dimension is only one possible way to measure the degree of complexity of a given set. In fact, in practice it is often extremely difficult to calculate it explicitly. There are other concepts which also give insight into the complexity of sets and which are often easier to compute. These include the *box-counting dimension* $\dim_{\mathrm{B}}$ and the *packing dimension* $\dim_{\mathrm{P}}$ (definitions will be given later). Our aim in the present paper is the determination of these two fractal dimensions for the limit sets of Kleinian groups. To be precise, we shall prove that if G is a geometrically finite Kleinian group, then

$$\dim_{\mathrm{B}}(L(G)) = \dim_{\mathrm{P}}(L(G)) = \dim_{\mathrm{H}}(L(G)) = \delta(G).$$

Our proof of this result was inspired by the proof of a similar result for the Julia sets of parabolic rational maps, given in [8]. There it was shown that the Hausdorff dimension and the box-counting dimension agree for the Julia set $J(T)$ of a parabolic rational map T. The proof in [8] is based on two general observations. On the one hand, it is possible to obtain a "general measure estimate" for the $\dim_{\mathrm{H}}(J(T))$-conformal measure on $J(T)$ (this was done in [7], and can be found more explicitly in [8]). On the other hand, there exists a well elaborated formalism for the "Schweiger transformation" T^* (sometimes also called "jump transformation"), which guarantees the existence of a unique T^*-invariant measure on $J(T)$ which is absolutely continuous with respect to the $\dim_{\mathrm{H}}(J(T))$-conformal measure ([1]).

For geometrically finite Kleinian groups there is no such well elaborated "Schweiger formalism" (but see [5]). Nevertheless, our proof here is also based on two general observations. On the one hand, there exists a "global measure formula" for the Patterson measure, which was given in [14] and which we recall in greater detail. On the other hand, we use a growth estimate for the counting function for "standard horoballs", which was also obtained in [14] and which we also recall in greater detail.

We remark that these two approaches seem to suggest that the T^*-invariance of the "Schweiger measure" in the Julia set case corresponds in some sense to the counting estimate for standard horoballs in the Kleinian group case.

Let us also give a sample application of our results. For this we recall from spectral theory the so-called *Weyl conjecture*, which states that the exponent of the second order term in the asymptotic expansion of the spectral counting function for the Laplacian for regions with fractal boundaries should be governed by the Hausdorff dimension of the fractal boundary ([10]). If in this conjecture "Hausdorff dimension" is replaced by "box-counting dimension", then we obtain the so-called *Weyl–Berry conjecture* ([10]). It is clear that the results in the present paper imply that for a region with boundary equal to the limit set of some geometrically finite group the Weyl conjecture and the Weyl–Berry conjecture coincide.

Finally, we remark that, after we had submitted the paper for publication, we received the preprint [3], in which the same result was obtained.

The authors would like to thank the Sonderforschungsbereich 170 at the University of Göttingen for support and hospitality. The second author would also like to thank the NSF for financial support.

2. Preliminaries

Counting standard horoballs. Let G be a geometrically finite subgroup of $\mathrm{Con}(N)$. We need to recall some facts about parabolic elements in G. For a more detailed discussion we refer to [14] and [17].

Let P denote a complete set of inequivalent parabolic fixed points of G. In particular, we shall assume that P is a subset of the closure of the fundamental polyhedron of G which contains the origin 0 in D^{N+1}. Then P is a finite set and $G(P)$ represents the countable set of all parabolic fixed points of G. For p in $G(P)$ let $G_p := \{g \in G : g(p) = p\}$ denote the stabilizer of p in G. The group G_p contains a free abelian subgroup G_p^* of finite index. The rank of G_p^* depends only on p and is referred to as the *rank* $k(p)$ of p.

A *horoball* at p is an open N-ball which is tangential to the boundary S^N at p. We choose a *complete set of standard horoballs*, that is, we assign to each $h(p)$ in $G(P)$ a horoball $H_{h(p)}$ such that the resulting family of horoballs is pairwise disjoint and $h(H_p) = H_{h(p)}$ for each p in P and h in G. With each standard horoball $H_{h(p)}$ we then associate a certain element g in $G_{h(p)}$ such that $g(0)$, if compared with all the other elements in the orbit $G_{h(p)}(0)$, lies closest to the origin 0 in D^{N+1}. The set of group elements g obtained in this fashion is called the *top representation* of the coset representatives of G_p in G and will be denoted by $\mathcal{T}_p$. In the following we shall assume that the standard horoballs are labelled by elements of $\mathcal{T}_p$. For p in P and g in $\mathcal{T}_p$ we let $H_{g(p)}(r_g) := H_{g(p)}$, where r_g denotes the Euclidean radius of the N-ball $H_{g(p)}$. Thus

$$\{H_{g(p)}(r_g) : p \in P,\ g \in \mathcal{T}_p\}$$

represents the complete set of standard horoballs. Also, for $0 < \varepsilon < 1$, we define $H_{g(p)}(\varepsilon r_g) \subseteq H_{g(p)}(r_g)$ to be the horoball at $g(p)$ of radius εr_g.

If we order the standard horoballs according to the sizes of their radii it becomes natural to ask for the cardinality of horoballs of a particular size. The following result was obtained in [14] (Corollary 4.1).

THEOREM 1. *There exist positive constants k_1, k_2, n_0 and ϱ, depending only on G, such that, for all $n > n_0$ and for all p in P,*

$$k_1 \varrho^{-n\delta} \le \operatorname{card} A_n(p) \le k_2 \varrho^{-n\delta},$$

where $A_n(p) := \{g \in \mathcal{T}_p : \varrho^{n+1} \le r_g < \varrho^n\}$, and δ denotes the exponent of convergence of G.

The global measure formula. We also need a formula obtained in [14] and [16]. This formula gives sufficiently good control on the Patterson measure of arbitrary balls centred around limit points. As it results from a slightly coarse study of the involved "geometrically finite geometry", it also only reflects the nature of the Patterson measure in a slightly coarse way, as should be expected from this kind of approach. Nevertheless, for the studies of various more advanced aspects of the "behaviour at infinity", this formula gives just the kind of control over the measure which is required (see e.g. [12]–[14]). In order to be able to state this formula explicitly, some further notation is needed.

For ξ in S^N let s_ξ denote the ray from the origin 0 in D^{N+1} to ξ. If A is a subset of D^{N+1} then $\Pi(A)$, the *shadow at infinity* of A, is defined by

$$\Pi(A) := \{\xi \in S^N : s_\xi \cap A \neq \emptyset\}.$$

For positive t, we let ξ_t denote that point on s_ξ which lies at hyperbolic distance t from the origin. Also, let $b(\xi_t)$ be the shadow at infinity of the N-dimensional hyperbolic hyperplane which is orthogonal to s_ξ and intersects s_ξ at ξ_t. Using elementary hyperbolic geometry it is easily seen that $b(\xi_t)$ is a Euclidean N-ball in S^N centred at ξ with Euclidean radius comparable to e^{-t}.

As already mentioned in the introduction, we denote by μ the Patterson measure. For a construction of this measure we refer to [11]. If G is a geometrically finite Kleinian group, it is well known that μ is an ergodic and "δ-conformal" probability measure supported on the limit set $L(G)$ and which has no atomic part. Here $\delta = \delta(G)$ is the *exponent of convergence* of G, i.e. δ is the exponent of convergence of the Dirichlet sum $\sum \exp(-s\cdot d(0, g0))$, where the sum is taken over all g in G. By *δ-conformal* we mean that for each Borel subset E of S^N and for each g in G, we have

$$\mu(g^{-1}(E)) = \int_E j(g,\xi)^{-\delta}\, d\mu(\xi),$$

where $j(g,\xi)$ denotes the conformal distortion of g at ξ.

The "global measure formula" for μ, announced above, can now be given. We point out that this formula reflects the δ-conformality of the Patterson measure in the presence of a geometrically finite group action. The existence of such a global estimate was certainly already known to Patterson and Sullivan and for its derivation we refer to the elementary proof in [14] (Theorem 2).

Theorem 2. *There exist positive constants k_3 and k_4, depending only on G, such that for each ξ in $L(G)$ and for all positive t,*

$$k_3 e^{-t\delta}(e^{-d(\xi_t, G(0))})^{\delta - k(\xi_t)} \le \mu(b(\xi_t)) \le k_4 e^{-t\delta}(e^{-d(\xi_t, G(0))})^{\delta - k(\xi_t)},$$

where $k(\xi_t)$ is equal to $k(p)$ if ξ_t lies in $H_{g(p)}(r_g)$ for some p in P and g in $\mathcal{T}_p$, and is equal to δ otherwise.

As an immediate consequence we obtain the following estimate for the measure of the shadow at infinity of arbitrarily "squeezed" standard horoballs ([14], Corollary 3.5).

Corollary 1. *There exist positive constants k_5 and k_6 such that for each p in P, g in $\mathcal{T}_p$ and for all positive $\theta < 1$,*

$$k_5 \theta^{2\delta - k(p)} r_g^\delta \le \mu(\Pi(H_{g(p)}(\theta r_g))) \le k_6 \theta^{2\delta - k(p)} r_g^\delta.$$

Hausdorff, packing and box-counting dimension. Finally, we recall a few well known measure-theoretic concepts and facts which have proved to be efficient tools for a classification of sets with an extremely complicated inner structure. For further details we refer to the nicely written book of Falconer [9].

Let a subset Λ of $\mathbb{R}^N$ be given. For positive ε let $\mathcal{C}_\varepsilon(\Lambda)$ denote the set of all coverings and $\mathcal{P}_\varepsilon(\Lambda)$ the set of all packings of Λ by open balls centred in Λ and with radii not exceeding ε. For positive s the *s-dimensional Hausdorff measure* $H_s(\Lambda)$ resp. *packing measure* $P_s(\Lambda)$ of Λ is defined by

$$H_s(\Lambda) := \lim_{\varepsilon\to 0} \inf_{\mathcal{U}\in\mathcal{C}_\varepsilon(\Lambda)} \sum_{u\in\mathcal{U}} (\mathrm{diam}(u))^s,$$

$$P_s(\Lambda) := \inf_{\Lambda=\bigcup_i \Lambda_i} \sum_i \lim_{\varepsilon\to 0} \sup_{\mathcal{U}\in\mathcal{P}_\varepsilon(\Lambda_i)} \sum_{u\in\mathcal{U}} (\mathrm{diam}(u))^s.$$

The *Hausdorff dimension* $\dim_{\mathrm{H}}(\Lambda)$ and the *packing dimension* $\dim_{\mathrm{P}}(\Lambda)$ of Λ are then defined by

$$\dim_{\mathrm{H}}(\Lambda) := \sup\{s : H_s(\Lambda) = \infty\} = \inf\{s : H_s(\Lambda) = 0\},$$

$$\dim_{\mathrm{P}}(\Lambda) := \sup\{s : P_s(\Lambda) = \infty\} = \inf\{s : P_s(\Lambda) = 0\}.$$

Let $\mathcal{P}^*_\varepsilon(\Lambda) \subset \mathcal{P}_\varepsilon(\Lambda)$ denote the set of all packings of Λ by open balls centred in Λ and with radii equal to ε. An element $\mathcal{U}$ of $\mathcal{P}^*_\varepsilon(\Lambda)$ is a *maximal ε-packing* of Λ if and only if $\mathrm{card}(\mathcal{U}) = \max\{\mathrm{card}(\mathcal{V}) : \mathcal{V} \in \mathcal{P}^*_\varepsilon(\Lambda)\}$. A maximal ε-packing of Λ will be denoted by $\mathcal{U}_\varepsilon(\Lambda)$. The *box-counting dimension* $\dim_{\mathrm{B}}(\Lambda)$ of Λ can then be defined by

$$\dim_{\mathrm{B}}(\Lambda) := \lim_{\varepsilon\to 0} \frac{\log \mathrm{card}\, \mathcal{U}_\varepsilon(\Lambda)}{-\log \varepsilon}$$

(if this limit exists).

For the purposes of this paper it is more convenient to define the box-counting dimension by using packings rather than, as is more common, by using coverings. It is easy to see that these two definitions are in fact equivalent ([9], p. 41).

It seems worth mentioning that the idea of using the concept of box-counting dimension in order to have a further method to distinguish between different highly complicated sets dates back to the beginning of this century. Since then various different terms for this concept were and unfortunately sometimes still are in use synonymously. For example, terms like "ε-entropy", "entropy dimension", "Minkowski dimension" or even "capacity" (distinct from "capacity" in potential theory) were in use to denote what we have introduced above as the box-counting dimension.

The following lemma relates the above defined three different notions of dimension. For the proof we refer to [9] (p. 43).

LEMMA 1. *If Λ is a subset of $\mathbb{R}^N$, then* $\dim_{\mathrm{H}}(\Lambda) \le \dim_{\mathrm{P}}(\Lambda) \le \dim_{\mathrm{B}}(\Lambda)$.

Finally, we state a useful method for obtaining upper bounds for the box-counting dimension of subsets in $\mathbb{R}^N$. This method is in a certain sense opposite to the method provided by "Frostman's lemma", which allows one to give lower bounds for the Hausdorff dimension of subsets of $\mathbb{R}^N$.

LEMMA 2. *Let Λ denote a subset of $\mathbb{R}^N$. Assume there exists a probability measure ν supported on Λ. Further, assume that there exist positive constants C and r_0 and a positive number α such that for all ξ in Λ and for all positive $r < r_0$ we have*

$$\nu(B_r(\xi)) \geq Cr^\alpha.$$

It then follows that $\dim_B(\Lambda) \leq \alpha$.

Proof. For positive $r < r_0$ we have

$$1 = \nu(\Lambda) \geq \sum_{u \in \mathcal{U}_r(\Lambda)} \nu(u) \geq Cr^\alpha \operatorname{card} \mathcal{U}_r(\Lambda).$$

Thus

$$\lim_{r \to 0} \frac{\log \operatorname{card} \mathcal{U}_r(\Lambda)}{-\log r} \leq \alpha. \ ■$$

COROLLARY 2. *If Λ is a subset of $\mathbb{R}^N$ such that there exist positive constants C^* and r_0^* so that, for all positive $r < r_0^*$,*

$$\operatorname{card} \mathcal{U}_r(\Lambda) \leq C^* r^{-\alpha},$$

then it follows that $\dim_B(\Lambda) \leq \alpha$.

3. The box-counting dimension of the limit set. In this section we prove the main result of this paper. The results presented in the previous section have the following immediate implication for the box-counting dimension of the limit sets of special geometrically finite Kleinian groups.

For a given group G, let $k_{\min}$ denote the minimal rank of parabolic fixed points of G. It is clear that $k_{\min}$ is a constant that depends on G only.

LEMMA 3. *If G is a geometrically finite Kleinian group which is either convex cocompact or, in the presence of cusps, has $k_{\min}$ greater than or equal to δ, then*

$$\dim_B(L(G)) = \dim_H(L(G)) = \delta.$$

Proof. For convex cocompact groups, as well as for geometrically finite groups with $k_{\min} \geq \delta$, the "fluctuation factor" $e^{-d(\xi_t, G(0))(\delta - k(\xi_t))}$ in Theorem 2 is always at least 1. An application of Lemma 2 then yields that $\dim_B(L(G)) \leq \delta$. The opposite inequality follows from Lemma 1 combined with a result of Sullivan stating that $\dim_H(L(G)) = \delta$ ([14], [15]). ■

THEOREM 3. *If G is a geometrically finite Kleinian group, then*

$$\dim_B(L(G)) = \dim_P(L(G)) = \dim_H(L(G)) = \delta.$$

Proof. The cases where G is either convex cocompact or, if there are cusps, has $k_{\min} \geq \delta$, have been dealt with in Lemma 3 above. Thus we assume in the following that $k_{\min} < \delta$.

It is clear that the non-trivial part of the theorem is the verification of the inequality $\dim_{\mathrm{B}}(L(G)) \le \delta$: the opposite inequality follows directly, as in the proof of Lemma 3, from Lemma 1 and from the fact that $\dim_{\mathrm{H}}(L(G)) = \delta$.

For t positive we consider $\mathcal{U}^I_{e^{-t}}(L(G))$, a maximal e^{-t}-packing of $L(G)$. Here I is some index set labelling the elements of the packing. In particular,

$$\mathcal{U}^I_{e^{-t}}(L(G)) = \{b(\xi^i_t) : i \in I\},$$

where $\xi^i \in L(G)$ for each i in I. In the following let a small, positive number ε be fixed. We now partition the index set I into pairwise disjoint sets as follows:

$$I = I_1 \cup I_2 \cup \bigcup_{\substack{p\in P\\ k(p)<\delta}} \bigcup_{n=1}^{\infty} J_n(p),$$

where

$$\begin{aligned}
I_1 &:= \{i \in I :\ \text{either } d(\xi^i_t, G(0)) \le C \text{ or } \xi^i_t \in H_{g(q)}(r_g) \text{ for some } q \in P\\
&\qquad\qquad \text{with } k(q) \ge \delta \text{ and for some } g \in \mathcal{T}_q\},\\
I_2 &:= \{i \in I : \xi^i_t \in H_{g(q)}(r_g) \text{ for some } q \in P \text{ with } k(q) < \delta\\
&\qquad\qquad \text{and for some } g \in \mathcal{T}_q, \text{ and } C < d(\xi^i_t, G(0)) \le \varepsilon t\},\\
J_n(p) &:= \{i \in I : d(\xi^i_t, G(0)) > \varepsilon t,\ n \le d(\xi^i_t, G(0)) < n+1\\
&\qquad\qquad \text{and } \xi^i_t \in H_{g(p)}(r_g) \text{ for some } g \in \mathcal{T}_p\}.
\end{aligned}$$

We remark that in this definition the positive constant C depends only on the group (C may be taken to be equal to the hyperbolic diameter of the "compact part" of the corresponding orbifold).

For the case $i \in I_1$ we argue as in the proof of Lemma 2, as follows:

$$1 \ge \mu\Big(\bigcup_{i\in I_1} b(\xi^i_t)\Big) \ge \operatorname{card} I_1 \cdot \min_{i\in I_1} \mu(b(\xi^i_t)) \ge k_3 \operatorname{card} I_1 \cdot e^{-t\delta},$$

and thus

$$\operatorname{card} I_1 \le k_3^{-1} e^{t\delta}. \tag{1}$$

For the case $i \in I_2$ we obtain in a similar way

$$\begin{aligned}
1 &\ge \mu\Big(\bigcup_{i\in I_2} b(\xi^i_t)\Big) \ge \operatorname{card} I_2 \cdot \min_{i\in I_2} \mu(b(\xi^i_t))\\
&\ge k_3 \operatorname{card} I_2 \cdot \min_{i\in I_2}(e^{-t\delta} e^{-d(\xi^i_t, G(0))\cdot(\delta - k(\xi^i_t))})\\
&\ge k_3 \operatorname{card} I_2 e^{-t\delta} e^{-\varepsilon t(\delta - k_{\min})},
\end{aligned}$$

and thus

$$\operatorname{card} I_2 \le k_3^{-1} e^{t(\delta + \varepsilon(\delta - k_{\min}))}. \tag{2}$$

Now we turn to the remaining case where i is an element of

$$\bigcup_{\substack{p\in P\\ k(p)<\delta}} \bigcup_{n=1}^{\infty} J_n(p).$$

For the moment fix n and p as above. We consider all possible locations for ξ_t^i with i in $J_n(p)$. An elementary geometric argument yields that the radii r_g of those standard horoballs $H_{g(p)}(r_g)$ with g in $\mathcal{T}_p$ which possibly contain elements ξ_t^i with i in $J_n(p)$ have the property that

$$r_g \geq e^{-t}.$$

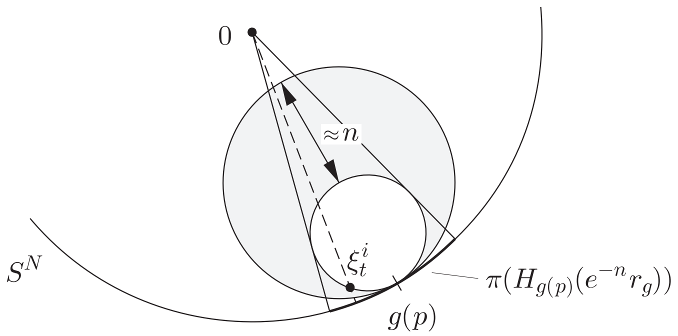

Fig. 1. Possible location of ξ_t^i, for $i \in J_n(p)$

In order to relate the underlying hyperbolic and Euclidean geometry, define n_t to be the smallest positive integer for which $\varrho^{n_t} \leq e^{-t}$, i.e.

(3) $$\frac{t}{-\log\varrho} \leq n_t \leq \frac{t}{-\log\varrho} + 1.$$

With this choice of n_t we obtain

(4) $$\{g \in \mathcal{T}_p : r_g \geq e^{-t}\} \subset \bigcup_{m=0}^{n_t} A_m(p).$$

We also observe that if ξ_t^i with i in $J_n(p)$ is given, then $\xi_t^i \in H_{g(p)}(r_g)$ for some g in $\mathcal{T}_p$, and it is geometrically evident that (see Fig. 1)

(5) $$b(\xi_t^i) \subset \Pi(H_{g(p)}(2e^{-n}r_g)).$$

From (3)–(5) and Theorems 1 and 2 it now follows that

$$\mu\Big(\bigcup_{i\in J_n(p)} b(\xi_t^i)\Big) \leq \mu\Big(\bigcup_{m=0}^{n_t} \bigcup_{g\in A_m(p)} \Pi(H_{g(p)}(2e^{-n}r_g))\Big)$$
$$\leq \sum_{m=0}^{n_t} \sum_{g\in A_m(p)} \mu(\Pi(H_{g(p)}(2e^{-n}r_g)))$$

$$\begin{aligned}
&\le 2^{2\delta-k_{\min}} k_4 \sum_{m=0}^{n_t} \sum_{g\in A_m(p)} r_g^\delta e^{-n(2\delta-k(p))} \\
&\le 2^{2\delta-k_{\min}} k_4 e^{-n(2\delta-k(p))} \sum_{m=0}^{n_t} \sum_{g\in A_m(p)} \varrho^{m\delta} \\
&\le 2^{2\delta-k_{\min}} k_4 k_2 e^{-n(2\delta-k(p))}(n_t+1) \\
&\le 2^{2\delta-k_{\min}+1} k_4 k_2 e^{-n(2\delta-k(p))}(-\log\varrho)^{-1} t.
\end{aligned}$$

For $i \in J_n(p)$ we have, by definition of $J_n(p)$,

$$t < \varepsilon^{-1} d(\xi_t^i, G(0)) \le \varepsilon^{-1}(n+1).$$

Hence, if we let $c_0 := 3\cdot 2^{2\delta-k_{\min}+1}k_4k_2$, we obtain

$$\mu\Big(\bigcup_{i\in J_n(p)} b(\xi_t^i)\Big) \le c_0\varepsilon^{-1} n e^{-n(2\delta-k(p))}. \tag{6}$$

On the other hand, we have, using Theorem 1 and the fact that $\mathcal{U}^I_{e^{-t}}(L(G))$ is an e^{-t}-packing,

$$\begin{aligned}
\mu\Big(\bigcup_{i\in J_n(p)} b(\xi_t^i)\Big) &\ge k_3 e^{-t\delta}\operatorname{card} J_n(p)\cdot \min_{i\in J_n(p)} e^{-d(\xi_t^i,G(0))(\delta-k(p))} \\
&\ge k_3 e^{k_{\min}-\delta} e^{-t\delta} e^{-n(\delta-k(p))}\operatorname{card} J_n(p).
\end{aligned}$$

Combining this estimate and (6), we obtain

$$\operatorname{card} J_n(p) \le c_0 k_3^{-1} e^{\delta-k_{\min}} \varepsilon^{-1} n e^{-n\delta} e^{t\delta}.$$

Hence, if we define $c_1 := c_0 k_3^{-1} e^{\delta-k_{\min}} \operatorname{card} P\cdot \sum_{n=1}^\infty n e^{-n\delta}$, it follows that

$$\operatorname{card}\Big(\bigcup_{\substack{p\in P\\ k(p)<\delta}} \bigcup_{n=1}^\infty J_n(p)\Big) \le c_1\varepsilon^{-1}e^{t\delta}. \tag{7}$$

Now (1), (2) and (7) imply that

$$\operatorname{card} I \le k_3^{-1}e^{t\delta} + k_3^{-1} e^{t(\delta+\varepsilon(\delta-k_{\min}))} + c_1\varepsilon^{-1}e^{t\delta}.$$

In order to summarize the above calculations, we remark that we have just shown that for each positive ε there exists a positive constant c such that for sufficiently large, positive t the index sets I of the maximal e^{-t}-packings $\mathcal{U}^I_{e^{-t}}(L(G))$ satisfy

$$\operatorname{card} I \le c e^{t(\delta+\varepsilon(\delta-k_{\min}))}.$$

From Corollary 2 it now follows that

$$\dim_{\mathrm B}(L(G)) \le \delta + \varepsilon(\delta - k_{\min}).$$

This holds for arbitrary small, positive ε, thus

$$\dim_{\mathrm B}(L(G)) \le \delta.$$

Using Sullivan's result ($\dim_{\mathrm{H}}(L(G)) \geq \delta$) and Lemma 1, we obtain

$$\dim_{\mathrm{B}}(L(G)) = \dim_{\mathrm{P}}(L(G)) = \dim_{\mathrm{H}}(L(G)) = \delta. \ \blacksquare$$

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Received 21 May 1995;
in revised form 7 August 1995