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On Auslander–Reiten components for quasitilted algebras

by

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Abstract. An artin algebra A over a commutative artin ring R is called quasitilted if $\mathrm{gl.dim}\, A \le 2$ and for each indecomposable finitely generated A-module M we have $\mathrm{pd}\, M \le 1$ or $\mathrm{id}\, M \le 1$. In [11] several characterizations of quasitilted algebras were proven. We investigate the structure and homological properties of connected components in the Auslander–Reiten quiver Γ_A of a quasitilted algebra A.

Let A be an artin algebra over a commutative artin ring R, that is, A is an R-algebra which is finitely generated as an R-module. Denote by $\mathrm{ind}\, A$ the category of indecomposable finitely generated right A-modules, by Γ_A the Auslander–Reiten quiver of A, and by τ_A the Auslander–Reiten translation in Γ_A. Following [10], the algebra A is called *tilted* if there exists a hereditary artin algebra H and a tilting H-module T such that $A = \mathrm{End}_H(T)$. Recall that a finitely generated H-module T is called *tilting* if $\mathrm{Ext}^1_H(T,T) = 0$ and there is an exact sequence $0 \to H_H \to T_0 \to T_1 \to 0$ with T_0 and T_1 in the additive category $\mathrm{add}\, T$, given by T.

The representation theory of tilted algebras is fairly well understood. In particular, we know the shape of all connected components of the Auslander–Reiten quivers of tilted algebras (see [8], [12], [13], [17]–[20], [27]). It is known that a tilted algebra A is of global dimension at most 2 and no module in $\mathrm{ind}\, A$ has both projective and injective dimension equal to 2. However, these properties do not characterize the tilted algebras. Happel, Reiten and Smalø have shown in [11] that they characterize the class of quasitilted algebras which are the artin algebras of the form $A = \mathrm{End}(T)$, where T is a tilting object in a hereditary abelian R-category $\mathcal{H}$.

Besides the tilted algebras, important classes of quasitilted algebras are provided by tubular algebras [19], canonical algebras [14], [19], [21], algebras with separating tubular families of modules [15], [25], and semiregular

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branch enlargements of such algebras (see [7]). Moreover, it is known that any representation-finite quasitilted algebra is tilted [11].

An important result proven in [11] is the following trisection of the category $\operatorname{ind} A$ of a quasitilted algebra A. Namely, let A be a quasitilted algebra, $\mathcal{R} = \mathcal{R}_A$ be the full subcategory of $\operatorname{ind} A$ formed by the modules all of whose successors in $\operatorname{ind} A$ have injective dimension at most one, and $\mathcal{L} = \mathcal{L}_A$ be the full subcategory of $\operatorname{ind} A$ formed by the modules all of whose predecessors in $\operatorname{ind} A$ have projective dimension at most one. Then we have a trisection

$$\operatorname{ind} A = (\mathcal{L} \setminus \mathcal{R}) \vee (\mathcal{L} \cap \mathcal{R}) \vee (\mathcal{R} \setminus \mathcal{L})$$

such that

$$\operatorname{Hom}_A(\mathcal{L} \cap \mathcal{R}, \mathcal{L} \setminus \mathcal{R}) = 0, \quad \operatorname{Hom}_A(\mathcal{R} \setminus \mathcal{L}, \mathcal{L} \cap \mathcal{R}) = 0,$$

and

$$\operatorname{Hom}_A(\mathcal{R} \setminus \mathcal{L}, \mathcal{L} \setminus \mathcal{R}) = 0.$$

Moreover, $\mathcal{L}$ (respectively, $\mathcal{R}$) contains all indecomposable projective (respectively, injective) A-modules.

We investigate here the structure of connected components in the Auslander–Reiten quiver of an arbitrary quasitilted algebra A. Good understanding of the Auslander–Reiten components seems to be the main step in describing the ring structure and representation theory of arbitrary quasitilted algebras (see [15], [24]–[26]). We are mainly interested in quasitilted algebras which are not tilted.

In order to state our main results, recall that a (connected) component Γ of Γ_A is called *regular* if Γ contains neither a projective module nor an injective module. Moreover, Γ is called *semiregular* if Γ does not contain a projective module and an injective module at the same time. We shall prove the following facts.

Theorem (A). *Let A be a quasitilted artin algebra, and Γ be a component of Γ_A containing an oriented cycle. Then Γ is a semiregular tube.*

We note that a semiregular tube is either regular (that is, of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^s)$, for some $s \geq 1$) or is obtained from a regular tube by a finite sequence of ray (or coray) insertions.

Theorem (B). *Let A be a quasitilted algebra, and Γ be a regular component of Γ_A.*

(i) *If $\Gamma \cap \mathcal{R} \neq \emptyset$, then Γ is contained in $\mathcal{R}$.*
(ii) *If $\Gamma \cap \mathcal{L} \neq \emptyset$, then Γ is contained in $\mathcal{L}$.*

Theorem (C). *Let A be a quasitilted algebra and Γ be a component of Γ_A with infinitely many τ_A-orbits or containing an oriented cycle.*

(i) *If Γ contains a projective module, then Γ is contained in $\mathcal{L} \setminus \mathcal{R}$.*
(ii) *If Γ contains an injective module, then Γ is contained in $\mathcal{R} \setminus \mathcal{L}$.*

THEOREM (D). *Let A be a quasitilted algebra which is not tilted, and Γ be a component of Γ_A.*

(i) *If Γ contains a projective module, then Γ is contained in $\mathcal{L} \setminus \mathcal{R}$.*
(ii) *If Γ contains an injective module, then Γ is contained in $\mathcal{R} \setminus \mathcal{L}$.*

We also get the following immediate consequences of the above theorems.

COROLLARY (E). *Let A be a quasitilted algebra which is not tilted. Then every component of Γ_A is semiregular.*

COROLLARY (F). *Let A be a quasitilted algebra which is not tilted. Then every component Γ of Γ_A having a module from $\mathcal{R} \cap \mathcal{L}$ is regular, and hence consists entirely of modules from $\mathcal{R} \cap \mathcal{L}$.*

Further consequences will be discussed in Section 6.

This paper is organized as follows. In Section 1 we prove preliminary results on the paths between indecomposable modules over artin algebras, playing a crucial role in our further investigations. In Section 2 we recall some facts on tilted and quasitilted algebras applied in the paper. Sections 3, 4 and 5 are devoted to the structure of components with oriented cycles, regular components, and nonregular components, respectively, in the Auslander–Reiten quivers of quasitilted algebras. In Section 6 we present some consequences of our main results.

1. Preliminary results

1.1. Let A be an artin algebra over a commutative artin ring R, that is, A is an R-algebra which is finitely generated as an R-module. Unless otherwise stated all algebras are assumed to be basic and connected. By an A-module is meant a finitely generated right A-module. We shall denote by $\operatorname{mod} A$ the category of all (finitely generated) A-modules, and by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ with one representative of each isomorphism class of indecomposable A-modules. Then $\operatorname{rad}(\operatorname{mod} A)$ denotes the *Jacobson radical* of $\operatorname{mod} A$, that is, the ideal in $\operatorname{mod} A$ generated by all noninvertible morphisms between indecomposable modules in $\operatorname{mod} A$. The *infinite radical* $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ of $\operatorname{mod} A$ is the intersection of all powers $\operatorname{rad}^{i}(\operatorname{mod} A)$, $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$.

1.2. We denote by Γ_A the Auslander–Reiten quiver of A, and by $\tau = D\operatorname{Tr}$ and $\tau^{-} = \operatorname{Tr} D$ the Auslander–Reiten translations in Γ_A. We identify the vertices of Γ_A with the corresponding A-modules in $\operatorname{ind} A$. By a *component* of Γ_A we mean a connected component. We observe that a morphism between indecomposable modules lying in different components of Γ_A belongs to $\operatorname{rad}^\infty(\operatorname{mod} A)$.

We frequently use the fact that, for an A-module X, $\operatorname{pd} X \leq 1$ if and only if $\operatorname{Hom}_A(D(A), \tau X) = 0$ (respectively, $\operatorname{id} X \leq 1$ if and only if $\operatorname{Hom}_A(\tau^- X, A) = 0$) (see [19, p. 74]).

Let Γ be a component of Γ_A. Then Γ is said to be *regular* if Γ contains neither a projective module nor an injective module, and *semiregular* if Γ does not contain a projective module and an injective module at the same time. Also, Γ is said to be *postprojective* (respectively, *preinjective*) if Γ contains no oriented cycles and each module in Γ belongs to the τ-orbit of a projective (respectively, an injective) module. We denote by ${}_l\Gamma$ the *left stable part* of Γ obtained from Γ by deleting the τ-orbits of projective modules, by ${}_r\Gamma$ the *right stable part* of Γ obtained from Γ by deleting the τ-orbits of injective modules, and by ${}_s\Gamma$ the *stable part* of Γ obtained from Γ by deleting the τ-orbits of both the projective and the injective modules.

A module $M \in \operatorname{ind} A$ is called *τ-periodic* if there exists an $m \geq 1$ such that $\tau^m M \simeq M$. Given $M \in \operatorname{ind} A$, we denote by $\mathcal{O}(M)$ the *τ-orbit* of M, that is,

$$\mathcal{O}(M) = \{\tau^m M : m \in \mathbb{Z}\}.$$

1.3. Let $M, N \in \operatorname{ind} A$. A *path* from M to N is given by a sequence of nonzero morphisms

$$M = X_0 \xrightarrow{f_1} X_1 \to \ldots \xrightarrow{f_t} X_t = N,$$

where, for each i, X_i is an indecomposable module and f_i is in $\operatorname{rad}(\operatorname{mod} A)$. We denote a path from M to N by $M \rightsquigarrow N$. If the morphisms f_i are in addition irreducible, then we call it a *path of irreducible maps*. An *oriented cycle* is a path of irreducible maps from a module to itself. A path of irreducible maps $X_0 \to X_1 \to \ldots \to X_t$ is called *sectional* if $X_i \not\simeq \tau X_{i+2}$ for each $i = 0, \ldots, t-2$.

Given a path $M \rightsquigarrow N$, M is said to be a *predecessor* of N and N a *successor* of M. The meaning of the terms "predecessor (or successor) by irreducible maps" should be clear. Finally, given $M, N \in \operatorname{ind} A$, we write M — N whenever there is either an irreducible map $M \to N$ or an irreducible map $N \to M$. For more details on the Auslander–Reiten theory we refer the reader to [3] and [19].

1.4. We now prove two lemmas needed later on.

Lemma. *Let A be an artin algebra,*

$$X = X_0 \to X_1 \to \ldots \to X_t = X \tag{$*$}$$

be an oriented cycle through indecomposable modules, *and* $r \geq 1$. *If* $\tau^i X_j \neq 0$ *for each* $1 \leq i \leq r$ *and each* $j = 0, \ldots, t$, *then there exists a path of irreducible maps from* X *to* $\tau^r X$.

Proof. We know, by [4], that the oriented cycle $(*)$ is not sectional. Therefore, there exists an l, $2 \leq l \leq t$, such that $\tau X_l \simeq X_{l-2}$. By hypothesis, one can apply τ to $(*)$ to get

$$(**) \qquad \tau X = \tau X_0 \to \tau X_1 \to \ldots \to \tau X_t = \tau X.$$

Observe that the module $\tau X_l \simeq X_{l-2}$ appears in both $(*)$ and $(**)$, and hence there exists a path from X to τX, namely

$$X = X_0 \to X_1 \to \ldots \to X_{l-2} \simeq \tau X_l \to \tau X_{l+1} \to \ldots \to \tau X_t = \tau X.$$

By applying τ and composing the paths, we get the desired result. ∎

1.5. The next result extends [22, Lemma 4].

Lemma. *Let* A *be an artin algebra and denote by* n *the rank of the Grothendieck group* $K_0(A)$ *of* A. *Let* Γ *be a connected component of* Γ_A *and* Γ' *be a connected component of* ${}_s\Gamma$. *Assume that* Γ' *has infinitely many* τ*-orbits and no oriented cycles. Let* M *be a module in* Γ' *such that the length of any walk in* Γ *from a nonstable module to the* τ*-orbit of* M *is at least* $2n$. *Then*, *for each* $s \geq 1$, *there exists a path*

$$M = X_0 \to X_1 \to \ldots \to X_l = \tau^s M$$

in $\operatorname{mod} A$ *with all* X_i *in* Γ.

Proof. It is enough to show that there exists a path in $\operatorname{mod} A$ from M to τM through modules in Γ, and then proceed inductively. By [22, Lemma 4], there is a path

$$M = X'_0 \to X'_1 \to \ldots \to X'_t = M$$

in $\operatorname{mod} A$ with $X'_1, \ldots, X'_t$ belonging to Γ. Since Γ has no oriented cycles, one of the maps in the above path should be in $\operatorname{rad}^\infty(\operatorname{mod} A)$. We infer that there exists a path

$$(*) \qquad M = Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} \ldots \to Y_r \xrightarrow{f_r} \ldots \to Y_{r+n} \xrightarrow{f_{r+n}} Y_{r+n+1} \xrightarrow{f_{r+n+1}} M,$$

where the morphisms $f_r, \ldots, f_{r+n+1}$ are irreducible maps (this is done by using the lifting properties of almost split sequences). Consider now the path of irreducible maps

$$(**) \qquad Y_r \xrightarrow{f_r} Y_{r+1} \to \ldots \to Y_{r+n} \xrightarrow{f_{r+n}} Y_{r+n+1}.$$

If $(**)$ is nonsectional, then there exists an i, $r \leq i \leq r+n-1$, such that $Y_i \simeq \tau Y_{i+2}$, and thus there exists a path of irreducible maps

$$Y_r \to \ldots \to Y_i \to \tau Y_{i+3} \to \ldots \to \tau M$$

(observe that the modules in $(**)$ are left stable, and hence one can apply τ to them). Therefore, there exists a path from M to τM passing through modules in Γ, namely

$$M = Y_0 \to Y_1 \to \ldots \to Y_r \to \ldots \to Y_i \to \tau Y_{i+3} \to \ldots \to \tau M.$$

Suppose now $(**)$ is sectional. Then, by [22, Lemma 2], there exist j and $l, r \le j, l \le r+n+1$, and a nonzero map $g \in \mathrm{Hom}_A(Y_j, \tau Y_l) = \mathrm{rad}^\infty(Y_j, \tau Y_l)$. Hence,

$$M = Y_0 \to Y_1 \to \ldots \to Y_j \xrightarrow{g} \tau Y_l \to \tau Y_{l+1} \to \ldots \to \tau M$$

gives the required path from M to τM passing through modules in Γ. Now, since for each $i \ge 1$, $\tau^i M$ is in the conditions of the Lemma, we can iterate the above procedure to get a path from M to each $\tau^r M$, $r \ge 1$, as required. ∎

1.6. COROLLARY. *Let A be an artin algebra and Γ be a regular component of Γ_A with infinitely many τ-orbits. Then, for each $M \in \Gamma$, and each $r \ge 1$, there exists a path in* mod$A$ *from M to $\tau^r M$.*

2. Quasitilted algebras

2.1. In this section we collect the results on quasitilted algebras needed along the paper. We start by recalling some facts on tilted algebras. For details on tilting theory we refer the reader to [10] and [19]. Let H be a hereditary algebra and let T be a tilting H-module, that is, a module such that $\mathrm{Ext}^1_H(T,T) = 0$ and there exists a short exact sequence $0 \to H \to T_0 \to T_1 \to 0$, where T_0 and T_1 are in $\mathrm{add}\, T$. The algebra $B = \mathrm{End}_H(T)$ is called a *tilted algebra*. An important fact about a tilted algebra B is that Γ_B contains a component, called *connecting*, which contains a so-called complete slice Σ which reproduces in a sense the structure of the hereditary algebra H. It is well known that all successors of such a Σ have injective dimension at most one, and all predecessors of Σ have projective dimension at most one. Recall that a subquiver Σ in a component Γ of Γ_A is called a *complete slice* if: (a) Σ is sincere; (b) Σ is path closed in mod$A$, and (c) Σ meets each τ-orbit of Γ exactly once.

Let $B = \mathrm{End}_H(T)$, where T is a tilting module over a representation-infinite hereditary algebra H. If T is a postprojective H-module (equivalently, $\mathrm{rad}^\infty(-,T) = 0$), then the algebra B is called *concealed*. It is well known that the Auslander–Reiten quiver of a tilted algebra B contains at most two connecting components, and it has exactly two if and only if B is concealed. Also, T is a regular H-module if and only if the connecting component of Γ_B is regular.

2.2. We now recall the definition of quasitilted algebras and some relevant results. We refer the reader to [11] for the proof of these results.

Definition. An algebra A is said to be *quasitilted* if $\mathrm{gl.dim}\, A \le 2$ and for each $X \in \mathrm{ind}\, A$, either $\mathrm{pd}\, X \le 1$ or $\mathrm{id}\, X \le 1$.

Tilted algebras are clearly examples of quasitilted algebras. However, as mentioned in the introduction the class of quasitilted algebras is much larger. It has been proven in [11, (II.3.6)] that representation-finite quasitilted algebras are tilted.

2.3. We mention the next result for later reference.

Theorem ([11, (II.1.14)]). *Let A be a quasitilted algebra. Then any path in $\mathrm{ind}\, A$ starting in an injective module and ending in a projective module has a refinement formed by irreducible maps and every such path is sectional.*

2.4. Let A be a quasitilted algebra. An important result is the existence of the following trisection of the category $\mathrm{ind}\, A$. Let

$$\mathcal{R} = \mathcal{R}_A = \{X \in \mathrm{ind}\, A : \text{for each } Y \text{ with } X \rightsquigarrow Y,\ \mathrm{id}\, Y \le 1\},$$
$$\mathcal{L} = \mathcal{L}_A = \{X \in \mathrm{ind}\, A : \text{for each } Y \text{ with } Y \rightsquigarrow X,\ \mathrm{pd}\, Y \le 1\}.$$

This induces a trisection

$$\mathrm{ind}\, A = (\mathcal{L} \setminus \mathcal{R}) \vee (\mathcal{L} \cap \mathcal{R}) \vee (\mathcal{R} \setminus \mathcal{L})$$

such that

$$\mathrm{Hom}_A(\mathcal{L} \cap \mathcal{R}, \mathcal{L} \setminus \mathcal{R}) = 0, \qquad \mathrm{Hom}_A(\mathcal{R} \setminus \mathcal{L}, \mathcal{L} \cap \mathcal{R}) = 0,$$

and

$$\mathrm{Hom}_A(\mathcal{R} \setminus \mathcal{L}, \mathcal{L} \setminus \mathcal{R}) = 0.$$

Moreover, $\mathcal{L}$ contains all the indecomposable projective modules and it is closed under predecessors, while $\mathcal{R}$ contains all the indecomposable injective modules and it is closed under successors.

2.5. The next result gives a criterion for a quasitilted algebra to be tilted.

Theorem ([11, (II.3.4)]). *Let A be a quasitilted algebra. If $\mathcal{R}$ contains a projective module, then A is tilted.*

2.6. Let A be a tilted algebra. Then, clearly, any complete slice in $\mathrm{mod}\, A$ is contained in $\mathcal{R} \cap \mathcal{L}$. In particular, for tilted algebras $\mathcal{R} \cap \mathcal{L}$ is nonempty. For quasitilted algebras which are not tilted, it is still an open question whether $\mathcal{R} \cap \mathcal{L}$ in nonempty. We shall show (Corollary (F)) that, if $\mathcal{R} \cap \mathcal{L}$ is nonempty for a quasitilted algebra which is not tilted, then $\mathcal{R} \cap \mathcal{L}$ is formed by modules lying in regular components.

3. Components with oriented cycles

3.1. Let A be a quasitilted algebra and Γ be a component of Γ_A containing oriented cycles. We shall show that Γ is in fact a semiregular tube, generalizing a result known for tilted algebras (see [12], [13], [17]). The main point in the proof is to show that such a Γ is semiregular. Semiregular components with oriented cycles have been described in [9], [16] and [28], and they are either of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^m)$ for some $m \geq 1$ (if regular), or obtained from it by a finite sequence of ray (or coray) insertions.

THEOREM (A). *Let A be a quasitilted algebra, and Γ be a component of Γ_A containing an oriented cycle. Then Γ is a semiregular tube.*

Proof. Let Γ be a component of Γ_A containing an oriented cycle. We first observe that Γ is infinite. Indeed, if Γ is finite, then A is representation-finite, and hence by (2.2) it is tilted and Γ $(= \Gamma_A)$ is a connecting component. It is well known that, in this case, Γ has no oriented cycles, a contradiction, and hence Γ is infinite.

If Γ is regular, then by [9] and [28], Γ is a stable tube. If Γ is semiregular but not regular, it follows from [16] that Γ is a semiregular tube.

Suppose then that Γ is not semiregular. We first claim that Γ has no τ-periodic modules. Suppose Γ has a τ-periodic module. Since Γ is not regular, we infer, using [2, (6.2)], that there exists an irreducible map $X \to Y$, where X is a τ-periodic module, and Y is neither left nor right stable, that is, there are m and m' such that $\tau^m Y$ is a projective module P and $\tau^{m'} Y$ is an injective module I. Therefore, there exists a nonsectional path from I to P, which contradicts our hypothesis that A is quasitilted (see (2.3)). This proves the claim.

Consider now the left and right stable parts ${}_l\Gamma$ and ${}_r\Gamma$ of Γ. Since Γ is infinite, either ${}_l\Gamma$ or ${}_r\Gamma$ is nonempty. Suppose ${}_l\Gamma \neq \emptyset$ and let Γ' be a connected component of ${}_l\Gamma$. Clearly, Γ' is infinite because otherwise it would contain a τ-periodic module, contradicting the above claim.

We now show that Γ' contains no oriented cycles. Suppose it contains oriented cycles. Then Γ' contains injective modules, because otherwise it would be a stable tube by [9] and [28], in particular, it would contain τ-periodic modules, a contradiction to the claim. Summing up, Γ' is an infinite connected component of ${}_l\Gamma$ with oriented cycles and containing injective modules. Then, by [16, (2.3)], there exists an infinite sectional path

$$\ldots \tau^{2t} X_1 \to \tau^t X_s \to \ldots \to \tau^t X_2 \to \tau^t X_1 \to X_s \to \ldots \to X_1$$

with $t > s$ such that $\{X_1, \ldots, X_s\}$ is a complete set of representatives of τ-orbits in Γ'. Since Γ' is a component of ${}_l\Gamma$ and Γ is not left stable, there exists an irreducible map $X' \to X''$ with X'' in the τ-orbit of a projective module and $X' \in \Gamma'$. By applying τ as many times as necessary, there exists an irreducible map $X \to P$ with $X \in \Gamma'$ and P an indecomposable projective module. Since $X \in \Gamma'$, we infer that $\tau^{m'} X \simeq \tau^{mt} X_j$ for some $1 \leq j \leq s$, and some $m, m' \geq 0$.

It follows from the hypothesis on Γ' that there exists an oriented cycle $(*)$ in Γ' containing an injective module I. By (1.4), there are paths from I to each $\tau^r I$, $r \geq 1$. Observe that I is in the τ-orbit of one of $X_1, \ldots, X_s$, say X_i. Thus, we get a path from I to $\tau^{(m+1)t} X_i$. Therefore, there exists a path

$$I \overset{(*)}{\rightsquigarrow} I \rightsquigarrow \tau^{(m+1)t} X_i \rightsquigarrow \tau^{mt} X_j \rightsquigarrow X \to P$$

from an injective module to a projective module which is not sectional because $(*)$ is not sectional by [4]. This contradicts A being quasitilted. Therefore, none of the components of ${}_l\Gamma$ contains an oriented cycle. Similarly, one can also show that ${}_r\Gamma$ contains no oriented cycles.

However, by hypothesis, Γ contains an oriented cycle

$$Y_0 \to Y_1 \to \ldots \to Y_u = Y_0.$$

By the above remarks, such an oriented cycle should contain a module which is not left stable and a module which is not right stable. By applying τ^{-1} we can assume that Y_0 is injective. Let Y_v and $l \geq 0$ be such that $\tau^l Y_v$ is projective, and assume that $\tau^i Y_j \neq 0$ for each $j \neq v$ and $i \leq l$. By (1.4), there exists a path from Y_0 to $\tau^l Y_0$ and then to $\tau^l Y_v$, which is projective, and this path can be chosen to be nonsectional, contradicting the fact that A is quasitilted, and the result is proven. ∎

4. Regular components

4.1. The main aim in this section is to prove Theorem (B) of the introduction, which concerns regular components of the Auslander–Reiten quiver of quasitilted algebras. We need the following lemma.

Lemma. *Let A be a quasitilted algebra and Γ be a component of Γ_A.*

(a) *If $\Gamma \cap \mathcal{R} \neq \emptyset$, then each τ-orbit of Γ contains a module from $\mathcal{R}$.*
(b) *If $\Gamma \cap \mathcal{L} \neq \emptyset$, then each τ-orbit of Γ contains a module from $\mathcal{L}$.*

Proof. We only prove (a) because the proof of (b) is similar.

Let Γ be a component of Γ_A containing a module from $\mathcal{R}$. If the right stable part ${}_r\Gamma$ is empty, or equivalently if each τ-orbit of Γ contains an injective module, then there is nothing to prove because $\mathcal{R}$ contains all the injective modules. Suppose ${}_r\Gamma \neq \emptyset$ and let Γ' be a connected component of ${}_r\Gamma$. We first claim that Γ' has a module from $\mathcal{R}$. If Γ has no injective modules then $\Gamma' = {}_r\Gamma = \Gamma$, and the claim is clear. Suppose that Γ contains an injective module. Then there exists an irreducible map $I \to X'$, where I is an injective module and $X' \in \Gamma'$. The claim now follows from the fact that $X' \in \mathcal{R}$, because $I \in \mathcal{R}$, and $\mathcal{R}$ is closed under successors.

Let $X \in \Gamma' \cap \mathcal{R}$ and let $Y \in \Gamma'$. We now show that $\mathcal{O}(Y) \cap \mathcal{R} \neq \emptyset$. Since Γ' is connected, there exists a walk

$$X = X_0 \longrightarrow X_1 \longrightarrow \ldots \longrightarrow X_s = Y$$

in Γ'. Observe that the modules X_i, $i = 1, \ldots, s$, are right stable and so one can apply τ^- as many times as necessary to get a path from X to some $\tau^{-m}Y$, $m \geq 0$. Since $\mathcal{R}$ is closed under successors we get $\tau^{-m}Y \in \mathcal{R}$, and hence each τ-orbit of ${}_r\Gamma$ has a module from $\mathcal{R}$. The result now follows from the fact that the τ-orbits which are not in ${}_r\Gamma$ contain an injective module, and hence a module from $\mathcal{R}$. ∎

4.2. We can now prove Theorem (B) of the introduction.

THEOREM (B). *Let A be a quasitilted algebra and Γ be a regular component of Γ_A.*

(a) *If $\Gamma \cap \mathcal{R} \neq \emptyset$, then Γ is contained in $\mathcal{R}$.*
(b) *If $\Gamma \cap \mathcal{L} \neq \emptyset$, then Γ is contained in $\mathcal{L}$.*

Proof. Again, we only prove (a). Let Γ be a regular component containing a module M from $\mathcal{R}$. If Γ has oriented cycles, then by [9], it is a stable tube and then clearly every module in Γ is a successor of M, and therefore belongs to $\mathcal{R}$ (see (2.4)).

Suppose from now on that Γ has no oriented cycles and let $N \in \Gamma$. We show that $N \in \mathcal{R}$. Suppose Γ has infinitely many τ-orbits. By (4.1) there exists an $m \in \mathbb{Z}$ such that $\tau^m N \in \mathcal{R}$. By (1.6), there exists a path from $\tau^m N$ to N, and hence N is also in $\mathcal{R}$. It remains to consider the case when Γ has only finitely many τ-orbits. If $N \notin \mathcal{R}$, then there exists a path

$$N = X_0 \xrightarrow{f_0} X_1 \to \ldots \xrightarrow{f_{t-1}} X_t = X,$$

where $\operatorname{id} X > 1$. It is well known that then $\operatorname{Hom}_A(\tau^- X, A) \neq 0$ (see (1.2)). Therefore there exists a path

$$N = X_0 \xrightarrow{f_0} X_1 \to \ldots \xrightarrow{f_{t-1}} X_t = X \xrightarrow{f_t} X_{t+1} \xrightarrow{f_{t+1}} \tau^- X \xrightarrow{f_{t+2}} P,$$

where P is an indecomposable projective module, and the morphisms f_t and f_{t+1} are irreducible.

Since Γ is regular, we deduce that $P \notin \Gamma$ and then at least one of the maps $f_0, f_1, \ldots, f_{t-1}, f_{t+2}$ is in $\operatorname{rad}^\infty(\operatorname{mod} A)$. Observe now that if $g : Y \to Y'$ is a map in $\operatorname{rad}^\infty(\operatorname{mod} A)$, then for each $r \geq 1$, there exist a chain of irreducible maps

$$Y = Y_0 \xrightarrow{g_1} Y_1 \to \ldots \xrightarrow{g_r} Y_r$$

and a morphism $h_r : Y_r \to Y'$ such that the composition $h_r g_r \ldots g_1$ is nonzero.

Suppose now that one of $f_0, \ldots, f_{t-1}$ is in $\operatorname{rad}^\infty(\operatorname{mod} A)$. By the above and (4.1), we infer that there exists a path from some module in Γ which belongs to $\mathcal{R}$ to X, and so $\operatorname{id} X = 1$, a contradiction. If none of $f_0, \ldots, f_{t-1}$ belongs to $\operatorname{rad}^\infty(\operatorname{mod} A)$, then $f_{t+2} \in \operatorname{rad}^\infty(\operatorname{mod} A)$. By similar arguments there exists a module $Z \in \Gamma$ such that τZ is in $\mathcal{R}$ and $\operatorname{Hom}_A(Z, P) \neq 0$, or equivalently, $\operatorname{id} \tau Z > 1$, a contradiction. ■

5. Nonregular components

5.1. Let A be a quasitilted algebra. We now concentrate on the study of nonregular components of Γ_A. In this section we prove Theorems (C) and (D) and establish some immediate consequences.

THEOREM (C). *Let A be a quasitilted algebra and Γ be a component of Γ_A with infinitely many τ-orbits or containing an oriented cycle.*

(a) *If Γ contains a projective module, then Γ is contained in $\mathcal{L} \setminus \mathcal{R}$.*
(b) *If Γ contains an injective module, then Γ is contained in $\mathcal{R} \setminus \mathcal{L}$.*

P r o o f. We only prove (a). Let Γ be a component of Γ_A containing a projective module. Suppose first that Γ has oriented cycles. Then, by (3.1), Γ is a ray tube. Suppose furthermore that there exists a module $M \in \Gamma \cap \mathcal{R}$. Since Γ is a ray tube, any module in Γ which belongs to a cycle is a successor of M, and hence belongs to $\mathcal{R}$ by (2.4). On the other hand, there exists a module X which is a nonprojective summand of the radical of some projective module which belongs to an oriented cycle. By (1.2), we infer that $\operatorname{id} \tau X > 1$, a contradiction to the fact that τX should be in $\mathcal{R}$ because it is a successor of M. Therefore, $\Gamma \cap \mathcal{R} = \emptyset$, and since $\operatorname{ind} A = \mathcal{R} \cup \mathcal{L}$, we conclude that $\Gamma \subset \mathcal{L} \setminus \mathcal{R}$.

Suppose now Γ has infinitely many τ-orbits but no oriented cycles, and that $\Gamma \cap \mathcal{R} \neq \emptyset$. Then there exists a connected component Γ' of ${}_r\Gamma$ with infinitely many τ-orbits. It now follows from (4.1) that there exists a module $M \in \Gamma' \cap \mathcal{R}$ such that the length of any walk from a nonstable module to the τ-orbit of M is at least $2n$, where n is the rank of $K_0(A)$. Let

$$M' = X_0 \text{ --- } X_1 \text{ --- } \ldots \text{ --- } X_t = P$$

be a walk in Γ' of minimal length from a module M' in the τ-orbit $\mathcal{O}(M)$ of M to a projective module P. Because of the minimality, all the modules $X_0, \ldots, X_{t-1}$ are left stable and then, by applying τ conveniently, there exist an $m \geq 0$ and a path of irreducible maps

$$\tau^m M = Y_0 \to Y_1 \to \ldots \to Y_t = P.$$

Since the modules $Y_0, \ldots, Y_{t-1}$ are left stable, we get a path of irreducible maps from $\tau^{m+1} M$ to τY_{t-1}. On the other hand, by (1.5), there exists a path from M to $\tau^{m+1} M$, and then τY_{t-1} is a successor of M, which implies that $\tau Y_{t-1} \in \mathcal{R}$ (by (2.4)). This, however, contradicts $\operatorname{id} \tau Y_{t-1} > 1$, because $\operatorname{Hom}_A(Y_{t-1}, A) \neq 0$. Therefore $\Gamma \cap \mathcal{R} = \emptyset$ and because $\operatorname{ind} A = \mathcal{R} \cup \mathcal{L}$, we have $\Gamma \subset \mathcal{L} \setminus \mathcal{R}$ as required. ■

5.2. For quasitilted algebras which are not tilted, the above result can be sharpened as follows.

THEOREM (D). *Let A be a quasitilted algebra which is not tilted, and Γ be a component of Γ_A.*

(a) *If Γ contains a projective module, then Γ is contained in $\mathcal{L} \setminus \mathcal{R}$.*
(b) *If Γ contains an injective module, then Γ is contained in $\mathcal{R} \setminus \mathcal{L}$.*

Proof. We only prove (a). Let Γ be a component containing a projective module. If Γ has oriented cycles or infinitely many τ-orbits, then the result follows from (5.1). Suppose then that Γ has no oriented cycles and only finitely many τ-orbits. Then there exists an indecomposable projective module P in Γ with no proper successors in Γ which are also projective modules. Since A is not tilted, $P \notin \mathcal{R}$ (by (2.5)). Therefore, there exists a path

$$P = X_0 \xrightarrow{f_0} X_1 \to \ldots \xrightarrow{f_{t-1}} X_t = X,$$

where $\operatorname{id} X > 1$, or equivalently, $\operatorname{Hom}_A(\tau^- X, A) \neq 0$ (by (1.2)). Hence, there exists a path

$$P = X_0 \xrightarrow{f_0} X_1 \to \ldots \xrightarrow{f_{t-1}} X_t = X \xrightarrow{f_t} X_{t+1} \xrightarrow{f_{t+1}} \tau^{-1} X \xrightarrow{f_{t+2}} P',$$

where P' is an indecomposable projective module, and f_t and f_{t+1} are irreducible maps. By our hypothesis on P, at least one of the maps $f_0, \ldots, f_{t-1}, f_{t+2}$ is in $\operatorname{rad}^\infty(\operatorname{mod} A)$. Suppose one of $f_0, \ldots, f_{t-1}$ is in $\operatorname{rad}^\infty(\operatorname{mod} A)$. Following the considerations in the proof of Theorem (B) we infer that there exists a path from some module in $\Gamma \cap \mathcal{R}$ to X, a contradiction because $\operatorname{id} X > 1$.

Thus, $f_{t+2} \in \operatorname{rad}^\infty(\operatorname{mod} A)$. Also, by similar considerations to those in the proof of Theorem (B), there exists a module $Z \in \Gamma$ such that $\tau Z \in \mathcal{R}$ and $\operatorname{Hom}_A(Z, P') \neq 0$, or equivalently, $\operatorname{id} \tau Z > 1$, a contradiction, and this finishes the proof. ∎

5.3. We have the following direct consequences of (5.2).

COROLLARY. *Let A be a quasitilted algebra, and Γ be a component of Γ_A. If Γ is not semiregular, then A is tilted and Γ is the* (*unique*) *connecting component of Γ_A.*

Note that the above corollary generalizes [11, (II.3.6)] which says that any representation-finite quasitilted algebra is tilted, because clearly the Auslander–Reiten quiver of any representation-finite algebra is not semiregular.

5.4. COROLLARY (E). *Let A be a quasitilted algebra which is not tilted. Then any component of Γ_A is semiregular.*

5.5. COROLLARY (F). *Let A be a quasitilted algebra which is not tilted. Then every component Γ of Γ_A having a module from $\mathcal{R} \cap \mathcal{L}$ is regular, and hence consists of modules from $\mathcal{R} \cap \mathcal{L}$.*

6. Some consequences

6.1. We say that a property holds for almost all modules if it holds for all but finitely many of them. In [1], I. Assem and the first named author have characterized the finite-dimensional algebras over algebraically closed fields which have the property that almost all of their indecomposable modules have injective (or projective) dimension at most one. These algebras are called *left* (respectively, *right*) *glueings of tilted algebras* (see [1] for details).

For an artin algebra A such that $\operatorname{id} X \leq 1$ for almost all $X \in \operatorname{ind} A$, it follows from [5], [6] and [23] that Γ_A contains a component Γ containing all the projective modules and such that: (i) almost all of its modules lie in the τ-orbits of projective modules; and (ii) there are at most finitely many modules in Γ belonging to oriented cycles.

6.2. We shall use the above fact to show the following result.

PROPOSITION. *Let A be a quasitilted algebra.*

(a) *The following are equivalent*:

- (i) $\operatorname{id} X \leq 1$ *for almost all* $X \in \operatorname{ind} A$.
- (ii) *A is tilted and Γ_A has a postprojective component with a complete slice.*
- (iii) $\mathcal{R}$ *is cofinite in* $\operatorname{ind} A$.

(b) *The following are equivalent*:

- (i) $\operatorname{pd} X \leq 1$ *for almost all* $X \in \operatorname{ind} A$.
- (ii) *A is tilted and Γ_A has a preinjective component with a complete slice.*
- (iii) $\mathcal{L}$ *is cofinite in* $\operatorname{ind} A$.

P r o o f. We only prove (a) because the proof of (b) is similar.

(i)⇒(ii). Suppose $\operatorname{id} X \leq 1$ for almost all $X \in \operatorname{ind} A$. By the above remarks, Γ_A has a component containing all the projective modules and such that almost all of its modules belong to the τ-orbits of projective modules and there are at most finitely many modules in Γ belonging to oriented cycles. Suppose Γ contains an injective module. Then Γ is a nonsemiregular component, and hence, by (5.3), A is tilted and Γ is a connecting component. Clearly, Γ is then postprojective.

If Γ contains no injective modules, then Γ is in fact a postprojective component (see [5, (6.7)] or [16, (2.1)]). Clearly, a postprojective component containing all projective modules and no injective modules is indeed connecting and (ii) follows.

(ii)⇒(iii). By (2.6), all modules which are successors of a complete slice belong to $\mathcal{R}$. Now, if Γ_A has a postprojective component Γ with a complete slice, then almost all modules in $\operatorname{ind} A$ are successors of a complete slice in Γ. This proves (iii).

(iii)$\Rightarrow$(i). Clear. ∎

6.3. Corollary. *Let A be a quasitilted algebra which is not tilted. Then there are infinitely many indecomposable modules X with* $\operatorname{pd} X = 2$ *and infinitely many indecomposable modules Y with* $\operatorname{pd} Y = 2$.

6.4. It has been shown independently in [1] and [23] that a representation-infinite algebra is concealed if and only if $\operatorname{pd} X \leq 1$ and $\operatorname{id} X \leq 1$ for almost all $X \in \operatorname{ind} A$. The next result is also a direct consequence of (6.2).

Corollary. *The following are equivalent for a representation-infinite artin algebra A*:

(a) $\operatorname{pd} X \leq 1$ *and* $\operatorname{id} X \leq 1$ *for almost all* $X \in \operatorname{ind} A$.

(b) *A is concealed.*

(c) *A is quasitilted and $\mathcal{R} \cap \mathcal{L}$ is cofinite in* $\operatorname{ind} A$.

6.5. The next two results are direct consequences of the previous sections.

Proposition. *Let A be a quasitilted algebra*, *and Γ be a component of Γ_A.*

(a) *If Γ contains a projective module*, *then $\Gamma \subset \mathcal{L} \setminus \mathcal{R}$ if and only if Γ has no complete slice.*

(b) *If Γ contains an injective module*, *then $\Gamma \subset \mathcal{R} \setminus \mathcal{L}$ if and only if Γ has no complete slice.*

Proof. We only prove (a). Let Γ be a component containing a projective module. If $\Gamma \subset \mathcal{L} \setminus \mathcal{R}$ then, clearly, Γ contains no complete slice (see (2.6)).

Suppose now that Γ has no complete slice. If A is not tilted, then by (5.2), $\Gamma \subset \mathcal{L} \setminus \mathcal{R}$. Moreover, if Γ contains oriented cycles or has infinitely many τ-orbits, then by (5.1), $\Gamma \subset \mathcal{L} \setminus \mathcal{R}$. It remains to show the result when A is tilted, and Γ is a component without oriented cycles and with only finitely many τ-orbits. Since by hypothesis, Γ is not a connecting component, we infer that Γ is postprojective and it does not contain injective modules. Clearly, $\Gamma \subset \mathcal{L}$.

Suppose now that $\Gamma \cap \mathcal{R}$ has a module X. Observe that Γ does not contain all the projective modules and in fact, since A is connected, there exist indecomposable projective modules $P \in \Gamma$ and $P' \notin \Gamma$ with $\operatorname{Hom}_A(P, P') \neq 0$. Since $\operatorname{Hom}_A(P, P') = \operatorname{rad}^{\infty}(P, P')$, we infer that for each $t \geq 1$, there exist a path of irreducible maps

$$P = Y_0 \to Y_1 \to \ldots \to Y_t$$

and a nonzero map $f_t \in \mathrm{Hom}(Y_t, P')$. Note that all the successors of X are in $\mathcal{R}$ and hence there are only finitely many modules in Γ which are not in $\mathcal{R}$. Therefore, there exists t such that $\tau Y_t \in \mathcal{R}$ and $\mathrm{Hom}_A(Y_t, P') \neq 0$, or equivalently $\mathrm{id}\, \tau Y_t > 1$ (by (1.2)), a contradiction. Therefore, $\Gamma \cap \mathcal{R} = \emptyset$ as required. ∎

6.6. PROPOSITION. *The following are equivalent for a quasitilted algebra A*:

(a) *Each nonregular component is contained either in $\mathcal{L} \setminus \mathcal{R}$ or in $\mathcal{R} \setminus \mathcal{L}$.*

(b) *A is either not tilted*, *or a tilted algebra of the form $A = \mathrm{End}_H(T)$*, *where T is a regular tilting module over a hereditary algebra H.*

Proof. (a)⇒(b). Suppose A is tilted. Then Γ_A contains a connecting component Γ. If Γ is nonregular, then by (a), it is contained either in $\mathcal{L} \setminus \mathcal{R}$ or in $\mathcal{R} \setminus \mathcal{L}$, a contradiction to the fact that Γ contains a complete slice lying in $\mathcal{R} \cap \mathcal{L}$ (2.6). Then Γ is regular and, by (2.1), $A = \mathrm{End}_H(T)$, where T is a regular tilting module over a hereditary algebra H.

(b)⇒(a). Let Γ be a nonregular component of Γ_A. If A is not tilted, then by (5.2), Γ is contained either in $\mathcal{L} \setminus \mathcal{R}$ or in $\mathcal{R} \setminus \mathcal{L}$. If now $A = \mathrm{End}_H(T)$, where T is a regular tilting module over a hereditary algebra H, then Γ is not the connecting component of Γ_A (by (2.1)), and hence it does not contain a complete slice. By [20], Γ is semiregular and by (5.1), Γ is contained in $\mathcal{L} \setminus \mathcal{R}$ in case it has projective modules, or in $\mathcal{R} \setminus \mathcal{L}$ in case it has injective modules. This proves the result. ∎

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